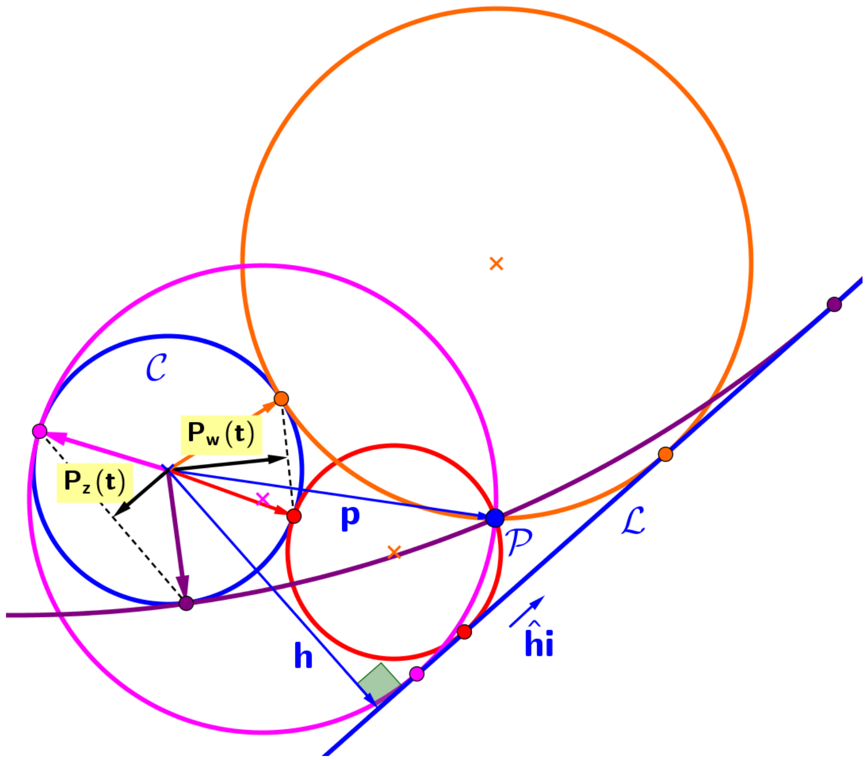
Solution of the Special Case "CLP" of the Problem of Apollonius

via Vector Rotations using Geometric Algebra



Geometric-Algebra Formulas for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
$\boldsymbol{b} \wedge \boldsymbol{a} = -\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ba} = \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} - \boldsymbol{a} \wedge \boldsymbol{b}$

$\boldsymbol{ab} + \boldsymbol{ba} = 2\boldsymbol{a} \cdot \boldsymbol{b}$
$\boldsymbol{ab} - \boldsymbol{ba} = 2\boldsymbol{a} \wedge \boldsymbol{b}$
$\boldsymbol{ab} = 2\boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{ba}$
$\boldsymbol{ab} = 2\boldsymbol{a} \wedge \boldsymbol{b} - \boldsymbol{ba}$

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)

<u>The inner product</u>
The inner product of a j-vector A and a k-vector B is
$A \cdot B = \langle AB \rangle_{k-j}$. Note that if $j>k$, then the inner product doesn't exist.
However, in such a case $B \cdot A = \langle BA \rangle_{j-k}$ does exist.

<u>The outer product</u>
The outer product of a j-vector A and a k-vector B is
$A \wedge B = \langle AB \rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, i.

For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$\boldsymbol{ia} = -\boldsymbol{ai}$
$\boldsymbol{a} \wedge \boldsymbol{b} = [(\boldsymbol{ai}) \cdot \boldsymbol{b}]\, \boldsymbol{i} = -\,[\boldsymbol{a} \cdot (\boldsymbol{bi})]\, \boldsymbol{i} = \boldsymbol{-b \wedge a}$

Equality of Multivectors

For any two multivectors $\mathcal{M}$ and $\mathcal{N}$,
$\mathcal{M} = \mathcal{N}$ if and only if for all k, $\langle \mathcal{M} \rangle_k = \langle \mathcal{N} \rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written in the form of "Fourier expansions" with respect to a third vector, $\boldsymbol{v}$:

$\boldsymbol{a} = (\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$ and $\boldsymbol{b} = (\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}$.

Using these expansions,

$$\boldsymbol{ab} = \{(\boldsymbol{a} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}\, \{(\boldsymbol{b} \cdot \hat{\boldsymbol{v}})\, \hat{\boldsymbol{v}} + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\, \hat{\boldsymbol{v}}\boldsymbol{i}\}$$

Equating the scalar parts of both sides of that equation,

$\boldsymbol{a} \cdot \boldsymbol{b} = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}] [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}] + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]$, and
$\boldsymbol{a} \wedge \boldsymbol{b} = \{[\boldsymbol{a} \cdot \hat{\boldsymbol{v}}] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] - [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})] [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]\} \boldsymbol{i}$.

Also, $a^2 = [\boldsymbol{a} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{a} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$, and $b^2 = [\boldsymbol{b} \cdot \hat{\boldsymbol{v}}]^2 + [\boldsymbol{b} \cdot (\hat{\boldsymbol{v}}\boldsymbol{i})]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators

For any vector $\boldsymbol{a}$, the product $\hat{\boldsymbol{v}}\boldsymbol{a}\hat{\boldsymbol{v}}$ is the reflection of $\boldsymbol{a}$ with respect to the direction $\hat{\boldsymbol{v}}$.
For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\hat{\boldsymbol{v}}\boldsymbol{a}\boldsymbol{b}\hat{\boldsymbol{v}} = \boldsymbol{b}\boldsymbol{a}$, and $\boldsymbol{v}\boldsymbol{a}\boldsymbol{b}\boldsymbol{v} = v^2\boldsymbol{b}\boldsymbol{a}$.
Therefore, $\hat{\boldsymbol{v}}e^{\theta\boldsymbol{i}}\hat{\boldsymbol{v}} = e^{-\theta\boldsymbol{i}}$, and $\boldsymbol{v}e^{\theta\boldsymbol{i}}\boldsymbol{v} = v^2e^{-\theta\boldsymbol{i}}$.

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Contents

1 Introduction

{*Author's note, 28 March 2016:*
This document has been prepared for two very different audiences: for my fellow students of GA, and for experts who are preparing materials for us, and need to know which GA concepts we understand and apply readily, and which ones we do not.

I confess that I had a terrible time finding the solution presented here! However, I'm happy to have had the opportunity to apply GA to this famous problem. Alternative solutions, obtained by using GA's capabilities for handling reflections, are in preparation.

Readers are encouraged to study the following documents, GeoGebra worksheets, and videos before beginning:

"Rotations of Vectors via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric "Construction" Problems"
https://drive.google.com/file/d/0B2C4TqxB32RRdE5KejhQTzMtN3M/view?usp=sharing

"Answering Two Common Objections to Geometric Algebra"
As GeoGebra worksheet
As YouTube video.

"Geometric Algebra: Find unknown vector from two dot products"
As GeoGebra worksheet
As YouTube video

For an more-complete treatment of rotations in plane geometry, be sure to read Hestenes D. 1999, pp. 78-92. His section on circles (pp. 87-89) is especially relevant to the present document. Macdonald A. 2010 is invaluable in many respects, and González Calvet R. 2001, Treatise of Plane Geometry through Geometric Algebra is a must-read.

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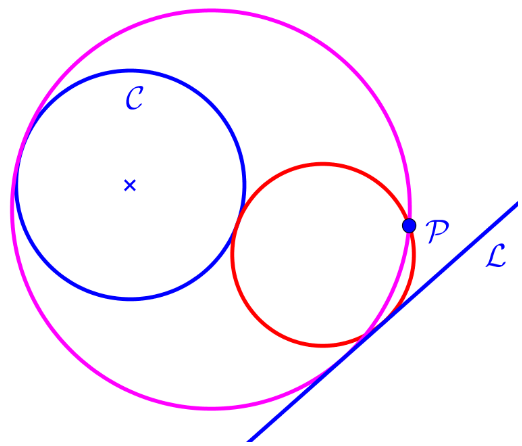
2 The Problem of Apollonius, and Its CLP Special Case

The famous "Problem of Apollonius", in plane geometry, is to construct all of circles that are tangent, simultaneously, to three given circles. In one variant of that problem, one of the circles has infinite radius (i.e., it's a line). The Wikipedia article that's current as of this writing has an extensive description of the problem's history, and of methods that have been used to solve it. As described in that article, one of the methods reduces the "two circles and a line" variant to the so-called "Circle-Line-Point" (CLP) special case:

Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.



2.1 Observations, and Potentially Useful Elements of the Problem

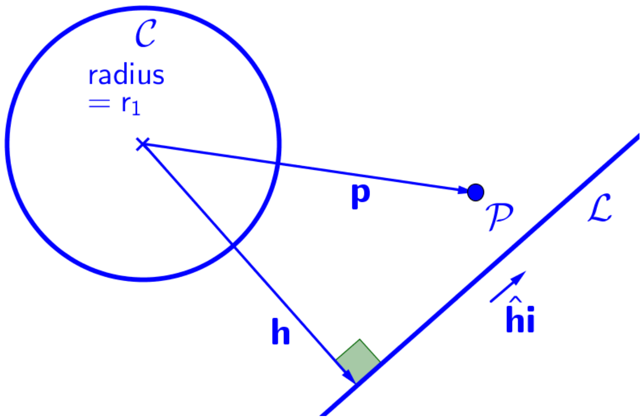
From the figure presented in the statement of the problem, we can see that there are two types of solutions. That is, two types of circles that satisfy the stated conditions:

- Circles that enclose $\mathcal{C}$;
- Circles that do not enclose $\mathcal{C}$.

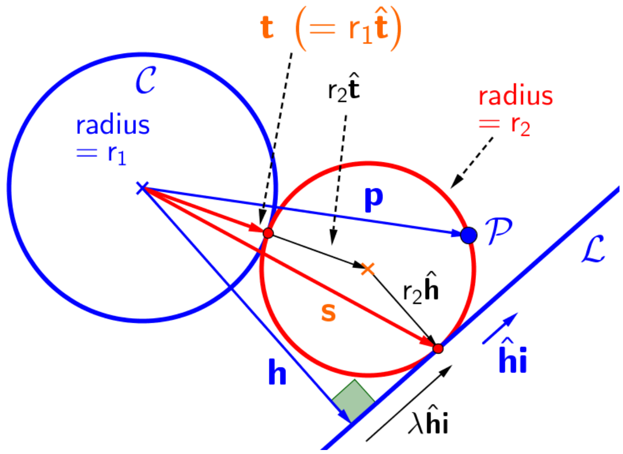
We'll begin by discussing circles that do not enclose $\mathcal{C}$. Most of our observations about that type will also apply, with little modification, to circles that do enclose $\mathcal{C}$.

Based upon our experience in solving other "construction problems involving tangency, a reasonable choice of elements for capturing the geometric content of the problem is as shown below:

- Use the center point of the given circle as the origin;



- Capture the perpendicular distance from $\boldsymbol{c_1}$'s center to the given line in the vector $\boldsymbol{h}$;
- Express the direction of the given line as $\pm\hat{\boldsymbol{hi}}$.
- Label the solution circle's radius and its points of tangency with $\mathcal{C}$ and $\mathcal{L}$ as shown below:



In deriving our solution, we'll use the same symbol —for example, $\boldsymbol{t}$ —to denote both a point and the vector to that point from the origin. We'll rely upon context to tell the reader whether the symbol is being used to refer to the point, or to the vector.

Now, we'll express key features of the problem in terms of the elements that we've chosen. First, we know that we can write the vector $\boldsymbol{s}$ as $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{hi}}$, where λ is some scalar. We also know that the points of tangency $\boldsymbol{t}$ and $\boldsymbol{s}$ are equidistant (by r_2) from the center point of the solution circle. Combining those observations, we can equate two expressions for the vector $\boldsymbol{s}$:

$$\boldsymbol{s} = (r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}} = \boldsymbol{h} + \lambda\hat{\boldsymbol{hi}}. \tag{1}$$

Examining that equation, we observe that we can obtain an expression for r_2 in terms of known quantities by "dotting" both sides with $\hat{\boldsymbol{h}}$:

$$\begin{aligned}
\left[(r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}}\right] \cdot \hat{\boldsymbol{h}} &= \left[\boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right] \cdot \hat{\boldsymbol{h}} \\
(r_1 + r_2)\,\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}} + r_2\hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{h}} &= \boldsymbol{h} \cdot \hat{\boldsymbol{h}} + \lambda\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right) \cdot \hat{\boldsymbol{h}} \\
(r_1 + r_2)\,\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}} + r_2 &= |\boldsymbol{h}| + 0; \\
\therefore r_2 &= \frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}{1 + \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}. \qquad (2)
\end{aligned}$$

The denominator of the expression on the right-hand side might catch our attention now because one of our two expressions for the vector $\boldsymbol{s}$, namely

$$\boldsymbol{s} = (r_1 + r_2)\,\hat{\boldsymbol{t}} + r_2\hat{\boldsymbol{h}}$$

can be rewritten as

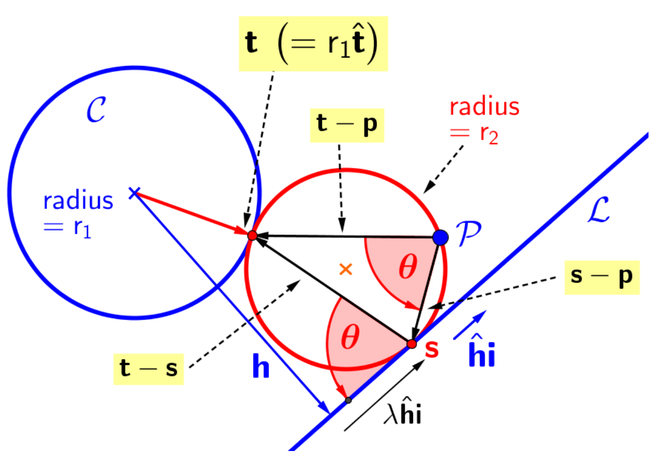
$$\boldsymbol{s} = r_1\hat{\boldsymbol{t}} + r_2\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right).$$

That fact becomes useful (at least potentially) when we recognize that $\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right)^2 = 2\left(1 + \hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}\right)$. Therefore, if we wish, we can rewrite Eq. (10) as

$$r_2 = 2\left[\frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{h}}}{\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}\right)^2}\right].$$

Those results indicate that we should be alert to opportunities to simplify expressions via appropriate substitutions invoving $\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}$ and $1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$

As a final observation, we note that when a circle is tangent to other objects, there will be many angles that are equal to each other. For example, the angles whose measures are given as θ in the following diagram:



We've seen in Smith J A 2016 that GA expressions for rotations involving angles like the two θ's often capture geometric content in convenient ways.

2.2 Identifying the Solution Circles that Don't Enclose $\mathcal{C}$

Many of the ideas that we'll employ here will also be used when we treat solution circles that do enclose $\mathcal{C}$.

2.2.1 Formulating a Strategy

Now, let's combine our observations about the problem in a way that might lead us to a solution. Our previous experiences in solving problems via vector rotations suggest that we should equate two expressions for the rotation $e^{\theta \boldsymbol{i}}$:

$$\begin{aligned} \left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right] &= \left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[-\hat{\boldsymbol{h}}\boldsymbol{i}\right] \\ &= \left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]. \end{aligned} \tag{3}$$

We've seen elsewhere that we will probably want to transform that equation into one in which some product of vectors involving our unknowns $\boldsymbol{t}$ and $\boldsymbol{s}$ is equal either to a pure scalar, or a pure bivector. By doing so, we may find some way of identifying either $\boldsymbol{t}$ or $\boldsymbol{s}$.

We'll keep in mind that although Eq. (3) has two unknowns (the vectors $\boldsymbol{t}$ and $\boldsymbol{s}$), our expression for r_2 (Eq. (10)) enables us to write the vector $\boldsymbol{s}$ in terms of the vector $\hat{\boldsymbol{t}}$.

Therefore, our strategy is to

- Equate two expressions, in terms of the unknown vectors $\boldsymbol{t}$ and $\boldsymbol{s}$, for the rotation $e^{\theta \boldsymbol{i}}$;

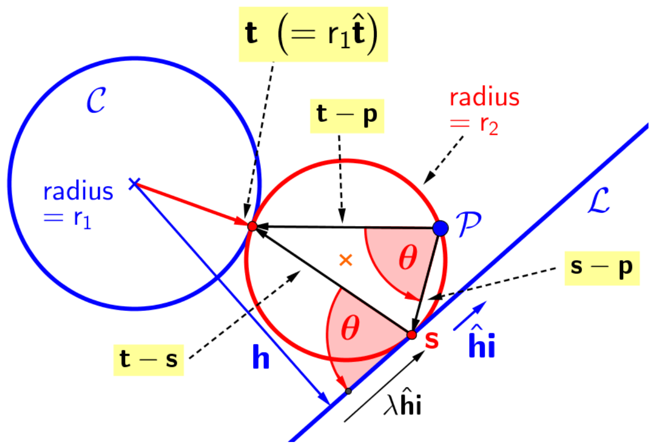
- Transform that equation into one in which on side is either a pure scalar or a pure bivector;

- Watch for opportunities to simplify equations by substituting for r_2; and

- Solve for our unknowns.

2.2.2 Transforming and Solving the Equations that Resulted from Our Observations and Strategizing

For convenience, we'll present our earlier figure again:



By examining that figure, we identified and equated two expressions for the rotation $e^{\theta \boldsymbol{i}}$, thereby obtaining Eq. (3):

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

We noted that we might wish at some point to make the substitution

$$\begin{aligned}\boldsymbol{s}&=[r_1+r_2]\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}\\&=\left[r_1+\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}\right]\hat{\boldsymbol{t}}+\left[\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}\right]\hat{\boldsymbol{h}}.\end{aligned}\tag{4}$$

We also noted that we'll want to transform Eq. (3) into one in which one side is either a pure scalar or a pure bivector. We should probably do that transformation before making the substitution for $\boldsymbol{s}$. One way to effect the transformation is by left-multiplying both sides of Eq. (3) by $\boldsymbol{s}-\boldsymbol{t}$, then by $\hat{\boldsymbol{h}}$, and then rearranging the result to obtain

$$\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]=|\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i}\tag{5}$$

This is the equation that we sought to obtain, so that we could now write

$$\langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\rangle_0=0.\tag{6}$$

Next, we need to expand the products on the left-hand side, but we'll want to examine the benefits of making a substitution for $\boldsymbol{s}$ first. We still won't, as yet, write $\boldsymbol{s}$ in terms of $\hat{\boldsymbol{t}}$. In hopes of keeping our equations simple enough for us to identify useful simplifications easily at this early stage, we'll make the substitution

$$\boldsymbol{s}=(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}},$$

rather than making the additional substitution (Eq. (10)) for r_2. Now, we can see that $\boldsymbol{s}-\boldsymbol{t}=r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$. Using this result, and $\boldsymbol{t}=r_1\hat{\boldsymbol{t}}$, Eq. (6) becomes

$$\langle\hat{\boldsymbol{h}}\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}-\boldsymbol{p}\right]\rangle_0=0.$$

Now here is where I caused myself a great deal of unnecessary work in previous versions of the solution by plunging in and expanding the product that's inside the $\langle\rangle_0$ without examining it carefully. Look carefully at the last factor in that product. Do you see that we can rearrange it to give the following?

$$\langle \hat{\boldsymbol{h}} \left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \right] \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \underbrace{\left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) + r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]}_{After\ rearrangement} \rangle_0 = 0.$$

That result is interesting, but is it truly useful to us? To answer that question, let's consider different ways in which we might expand the product, then find its scalar part.

If we effect the multiplications in order, from left to right, we're likely to end up with a confusing mess. However, what if we multiply the last three factors together? Those three factors, together, compose a product of the form $\boldsymbol{ab}\left[\boldsymbol{a} + \boldsymbol{b}\right]$:

$$\underbrace{\left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \right]}_{\boldsymbol{a}} \underbrace{\left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]}_{\boldsymbol{b}} \left[\underbrace{r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)}_{\boldsymbol{a}} + \underbrace{r_1 \hat{\boldsymbol{t}} - \boldsymbol{p}}_{\boldsymbol{b}} \right].$$

The expansion of $\boldsymbol{ab}\left[\boldsymbol{a} + \boldsymbol{b}\right]$ is

$$\begin{aligned} \boldsymbol{ab}\left[\boldsymbol{a} + \boldsymbol{b}\right] &= \boldsymbol{aba} + b^2\boldsymbol{a} \\ &= 2\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)\boldsymbol{a} - a^2\boldsymbol{b} + b^2\boldsymbol{a} \quad \text{(among other possibilites).} \end{aligned}$$

That expansion evaluates to a vector, of course. Having obtained the corresponding expansion of the product $\left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \right] \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) + r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]$, we'd then "dot" the result with $\hat{\boldsymbol{h}}$ to obtain $\langle \hat{\boldsymbol{h}} \left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \right] \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) + r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \rangle_0$. We know, from the solutions to Problem 6 in Smith J A 2016 , that such a maneuver can work out quite favorably. So, let's try it.

Expanding $\left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \right] \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \left[r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) + r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]$ according to the identity $\boldsymbol{ab}\left[\boldsymbol{a} + \boldsymbol{b}\right] = 2\left(\boldsymbol{a}\cdot\boldsymbol{b}\right)\boldsymbol{a} - a^2\boldsymbol{b} + b^2\boldsymbol{a}$, we obtain, initially,

$$2\left\{ {r_2}^2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \cdot \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \right\} \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) - {r_2}^2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)^2 \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] + r_2 \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]^2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)$$

When we've completed our expansion and dotted it with $\hat{\boldsymbol{h}}$, we'll set the result to zero, so let's divide out the common factor r_2 now:

$$2\left\{ r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \cdot \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \right\} \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) - r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)^2 \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] + \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]^2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)$$

Recalling that $\left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right)^2 = 2\left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \right)$, the preceding becomes

$$2\left\{ r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \cdot \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \right\} \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) - 2r_2 \left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \right) \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] + \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]^2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right).$$

This is the form that we'll dot with $\hat{\boldsymbol{h}}$. Having done so, the factor $\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}}$ becomes $1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$. Then, as planned, we set the resulting expression equal to zero:

$$2\left\{ r_2 \left(\hat{\boldsymbol{t}} + \hat{\boldsymbol{h}} \right) \cdot \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right] \right\} \left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \right) - 2r_2 \left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \right) \left[r_1 \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} - \hat{\boldsymbol{h}} \cdot \boldsymbol{p} \right] + \left[r_1 \hat{\boldsymbol{t}} - \boldsymbol{p} \right]^2 \left(1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}} \right) = 0.$$

Next, we'll rearrange that equation to take advantage of the relation $r_2 = \frac{|\boldsymbol{h}| - r_1 \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}{1 + \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}}$ (see Eq. (10)). We'll show the steps in some detail:

$$\begin{aligned}
2\left\{r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\right\}\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)-2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left[r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right]+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0.\\
2r_2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[\frac{|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}{1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}}\right]\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\cdot\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-\hat{\boldsymbol{h}}\cdot\boldsymbol{p}-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\cdot\boldsymbol{p}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right\}+\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0.
\end{aligned}$$

Now that the dust has settled from the r_2 substitution, we'll expand $\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]^2$, then simplify further:

$$\begin{aligned}
2\left[|\boldsymbol{h}|-r_1\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right]\left\{r_1-\boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right\}+\left[{r_1}^2-2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+p^2\right]\left(1+\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}\right)&=0\\
\left[-2{r_1}^2+2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+{r_1}^2-2r_1\hat{\boldsymbol{t}}\cdot\boldsymbol{p}+p^2\right]\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-2\,|\boldsymbol{h}|\,\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+2\,|\boldsymbol{h}|\,r_1+{r_1}^2-2r_1\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+p^2&=0\\
\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}}-2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}\cdot\hat{\boldsymbol{t}}+{r_1}^2+p^2+2\,|\boldsymbol{h}|\,r_1&=0.
\end{aligned}$$

We saw equations like this last one many times in Smith J A 2016. There, we learned to solve those equations by grouping the dot products that involve $\boldsymbol{t}$ into a dot product of $\boldsymbol{t}$ with a linear combination of known vectors:

$$\underbrace{\left[2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}\right]}_{\text{A linear combination of }\hat{\boldsymbol{h}}\text{ and }\boldsymbol{p}}\cdot\hat{\boldsymbol{t}}=2\,|\boldsymbol{h}|\,r_1+{r_1}^2+p^2. \tag{7}$$

The geometric interpretation of Eq. (7) is that $2\,|\boldsymbol{h}|\,r_1+{r_1}^2+p^2$ is the projection of the vector $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$ upon $\hat{\boldsymbol{t}}$. Because we want to find $\boldsymbol{t}$, and know $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$, we'll transform Eq. (7) into a version that tells us the projection of the vector $\boldsymbol{t}$ upon $2\left(r_1+|\boldsymbol{h}|\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\hat{\boldsymbol{h}}$.

First, just for convenience, we'll multiply both sides of Eq. (7) by $r_1\,|\boldsymbol{h}|$:

$$\left[2\left(r_1\,|\boldsymbol{h}|+h^2\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\boldsymbol{h}\right]\cdot\boldsymbol{t}=2h^2{r_1}^2+r_1\,|\boldsymbol{h}|\left({r_1}^2+p^2\right).$$

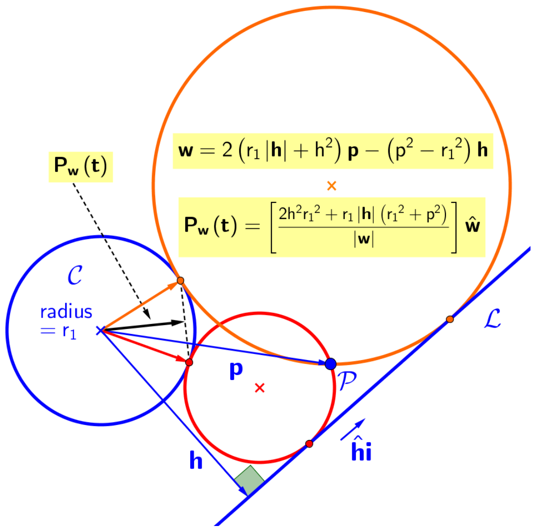
Next, we'll use the symbol "$\boldsymbol{w}$" for the vector $\left[2\left(r_1\,|\boldsymbol{h}|+h^2\right)\boldsymbol{p}-\left(p^2-{r_1}^2\right)\boldsymbol{h}\right]$, and write

$$\boldsymbol{w}\cdot\boldsymbol{t}=2h^2{r_1}^2+r_1\,|\boldsymbol{h}|\left({r_1}^2+p^2\right).$$

Finally, because $\boldsymbol{P}_{\boldsymbol{w}}\left(\boldsymbol{t}\right)$, the projection of the vector $\boldsymbol{t}$ upon $\boldsymbol{w}$ is $\left(\boldsymbol{t}\cdot\hat{\boldsymbol{w}}\right)\hat{\boldsymbol{w}}$, we have

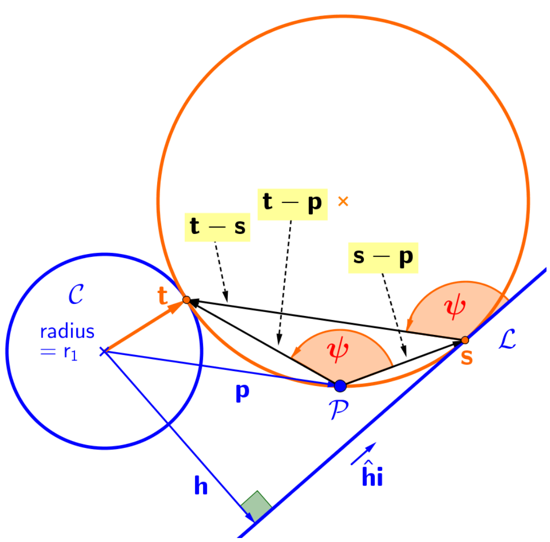
$$\boldsymbol{P}_{\boldsymbol{w}}\left(\boldsymbol{t}\right)=\left[\frac{2h^2{r_1}^2+r_1\,|\boldsymbol{h}|\left({r_1}^2+p^2\right)}{|\boldsymbol{w}|}\right]\hat{\boldsymbol{w}}. \tag{8}$$

As we learned in Smith J A 2016, Eq. (8) tells us that Eq (7) has two solutions. That is, there are two circles that are tangent to $\mathcal{L}$ and pass through the point $\mathcal{P}$, and are also tangent to $\mathcal{C}$ without enclosing it:



Having identified $\boldsymbol{P_w}(\boldsymbol{t})$, the points of tangency with $\mathcal{C}$ and $\mathcal{L}$ can be determined using methods shown in Smith J A 2016, as can the equations for the corresponding solution circles.

To round off our treatment of solution circles that don't enclose $\mathcal{C}$, we should note that we derived our solution starting from equations that express the relationship between $\mathcal{C}$, $\mathcal{L}$, $\mathcal{P}$, and the smaller of the two solution circles. You may have noticed that the larger solution circle does not bear quite the same relationship to $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{C}$ as the smaller one. To understand in what way those relationships differ, let's examine the following figure.



By equating two expressions for the rotation $e^{\psi \boldsymbol{i}}$, we'd find that

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]=\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right].$$

Compare that result to the corresponding equation for the smaller of the solution circles:

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

We followed up on that equation by transforming it into one in which $\hat{\boldsymbol{h}}$ was at one end of the product on the left-hand side. The result was Eq. (5):

$$\hat{\boldsymbol{h}}\left[\boldsymbol{s}-\boldsymbol{t}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{s}-\boldsymbol{p}\right]=|\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i}.$$

We saw the advantages of that arrangement when we proceeded to solve for $\boldsymbol{t}$. All we had to do in order to procure that arrangement was to left-multiply both sides of the equation

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]$$

by $\boldsymbol{s}-\boldsymbol{t}$, and then by $\hat{\boldsymbol{h}}$.

To procure a similar arrangement starting from the equation that we wrote for the larger circle, using the angle ψ ,

$$\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]=\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right]\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right],$$

we could left-multiply both sides of the equation by $\hat{\boldsymbol{h}}$, then right-multiply by $\boldsymbol{s}-\boldsymbol{t}$, giving

$$\hat{\boldsymbol{h}}\left[\boldsymbol{s}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{s}\right]=|\boldsymbol{s}-\boldsymbol{p}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{t}-\boldsymbol{s}|\,\boldsymbol{i}.$$

Using the same ideas and transformations as for the smaller circle, we'd then transform the product $\left[\boldsymbol{s}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{p}\right]\left[\boldsymbol{t}-\boldsymbol{s}\right]$ into

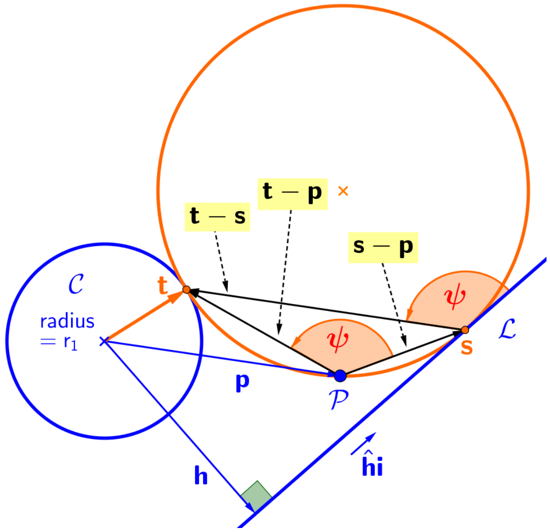
$$\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right].$$

By comparison, the product that we obtained for the smaller circle was

$$\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\right]\left[r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right]\left[r_2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)+r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right].$$

In both cases, the products that result from the expansion have the forms $\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)\left(r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right)\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$ and $\left(r_1\hat{\boldsymbol{t}}-\boldsymbol{p}\right)^2\left(\hat{\boldsymbol{t}}+\hat{\boldsymbol{h}}\right)$, so the same simplifications work in both, and give the same results when the result is finally dotted with $\hat{\boldsymbol{h}}$ and set to zero.

The above having been said, let's look at another way of obtaining an equation, for the larger circle, that has the same form as Eq. (5). For convenience, we'll present the figure for the larger circle again:

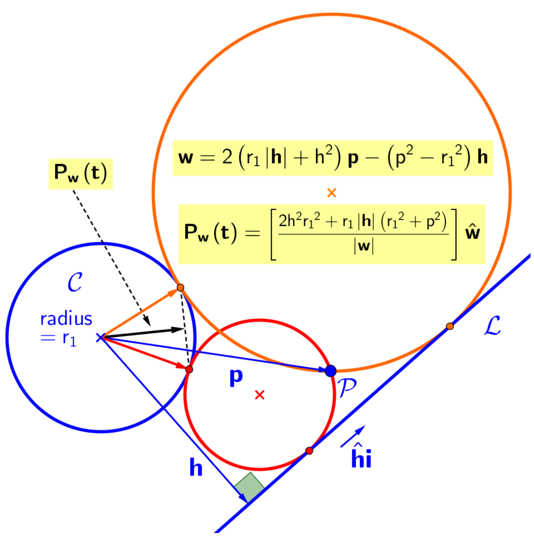


Instead of beginning by equating two expressions for the rotation $e^{\psi \boldsymbol{i}}$, we'll equate two expressions for $e^{-\psi \boldsymbol{i}}$:

$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right]=\left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right].$$

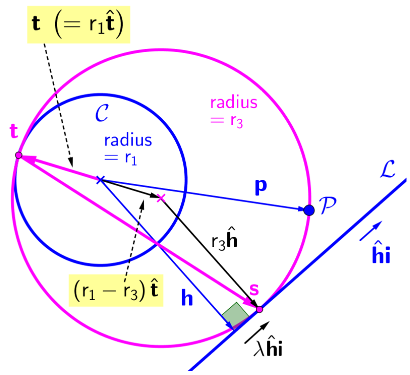
After left-multiplying both sides by $\boldsymbol{t}-\boldsymbol{s}$, then by $\hat{\boldsymbol{h}}$, and rearranging, the result would be identical to Eq. (5), except for the algebraic sign of the right-had side, which —because it's a bivector—would drop out when we took the scalar part of both sides.

However, the difference in the sign of that bivector captures the geometric nature of the difference between the relationships of the large and small circles to $\mathcal{L}$, $\mathcal{P}$, and $\mathcal{C}$. That difference in sign is also reflected in the positions, with respect to the vector $\boldsymbol{w}$, of the solution circles' points of tangency $\boldsymbol{t}$:



2.3 Identifying the Solution Circles that Enclose $\mathcal{C}$

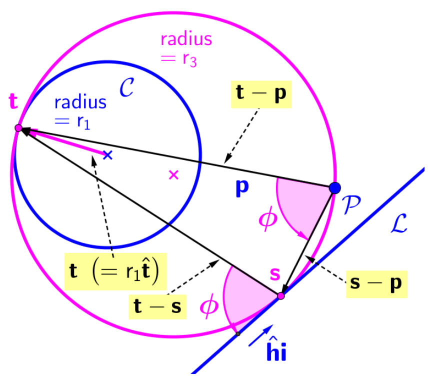
These solution circles can be found by modifying, slightly, the ideas that we used for finding solution circles that don't enclose $\mathcal{C}$. Here, too, we'll want to express the radius r_3 of the solution circle in terms of the vector $\boldsymbol{t}$. Examining the next figure,



we see that $\boldsymbol{s} = (r_1 - r_3)\,\hat{\boldsymbol{t}} + r_3\hat{\boldsymbol{h}}$, and also that $\boldsymbol{s} = \boldsymbol{h} + \lambda\hat{\boldsymbol{h}}\boldsymbol{i}$. By equating those two expressions, dotting both sides with $\hat{\boldsymbol{h}}$, and then solving for r_3, we find that

$$r_3 = \frac{|\boldsymbol{h}| - r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1 - \hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}.$$

As was the case when we found solution circles that didn't enclose $\mathcal{C}$, we'll want to equate expressions for two rotations that involve the unknown points of tangency $\boldsymbol{t}$ and $\boldsymbol{s}$. For example, through the angles labeled ϕ, below:



$$\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{|\boldsymbol{t}-\boldsymbol{p}|}\right]\left[\frac{\boldsymbol{s}-\boldsymbol{p}}{|\boldsymbol{s}-\boldsymbol{p}|}\right] = \left[\frac{\boldsymbol{t}-\boldsymbol{s}}{|\boldsymbol{t}-\boldsymbol{s}|}\right]\left[-\hat{\boldsymbol{h}}\boldsymbol{i}\right]$$
$$= \left[\frac{\boldsymbol{s}-\boldsymbol{t}}{|\boldsymbol{s}-\boldsymbol{t}|}\right]\left[\hat{\boldsymbol{h}}\boldsymbol{i}\right],$$

Left-multiplying that result by $\boldsymbol{s} - \boldsymbol{t}$, and then by $\hat{\boldsymbol{h}}$,

$$\begin{aligned} \hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}] &= |\boldsymbol{s}-\boldsymbol{t}|\,|\boldsymbol{t}-\boldsymbol{p}|\,|\boldsymbol{s}-\boldsymbol{p}|\,\boldsymbol{i} \\ \therefore\ \langle\hat{\boldsymbol{h}}\,[\boldsymbol{s}-\boldsymbol{t}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{s}-\boldsymbol{p}]\rangle_0 &= 0, \end{aligned}$$

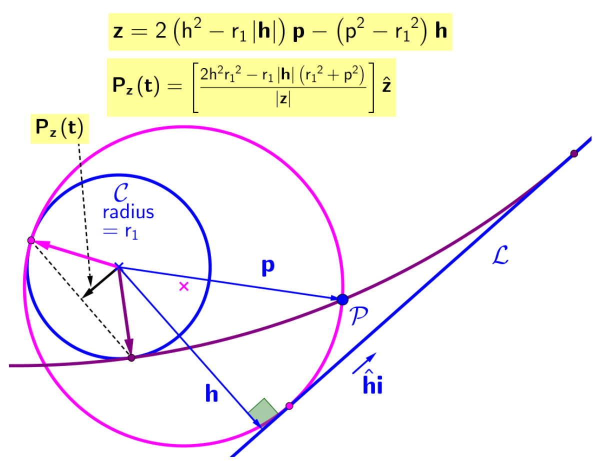
which is identical to Eq. (6). To solve for $\boldsymbol{t}$, we use exactly the same technique that we did when we identified the solution circles that don't surround $\mathcal{C}$, with r_3 and $1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$ taking the place of r_2 and $1 - \hat{\boldsymbol{h}} \cdot \hat{\boldsymbol{t}}$, respectively. The result is

$$\boldsymbol{z} \cdot \boldsymbol{t} = 2h^2{r_1}^2 - r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right),$$

where $\boldsymbol{z} = \left[2\left(h^2 - r_1\,|\boldsymbol{h}|\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right]$. Thus,

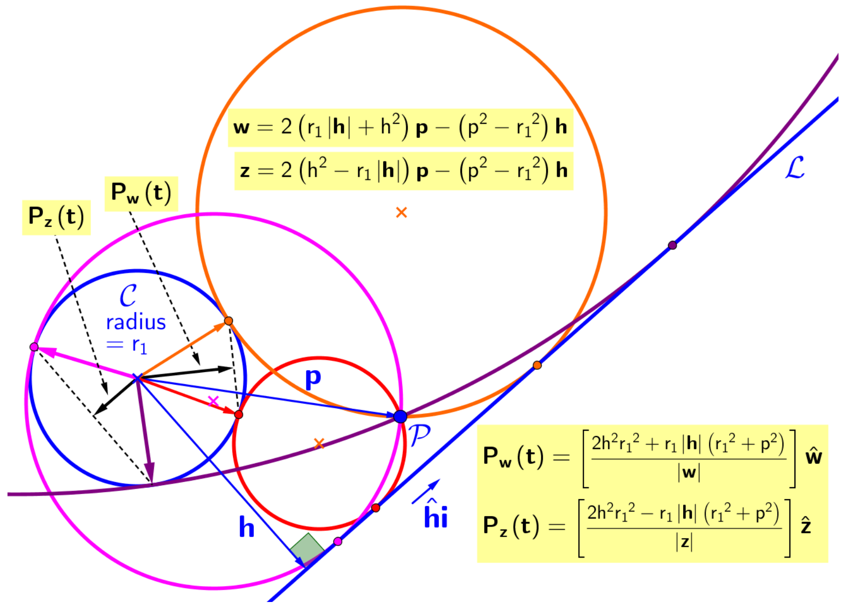
$$\boldsymbol{P}_{\boldsymbol{z}}(\boldsymbol{t}) = \left[\frac{2h^2{r_1}^2 - r_1\,|\boldsymbol{h}|\left({r_1}^2 + p^2\right)}{|\boldsymbol{z}|}\right]\hat{\boldsymbol{z}}. \tag{9}$$

Again, there are two solution circles of this type:



2.4 The Complete Solution, and Comments

There are four solution circles: two that enclose $\mathcal{C}$, and two that don't:



I confess that I don't entirely care for the solution presented in this document. The need to identify r_2, in order to eliminate it later, makes me suspect that I did not make good use of GA's capabilities. A solution that uses reflections in addition to rotations is in preparation, and is arguably more efficient.

3 Literature Cited

GeoGebra Worksheets and Related Videos (by title, in alphabetical order):

"Answering Two Common Objections to Geometric Algebra"
GeoGebra worksheet: http://tube.geogebra.org/m/1565271
YouTube video: https://www.youtube.com/watch?v=oB0DZiF86Ns.

"Geometric Algebra: Find unknown vector from two dot products"
GeoGebra worksheet:
http://tube.geogebra.org/material/simple/id/1481375
YouTube video:
https://www.youtube.com/watch?v=2cqDVtHcCoE

Books and articles (according to author, in alphabetical order):

González Calvet R. 2001. Treatise of Plane Geometry through Geometric Algebra. 89.218.153.154:280/CDO/BOOKS/Matem/Calvet.Treatise.pdf. Retrieved 30 December 2015.

Hestenes D. 1999. *New Foundations for Classical Mechanics* (Second Edition). Kluwer Academic Publishers, Dordrecht/Boston/London.

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Smith J. A. 2014. "Resoluciones de 'problemas de construcción' geométricos por medio de la geometría clásica, y el Álgebra Geométrica", `http://quelamatenotemate.webs.com/3%20soluciones %20construccion%20geom%20algebra%20Cliffod.xps`

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4 APPENDIX: Improved Solutions

We'll show two ways of solving the problem; the second takes advantage of observations made during the first.

As noted in the main text, the CLP limiting case reads,

> *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

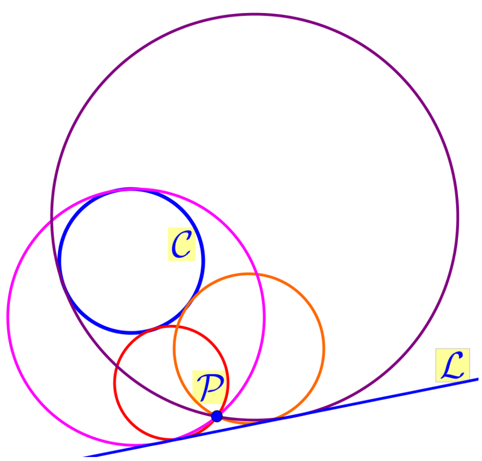

Figure 1: The CLP Limiting Case of the Problem of Apollonius: *Given a circle $\mathcal{C}$, a line $\mathcal{L}$, and a point $\mathcal{P}$, construct the circles that are tangent to $\mathcal{C}$ and $\mathcal{L}$, and pass through $\mathcal{P}$.*

The problem has two types of solutions:

- Circles that enclose $\mathcal{C}$;
- Circles that do not enclose $\mathcal{C}$.

There are two solution circles of each type. In this document, we'll treat only those that do not enclose the given circle.

4.1 The First Solution

Fig. 2 shows how we will capture the geometric content of the problem. An important improvement, compared to the solution technique presented in [**?**], is that we will use rotations with respect to the vector from the given point $\mathcal{P}$ to the still-unidentified center point ($\boldsymbol{c}_2$) of the solution circle.

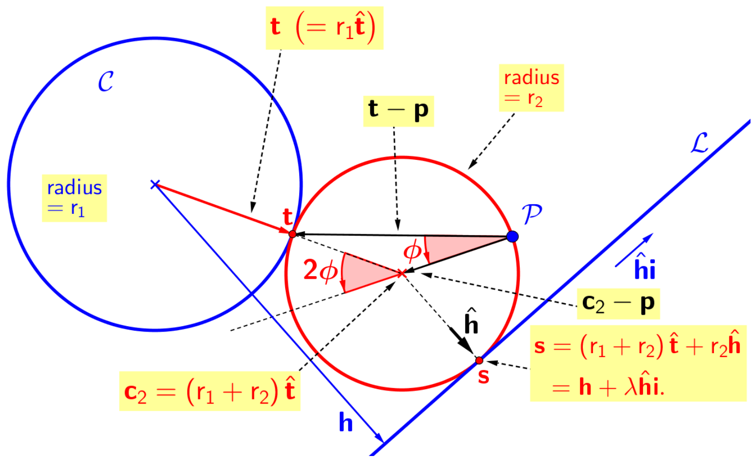

Figure 2: Elements used in the first solution of the CLP limiting case.

In deriving our solution, we'll use the same symbol —for example, $\boldsymbol{t}$ —to denote both a point and the vector to that point from the origin. We'll rely upon context to tell the reader whether the symbol is being used to refer to the point, or to the vector.

We'll begin our solution by deriving an expression for r_2 in terms of $\hat{\boldsymbol{t}}$. We´ll do so by equating two independent expressions for $\boldsymbol{s}$, then "dotting" both sides with $\hat{\boldsymbol{h}}$, after which we'll solve for r_2:

$$\begin{aligned}
(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}} &= \boldsymbol{h}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\\
\left[(r_1+r_2)\,\hat{\boldsymbol{t}}+r_2\hat{\boldsymbol{h}}\right]\cdot\hat{\boldsymbol{h}} &= \left[\boldsymbol{h}+\lambda\hat{\boldsymbol{h}}\boldsymbol{i}\right]\cdot\hat{\boldsymbol{h}}\\
(r_1+r_2)\,\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}+r_2\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{h}} &= \boldsymbol{h}\cdot\hat{\boldsymbol{h}}+\lambda\left(\hat{\boldsymbol{h}}\boldsymbol{i}\right)\cdot\hat{\boldsymbol{h}}\\
(r_1+r_2)\,\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}+r_2 &= \|\boldsymbol{h}\|+0;\\
\therefore r_2 = \frac{\|\boldsymbol{h}\|-r_1\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}},\ \text{and}\ \ r_1+r_2 &= \frac{\|\boldsymbol{h}\|+r_1}{1+\hat{\boldsymbol{t}}\cdot\hat{\boldsymbol{h}}}.
\end{aligned} \tag{10}$$

Next, we equate two expressions for the rotation $e^{\boldsymbol{i}2\phi}$:

$$\underbrace{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}\phi}}\underbrace{\left[\frac{\boldsymbol{t}-\boldsymbol{p}}{\|\boldsymbol{t}-\boldsymbol{p}\|}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}\phi}}=\underbrace{\left[-\hat{\boldsymbol{t}}\right]\left[\frac{\boldsymbol{c}_2-\boldsymbol{p}}{\|\boldsymbol{c}_2-\boldsymbol{p}\|}\right]}_{=e^{\boldsymbol{i}2\phi}},$$

from which

$$\begin{aligned} [\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] &= \text{some scalar}, \\ \therefore\ \langle[\boldsymbol{t}-\boldsymbol{p}]\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 &= 0. \end{aligned} \tag{11}$$

Using the identity $\boldsymbol{ab} \equiv 2\boldsymbol{a}\wedge\boldsymbol{b} + \boldsymbol{ba}$, we rewrite 11 as

$$\begin{gathered} \langle(2\,[\boldsymbol{t}-\boldsymbol{p}]\wedge[\boldsymbol{c}_2-\boldsymbol{p}] + [\boldsymbol{c}_2-\boldsymbol{p}]\,[\boldsymbol{t}-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 = 0, \\ \langle(2\,[\boldsymbol{t}-\boldsymbol{p}]\wedge[\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}] + [\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 = 0, \text{ and} \\ \langle(2\,[\boldsymbol{t}-\boldsymbol{p}]\wedge[\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 + \langle[\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 = 0. \end{gathered} \tag{12}$$

Now, we note that

$$\begin{aligned} \langle(2\,[\boldsymbol{t}-\boldsymbol{p}]\wedge[\boldsymbol{c}_2-\boldsymbol{p}])\,[\boldsymbol{t}-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 &= 2\,([\boldsymbol{t}-\boldsymbol{p}]\wedge[\boldsymbol{c}_2-\boldsymbol{p}])\left([\boldsymbol{t}-\boldsymbol{p}]\cdot[\hat{\boldsymbol{t}}]\right), \\ \text{and } \langle[\boldsymbol{t}-\boldsymbol{p}]^2\,[\boldsymbol{c}_2-\boldsymbol{p}]\,[\hat{\boldsymbol{t}}]\rangle_2 &= [\boldsymbol{t}-\boldsymbol{p}]^2\left([\boldsymbol{c}_2-\boldsymbol{p}]\wedge[\hat{\boldsymbol{t}}]\right). \end{aligned}$$

Because $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$ and $\boldsymbol{c}_2 = (r_1+r_2)\,\hat{\boldsymbol{t}}$, $\boldsymbol{t}\wedge\boldsymbol{c}_2 = 0$. We can expand $[\boldsymbol{t}-\boldsymbol{p}]^2$ as ${r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2$. Using all of these ideas, (12) becomes (after simplification)

Note how the factor $\boldsymbol{p}\wedge\hat{\boldsymbol{t}}$ canceled out in Eq. (13). That cancellation suggests an improvement that we'll see in our second solution of the CLP case.

$$2r_2\left(r_1 - \boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right)\boldsymbol{p}\wedge\hat{\boldsymbol{t}} + \left({r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2\right)\boldsymbol{p}\wedge\hat{\boldsymbol{t}} = 0. \tag{13}$$

For $\boldsymbol{p}\wedge\hat{\boldsymbol{t}} \neq 0$, that equation becomes

$$2r_2\left(r_1 - \boldsymbol{p}\cdot\hat{\boldsymbol{t}}\right) + {r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + p^2 = 0.$$

Substituting the expression that we derived for r_2 in (10), then expanding and simplifying,

$$2\left(\|\boldsymbol{h}\| + r_1\right)\boldsymbol{p}\cdot\hat{\boldsymbol{t}} - \left(p^2 - {r_1}^2\right)\hat{\boldsymbol{h}}\cdot\hat{\boldsymbol{t}} = 2\|\boldsymbol{h}\|r_1 + {r_1}^2 + p^2.$$

Finally, we rearrange that result and multiply both sides by $r_1\|\boldsymbol{h}\|$, giving the equation that we derived in [**?**]:

$$\left\{2\left(r_1\|\boldsymbol{h}\| + h^2\right)\boldsymbol{p} - \left(p^2 - {r_1}^2\right)\boldsymbol{h}\right\}\cdot\boldsymbol{t} = 2h^2{r_1}^2 + r_1\|\boldsymbol{h}\|\left({r_1}^2 + p^2\right). \tag{14}$$

4.2 The Second Solution: Learning From and Building Upon the First

In Eq. (13), we saw how the factor $\boldsymbol{p}\wedge\hat{\boldsymbol{t}}$ canceled out. That cancellation suggests that we might solve the problem more efficiently by expressing rotations with respect to the unknown vector $\hat{\boldsymbol{t}}$, rather than to a vector from $\mathcal{P}$ to $\boldsymbol{c}_2$ (Fig. 3).

For this new choice of vectors, our equation relating two expressions for the rotation $e^{\boldsymbol{i}2\phi}$ is:

$$\underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{t}}{\|\boldsymbol{p}-\boldsymbol{t}\|}\right]}_{=e^{\boldsymbol{i}\phi}}\underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{t}}{\|\boldsymbol{p}-\boldsymbol{t}\|}\right]}_{=e^{\boldsymbol{i}\phi}} = \underbrace{[\hat{\boldsymbol{t}}]\left[\frac{\boldsymbol{p}-\boldsymbol{c}_2}{\|\boldsymbol{p}-\boldsymbol{c}_2\|}\right]}_{=e^{\boldsymbol{i}2\phi}},$$

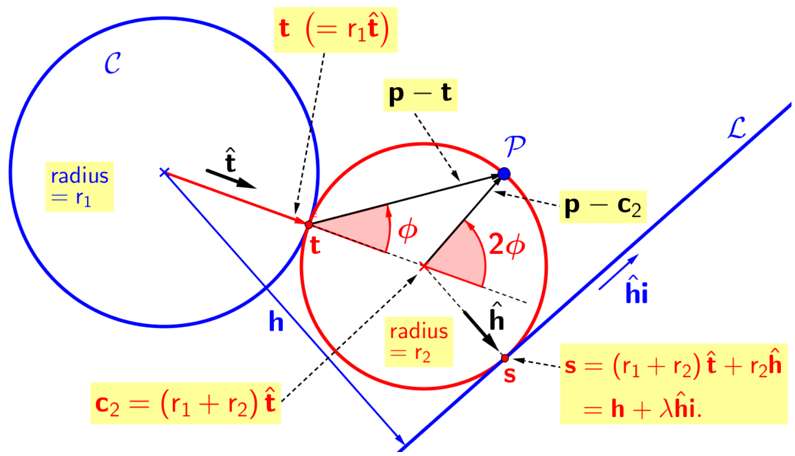

Figure 3: Elements used in the second solution of the CLP Limiting Case: rotations are now expressed with respect to the unknown vector $\hat{\boldsymbol{t}}$, rather than to a vector from $\mathcal{P}$ to $\boldsymbol{c}_2$.

from which

$$\begin{aligned} &[\boldsymbol{p}-\boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2] = \text{some scalar}, \\ \therefore\ &\langle [\boldsymbol{p}-\boldsymbol{t}]\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 = 0. \end{aligned} \tag{15}$$

Using the identity $\boldsymbol{ab} \equiv 2\boldsymbol{a}\wedge\boldsymbol{b} + \boldsymbol{ba}$, we rewrite 15 as

$$\begin{aligned} \langle \left(2\,[\boldsymbol{p}-\boldsymbol{t}]\wedge[\hat{\boldsymbol{t}}] + [\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{t}]\right)[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 = 0, \text{ and} \\ \langle \left(2\,[\boldsymbol{p}-\boldsymbol{t}]\wedge[\hat{\boldsymbol{t}}]\right)[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 + \langle [\boldsymbol{p}-\boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 = 0. \end{aligned} \tag{16}$$

Now, we note that

$$\begin{aligned} \langle \left(2\,[\boldsymbol{p}-\boldsymbol{t}]\wedge[\hat{\boldsymbol{t}}]\right)[\boldsymbol{p}-\boldsymbol{t}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 &= \left(2\,[\boldsymbol{p}-\boldsymbol{t}]\wedge[\hat{\boldsymbol{t}}]\right)[\boldsymbol{p}-\boldsymbol{t}]\cdot[\boldsymbol{p}-\boldsymbol{c}_2], \\ \text{and } \langle [\boldsymbol{p}-\boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}]\,[\boldsymbol{p}-\boldsymbol{c}_2]\rangle_2 &= [\boldsymbol{p}-\boldsymbol{t}]^2\,[\hat{\boldsymbol{t}}]\wedge[\boldsymbol{p}-\boldsymbol{c}_2]. \end{aligned}$$

Because $\boldsymbol{t} = r_1\hat{\boldsymbol{t}}$ and $\boldsymbol{c}_2 = (r_1+r_2)\,\hat{\boldsymbol{t}}$, $\boldsymbol{t}\wedge\boldsymbol{c}_2 = 0$. We can expand $[\boldsymbol{p}-\boldsymbol{t}]^2$ as $p^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + {r_1}^2$. Using all of these ideas, (16) becomes (after simplification)

$$2\left([\boldsymbol{p}-\boldsymbol{t}]\cdot[\boldsymbol{p}-\boldsymbol{c}_2]\right)\boldsymbol{p}\wedge\boldsymbol{t} - \left(p^2 - 2\boldsymbol{p}\cdot\boldsymbol{t} + {r_1}^2\right)\boldsymbol{p}\wedge\boldsymbol{t} = 0.$$

For $\boldsymbol{p}\wedge\hat{\boldsymbol{t}} \neq 0$, that equation becomes, after expanding $[\boldsymbol{p}-\boldsymbol{t}]\cdot[\boldsymbol{p}-\boldsymbol{c}_2]$ and further simplifications,

$$p^2 - {r_1}^2 - 2\boldsymbol{p}\cdot\boldsymbol{c}_2 + 2\boldsymbol{t}\cdot\boldsymbol{c}_2 = 0.$$

Now, recalling that $\boldsymbol{c}_2 = (r_1+r_2)\,\hat{\boldsymbol{t}}$, we substitute the expression that we derived for $r_1 + r_2$ in (10), then expand and simplify to obtain (14). This solution process has been a bit shorter than the first because (16) was so easy to simplify.