Mathematical Formulas:Part 1

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abstract

In this paper we give some formulas related with the constant pi:

$$\pi = 3.1415926535 \ldots$$

I. Introducción

Notación

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}, \ \mathbb{N}_0 = \mathbb{N} \cup \{0\} \quad (1)$$

función zeta de Riemann:

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}, \ x > 1 \quad (2)$$

función zeta alternada:

$$\zeta^*(x) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-x}, \ x > 0 \quad (3)$$

$$s_k = \sqrt{2 - \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}, \ k - \text{radicales}, \ k \in \mathbb{N} \quad (4)$$

$$c_k = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}, \ k - \text{radicales}, \ k \in \mathbb{N} \quad (5)$$

$$(a)_0 = 1, \ (a)_n = a(a+1)(a+2)\ldots(a+n-1), \ n \in \mathbb{N} \quad (6)$$

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182 \ldots \quad (7)$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n\, 2^n} \sum_{k=1}^{n} \binom{n}{k} \frac{z^k}{k^2} = z \int_0^1 \frac{\ln(u)}{1 - z u}\, du, \ 0 < z < 1 \quad (8)$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \ n \in \mathbb{N} \quad (9)$$

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}, \ n, m \in \mathbb{N} \quad (10)$$

números de Bernoulli:

$$B_k = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \ldots\right\} \tag{11}$$

constante de Catalan:

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \tag{12}$$

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right) = 0.5772\ldots \tag{13}$$

función gamma:

$$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1}\, dx \;,\; x > 0 \tag{14}$$

función Psi:

$$\psi(x) = \Gamma'(z)/\Gamma(z) = \frac{d}{dx}\ln(\Gamma(x)) \tag{15}$$

II. Fórmulas

$$\pi = 3 + \sum_{k=0}^{m} 6^{-2k-1}\, \zeta^*(2k+2) + 6^{-2m-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n^{-2m-2}}{36n^2 - 1} \;,\; m \in \mathbb{N}_0 \tag{16}$$

$\zeta^*(x)$ es la función zeta alternada

$$\pi\left(1 + \sum_{k=1}^{m} \frac{s_{k+1}}{c_{k+1}}\right) = 2^{m+3} \sum_{n=1}^{\infty} \frac{(2^{2n}-1)\left(2^{(2n-1)(m+1)}-1\right)\zeta(2n)}{2^{2n(m+2)}\,(2^{2n-1}-1)} \;,\; m \in \mathbb{N}_0 \tag{17}$$

$\zeta(x)$ es la función zeta de Riemann

$$\pi\left(1 + \sum_{k=1}^{\infty} \frac{s_{k+1}}{c_{k+1}}\right) = 4 \sum_{n=1}^{\infty} \frac{(2^{2n}-1)\,\zeta(2n)}{2^{2n}\,(2^{2n-1}-1)} \tag{18}$$

$$\frac{70}{2197}\pi^3 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{7\pi n}{13}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{8\pi n}{13}\right) \tag{19}$$

$$\frac{10}{343}\pi^3 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{3\pi n}{7}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{5\pi n}{7}\right) \tag{20}$$

$$\frac{140}{6859}\pi^3 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{5\pi n}{19}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{16\pi n}{19}\right) \tag{21}$$

$$\frac{2^8}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^3\,(2n+1)\,(84n^2+164n+81)}{2^{6n}\,(n!)^3\,(n+1)^2} \tag{22}$$

$$\frac{\sqrt[7]{2}}{\pi}\left(\frac{(\Gamma(1/14))^2}{\Gamma(1/7)}\right) = 11 - \sum_{n=1}^{\infty} \frac{(-3/7)_n^2\,(1/2)_n\,(10/7)_n\,(343n^2-11)}{2^{6n}\,(n!)^3\,(1/28)_n\,(15/28)_n\,(28n+1)} \tag{23}$$

$$\frac{\pi}{4}+\sum_{n=2}^{\infty}\tan^{-1}\left(\frac{1}{n^4}\right)=\tan^{-1}\left(\frac{\operatorname{sen}(a)\operatorname{sh}(b)+\cos(a)\operatorname{ch}(b)+\operatorname{sen}(b)\operatorname{sh}(a)-\cos(b)\operatorname{ch}(a)}{\operatorname{sen}(a)\operatorname{sh}(b)-\cos(a)\operatorname{ch}(b)+\operatorname{sen}(b)\operatorname{sh}(a)+\cos(b)\operatorname{ch}(a)}\right) \tag{24}$$

$$a=\frac{\pi\sqrt{4+2\sqrt{2}}}{2}\ ,\ \ b=\frac{\pi\sqrt{4-2\sqrt{2}}}{2} \tag{25}$$

$$\frac{\pi^2}{32}=\lim_{n\to\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n-1}\sum_{m=k+1}^{n}\sqrt{1-\left(\frac{k}{n+1}\right)^2}\ \sqrt{1-\left(\frac{m}{n+1}\right)^2} \tag{26}$$

Sean:

$$F(a)=\int_0^1\int_0^1\frac{1-x}{(1+x^2y^2a^2)(-\ln(x\,y))}\,dx\,dy\ ,\ \ 0<a<1 \tag{27}$$

$$G(a)=\prod_{n=0}^{\infty}\left(\frac{4n+1}{4n+2}\right)^{a^{4n+1}}\left(\frac{4n+4}{4n+3}\right)^{a^{4n+3}}\ ,\ \ 0<a<1 \tag{28}$$

se tiene:

$$\frac{1}{2}F\left(\frac{1}{2}\right)+\frac{1}{3}F\left(\frac{1}{3}\right)=\frac{\pi}{4}+\ln\left(G\left(\frac{1}{2}\right)G\left(\frac{1}{3}\right)\right) \tag{29}$$

$$\frac{2}{3}F\left(\frac{1}{3}\right)+\frac{1}{7}F\left(\frac{1}{7}\right)=\frac{\pi}{4}+\ln\left(\left(G\left(\frac{1}{3}\right)\right)^2G\left(\frac{1}{7}\right)\right) \tag{30}$$

$$\frac{5}{7}F\left(\frac{1}{7}\right)+\frac{6}{79}F\left(\frac{3}{79}\right)=\frac{\pi}{4}+\ln\left(\left(G\left(\frac{1}{7}\right)\right)^5\left(G\left(\frac{3}{79}\right)\right)^2\right) \tag{31}$$

$$\frac{2}{3\sqrt{3}}F\left(\frac{1}{3\sqrt{3}}\right)+\frac{1}{4\sqrt{3}}F\left(\frac{1}{4\sqrt{3}}\right)=\frac{\pi}{6}+\ln\left(\left(G\left(\frac{1}{3\sqrt{3}}\right)\right)^2G\left(\frac{1}{4\sqrt{3}}\right)\right) \tag{32}$$

$$\frac{5}{7}F\left(\frac{1}{7}\right)+\frac{4}{53}F\left(\frac{1}{33}\right)+\frac{2}{4443}F\left(\frac{1}{4443}\right)=\frac{\pi}{4}+\ln\left(\left(G\left(\frac{1}{7}\right)\right)^5\left(G\left(\frac{1}{53}\right)\right)^4\left(G\left(\frac{1}{4443}\right)\right)^2\right) \tag{33}$$

$$\frac{4}{5}F\left(\frac{1}{10}\right)-\frac{1}{239}F\left(\frac{1}{239}\right)-\frac{4}{515}F\left(\frac{1}{515}\right)=\frac{\pi}{4}+\ln\left(\left(G\left(\frac{1}{10}\right)\right)^8\left(G\left(\frac{1}{239}\right)\right)^{-1}\left(G\left(\frac{1}{515}\right)\right)^{-4}\right) \tag{34}$$

$$\frac{44}{57}F\left(\frac{1}{57}\right)+\frac{7}{239}F\left(\frac{1}{239}\right)-\frac{6}{341}F\left(\frac{1}{682}\right)+\frac{24}{12\,943}F\left(\frac{1}{12\,943}\right)=\frac{\pi}{4}+\ln\left(\left(G\left(\frac{1}{57}\right)\right)^{44}\left(G\left(\frac{1}{239}\right)\right)^7\left(G\left(\frac{1}{682}\right)\right)^{-12}\left(G\left(\frac{1}{12\,943}\right)\right)^{24}\right) \tag{35}$$

$$\frac{\pi}{4}s_n+\sum_{k=1}^{n-1}\frac{c_{n-k}}{2^{2k}-1}-\frac{2}{2^{2n+2}-1}+\sum_{k=n+2}^{\infty}\frac{2}{2^{2k}-1}=1-2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\left((2m+1)\,2^{k-n-1}\pi\right)}{2^{2k}(2m+1)^2-1}\ ,\ \ n\in\mathbb{N} \tag{36}$$

Ejemplos:

$$\frac{\pi}{4}\sqrt{2-\sqrt{2}}+\frac{\sqrt{2}}{3}-\frac{2}{63}+\sum_{k=4}^{\infty}\frac{2}{2^{2k}-1}=1-2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\left((2m+1)\,2^{k-3}\pi\right)}{2^{2k}(2m+1)^2-1} \tag{37}$$

$$\frac{\pi}{4}\sqrt{2-\sqrt{2+\sqrt{2}}}+\frac{\sqrt{2+\sqrt{2}}}{3}+\frac{\sqrt{2}}{15}-\frac{2}{255}+\sum_{k=5}^{\infty}\frac{2}{2^{2k}-1}=1-2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\left((2m+1)\,2^{k-4}\pi\right)}{2^{2k}(2m+1)^2-1} \tag{38}$$

$$\frac{\pi}{4}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}+\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{3}+\frac{\sqrt{2+\sqrt{2}}}{15}+\frac{\sqrt{2}}{63}-\frac{2}{1023}+\sum_{k=6}^{\infty}\frac{2}{2^{2k}-1}=$$

$$1-2\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\left((2m+1)\,2^{k-5}\,\pi\right)}{2^{2k}\,(2m+1)^2-1}$$

$$\frac{\Gamma(1+b)\,\Gamma((n+c+1)/a)}{a\,\Gamma(1+b+(n+c+1)/c)}=\sum_{k=0}^{\infty}\frac{(-b)_k}{k!\,(a\,k+n+c+1)} \tag{40}$$

$$a>0,\ b>-1,\ c>-1,\ n\in\mathbb{N}_0$$

$$\frac{\sqrt{\pi}\ \Gamma((n+c+1)/a)}{a\,\Gamma\left(m+\frac{3}{2}+\frac{n+c+1}{a}\right)}=\frac{2^{2m+2}\,(m+1)!}{(2m+2)!}\sum_{k=0}^{\infty}\frac{(-(2m+1)/2)_k}{k!\,(a\,k+n+c+1)} \tag{41}$$

$$a>0,\ c>-1,\ n\in\mathbb{N}_0,\ m\in\mathbb{N}_0\bigcup\{-1\}$$

$$\frac{\sqrt{\pi}\ \Gamma((n+c+1)/a)}{a\,\Gamma\left(\frac{1}{2}+\frac{n+c+1}{a}\right)}=\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!\,(a\,k+n+c+1)} \tag{42}$$

$$a>0,\ c>-1,\ n\in\mathbb{N}_0$$

$$\frac{\sqrt{\pi}\ \Gamma(1+b)}{\Gamma\left(\frac{3}{2}+m+b\right)}=\frac{2^{2m+1}\,m!}{(2m)!}\sum_{k=0}^{\infty}\frac{(-b)_k}{k!\,(2k+2m+1)} \tag{43}$$

$$b>-1,\ m\in\mathbb{N}_0$$

$$\frac{\pi\,\Gamma\left(\frac{n_1+c_1+1}{a_1}\right)\Gamma\left(\frac{n_2+c_2+1}{a_2}\right)}{a_1\,a_2\,\Gamma\left(\frac{1}{2}+\frac{n_1+c_1+1}{a_1}\right)\Gamma\left(\frac{1}{2}+\frac{n_2+c_2+1}{a_2}\right)}=\sum_{k=0}^{\infty}2^{-2k}\sum_{m=0}^{k}\binom{2m}{m}\binom{2k-2m}{k-m}\frac{1}{(a_1\,m+n_1+c_1+1)\,(a_2(k-m)+n_2+c_2+1)} \tag{44}$$

$$a_1,\ a_2>0\,,\ \ c_1,\ c_2>-1\,,\ \ n_1,\ n_2\in\mathbb{N}_0$$

$$\frac{\pi\,(\Gamma((n+c+1)/a))^2}{a^2\left(\Gamma\left(\frac{1}{2}+\frac{n+c+1}{a}\right)\right)^2}=\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{2^{-4k}}{(a\,k+n+c+1)^2}+\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{2k+2}{k+1}\binom{2m}{m}\frac{2^{-2k-2m-1}}{(a\,k+a+n+c+1)\,(a\,m+n+c+1)} \tag{45}$$

$$a>0\,,\ \ c>-1\,,\ \ n\in\mathbb{N}_0$$

$$\pi=\frac{2^{2(n+m)+1}\,n!\,m!\,(n+m)!}{(2n)!\,(2m)!}\sum_{k=0}^{\infty}\frac{(-(2n-1)/2)_k}{k!\,(2k+2m+1)} \tag{46}$$

$$n,\ m\in\mathbb{N}_0$$

$$\frac{\Gamma(1+b)\,\Gamma(m-b-1/2)}{\sqrt{\pi}}=\frac{(2m)!}{2^{2m-1}\,m!}\sum_{k=0}^{\infty}\frac{(-b)_k}{k!\,(2k+2m-2b-1)} \tag{47}$$

$$b>-1\,,\ \ m-b>1/2\,,\ \ m\in\mathbb{N}_0$$

$$\frac{\Gamma(x_1)\,\Gamma(x_2)\ \ldots\,\Gamma(x_n)}{\sqrt{\pi}}=\frac{1/2}{x_1\,x_2\,\ldots\,x_n}\prod_{k=1}^{\infty}\frac{k^{n-1}(2k+1)}{2\,(k+x_1)\ldots(k+x_n)} \tag{48}$$

$$n\in\mathbb{N}\,,\ \ x_1,\ x_2,\ \ldots,\ x_n>0\,,\ \ x_1+\ldots+x_n=1/2$$

$$\pi = 11 \prod_{n=1}^{\infty} \frac{n^2 (121\,(2n-1)^2 - 81)}{(2n-1)^2\,(121\,n^2 - 9)} - \frac{44}{3} \prod_{n=1}^{\infty} \frac{n^2 (121\,(2n-1)^2 - 81)^3}{11^4\,(2n-1)^6\,(121\,n^2 - 9)} \tag{49}$$

$$\frac{1}{\pi} = \frac{3}{11} \prod_{n=1}^{\infty} \frac{(2n-1)^2\,(121\,n^2 - 1)}{n^2 (121\,(2n-1)^2 - 25)} - \frac{4}{11} \prod_{n=1}^{\infty} \frac{(121\,n^2 - 1)\,(121\,(2n-1)^2 - 81)^2}{11^4\,n^2 (2n-1)^2\,(121\,(2n-1)^2 - 25)} \tag{50}$$

$$\pi = \frac{(n+2)}{2} \operatorname{sen}\left(\frac{2\pi}{n+2}\right) + 2\,(n+2) \int_0^{\operatorname{sen}(\pi/(n+2))} \frac{x^2}{\sqrt{1-x^2}}\, dx\ ,\ n \in \mathbb{N} \tag{51}$$

$$\pi = (n+2) \tan\left(\frac{\pi}{n+2}\right) - (n+2) \int_0^{\operatorname{sen}(\pi/(n+2))} \frac{x^2}{(1-x^2)^{3/2}}\, dx\ ,\ n \in \mathbb{N} \tag{52}$$

$$\pi = \frac{(n+2)}{2} \operatorname{sen}\left(\frac{2\pi}{n+2}\right) + 2\,(n+2) \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!\,(2k+3)} \left(\operatorname{sen}\left(\frac{\pi}{n+2}\right)\right)^{2k+3}\ ,\ n \in \mathbb{N} \tag{53}$$

$$\pi = (n+2) \tan\left(\frac{\pi}{n+2}\right) - (n+2) \sum_{k=0}^{\infty} \frac{(3/2)_k}{k!\,(2k+3)} \left(\operatorname{sen}\left(\frac{\pi}{n+2}\right)\right)^{2k+3}\ ,\ n \in \mathbb{N} \tag{54}$$

$$\pi = 3\sqrt{3} \left(\frac{1}{4} + \sum_{k=0}^{\infty} \frac{(1/2)_k\,(3/4)^{k+1}}{k!\,(2k+3)}\right) \tag{55}$$

$$\pi = 2^n\,s_n + 2^n \sum_{k=0}^{\infty} \frac{(1/2)_k\,s_{n+1}^{2k+3}}{k!\,(2k+3)\,2^{2k}}\ ,\ n \in \mathbb{N} \tag{56}$$

$$\pi = \frac{7n}{2} \prod_{k=1}^{\infty} \left(1 - \left(\frac{7n-4}{7n\,(2k-1)}\right)^2\right) + 14\,n \sum_{k=0}^{\infty} \frac{(1/2)_k\,a_n^{2k+3}}{k!\,(2k+3)}\ ,\ n \in \mathbb{N} \tag{57}$$

$$a_n = \prod_{k=1}^{\infty} \left(1 - \left(\frac{7n-2}{7n\,(2k-1)}\right)^2\right)\ ,\ n \in \mathbb{N}$$

$$\frac{\pi^3}{\lambda^4} = 36 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \operatorname{sen}\left(\frac{n\pi}{\lambda}\right)\ ,\ \lambda = \sqrt[3]{\frac{1}{3} + \sqrt[3]{\frac{1}{3} + \sqrt[3]{\frac{1}{3} + \ldots}}} \tag{58}$$

Sean

$$a = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\operatorname{sh}(\pi)} = 0.2720290549\ \ldots \tag{59}$$

$$a = 1 + \frac{i}{2}\left(\psi\left(1 - \frac{i}{2}\right) - \psi\left(\frac{1-i}{2}\right)\right) - \frac{i}{2}\left(\psi\left(1 + \frac{i}{2}\right) - \psi\left(\frac{1+i}{2}\right)\right) \tag{60}$$

$$a = 2 \sum_{n=1}^{\infty} (-1)^{n-1} \left(1 - \left(1 - 2^{1-2n}\right) \zeta(2n)\right) \tag{61}$$

$$b = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{\pi}{\operatorname{th}(\pi)} = 3.1533480949\ \ldots \tag{62}$$

$$b = 1 - i(\psi(1+i) - \psi(1-i)) \tag{63}$$

$$b = 2 + 2 \sum_{n=1}^{\infty} (-1)^{n-1} (\zeta(2n) - 1) \tag{64}$$

se tiene:

$$x_{n+1} = \ln\left(\frac{b}{a} + \frac{1}{a} x_n\right) \ , \ x_1 = 0 \implies \lim_{n\to\infty} x_n = \pi \tag{65}$$

$$y_{n+1} = b - a\, e^{-y_n} \ , \ y_1 = 0 \implies \lim_{n\to\infty} y_n = \pi \tag{66}$$

$$z_{n+1} = b\, \text{th}(z_n) \ , \ z_1 = b \implies \lim_{n\to\infty} z_n = \pi \tag{67}$$

$$\pi = \ln\left(\frac{b}{a} + \frac{1}{a} \ln\left(\frac{b}{a} + \frac{1}{a} \ln\left(\frac{b}{a} + \ldots\right)\right)\right) \tag{68}$$

$$\pi = b - a\, e^{-b + a\, e^{-b+a\, e^{-b+\ldots}}} \tag{69}$$

$$a\, e^{\pi} = b + \pi \tag{70}$$

$$a + b = \frac{\pi}{\text{th}(\pi/2)} \tag{71}$$

$$b - a = \pi\, \text{th}\left(\frac{\pi}{2}\right) \tag{72}$$

$$b^2 - a^2 = (b + a)\,(b - a) = \pi^2 \tag{73}$$

$$\frac{b}{a} = \text{ch}(\pi) \tag{74}$$

$$a\, b = \frac{\pi^2\, \text{ch}(\pi)}{(\text{sh}(\pi))^2} \tag{75}$$

$$\frac{\pi^2}{4} = \sum_{n=0}^{\infty}\left[\frac{1}{2\, n(n+1) + 1} + 2 \sum_{k=0}^{n} \frac{1}{((2\, k + 2)^2 + 1)\,(2\,(n - k)\,(n - k + 1) + 1)}\right] \tag{76}$$

Sea

$$\alpha = \prod_{n=1}^{\infty} \frac{n(n+1)\,((4\, n - 2)^2 + 1)}{(4\, n^2 - 1)^2} + \prod_{n=1}^{\infty} \frac{(n+1)^2\,(16\, n^2 + 1)}{(2\, n + 1)^4} \tag{77}$$

$$\alpha = \frac{1}{2} \prod_{n=1}^{\infty} \frac{n^2((4\, n - 2)^2 + 1)}{(2\, n - 1)^3\,(2\, n + 1)} + \frac{1}{4} \prod_{n=1}^{\infty} \frac{n^2(16\, n^2 + 1)}{(4\, n^2 - 1)^2} \tag{78}$$

$$\alpha = \frac{\pi}{4} e^{\pi/4} = 1.7225981236 \ldots \tag{79}$$

se tiene:

$$\pi = 4\, \alpha\, e^{-\alpha\, e^{-\alpha\, e^{-\alpha \ldots}}} \tag{80}$$

$$\pi^2 = 6\, \text{Li}_2\left(e^{-1}\right) + 6\, \text{Li}_2\left(1 - e^{-1}\right) - 6 \ln\left(1 - e^{-1}\right) \tag{81}$$

$$\pi^2 = 6\, \text{Li}_2(a) + 6\, \text{Li}_2(1 - a) + 6\, a \ln(1 - a) \tag{82}$$

$$a = e^{-e^{-e^{-\cdots}}} = 0.567143 \ldots, \ a\, e^a = 1 \tag{83}$$

$$\pi^2 = 6\, \text{Li}_2(z) + 6\, \text{Li}_2(1 - z) + 18\, (\ln z)^2 \tag{84}$$

$$z = \frac{1}{6}\left(108 + 12\sqrt{93}\right)^{1/3} - 2\left(108 + 12\sqrt{93}\right)^{-1/3} \tag{85}$$

$$\pi^2 = 6\, \text{Li}_2(z) + 6\, \text{Li}_2(1 - z) + 2\, (\ln z)^2 \tag{86}$$

$$z = 1 - \frac{1}{6}\left(108 + 12\sqrt{93}\right)^{1/3} + 2\left(108 + 12\sqrt{93}\right)^{-1/3} \tag{87}$$

$$\pi^2 = 6\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_2(1 - z) + 9\,(\ln z)^2 \tag{88}$$

$$z = \frac{1}{3} + \frac{1}{6}\left(44 + 12\sqrt{69}\right)^{1/3} - \frac{10}{3}\left(44 + 12\sqrt{69}\right)^{-1/3} \tag{89}$$

$$\pi^2 = 6\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_2(1 - z) + 4\,(\ln z)^2 \tag{90}$$

$$z = \frac{2}{3} - \frac{1}{6}\left(44 + 12\sqrt{69}\right)^{1/3} + \frac{10}{3}\left(44 + 12\sqrt{69}\right)^{-1/3} \tag{91}$$

$$\pi^2 = 6\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_2(1 - z) + \frac{6\,x}{y}\,(\ln z)^2 \tag{92}$$

$$x > 0\,,\ \ y > 0\,,\ \ 0 < z < 1\,,\ \ z^x = (1 - z)^y \tag{93}$$

$$\pi^2 = 24\,\ln(\mathrm{sen}(\theta))\,\ln(\cos(\theta)) + 6 + 6\sum_{n=2}^{\infty}\frac{(\mathrm{sen}(\theta))^{2\,n} + (\cos(\theta))^{2\,n}}{n^2}\,,\ \ 0 < \theta < \frac{\pi}{2} \tag{94}$$

$$\pi^2 = 6\,\mathrm{Li}_2(z) + 6\,\mathrm{Li}_2(z^m) + 6\,m\,(\ln z)^2 \tag{95}$$

$$m > 0\,,\ \ 0 < z < 1\,,\ \ z^m = 1 - z \tag{96}$$

$$\pi^2 = 6\,\mathrm{Li}_2\!\left(\frac{1}{1 + x}\right) + 6\,\mathrm{Li}_2\!\left(\frac{x}{1 + x}\right) + 6\,m\,(m - 1)\,(\ln x)^2 \tag{97}$$

$$m = 2,\ 3,\ 4,\ \ldots,\ \ x = \sqrt[m]{1 + \sqrt[m]{1 + \sqrt[m]{1 + \ldots}}} \tag{98}$$

Sea

$$a = \frac{1}{2}\,e^{-\frac{1}{2}e^{-\frac{1}{2}e^{-\cdots}}} = 0.3517337\ldots,\ \ 2\,a = e^{-a} \tag{99}$$

se tiene:

$$\pi = \int_0^{\infty}\frac{\cos x}{a^2 + x^2}\,dx \tag{100}$$

$$\pi = \frac{1}{a}\int_0^{\infty}\frac{\cos(a\,x)}{1 + x^2}\,dx \tag{101}$$

$$\pi = \frac{2\,a}{1 - 2\,a}\int_0^{\infty}\frac{1 - \cos(a\,x)}{a^2 + x^2}\,dx = \frac{4\,a}{1 - 2\,a}\int_0^{\infty}\frac{(\mathrm{sen}(x/2))^2}{a^2 + x^2}\,dx \tag{102}$$

$$\pi = \frac{2}{1 - 2\,a}\int_0^{\infty}\frac{1 - \cos(a\,x)}{1 + x^2}\,dx = \frac{4}{1 - 2\,a}\int_0^{\infty}\frac{(\mathrm{sen}(a\,x/2))^2}{1 + x^2}\,dx \tag{103}$$

$$\pi = 2\int_0^{\infty}\frac{x\,\mathrm{sen}\,x}{(a^2 + x^2)^2}\,dx \tag{104}$$

$$\pi = \frac{1}{a^2} - 2\int_0^{\infty}\frac{x\,(1 - \mathrm{sen}\,x)}{(a^2 + x^2)^2}\,dx \tag{105}$$

$$\pi = 2\int_0^{\infty}\frac{x\cos(x^2)}{a^2 + x^4}\,dx \tag{106}$$

$$\pi\left(\frac{3}{4}-\frac{1}{16\,a^2}\right)=-\sum_{n=1}^{\infty}\frac{a^{2n-1}}{(2\,n)\,!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2\,k-1}+\frac{1}{a}\int_1^{\infty}\frac{\cos(a\,x)}{1+x^2}\,dx \tag{107}$$

$$\pi=2\int_0^{\infty}\int_0^{\infty}\frac{y\cos x}{(x^2+y^2)^2}\,dy\,dx \tag{108}$$

$$\pi=\frac{2}{1-2\,a}\int_0^{\infty}\int_0^{a}\frac{x\,\mathrm{sen}(x\,y)}{1+x^2}\,dy\,dx \tag{109}$$

$$\pi=\frac{1}{a}\int_0^{\infty}\int_0^{\infty}e^{-x\,y}\,\mathrm{sen}(a\,y)\cos(x)\,dy\,dx \tag{110}$$

fórmulas con números de Stirling de primera clase:

$$(-1)^m\left(\frac{\pi}{4}\right)^{2m}=\sum_{n=2m}^{\infty}\frac{s(n,\,2\,m)}{n\,!\,2^n}\left(a_n+b_n\sqrt{2}\right),\quad m\in\mathbb{N} \tag{111}$$

$$(-1)^{m-1}\left(\frac{\pi}{4}\right)^{2m-1}=\sum_{n=2m-1}^{\infty}\frac{s(n,\,2\,m-1)}{n\,!\,2^n}\left(c_n+d_n\sqrt{2}\right),\quad m\in\mathbb{N} \tag{112}$$

donde

$$s(n,\,k)=0\;,\;\;\forall\,n\le k-1 \tag{113}$$

$$s(n+1,\,k)=s(n,\,k-1)-n\,s(n,\,k)\;,\;\;1\le k\le n \tag{114}$$

$$s(n+1,\,1)=-n\,s(n,\,1)\;,\;\;s(n+1,\,n+1)=s(n,\,n) \tag{115}$$

$$x(x-1)\,\ldots\,(x-n+1)=\sum_{k=0}^{n}s(n,\,k)\,x^k \tag{116}$$

$$s(1,\,1)=1,\;s(2,\,1)=-1,\;s(2,\,2)=1,\;s(3,\,1)=2,\;s(3,\,2)=-3,\;s(3,\,3)=1,\;\ldots \tag{117}$$

$$x_n=\begin{pmatrix}a_n\\b_n\\c_n\\d_n\end{pmatrix},\quad A=\begin{pmatrix}2&-2&1&0\\-1&2&0&1\\0&-2&2&-2\\-1&0&-1&2\end{pmatrix},\quad x_0=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \tag{118}$$

$$x_{n+1}=A\,x_n\;,\;\;n=0,\,1,\,2,\,3,\,\ldots \tag{119}$$

$$x_n=A^n\,x_0\;,\;\;n=0,\,1,\,2,\,3,\,\ldots \tag{120}$$

$$G+\frac{\pi}{4}=\int_0^1\frac{\ln(e/x)}{1+x^2}\,dx \tag{121}$$

$$G-\frac{\pi}{4}=-\int_0^1\frac{\ln(e\,x)}{1+x^2}\,dx \tag{122}$$

$$G-\frac{\pi}{4}=-e\int_0^{e}\frac{\ln(x)}{e^2+x^2}\,dx \tag{123}$$

$$G+\frac{\pi}{4}=e\int_e^{\infty}\frac{\ln(x)}{e^2+x^2}\,dx \tag{124}$$

$$\frac{\pi}{4}=\ln\left(\prod_{n=0}^{\infty}\mathrm{ch}\left(\frac{1}{2\,n+1}\right)\right)+\ln\left(\prod_{n=0}^{\infty}\left(1+(-1)^n\,\mathrm{th}\left(\frac{1}{2\,n+1}\right)\right)\right) \tag{125}$$

$$\frac{1-x^2}{2\,(1+x^2)}\ln\left(\frac{1-x}{1+x}\right)+\frac{x\ln x}{1+x^2}+\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{1}{(4\,n+1)\,(4\,n+3)}\left(\frac{4\,x^3}{1+x^2}\,x^{4n}+\frac{2\,(1-x)^3}{(1+x)\,(1+x^2)}\left(\frac{1-x}{1+x}\right)^{4n}\right),\quad 0\le x\le 1 \tag{126}$$

$$\frac{5}{8}\pi - \frac{3}{4}\ln 3 - \ln 2 = \sum_{n=0}^{\infty} \frac{2^{-4n} + 3^{-4n-1}}{(4n+1)(4n+3)} \tag{127}$$

$$\pi \cos^{-1} x = \sum_{n=0}^{\infty} \frac{2^{n+1}(n!)^2}{(2n+2)!}\left(\left(1+\sqrt{1-x^2}\right)^{n+1} - \left(1-\sqrt{1-x^2}\right)^{n+1}\right), \quad 0 < x < 1 \tag{128}$$

$$\frac{\pi}{\sqrt{k^2-m^2}} \cos^{-1}\left(\frac{m}{k}\right) = \sum_{n=0}^{\infty} \frac{2^{n+1}(n!)^2}{(2n+2)!\,k^{n+1}} u_n(m, k) \tag{129}$$

$$u_{n+2}(m, k) = 2k\,u_{n+1}(m, k) - m^2\,u_n(m, k) \tag{130}$$

$$u_0(m, k) = 2, \quad u_1(m, k) = 4k, \quad m, k \in \mathbb{N}, \quad m < k \tag{131}$$

$$\frac{2^{2k-1} B_k}{(2k)!}\pi^{2k} = \sum_{n=1}^{\infty}\left(-\frac{1}{2k}\right)^{n-1} \zeta(2k(n+1)) + \sum_{n=1}^{\infty} \frac{2k}{2k\,n^{2k}+1}, \quad k \in \mathbb{N} \tag{132}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1} \zeta(2n+2) + \sum_{n=1}^{\infty} \frac{2}{2n^2+1} \tag{133}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty}\left(-\frac{1}{4}\right)^{n-1} \zeta(4n+4) + \sum_{n=1}^{\infty} \frac{4}{4n^4+1} \tag{134}$$

$$\frac{e^{-(2k+1)\gamma}}{\pi} = \left(\frac{1}{2}\right)_{k+1}^{2} \left(k+\frac{1}{2}\right)_{n+1}^{2} * \frac{e^{-(2k+1)H_n}}{(n!)^2} \prod_{m=n+1}^{\infty}\left(1+\frac{2k+1}{2m}\right)^2 e^{-(2k+1)/m} \tag{135}$$

$$n \in \mathbb{N}, \quad k \in \mathbb{N}_0$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n (1/3)_n (2/3)_n 3^{3n}}{(3/2)_n\, n!\, 2^{2n}} P(n) \tag{136}$$

$$P(n) = \sum_{k=0}^{2n} \frac{\binom{2n}{k}}{2n+2k+1}\left(2\left(\frac{1}{5}\right)^{2n+2k+1} + \left(\frac{1}{7}\right)^{2n+2k+1} + 2\left(\frac{1}{8}\right)^{2n+2k+1}\right) \tag{137}$$

$$\frac{\pi}{4} = -\tan^{-1}\left(\frac{1}{x_1}\right) - \tan^{-1}\left(\frac{1}{x_2}\right) - \tan^{-1}\left(\frac{1}{x_3}\right) \tag{138}$$

$$x_1 = \sqrt[3]{11 + 10\sqrt[3]{11 + 10\sqrt[3]{11 + \ldots}}} \tag{139}$$

$$x_2 = -\frac{x_1}{2} + \frac{1}{2}\sqrt{10 - \frac{33}{x_1}} \tag{140}$$

$$x_3 = -\frac{x_1}{2} - \frac{1}{2}\sqrt{10 - \frac{33}{x_1}} \tag{141}$$

$$\frac{\pi^2}{6} + \zeta(4m) = 2\zeta(2m+1) + \sum_{n=2}^{\infty}\left(\frac{n^{2m}-n}{n^{2m+1}}\right)^2, \quad m \in \mathbb{N} \tag{142}$$

$$G + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} \beta(n+1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \ln\left(\frac{2n+2}{2n+1}\right) \tag{143}$$

$\beta(x)$, función beta de dirichlet

$$\frac{\pi}{4} = \int_0^a \frac{(1+x)\,e^x}{1+x^2\,e^{2x}}\,dx\,,\quad a = e^{-e^{-e^{-\cdots}}} = 0.567143\ldots \tag{144}$$

$$\frac{\pi}{4} = \int_1^e \frac{1}{x\,(1+(\ln x)^2)}\,dx \tag{145}$$

$$\frac{\pi}{\sqrt{6}} = 1 + \sum_{n=1}^{\infty} \frac{n!}{(n+1)\left((n+1)\sqrt{h_n} + \sqrt{h_{n+1}}\right)} = 1 + \frac{1!}{2\left(2\sqrt{1} + \sqrt{5}\right)} + \frac{2!}{3\left(3\sqrt{5} + \sqrt{49}\right)} + \ldots \tag{146}$$

$$h_n = \sum_{k=1}^{n} \left(\frac{n!}{k}\right)^2\,,\quad h_{n+1} = (n+1)^2\,h_n + (n!)^2\,,\quad h_1 = 1 \tag{147}$$

$$\frac{\pi\sqrt{3}}{16} = \sum_{n=0}^{\infty} \frac{(1/5)_n\,(2/5)_n\,(3/5)_n\,(4/5)_n}{(1/2)_n\,(3/4)_n\,(5/4)_n\,n!} \left(\frac{3125}{2304}\right)^n a_n \tag{148}$$

$$a_n = \sum_{k=0}^{4n} \binom{4n}{k} \frac{(-1)^k\,(n+k+1)}{3^{2k}\,(4n+4k+1)\,(4n+4k+3)} \tag{149}$$

$$\frac{\pi}{4} = \int_b^{\infty} \frac{e^x - 1}{e^{2x} - 2\,e^x(a+x) + (a+x)^2 + 1}\,dx\,,\quad a > 0,\ b = \ln(1+a+b) \tag{150}$$

$$b = \ln(1 + a + \ln(1 + a + \ln(1 + a + \ldots))) \tag{151}$$

$$\frac{\pi}{2} + \ln\left(\frac{1+x}{x(1-x)}\right) = 4\sum_{n=1}^{\infty} \frac{1}{4n-3}\left(x^{4n-3} + \left(\frac{1-x}{1+x}\right)^{4n-3}\right),\quad 0 < x < 1 \tag{152}$$

$$\frac{\pi}{2} + \ln(6) = 4\sum_{n=1}^{\infty} \frac{1}{4n-3}\left(2^{-(4n-3)} + 3^{-(4n-3)}\right) \tag{153}$$

$$\ln 2 = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{n+1}{(2n-1)\,(2n^2+1)+n}\right) \tag{154}$$

$$\frac{\pi}{4} - \ln 2 = \tan^{-1}\left(\frac{1}{3}\right) - \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{n+1}{(2n-1)\,(2n^2+1)+n}\right) \tag{155}$$

$$\frac{\pi}{4} - \ln 2 = \sum_{n=2}^{\infty} (-1)^n \tan^{-1}\left(\frac{n}{n^2+1+(-1)^n}\right) \tag{156}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}\left((x+i(1-x))^{2^n}\right)}{1+\mathrm{Re}\left((x+i(1-x))^{2^n}\right)}\right),\quad 0 < x < 1 \tag{157}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}\left((p+i(q-p))^{2^n}\right)}{q^{2^n}+\mathrm{Re}\left((p+i(q-p))^{2^n}\right)}\right),\quad 0 < p < q\,;\ p, q \in \mathbb{N} \tag{158}$$

$$\frac{2^{m+1}}{\pi}\,\frac{s_m}{c_{m+1}} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^{2m+4}\,n^2}\right),\quad m \in \mathbb{N} \tag{159}$$

$$\pi^4 \prod_{n=1}^{\infty}\left(1 - \frac{1}{2^{2m+6}\,n^2}\right)^4 = 2^{4m+8}\,(6 + c_m - 4\,c_{m+1})\,,\quad m \in \mathbb{N} \tag{160}$$

$$\pi^4\left(2^{m+2}-1\right)^4 \prod_{n=1}^{\infty}\left(1 - \frac{(2^{m+2}-1)^2}{2^{2m+6}\,n^2}\right)^4 = 2^{4m+8}\,(6 + c_m + 4\,c_{m+1})\,,\quad m \in \mathbb{N} \tag{161}$$

$$\pi = \frac{1}{2}\left(\frac{(m-1)!}{(1/2)_m}\right)^2 \prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2m)^{(-1)^k\binom{n}{k}}\right)^{2^{-n}}, \quad m \in \mathbb{N} \tag{162}$$

$$\frac{1}{\pi} = \frac{1}{2}\left(\frac{(1/2)_{m-1}}{(m-1)!}\right)^2 \prod_{n=0}^{\infty}\left(\prod_{k=0}^{n}(k+2m-1)^{(-1)^k\binom{n}{k}}\right)^{2^{-n}}, \quad m \in \mathbb{N} \tag{163}$$

$$G = \cfrac{1}{1+\cfrac{1^4}{8+\cfrac{3^4}{16+\cfrac{5^4}{24+\cfrac{7^4}{32+\ldots}}}}} \tag{164}$$

$$\frac{\pi^2}{16}+\frac{G}{2} = \cfrac{1}{1-\cfrac{1^4}{1^2+5^2-\cfrac{5^4}{5^2+9^2-\cfrac{9^4}{9^2+13^2-\ldots}}}} \tag{165}$$

$$\frac{\pi^2}{16}-\frac{G}{2} = \cfrac{1}{1-\cfrac{3^4}{3^2+7^2-\cfrac{7^4}{7^2+11^2-\cfrac{11^4}{11^2+15^2-\ldots}}}} \tag{166}$$

$$\frac{\pi}{4} = \sum_{n=0}^{m}\frac{(-1)^n}{2n+1} - \cfrac{(-1)^m}{1+\cfrac{(3+2m)^2}{2+\cfrac{(5+2m)^2}{2+\cfrac{(7+2m)^2}{2+\ldots}}}}, \quad m \in \mathbb{N} \tag{167}$$

$$\frac{\pi}{3} = \ln\left(\frac{(1+a)\left(1+a+\sqrt{3}\,(a-1)\right)}{(1-a)\left(1-a+\sqrt{3}\,(a+1)\right)}\right) - \sum_{n=3}^{\infty}2^{n-1}\int_a^u \frac{1}{1+x^{2^{n-1}}}\,dx \tag{168}$$

$$u = \frac{1+a\sqrt{3}}{\sqrt{3}-a}, \quad 1 < a < \sqrt{3} \tag{169}$$

$$\pi = 2\ln\left(\frac{(a+1)^2(2a-1)(3a-1)}{(a-1)^2(2+a)(3+a)}\right) - 4\sum_{n=1}^{\infty}2^n\left(\int_a^u \frac{1}{1+x^{2^{n+1}}}\,dx + \int_a^v \frac{1}{1+x^{2^{n+1}}}\,dx\right) \tag{170}$$

$$u = \frac{1+2a}{2-a}, \quad v = \frac{1+3a}{3-a}, \quad 1 < a < 2 \tag{171}$$

$$\pi = 4\ln\left(\frac{10}{3}\right) - 4\sum_{n=1}^{\infty}2^n\left(\int_{3/2}^{8} \frac{1}{1+x^{2^{n+1}}}\,dx + \int_{3/2}^{11/3} \frac{1}{1+x^{2^{n+1}}}\,dx\right) \tag{172}$$

$$\tan^{-1}\left(\frac{\sqrt{7}-\sqrt{3}}{2}\right) = \frac{\pi}{6} + \sum_{n=1}^{\infty}(-1)^n \tan^{-1}\left(\frac{\left(\left(\sqrt{7}-\sqrt{3}\right)/2\right)^{3^n}}{1-\left(\left(\sqrt{7}-\sqrt{3}\right)/2\right)^{2\cdot 3^n}}\right) \tag{173}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty}3^n\int_0^{1/\sqrt{3}} \frac{\left(1-2x^{2\cdot 3^n}\right)x^{2(3^n-1)}}{1-x^{2\cdot 3^n}+x^{4\cdot 3^n}}\,dx \tag{174}$$

$$\tan^{-1}\left(\frac{2\,\mathrm{sh}(\pi/3)}{\sqrt{3}}\right) = \frac{\pi}{4} - \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{18n^2}\right) + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2}{9(2n-1)^2}\right) \tag{175}$$

$$\tan^{-1}\left(\sqrt{2}\ \mathrm{sh}(\pi/4)\right) = \frac{\pi}{4} - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{32\,n^2}\right) + \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{8\,(2\,n-1)^2}\right) \tag{176}$$

$$\tan^{-1}(2\ \mathrm{sh}(\pi/6)) = \frac{\pi}{4} - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{72\,n^2}\right) + \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{2}{18\,(2\,n-1)^2}\right) \tag{177}$$

$$\tan^{-1}\left(\frac{2\ \mathrm{sh}\left(\pi\, 2^{-k-1}\right)}{s_k}\right) = \frac{\pi}{4} - \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{2^{2k+3}\, n^2}\right) + \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{2^{2k+1}\,(2\,n-1)^2}\right),\ k \in \mathbb{N} \tag{178}$$

$$\frac{\pi}{6} + \sum_{n=2}^{\infty} (-1)^{n-1} \tan^{-1}\left(\frac{\sqrt{3}}{3^n}\right) = \sqrt{3} \sum_{n=1}^{\infty} \left(\frac{3^{2n-2}}{(4\,n-3)\,(3^{4n-3}+1)} - \frac{3^{2n-1}}{(4\,n-1)\,(3^{4n-1}+1)}\right) \tag{179}$$

$$\frac{\pi}{6\sqrt{3}} + \frac{1}{\sqrt{3}} \sum_{n=2}^{m} \tan^{-1}\left(\frac{1}{n\sqrt{3}}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2\,n-1)\,3^n} H_{m,2n-1}\ ,\ \ m \in \mathbb{N} \tag{180}$$

$$\frac{\pi}{4} + \sum_{n=2}^{m} \tan^{-1}\left(\frac{5\,n}{6\,n^2-1}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(2^{-(2n-1)} + 3^{-(2n-1)}\right)}{(2\,n-1)} H_{m,2n-1}\ ,\ \ m \in \mathbb{N} \tag{181}$$

$$\frac{\pi}{4} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((x-1+i\,x)^n)}{1+\mathrm{Re}((x-1+i\,x)^n)}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\ \mathrm{Im}((x-1+i\,x)^n)}{n\left(1+(1-2\,x+2\,x^2)^2 - 2\ \mathrm{Re}((x-1+i\,x)^n)\right)} \tag{182}$$

$$0 < x < 1$$

$$\frac{\pi\sqrt{2}}{6} = \sqrt{2}\ \ln\left(2+\sqrt{3}\right) - \sum_{n=0}^{\infty} \binom{2\,n}{n} 2^{-3n} \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{(-1)^k \binom{n}{k} \binom{k}{m}}{(2\,m+1)\,(2\,k-2\,m+1)} \tag{183}$$

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{\sqrt{2\sqrt{2+\sqrt{2}} - 2 - \sqrt{2}}}{\sqrt{2+\sqrt{2}} - 1}\right) + \sum_{n=1}^{\infty} \binom{2\,n}{n} \frac{(-1)^{n-1}}{n\,2^{2n}}\ \mathrm{sen}\left(\frac{n\,\pi}{4}\right) \tag{184}$$

$$\frac{\pi}{3} = \tan^{-1}\left(\frac{\sqrt{2\sqrt{3} - 3}}{\sqrt{3} - 1}\right) + \sum_{n=1}^{\infty} \binom{2\,n}{n} \frac{(-1)^{n-1}}{n\,2^{2n}}\ \mathrm{sen}\left(\frac{n\,\pi}{3}\right) \tag{185}$$

$$\frac{\pi}{4} = \sqrt{2\sqrt{2+\sqrt{2}} - 2 - \sqrt{2}} + \tan^{-1}\left(\frac{\sqrt{2\sqrt{2+\sqrt{2}} - 2 - \sqrt{2}}}{\sqrt{2+\sqrt{2}} - 1}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(-1/2)_n}{n!\,n}\ \mathrm{sen}\left(\frac{n\,\pi}{4}\right) \tag{186}$$

$$\frac{\pi}{3} = \sqrt{2\sqrt{3} - 3} + \tan^{-1}\left(\sqrt{\frac{\sqrt{3}}{2}}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(-1/2)_n}{n!\,n}\ \mathrm{sen}\left(\frac{n\,\pi}{3}\right) \tag{187}$$

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{\left(\sqrt{3}\right)^{3^n}}{3^{3^n} - 1}\right) \tag{188}$$

$$\frac{\pi}{12} = \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{\left(2-\sqrt{3}\right)^{3^n}}{1-\left(2-\sqrt{3}\right)^{2\cdot 3^n}} \right) \tag{189}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \left(\tan^{-1}\left(\frac{2^{-3^n}}{1-4^{-3^n}} \right) + \tan^{-1}\left(\frac{3^{-3^n}}{1-9^{-3^n}} \right) \right) \tag{190}$$

$$\frac{\pi}{4} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((-1+i)^n)}{2^n + \mathrm{Re}((-1+i)^n)} \right) = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((-1+i)^{2n-1})}{2^{2n-1} - \mathrm{Re}((-1+i)^{2n-1})} \right) \tag{191}$$

$$\frac{\pi}{4} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((-1+(k-1)\,i)^n)}{k^n + \mathrm{Re}((-1+(k-1)\,i)^n)} \right) = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((-1+(k-1)\,i)^{2n-1})}{k^{2n-1} - \mathrm{Re}((-1+(k-1)\,i)^{2n-1})} \right) \tag{192}$$

$$k \in \mathbb{N} - \{1\}$$

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^n)}{\left(\sqrt{3}+1\right)^n + \mathrm{Re}((1+i)^n)} \right) = \frac{\pi}{6} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^{2n-1})}{\left(\sqrt{3}+1\right)^{2n-1} - \mathrm{Re}((1+i)^{2n-1})} \right) \tag{193}$$

$$\begin{gathered} \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^n)}{3^n + \mathrm{Re}((1+i)^n)} \right) + \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^n)}{4^n + \mathrm{Re}((1+i)^n)} \right) = \\ \frac{\pi}{4} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^{2n-1})}{3^{2n-1} - \mathrm{Re}((1+i)^{2n-1})} \right) + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\mathrm{Im}((1+i)^{2n-1})}{4^{2n-1} - \mathrm{Re}((1+i)^{2n-1})} \right) \end{gathered} \tag{194}$$

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{\left(3-\sqrt{3}\right)^n \mathrm{Im}((1+i)^n)}{2^n + \left(3-\sqrt{3}\right)^n \mathrm{Re}((1+i)^n)} \right) = \frac{\pi}{3} + \sum_{n=2}^{\infty} \tan^{-1}\left(\frac{\left(3-\sqrt{3}\right)^{2n-1} \mathrm{Im}((1+i)^{2n-1})}{2^{2n-1} - \left(3-\sqrt{3}\right)^{2n-1} \mathrm{Re}((1+i)^{2n-1})} \right) \tag{195}$$

$$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{1 + (-1)^n\, 2^{(3^n-1)/2}}{2^{3^n} + (-1)^n\, 2^{(3^n-1)/2}} \right) \tag{196}$$

$$\pi = 4 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n}{2n-2k+1} \left(\frac{1 + (-1)^k\, 2^{\left(3^k-1\right)/2}}{2^{3^k} + (-1)^k\, 2^{\left(3^k-1\right)/2}} \right)^{2n-2k+1} \tag{197}$$

$$\pi = 4 \sum_{k=0}^{\infty} \sum_{n=0}^{k} \frac{(-1)^k}{2n+1} \left(\frac{1 + (-1)^{k-n}\, 2^{\left(3^{k-n}-1\right)/2}}{2^{3^{k-n}} + (-1)^{k-n}\, 2^{\left(3^{k-n}-1\right)/2}} \right)^{2n+1} \tag{198}$$

$$\pi = 4 \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{b^{3^n} B_n + 2 A_n B_n}{b^{2\cdot 3^n} + b^{3^n} A_n + A_n^2 - B_n^2} \right) \tag{199}$$

donde

$$A_{n+1} = A_n^3 - 3 A_n B_n^2 \,,\; B_{n+1} = 3 A_n^2 B_n - B_n^3 \,,\; A_0 = a \,,\; B_0 = b - a \tag{200}$$

$$a\,, b \in \mathbb{N} \,,\; a < b \tag{201}$$

$$\pi = 8 \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{\left(\sqrt{2}\right)^{3^n}}{\left(2+\sqrt{2}\right)^{3^n} - \left(2-\sqrt{2}\right)^{3^n}} \right) \tag{202}$$

$$\pi = 16 \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{\left(\sqrt{2-\sqrt{2}}\right)^{3^n}}{\left(2+\sqrt{2+\sqrt{2}}\right)^{3^n} - \left(2-\sqrt{2+\sqrt{2}}\right)^{3^n}}\right) \tag{203}$$

$$\pi = 2^{k+2} \sum_{n=0}^{\infty} (-1)^n \tan^{-1}\left(\frac{(s_k)^{3^n}}{(2+c_k)^{3^n} - (2-c_k)^{3^n}}\right), \quad k \in \mathbb{N} \tag{204}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 \left(\ln\left(1+x^2\right)\right)^n dx \tag{205}$$

$$\int_0^1 \left(\ln\left(1+x^2\right)\right)^n dx = \frac{1}{2}\int_0^1 \frac{(\ln(1+x))^n}{\sqrt{x}} dx = \frac{1}{2}\int_1^2 \frac{(\ln x)^n}{\sqrt{x-1}} dx = \frac{1}{2}\int_0^{\ln 2} \frac{e^x x^n}{\sqrt{e^x - 1}} dx \tag{206}$$

$$n \in \mathbb{N}_0$$

$$\pi = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_1^{\infty} \left(e^{(1+x^2)^{-1}} - 1\right)^n dx \tag{207}$$

$$\int_1^{\infty} \left(e^{(1+x^2)^{-1}} - 1\right)^n dx = \frac{1}{2}\int_1^{\infty} \frac{\left(e^{(1+x)^{-1}} - 1\right)^n}{\sqrt{x}} dx = \frac{1}{2}\int_2^{\infty} \frac{\left(e^{x^{-1}} - 1\right)^n}{\sqrt{x-1}} dx = \frac{1}{2}\int_0^{1/2} \frac{(e^x - 1)^n}{x\sqrt{x(1-x)}} dx, \quad n \in \mathbb{N} \tag{208}$$

$$\pi = 4 \sum_{n=1}^{\infty} (-1)^{n-1} H_n \int_0^1 \frac{x^{2n}}{\ln(1+x^2)} dx \tag{209}$$

$$\int_0^1 \frac{x^{2n}}{\ln(1+x^2)} dx = \frac{1}{2}\int_0^1 \frac{x^n}{\sqrt{x}\,\ln(1+x)} dx = \frac{1}{2}\int_1^2 \frac{(x-1)^{n-1/2}}{\ln x} dx = \int_0^{\ln 2} \frac{(e^x - 1)^{n-1/2} e^x}{2x} dx, \quad n \in \mathbb{N} \tag{210}$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^1 \left(\ln\left(\frac{1+\sqrt{2+2x^2+x^4}}{1+x^2}\right)\right)^{2n+1} dx \tag{211}$$

$$\begin{gathered}\int_0^1 \left(\ln\left(\frac{1+\sqrt{2+2x^2+x^4}}{1+x^2}\right)\right)^{2n+1} dx = \\ \frac{1}{2}\int_0^1 \left(\ln\left(\frac{1+\sqrt{2+2x+x^2}}{1+x}\right)\right)^{2n+1} \frac{1}{\sqrt{x}} dx = \frac{1}{2}\int_1^2 \left(\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\right)^{2n+1} \frac{1}{\sqrt{x-1}} dx\end{gathered} \tag{212}$$

$$n \in \mathbb{N}_0$$

$$\pi = 4 \sum_{n=0}^{\infty} \frac{1}{(2n)!} \int_0^1 \left(\ln\left(\frac{2+x^2+\sqrt{3+2x^2}}{1+x^2}\right)\right)^{2n} dx \tag{213}$$

$$\begin{gathered}\int_0^1 \left(\ln\left(\frac{2+x^2+\sqrt{3+2x^2}}{1+x^2}\right)\right)^{2n} dx = \\ \frac{1}{2}\int_0^1 \left(\ln\left(\frac{2+x+\sqrt{3+2x}}{1+x}\right)\right)^{2n} \frac{1}{\sqrt{x}} dx = \frac{1}{2}\int_1^2 \left(\ln\left(\frac{1+x+\sqrt{1+2x}}{x}\right)\right)^{2n} \frac{1}{\sqrt{x-1}} dx\end{gathered} \tag{214}$$

$$n \in \mathbb{N}_0$$

$$\pi = 4\sum_{n=0}^{\infty}\frac{1}{2n+1}\int_0^1\left(\operatorname{th}\left(\frac{1}{1+x^2}\right)\right)^{2n+1}dx \tag{215}$$

$$\int_0^1\left(\operatorname{th}\left(\frac{1}{1+x^2}\right)\right)^{2n+1}dx = \frac{1}{2}\int_0^1\left(\operatorname{th}\left(\frac{1}{1+x}\right)\right)^{2n+1}\frac{1}{\sqrt{x}}\,dx = \frac{1}{2}\int_1^2\left(\operatorname{th}\left(\frac{1}{x}\right)\right)^{2n+1}\frac{1}{\sqrt{x-1}}\,dx = \frac{1}{2}\int_{1/2}^1(\operatorname{th}(x))^{2n+1}\frac{1}{x\sqrt{x(x-1)}}\,dx,\ n\in\mathbb{N}_0 \tag{216}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1\frac{(1+x^2)^{n-1}}{e^{1+x^2}-1}\,dx \tag{217}$$

$$\int_0^1\frac{(1+x^2)^{n-1}}{e^{1+x^2}-1}\,dx = \int_0^1\frac{(1+x)^{n-1}}{2\,(e^{1+x}-1)\sqrt{x}}\,dx = \int_1^2\frac{x^{n-1}}{2\,(e^x-1)\sqrt{x-1}}\,dx,\ n\in\mathbb{N} \tag{218}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}\int_0^1\frac{(1+x^2)^{n-1}}{1-e^{-1-x^2}}\,dx \tag{219}$$

$$\int_0^1\frac{(1+x^2)^{n-1}}{1-e^{-1-x^2}}\,dx = \int_0^1\frac{(1+x)^{n-1}}{2\,(1-e^{-1-x})\sqrt{x}}\,dx = \int_1^2\frac{x^{n-1}}{2\,(1-e^{-x})\sqrt{x-1}}\,dx,\ n\in\mathbb{N} \tag{220}$$

$$\pi = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(4n-2)!} + 4\int_0^1\frac{(\operatorname{ch}1)\cos x + (\operatorname{sh}1)\,x\operatorname{sen}x}{1+x^2}\,dx \tag{221}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(4n-1)!} + 4\int_0^1\frac{(1-x^2)\,(\operatorname{sen}1)\operatorname{ch}x + 2\,(\cos 1)\,x\operatorname{sh}x}{(1+x^2)^2}\,dx \tag{222}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(4n-2)!} + 4\int_0^1\frac{(\cos 1)\operatorname{ch}x - (\operatorname{sen}1)\,x\operatorname{sh}x}{1+x^2}\,dx \tag{223}$$

$$\pi = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(4n-1)!} + 4\int_0^1\frac{(1-x^2)\,(\operatorname{sh}1)\cos x + 2\,(\operatorname{ch}1)\,x\operatorname{sen}x}{(1+x^2)^2}\,dx \tag{224}$$

$$\pi = -2\sum_{n=1}^{\infty}(-1)^{n-1}\,4^n\left(\frac{1}{(4n-1)\,(4n-1)!} + \frac{1}{(4n-2)\,(4n-2)!} + \frac{1/2}{(4n-3)\,(4n-3)!}\right) + 4\,e\int_0^1\frac{\cos x + x\operatorname{sen}x}{1+x^2}\,dx \tag{225}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(2n-1)!} + \frac{4}{e}\int_0^1\frac{(\cos(2x)\ - x\operatorname{sen}(2x))\,e^{x^2}}{1+x^2}\,dx \tag{226}$$

$$\pi = -\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{2n}}{(2n-1)\,(2n-1)!} + 4\,e\int_0^1\frac{(\cos(2x)\ + x\operatorname{sen}(2x))\,e^{-x^2}}{1+x^2}\,dx \tag{227}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,(m)_n}{n!\,n\,2^n}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1} + 2^{m+2}\int_0^1\frac{p_m(x)}{(1+x^2)\,(9+x^2)^m}\,dx \tag{228}$$

$$p_m(x) = \sum_{k=0}^{[m/2]}(-1)^k\binom{m}{2k}3^{m-2k}\,x^{2k} - \sum_{k=0}^{[(m-1)/2]}(-1)^k\binom{m}{2k+1}3^{m-2k-1}\,x^{2k+2},\ m\in\mathbb{N} \tag{229}$$

$$\pi = 2\int_0^1\frac{\operatorname{sh}2 - x\operatorname{sen}(2x)}{(1+x^2)\,((\operatorname{sh}1)^2 + (\operatorname{sen}x)^2)}\,dx - 8\sum_{n=1}^{\infty}e^{-2n}\int_0^1\frac{\cos(2nx) - x\operatorname{sen}(2nx)}{1+x^2}\,dx \tag{230}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\,2^{-2n}}{(2n-1)\,(4n-1)!} + 4\int_0^1\operatorname{sen}\left(\frac{1}{1+x^2}\right)\operatorname{ch}\left(\frac{x}{1+x^2}\right)dx \tag{231}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^n 2^{-2n}}{(2n-1)(4n-1)!} + 4\int_0^1 \operatorname{sh}\left(\frac{1}{1+x^2}\right)\cos\left(\frac{x}{1+x^2}\right)dx \tag{232}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n+1}}{(2n-1)(4n)!} + 8\int_0^1 \frac{(1-3x^2)((\operatorname{ch}1)\cos x - 1) - (x^3-3x)(\operatorname{sh}1)\operatorname{sen}x}{(1+x^2)^3}dx \tag{233}$$

$$\pi = -\frac{4}{5}\sum_{n=1}^{\infty}\frac{2^{-3n}}{n}\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1} + 128\int_0^1 \frac{33-15x^2+15x^4-x^6}{(1+x^2)(1089-635x^2+330x^4+10x^6+5x^8+x^{10})}dx \tag{234}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}}{(2n-1)(8n-3)(4n-2)!} + 4\int_0^1\int_0^1 \frac{\operatorname{ch}(xy^2)\cos(y^2) - x\operatorname{sh}(xy^2)\operatorname{sen}(y^2)}{1+x^2}dx\,dy \tag{235}$$

$$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{-2n+2}}{(2n-1)(4n-1)} + 16\int_0^1\int_0^1 \frac{4+(1-3x^2)y^2}{(1+x^2)\left((4+(1-x^2)y^2)^2+4x^2y^4\right)}dx\,dy \tag{236}$$

$$\pi^2 = -16\sum_{n=1}^{\infty}\frac{1}{n!\,n^2}\left(\sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}\right)^2 + 16e\int_0^1\int_0^1 \frac{f(x,y)}{(1+x^2)(1+y^2)}dx\,dy \tag{237}$$

$$f(x,y) = \operatorname{ch}(xy)(\cos x + x\operatorname{sen}x)(\cos y + y\operatorname{sen}y) + \operatorname{sh}(xy)(x\cos x - \operatorname{sen}x)(y\cos y - \operatorname{sen}y) \tag{238}$$

$$\pi = 4\tan^{-1}(\operatorname{th}x_1) + 4\sum_{n=1}^{m-1}\tan^{-1}\left(\frac{\operatorname{sh}(x_{n+1}-x_n)}{\operatorname{ch}(x_{n+1}+x_n)}\right) + 4\tan^{-1}\left(e^{-2x_m}\right) \tag{239}$$

$$m\in\mathbb{N},\ 0<x_1<x_2<---<x_m$$

$$\pi = 12\tan^{-1}\left(\frac{\sqrt{3}-e^{2mx}}{1+\sqrt{3}\,e^{2mx}}\right) + 12\sum_{n=1}^{m}\tan^{-1}\left(\frac{\operatorname{sh}x}{\operatorname{ch}((2n-1)x)}\right) \tag{240}$$

$$m\in\mathbb{N},\ 0<x<\frac{\ln 3}{4m}$$

$$\pi = 4 - 4\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{(2n-2k+2)^{k+1}} \tag{241}$$

$$\pi = 4\tan^{-1}\left(e^{-2mx}\right) + 4\sum_{n=0}^{m-1}\tan^{-1}\left(\frac{\operatorname{sh}x}{\operatorname{ch}((2n+1)x)}\right) \tag{242}$$

$$m\in\mathbb{N},\ x>0$$

$$\pi = 2\tan^{-1}\left(\sqrt{2}\operatorname{sh}x\right) + 2\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^{[n/2]}e^{-(2n+1)x}}{2n+1},\ x>0 \tag{243}$$

$$\pi = \sum_{n=0}^{\infty}\frac{(-1)^{[n/2]}2^{n+3}}{(2n+1)\left(1+\sqrt{3}\right)^{2n+1}} \tag{244}$$

$$\pi = 4\tan^{-1}\left(\frac{\sqrt{2}\operatorname{sh}x-1}{\sqrt{2}\operatorname{sh}x+1}\right) + 4\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^{[n72]}e^{-(2n+1)x}}{2n+1} \tag{245}$$

$$x>\ln\left(\frac{\sqrt{2}+\sqrt{6}}{2}\right)$$

$$\pi = 2\tan^{-1}(e^x) + \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(\operatorname{ch} x)^{-2n-1}}{2^{2n}(2n+1)}, \quad x \geq 0 \tag{246}$$

$$\pi = 4\tan^{-1}(e^x) - 2\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(\operatorname{th} x)^{2n+1}}{2^{2n}(2n+1)}, \quad x \in \mathbb{R} \tag{247}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \coth\left(\frac{(2n+1)x}{2}\right) - 8\sum_{n=1}^{\infty} \tan^{-1}(e^{-nx}), \quad x > 0 \tag{248}$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{th}\left(\frac{(2n+1)x}{2}\right) + 8\sum_{n=1}^{\infty} (-1)^{n-1}\tan^{-1}(e^{-nx}), \quad x > 0 \tag{249}$$

$$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n}{2k+1} \left(\frac{1}{2^{2k}\left(2+\sqrt{3}\right)^{2n-2k+1}} + \frac{1}{3^{2k}\left(3+2\sqrt{2}\right)^{2n-2k+1}}\right) \tag{250}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n}{(2k+1)(n-k)!} \left(\frac{(\ln 2)^{n-k}}{2^{2k}} + \frac{(\ln 3)^{n-k}}{3^{2k}}\right) \tag{251}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{n+k}}{(2k+1)!\,(2n-2k+1)} \left(\left(\frac{1}{2}\right)^{2n-2k} \left(\ln\left(\frac{1+\sqrt{5}}{2}\right)\right)^{2k+1} + \left(\frac{1}{3}\right)^{2n-2k} \left(\ln\left(\frac{1+\sqrt{10}}{3}\right)\right)^{2k+1}\right) \tag{252}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2n}{n}\binom{2n+1}{2k+1} \frac{(-1)^{n+k}}{2^{2n}(2n+1)} \left(\frac{2x^2-1}{4x}\right)^{2n-2k} \left(\frac{2x^2+1}{4x}\right)^{2k+1} \tag{253}$$

$$\frac{\sqrt{3}-1}{2} < x < \frac{\sqrt{3}+1}{2}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{2n}{n}\binom{2n+1}{2k+1} \frac{(-1)^{n+k}}{2^{2n}(2n+1)} \left(\left(\frac{1}{10}\right)^{2n-2k} \left(\frac{9}{20}\right)^{2k+1} + \left(\frac{1}{20}\right)^{2n-2k} \left(\frac{19}{60}\right)^{2k+1}\right) \tag{254}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n}{2k+1} \left(\frac{4}{4^{n-k+1}\,5^{2k}} - \frac{1}{238^{n-k+1}\,239^{2k}}\right) \tag{255}$$

$$\pi = 4\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k}{2k+1} \binom{n}{2k} (1-x)^{n-2k} x^{2k+1}, \quad 0 < x < 1 \tag{256}$$

$$\pi = 2\sqrt{2}\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{[(n-1)/4]} (-1)^k \binom{n}{4k+1} x^{4k+1} - \sqrt{2}\sum_{k=0}^{[(n-2)/4]} (-1)^k \binom{n}{4k+2} x^{4k+2} + \sum_{k=0}^{[(n-3)/4]} (-1)^k \binom{n}{4k+3} x^{4k+3}\right) \tag{257}$$

$$0 < x < \sqrt{2}$$

$$\pi = \frac{3}{2}\sqrt{3}\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{[(n-1)/3]} (-1)^k \binom{n}{3k+1} x^{3k+1} - \sum_{k=0}^{[(n-2)/3]} (-1)^k \binom{n}{3k+2} x^{3k+2}\right) \tag{258}$$

$$0 < x < 1$$

to be continued

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