A higher language for mathematics.

Johan Noldus*

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Abstract

We extend the language of mathematics by generalizing the logical implication and negation. As such, our framework serves as an extension of intuitionistic logic.

1 Introduction

It is true that one's knowledge depends upon the language one speaks and the language of mathematics is not a universally given entity. Rather it is a primitive way of expressing our most basic observations and the art of science consists in finding out how far one can get with these primates. Changing the axioms, or irreducible rules, leads to another spectrum of knowledge and therefore broadens our horizons. The aim of this paper is to deal with such extensions, motivate them and in the line of the best tradition, start with an example. In this way, we construct a framework which brings spoken language closer to mathematics meaning we avoid the usual paradoxes of logic by making a difference between sentences from which one can establish the truth and those which do not allow for such verification. In this way, the contradictory sentences "a spirit exists is true" and "a spirit does not exist is true" can live piecefully together; in other words, the logical mapping "is true" does have a nontrivial domain, this statement holds irrespective of wether one considers classical or intuitionistic logic. Indeed, not only does mathematics require an extension, logic does too and we shall start this paper by a more humble effort, which is to give an extension of cohomology theory.

2 An extension of Cohomology Theory and sequential reasoning.

Traditional cohomology theory started out by an observation of Cartan in the context of simplicial complexes or manifolds with a boundary regarding the "boundary operation" which maps a $n+1$-dimensional simplicial complex to an n-dimensional one or an $n+1$ dimensional manifold with boundary to its n dimensional boundary. This operation turned out to be nilpotent of order two: that is, taking the boundary of the boundary results in nothing, that

*Relativity Group,departement of mathematics, university of Ghent, email: johan.noldus@gmail.com

is, the empty complex. Of course, this result depends heavily on definition of simplicial complexes or, as a matter of fact, manifolds with a boundary. In the latter case, there is no good reason why one should restrict the definition in the way it is framed and it is quite easy to think of non-closed sets whose interior is an ordinary manifold such that the boundary of the boundary does not vanish. For such n dimensional spaces, it is nevertheless still true that the $n+1$'th boundary vanishes and what follows has this limitation build in (but it is easy to generalize). In general, consider a sequence of objects A_n with a distinguished "initial" object 0 and mappings $\delta_n : A_n \to A_{n-1}$ which terminates at $A_{-1} = 0$; then, one can define for $0 \leq k \leq n+1$ the k'th terminator set Z_n^k of A_n which is defined as

$$Z_n^k = (\delta_n)^{-1} \circ (\delta_{n-1})^{-1} \circ \ldots \circ (\delta_{n-k+1})^{-1}(0)$$

where $0 \in A_{n-k}$. Then, it is obvious that $Z_n^k \subseteq Z_n^{k+1}$ where the operation $\subseteq$ may have a more general definition as is the case in set theory. Also $\delta_{n+1} : Z_{n+1}^{k+1} \to Z_n^k$ so that we can speak about k-cohomology which is defined as

$$H_n^k = \frac{Z_n^k}{\delta_{n+1}\left(Z_{n+1}^{k+1}\right)}$$

for $k \geq 0$ and $n \geq k-1$ where the quotient is defined independently. For example, in the case of modules, it is the standard module quotient. In traditional Cohomology theory, the sequence of A_n and the associated mappings δ_n is called a long exact sequence and one can speak about a homomorphism ζ between two long exact sequences when there exist mappings $\zeta_n : A_n \to A'_n$ such that $\zeta_n \circ \delta_{n+1} = \delta'_{n+1} \circ \zeta_{n+1}$ preserving the initial object, in either when one can draw a commuting diagram. It is easy to derive that

$$\zeta_n\left(Z_n^k\right) \subseteq Z_n'^k$$

and that therefore

$$\zeta_n\left(H_n^k\right) \subseteq H_n'^k.$$

These constitute the usual generalizations of ordinary Cohomology theory where $k = 1$ and, as is usual the case,

$$\delta_{n+1} : H_{n+1}^{k+1} \to 0 \in H_n^k.$$

The attentive reader may notice that this is not all one can do and one can introduce a further parameter $r \in \mathbb{N}_0$, defining the objects:

$$H_n^{k,r} = \frac{Z_n^k}{\delta_{n+1} \circ \ldots \circ \delta_{n+r}\left(Z_{n+r}^{k+r}\right)}$$

so that our previous objects corresponded to $r = 1$; one would expect mappings to exist between $H_n^{k,r}$ and $H_n^{k,s}$ with $r \geq s$ but we shall not go into that matter here.

3 Higher reasoning and incomplete knowledge.

In spoken language, one has reasons of reasons and the latter are sometimes infinite and circular. Let us give an example: "an apple falls down on the earth"

WHY? "because of the gravitational field" WHY? "Because the laws of physics are background independent and involve a (pre) geometry" WHY? "because you need to be able to say what a straight line is and because nothing is static in this world" WHY? "If you could not, then you cannot tell the difference between a free object and one which is acted upon and because the notion of freedom isn't absolute" WHY? ... Obviously, this sequence is either never ending or ends with a dogmatic statement we suppose to be true. In mathematics, we have allowed for such argumentation by means of the logical operator "implies" of the words "if A then B" and every situation of this kind can be written as a sequence of such sentences. Also, we have in spoken language the words "by means of" for example: a chemical substance A changes into B by means of the catalyst C. Here, a mathematician would try to say that there is an equivalence with the statement "if A and C react then B and some leftover D remain". However, sometimes it is just not known what the leftovers are and neither is the mechanism by which C serves as a catalyst for A to change. The attitude of the mathematician is that this imperfect knowledge is just due to a limitation of our knowledge and is not intrinsic by any means and he would still write

$$A + C \to B + D.$$

But what if *nature* would be such that no precise statements can be made, not even probabilistically? Then we could not write it down in this way and we would have to invent a new logical operator in order to accomodate for the meaning of this sentence; the latter is noted down as

$$A \overset{C}{\to} B$$

where we ignore the leftovers. This operation is intended to mean "A evolves into B by means of the catalyst C" or "A implies B if C amongst others holds" where we just don't know the others or perhaps don't know if others are needed in the first place. This is the principle of incomplete knowledge which I think one needs to import into mathematics because as far as I know nature operates in this way by means of our free will. The reader automatically notices that it is possible for

$$A \overset{C}{\to} B \,\text{and}\, A \overset{C}{\to} \neg B$$

to hold where $\neg$ can be interpreted in the classical or intuitionistic sense. A mistake which is commonly made in science is that $A \to B$ is interpreted to mean that A is a cause for B, or that A is a reason for B to hold. Such interpretations however are wrong since it is not (experimentally) possible to verify a reason for something to occur; the only thing we can measure are coincidences. For example, it is equally possible for angels to move the planets the way they do than it is for gravity to do the job; the laws of gravity merely establish the way in which the motion of the earth around the sun occurs but it provides no reason or cause for it. One expects the following rule to hold

$$\left(A \overset{C,D}{\to} B\right) \to \left(A \overset{C}{\to} B\right)$$

in either further specification narrows down the implication. One does not have that

$$(A \to B) \to \left(A \overset{C}{\to} B\right)$$

is true since for $C = \neg B$, the right hand side is always false. We call C *compatible* or of no influence if this sentence is true. In case also the reverse implication holds

$$\left(A \overset{C}{\to} B\right) \to (A \to B)$$

we call C *redundant* or unnecessary. For example, A is H_2O and B liquid water and C is 50 degrees centigrade; since the molecule H_2O is always liquid water at this temperature, this information is redundant. In case C is an influence, we call the latter maximal or complete if

$$\left(A \overset{C}{\to} B\right) \to (A \wedge C \to B)$$

is true while the implication

$$(A \wedge C \to B) \to \left(A \overset{C}{\to} B\right)$$

is true by definition. While the sentence

$$(A \wedge C \to B) \to (A \wedge \neg C \to B)$$

is not always true, it can be that

$$\left(A \overset{C}{\to} B\right) \to \left(A \overset{\neg C}{\to} B\right)$$

is true and the reader should give an example of this (for example when C and $\neg C$ are redundant). In the next section, we develop an arithmetic application of these logical rules.

4 Generalized arithmetic.

As an example, we consider a real measurement of the length of a straight line segment; the latter is never perfect and in principle it is impossible to know the probability distribution of the error margin, if such distribution would exist in the first place. So a number 1.2 could mean exactly that or it might stand for any real number with further digits following the 2. Hence, $1.2 + 1$ could be equal to 2.2 or 2.3 or ... or 3.2 and the probabilities for this to happen are unknown. It could of course be also 2.33 and so on, so we have sentences

$$(x = 1.2 \wedge y = 1) \overset{+}{\to} x + y = 2.2$$

and

$$(x = 1.2 \wedge y = 1) \overset{+}{\to} x + y = \neg 2.2$$

where

$$\neg 2.2 = \text{ any two digit number of the form } 2.a \text{ with } 9 > a > 2.$$

It is a piece of cake to verify that both are correct and therefore embody the very essence of our logic. One could develop further, more radical, examples but we leave this for future work; the work in here merely served as an introduction to the idea.