Kochen-Specker theorem in almost all the two-dimensional states

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We present the Kochen-Specker (KS) theorem in almost all the two-dimensional states. We consider whether we can simulate the double-slit experiment in a state by a realistic theory of the KS type. It turns out that we cannot simulate the double-slit experiment in almost all the states by a realistic theory of the KS type. An exception is an eigenvector of a measured Pauli observable.

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I. INTRODUCTION

Quantum mechanics (cf. [1–5]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data has fit to the quantum predictions for long time.

Kochen and Specker present the no-hidden-variables theorem (the KS theorem) [6]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [7, 8] the so-called GHZ theorem for four-partite GHZ state. And, the KS theorem becomes very simple form (see also Refs. [9–13]).

It is begun to research the validity of the KS theorem by using inequalities (see Refs. [14–17]). To find such inequalities to test the validity of the KS theorem is particularly useful for experimental investigation [18]. One of authors derives an inequality [17] as tests for the validity of the KS theorem. The quantum predictions violate the inequality when the system is in an uncorrelated state. An uncorrelated state is defined in Ref. [19]. The quantum predictions by n-partite uncorrelated state violate the inequality by an amount that grows exponentially with n.

The double-slit experiment is an illustration of wave-particle duality. In it, a beam of particles (such as photons) travels through a barrier with two slits removed. If one puts a detector screen on the other side, the pattern of detected particles shows interference fringes characteristic of waves; however, the detector screen responds to particles. The system exhibits behaviour of both waves (interference patterns) and particles (dots on the screen).

If we modify this experiment so that one slit is closed, no interference pattern is observed. Thus, the state of both slits affects the final results. We can also arrange to have a minimally invasive detector at one of the slits to detect which slit the particle went through. When we do that, the interference pattern disappears [20]. An analysis of a two-atom double-slit experiment based on environment-induced measurements is reported [21].

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes another slit, then the value of the result of measurement is -1. This is easy detector model for Pauli observable.

Here we consider whether we can simulate a state by a realistic theory of the KS type. So, we investigate the relation between easy detector model to a Pauli observable and the KS theorem.

In this paper, we show we cannot simulate the double-slit experiment in almost all the states. We assume an implementation of the double-slit experiment. We assume that a source of spin-carrying particles emits them in a state. We consider a single expected value of Pauli observable σ_z in the double-slit experiment. A wave function analysis says some assumption concerning the quantum expected value. However, the realistic theory of the KS type cannot coexist with the assuption concerning the expected value when the state is not an eigenvector of σ_z. Hence, we cannot simulate almost all the states by the realistic theory of the KS type. when the state is not an eigenvector of σ_z.

II. THE DOUBLE-SLIT EXPERIMENT AND THE KS THEOREM

In this section, by using the double-slit experiment, we present the KS theorem with almost all the two-dimensional states. Especially, we systematically describe our assertion based on more mathematical analysis using raw data in a thoughtful experiment.

We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are either +1 or −1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is +1. If a particle passes another slit, then the value of the result of measurement is −1. This is an easy detector model of a single Pauli observable.

A. A wave function analysis

Let σ_z be a single Pauli observable. Here,

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

We assume that a source of a spin-carrying particle emits them in a state ρ. ρ is not an eigenvector of σ_z. Thus,

$$\rho \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

and

$$\rho \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

We consider a quantum expected value $\mathrm{Tr}[\rho\sigma_z]$. If we consider only a wave function analysis, the possible values of the square of the quantum expected value are

$$(\mathrm{Tr}[\rho\sigma_z])^2 = Z, \ (0 \leq Z < 1). \tag{4}$$

We define $\|E_{\mathrm{QM}}\|^2$ as

$$\|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_z])^2. \tag{5}$$

We have

$$\|E_{\mathrm{QM}}\|^2 \leq Z. \tag{6}$$

Thus,

$$\|E_{\mathrm{QM}}\|^2_{\max} = Z \tag{7}$$

where $\|E_{\mathrm{QM}}\|^2_{\max}$ is the maximal possible values of the product. Hence we have

$$\|E_{\mathrm{QM}}\|^2_{\max} = Z. \tag{8}$$

B. The realistic theory of the KS type

A mean value E satisfies the realistic theory of the KS type if it can be written as

$$E = \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m}, \tag{9}$$

where l denotes a notation and r is the result of the measurement of the Pauli observable σ_z. We assume the values of r are either +1 or −1 (in $\hbar/2$ unit). Assume the quantum mean values with the system in a state admits the realistic theory of the KS type. One has the following proposition concerning the realistic theory of the KS type

$$\mathrm{Tr}[\rho\sigma_z](m) = \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m}. \tag{10}$$

We can assume the following by Strong Law of Large Numbers [22],

$$\mathrm{Tr}[\rho\sigma_z](+\infty) = \mathrm{Tr}[\rho\sigma_z]. \tag{11}$$

We define $\|E_{\mathrm{QM}}\|^2(m)$ as

$$\|E_{\mathrm{QM}}\|^2(m) = (\mathrm{Tr}[\rho\sigma_z](m))^2. \tag{12}$$

We can assume the following by Strong Law of Large Numbers,

$$\|E_{\mathrm{QM}}\|^2(+\infty) = \|E_{\mathrm{QM}}\|^2 = (\mathrm{Tr}[\rho\sigma_z])^2. \tag{13}$$

In what follows, we show that we cannot accept the relation (10) concerning the realistic theory of the KS type. Assume the proposition (10) is true. By changing the notation l into l', we have same quantum mean value as follows

$$\mathrm{Tr}[\rho\sigma_z](m) = \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_z)}{m}. \tag{14}$$

We introduce an assumption that Sum rule and Product rule commute with each other [23]. We do not pursue the details of the assumption. To pursue the details is an interesting point. It is suitable to the next step of researches. We have the following

$$\begin{aligned} &\|E_{\mathrm{QM}}\|^2(m) \\ &= \frac{\sum_{l=1}^{m} r_l(\sigma_z)}{m} \times \frac{\sum_{l'=1}^{m} r_{l'}(\sigma_z)}{m} \\ &\leq \frac{\sum_{l=1}^{m}}{m} \cdot \frac{\sum_{l'=1}^{m}}{m} |r_l(\sigma_z) r_{l'}(\sigma_z)| \\ &= \frac{\sum_{l=1}^{m}}{m} \times \frac{\sum_{l'=1}^{m}}{m} = 1. \end{aligned} \tag{15}$$

Clearly, the above inequality can have the upper limit since the following case is possible:

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_z) = 1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_z) = 1\}\|, \tag{16}$$

and

$$\|\{l | l \in \mathbf{N} \wedge r_l(\sigma_z) = -1\}\| = \|\{l' | l' \in \mathbf{N} \wedge r_{l'}(\sigma_z) = -1\}\|. \tag{17}$$

Thus we derive a proposition concerning the quantum mean value under the assumption that the realistic theory of the KS type is true (in a spin-1/2 system), that is

$$\|E_{\mathrm{QM}}\|^2(m) \leq 1. \tag{18}$$

From Strong Law of Large Numbers, we have

$$\|E_{\mathrm{QM}}\|^2 \leq 1. \tag{19}$$

Hence we derive the following proposition concerning the realistic theory of the KS type

$$\|E_{\mathrm{QM}}\|^2_{\max} = 1. \tag{20}$$

We cannot accept the two relations (8) (concerning the wave function analysis) and (20) (concerning the realistic theory of the KS type), simultaneously. Hence we are in the KS contradiction. The realistic theory of the KS type does not meet the wave function analysis and cannot simulate almost all the two-dimensional states. An exception is an eigenstate of the measured spin observable.

III. CONCLUSIONS

In conclusion, we have considered whether we can simulate a state by a realistic theory of the KS type. We have assumed an implementation of double-slit experiment. There has been a detector just after each slit. Thus interference figure has not appeared, and we do not have considered such a pattern. We have assumed that a source of spin-carrying particles emits them in the state. We have considered a single expected value of a Pauli observable σ_z in the double-slit experiment. A wave function analysis has said some assumption concerning the quantum expected value. However, the realistic theory of the KS type cannot have coexisted with the assumption concerning the expected value when the state is not an eigenvector of σ_z. Hence, we cannot have simulated the double-slit experiment by the realistic theory of the KS type when the state is not an eigenvector of σ_z.

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[22] In probability theory, the law of large numbers is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed. The strong law of large numbers states that the sample average converges almost surely to the expected value.

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