Stabilization Principle in dynamics.

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Abstract.

This paper introduces and illustrates the Stabilization Principle that provides a strategy for modeling post instability behavior in dynamics, including turbulence and chaos. It starts with investigation of different types of instability with the objective to demonstrate that stability is not a physical invariant since it depends upon the frame to which the motion of fluid is referred, upon the class of functions in which the governing equations are derived, est. The application of the Stabilization Principle to the Navier-Stokes equations is illustrated by closure of the Reynolds equations for the Poiseuille flow.

Introduction.

One of the most persisting believes among physicists and engineers is that stability is an invariant of the dynamical system under consideration. As a result of that, inability of the Newton's laws to discriminate between stable and unstable motions is considered as a fundamental limitation of classical mechanics. The primary objection of this paper is to clear up that misconception and demonstrate that stability/instability are not Physical invariants: they are rather attributes of the mathematical model since they depend upon the frame to which the model is referred, to the class of functions in which the governing equations are derived, the metric of the configuration space etc. In other words, they depends upon a mathematical formulation of the model. Departure from this misconception opened up a new avenue for the establishment of the Stabilization Principle that is based upon the freedom to choose the frame of reference or the class of functions that provide the "best view" of the system. According to this Principle, the Reynolds equations can be interpreted as the Navier-Stokes equations referred to a fast oscillating frame of reference, and the Stabilization Principle formulates the appropriate choice of such a non-inertial frame that leads to the closure of the Reynolds equations. The application of the Stabilization Principle to the Navier-Stokes equations is illustrated by closure of the Reynolds equations for the Poiseuille flow.

1. Existence, uniqueness and stability of solutions.

Any mathematical model of a continuum should be tested for three properties: existence, uniqueness and stability of its solutions. However, none of these properties are physical invariants since they depend upon a mathematical setting of the corresponding model. As an example, consider a vertical, ideally flexible filament OA with a free lower end A suspended in the gravity field at the point O, Fig. 1.

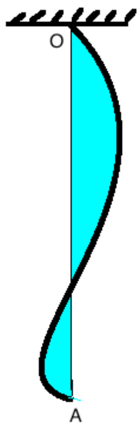


Figure 1. Snap of a whip – violation of the Lipchitz condition at the free and.

As shown in textbooks on analytical mechanics, [1], the problem of small oscillations of a filament with respect to its vertical position is described by a unique and stable solution ***if the Lipchitz condition is enforced***. However this condition suppresses the snap at the free end that is well known from simple experiments, and therefore, it is far from physical reality. Revision of this solution was made in [2]. Omitting mathematical details, we will describe here the physical argumentation that leads to a snap at the free end.

The tension T of the filament due to gravity is the following

$$T = \gamma(L - x) \qquad (1)$$

where γ and L are the specific weight and Length of the filament.

Since the characteristic speed λ of a transverse wave in ideal filaments is

$$\lambda = \sqrt{\frac{T}{\rho}} \qquad (2)$$

this speed vanishes at the free end

$$T|_{x=L} = 0, \quad \lambda = 0 \quad at \quad x \to 0 \qquad (3)$$

In other words, for small transverse displacements of the filament, the governing equations is of hyperbolic type only in the open interval that excludes the free end

$$0 \le x < L \qquad (4)$$

As shows in [2], in this ***open*** interval there exists a unique stable solution. However, in the ***closed*** interval that includes the free end

$$0 \le x \le L \qquad (5)$$

the solution is not unique, and there are unstable solutions since the improper integral

$$\int_0^x \frac{d\xi}{\sqrt{T(\xi)/\rho}} \qquad (6)$$

converges for $x \to L$.

This result has a clear physical interpretation: suppose that an isolated transverse wave of small amplitude was generated at the point of suspension O, Fig. 1. Then the speed of propagation of its leading front will be smaller than the speed of propagation of the trailing front because the tension decreases from the point of suspension to the free end (see Eq.1)). Hence the length of the wave will decrease and vanish ***at*** the free end. Then according to the law of conservation of energy, the kinetic energy per unit of length will tend to infinity and produce a snap. It can be verified that the Lipchitz condition ***at*** the free end is violated

$$\frac{d\dot{x}}{dx} \to \infty \quad at \quad x \to L \tag{7}$$

in the closed interval (5).
Thus it turns out that the unique stable solution exists in the class of functions satisfying the Lipchitz condition. However despite its "nice" mathematical properties, this solution is in contradiction with experiments: the cumulative effect – snap of a whip – is lost. At the same time, the removal of the Lipchitz conditions leads to non-unique unstable solutions that perfectly describe the snap of a whip.

This trivial example leads to an important conclusion: existence, uniqueness and stability of solutions of PDE describing dynamics of a continuum are ***not*** physical invariants: they are attributes of underlying ***mathematical model***. It also becomes clear that in some cases, unstable and non-unique solutions are closer to reality than a unique and stable one. However this is ***not*** the case in models of fluid: as will be shown below, instability of models of fluid demonstrates inadequacy of mathematical restrictions imposed upon the models. However as shown in [3], relaxation of the Lipchitz conditions leads to important singular effects in viscose flows that include the source of randomness in solutions of the Navier-Stokes equations.

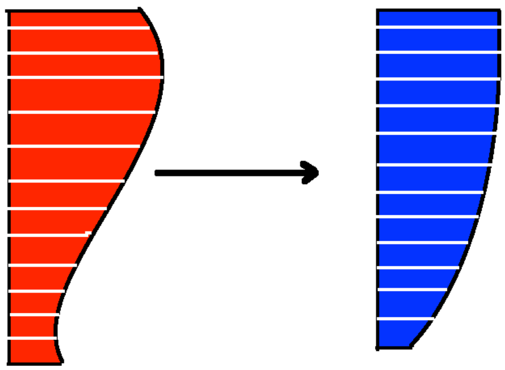
Another example, [4], illustrates the dependence of stability of the solution upon the frame of reference: consider an inviscid stationary flow with a *smooth* velocity field
$v_x = A\sin z + C\cos y,\ v_y = B\sin x + A\cos z,\ v_z = C\sin y + b\cos x$
Surprisingly, the trajectories of this flow are unstable (Lagrangian turbulence). It means that this flow is stable in the Eulerian coordinates, but is unstable in the Lagrangian coordinates.

2. Instability in models of fluid.

Most processed in fluid mechanics are so complex that their universal theory that would capture all the details is unimaginable. That is why the purpose of mathematical modeling is to extract only fundamental aspects of the processes under consideration, and neglect insignificant features without loosing core information. But identification of "insignificant features" is an art rather than science: in many cases even vanishingly small forces can cause large changes in state variables, and such situation is associated with instability. Obviously destabilizing forces cannot be considered as "insignificant features", and therefore, they cannot be ignored. But since they could be humanly indistinguishable in the very beginning, there is no way to incorporate them into a model. This simply means that the model is not adequate for a quantitative description of the underlying dynamical process: it must be changed or modified. However instability deliver an important qualitative message: it manifests the ***boundaries of applicability of the original model.***

It is important to distinguish short- and long-term instabilities.

Short-term instability occurs when the system has alternative stable states (inverted pendulum); it is characterized by bounded deviations of position coordinates whose change affects the energy of the system, and therefore this type of instability does not require a model modification. In order to illustrate this statement by an example from fluid mechanics, consider a parallel iviscid shear flow, Fig. 11.



Unstable flow Stable flow

Figure 2. Transition from an unstable flow profile to the stable flow profile.

As follows from Eqs. (2),(3), the Euler's equations are satisfied by any profiles of flows no matter whether they are stable or unstable, while the choice of the actual profile is made by the mechanism of instability: driven by this mechanism, the flow departs from the unstable profile and approached the "closest" stable profile in the same way as an inverted pendulum approach its minimum-potential-energy position.

The long-term instability occurs when the system does not have an alternative stable state. Such instability can involve only ignorable coordinates since these coordinates do not affect the energy of the system. That is why the long-term instability, from physical viewpoint, can be associated with chaos, and from mathematical viewpoint – with the loss of smoothness, or with the loss of differentiability. And that is why the long-term instability requires a modification of the model. Since the Euler's model of inviscid fluid abounds with chaotic instabilities with no alternative stable states, modification of this model is the main subject of this book.

It should be mentioned that the long-term instability is subdivided, at least, in two different groups: Liapunov instability that is associated with unbounded growth of some selected modes, and Hadamard instability that results from degeneration of a hyperbolic PDE into an elliptic PDE, while all the modes grow unboundedly. That is why the Hadamard instability is based upon local relationships that do not explicitly depend upon boundary conditions. In addition to that, in case of Hadamard instability, an infinitesimal initial disturbance becomes finite in finite time period, while in Liapunov instability case this period must be infinite.

In the Euler's model of inviscid fluid, both type of long-term instability occurs: vortices are Liapunov -unstable, and tangential discontinuities of velocity are Hadamard-unstable. The last statement we will illustrate by the following example.

Example.Consider a surface of a tangential jump of velocity $V_2 - V_1$ in a horizontal unidirectional flow of an inviscid incompressible fluid. Applying the principle of virtual work to a small volume *V* containing both flows of the fluid as well as the surface of the tangential jump of velocities separating the flows, one obtains

$$\int_V (\rho a_1 \bullet \delta U_1 + \rho a_2 \bullet \delta U_2) dV = 0 \qquad (8)$$

where ρ, U_1, U_2, a_1, a_2 are density, displacements, and accelerations of the fluid. The displacements U_1 and U_2 are mutually independent in the region that does not contain a separating surface, but they are dependent at the surface due to its impenetrability

$$U_1 \bullet n = U_2 \bullet n = U, \qquad (9)$$

Hence, as follows from Eq. (8), at the surface the following equality holds:

$$(a_1 + a_2) \bullet n = 0, \quad i.e. \quad [(\frac{\partial}{\partial t} + V_1 \frac{\partial}{\partial S})^2 + (\frac{\partial}{\partial t} + V_2 \frac{\partial}{\partial S})^2]U + \alpha = 0 \qquad (10)$$

where α is the term that does not contain the second order derivatives of *U*, and therefore, it does not effect the characteristic equation

$$(\lambda - V_1)^2 + (\lambda - V_2)^2 = 0 \qquad (11)$$

and its characteristic roots

$$\lambda = \frac{1}{2}[(V_2 + V_1) \pm i(V_2 - V_1)] \qquad (12)$$

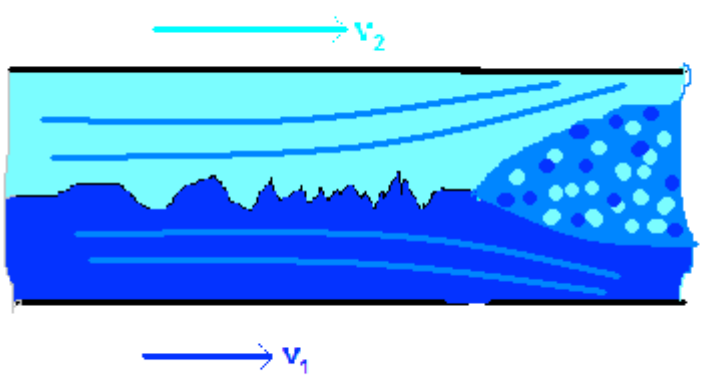


Figure 3. Loss of differentiability in fluid mechanics.

Let us start with studying propagation of high frequency oscillations of the transverse displacements *U*. Recall that Eq. (10) being linear with respect to the second order time-space derivatives, strictly speaking, is nonlinear with respect to *U* and its first time/space derivatives that are contained in the term *α*. For small amplitudes and their first derivatives, this term can be linearized:

$$\alpha = \alpha_1 \frac{\partial U}{\partial t} + \alpha_2 \frac{\partial U}{\partial S} + \alpha_3 U + \alpha_4 \qquad (13)$$

For further simplifications, all the coefficients in Eq. (10) can be linearized with respect to an arbitrarily selected point S_0 and instant of time $t_0 = 0$. Then Eq. (10) takes form of a linear elliptic PDE with constant coefficients

$$[(\frac{\partial}{\partial t} + V_1^0 \frac{\partial}{\partial S})^2 + (\frac{\partial}{\partial t} + V_2^0 \frac{\partial}{\partial S})^2 + \alpha^0{}_1 \frac{\partial}{\partial t} + \alpha^0{}_2 \frac{\partial}{\partial S} + \alpha^0{}_3]U + \alpha^0{}_4 = 0 \qquad (14)$$

Obviously this equation is valid only for small amplitudes with small first derivatives, the small area around the above selected point S_0, and within a small period of time Δt.

Let us derive the solution to Eq. (214) subject to the following initial conditions

$$U^*{}_0 = \frac{1}{\lambda_0} e^{-\lambda_0 S i}, \ at \qquad t = 0 \qquad (15)$$

assuming that λ_0 can be made as large as desired, i.e. $\lambda_0 > N \to \infty$. Consequently, the initial disturbances can be made as small as desired, i.e. $U^*{}_0 < N^{-1} \to 0$. The corresponding solution is written in the form

$$U^* = C_1 e^{-\lambda_0(\lambda_1 t - S)i} + C_2 e^{-\lambda_0(\lambda_2 t - S)i} \qquad (16)$$

where λ_1, λ_2 are the roots of the characteristic equation (11) (see Eq. (12). Since the characteristic roots are complex, the solution (16) will contain the term

$$\frac{1}{\lambda_0} e^{|\text{Im}\,\lambda_{1,2}|\Delta t} \sin \lambda_0 S, \qquad \lambda_0 \to \infty \qquad (17)$$

that leads to infinity within an arbitrarily short period of time *Δt* and within an infinitesimal area around the point S_0 .

Hence one arrived at the following situation: $|U^*| \to \infty$ in spite the fact that $|U_0| \to 0$. In order to obtain a geometrical interpretation of the above described instability, let's note that if the second derivatives in Eq. (14) are of order λ_0, then the first derivatives are of order of *1*, and *U* is of order of $1/\lambda_0$. Hence, the period of time *Δt* can be selected in such a way that the second derivatives will be as large as desired, but *U* and its first derivatives are still sufficiently small. Taking into account that the original governing equation (10) is quasi-linear with respect to the second derivatives, and therefore, the linearization does not impose any restrictions on their values, one concludes that the linearized equation (14) is valid for the solution during the above mentioned period of time *Δt*. Turning to the term (17) of the solution (16), one can now interpret it as being represented by a function having an infinitesimal amplitude and changing its sign with an infinite frequency ($\lambda_0 \to \infty$). The first derivatives of this function can be small and change their signs by finite jumps (with the same infinite frequency), so that the second derivatives at the points of such jumps are infinite. From the mathematical viewpoint, this kind of function is considered as continuous, but *non-differentiable.*

The result formulated above was obtained under specially selected initial conditions (15), but it can be generalized to include any initial conditions. Indeed, let the initial conditions be defined as

$$U|_{t=0} = U^{**}_0(X) \tag{18}$$

and the corresponding solution to Eq. (10) is

$$U = U^{**}(X,t) \tag{19}$$

Then, by altering the initial conditions to

$$U|_{t=0} = U_0{}^*(X) + U_0{}^{**}(X) \tag{20}$$

where U^*_0 is defined by Eq. (15), one observes the preceding argument by superposition that vanishingly small change in the initial condition (20) leads to unboundedly large change in the solution (19) that occurs during an infinitesimal period of time. That makes the solution (19) *non-differentiable*, but still continuous. However, the Euler's equation that governs the flow discussed above was derived under condition ***of space/time differentiability*** of the velocity field. This discrepancy manifests inapplicability of the Euler's equation for description of ***postinstability*** motion of an inviscid fluid, and therefore, for a developed turbulence. But this does not imply the incompleteness of Newtonian mechanics: it only means that the mathematical formalism that expresses the Newton's laws should include non-differentiable components of the velocity field.

Incorporation of thermodynamics into Newtonian mechanics does not eliminate the loss of differentiability: the Navier-Stokes equations appear to have even more sophisticated patterns of instability than its particular case – the Euler equations, if the Reynolds number is supercritical, and from the point of view of mathematical formalism, it leads to the phenomenon described above. As a matter of fact, the Navier-Stokes equations impose even stronger mathematical restriction on the velocity field requiring it's twice differentiability with respect to space coordinates. Therefore, the Navier-Stokes equations require a modification that would allow one to include non-differentiable velocity field similar to those in the Euler's equations.

It should be noticed that the same kind of phenomena occurs in other branches of continuum mechanics, and in particular, in theory of flexible bodies. Indeed, consider an ideal filament in the gravity field stretched in the vertical direction with the additional tension T_0 as shown in Fig.4. Let us cut it at the middle point and observe a behavior of its lower part. As shown in [4], the lower part has imaginary characteristic speeds

$$\lambda = \pm i\sqrt{gL} \tag{21}$$

where L is the current length (see Fig. 4), and the solution to the corresponding governing equation contains the same destabilizing term as that in Eq. (16).

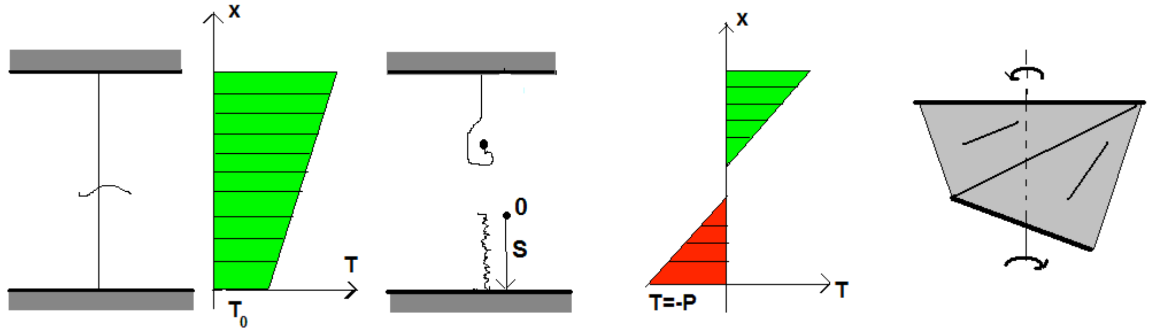


Figure 4. Loss of differentiability in flexible bodies.

As shown in [4], the same loss of differentiability occurs in two- and three-dimensional flexible bodies (wrinkles in films, buckling in soft shells, and fractures in composite materials). In mathematics, all these phenomena are associated with the Hadamard's instability.

3. Stability in non-inertial frames of reference.

As stated and illustrated in the previous Section, stability is not a physical invariant since it depends upon the class of functions in which the solution is sought. In this section, we concentrate on dependence of stability upon frames of reference and start with the following example.

Example. Consider the motion of a mechanical system in a constant potential field *U* subjected to a fast oscillating force

$$f_i = f_i^{(1)} \cos \omega t + f_i^{(2)} \sin \omega t \qquad (22)$$

applied to the i^{th} particle . It is assumed that f_i are functions only on the generalized coordinates q_i *(i=1,2…n),* and $\omega >> 1/T$ where *T* is the order of a period of motion of the system had it be only under the action of the potential field Π .

Under these conditions, the contribution of the forces (22) will be represented by an additional term in the potential energy

$$\tilde{\Pi} = \Pi + \frac{1}{2\omega^2} \sum_{i,j} a_{ij}^{-1} < f_i f_j > \qquad (23)$$

Here a_{ij}^{-1} are the elements of the matrix inverse to the matrix of the kinetic energy *K* of the system

$$K = \frac{1}{2}\sum_{i,j} a_{i,j}(q)\dot{q}_i\dot{q}_j \tag{24}$$

and $< f_i f_j >$ is the averaged over time the product $f_i f_j$.

So far we have not discussed the origin of the oscillating forces (22). One of the possibilities that we will pursue here is referring the governing equations of the system under consideration to a non-inertial frame that perform oscillations with the accelerations

$$w_i = -f / m_i, \quad f = f_1 = f_2 = ... = f_n \tag{25}$$

Then the forces (22) are interpreted as inertia forces generated by the acceleration of the transport motion of the corresponding non-inertial frame.

Let us turn to a particular system: an inverted pendulum on a fast oscillating support, Fig. 5.

Assuming that $\omega >> \sqrt{g / L}$ where *L* in the length of the pendulum, one obtains the potential energy in the non-inertial fast oscillating system as a function of φ

$$\tilde{\Pi} = mgL(-\cos\varphi + \frac{a^2\omega^2}{4gL}\sin^2\varphi) \tag{26}$$

in which *a* is amplitude of the oscillations, and φ is angle between the pendulum and the vertical. The potential energy (26) has two equilibrium points: $\varphi=0$ when the pendulum is in its lowest position, and $\varphi = \pi$ when the pendulum is in its upper position. The lowest position in always stable, but the upper position is stable only if

$$a^2\omega^2 > 2gL \tag{27}$$

Thus as follows from Eqs. (26) and (27), both vertical positions could be stable if one chooses the amplitude and frequency of the oscillation sufficiently large. It should be emphasized that the inverted position of the pendulum will be stable ***with respect to the non-inertial frame,*** while it is obviously unstable with respect to any inertial frame.

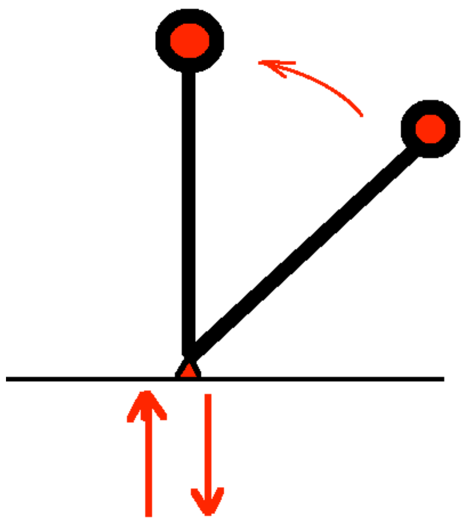


Figure 5. Stability of inverted pendulum in non-inertial frame.

4. Alternative interpretation of the Reynolds equations.

In this section we will try to find such a non-inertial frame of reference that provides the **'best view"** for the Navier-Stokes equations when they are unstable in inertial frames of reference. Although we will not be able to apply the result of the previous Section directly (since a viscose fluid is not a potential system), nevertheless we will use a similar strategy for presenting the Navier-Stokes equation in the fast oscillating frame.

Let us start with the Navier-Stokes equations in the Cartesian coordinates

$$\frac{\partial u}{\partial t}+\frac{\partial (u^2)}{\partial x}+\frac{\partial (uv)}{\partial y}+\frac{\partial (uw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta u \tag{28}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial (vu)}{\partial x}+\frac{\partial (v^2)}{\partial y}+\frac{\partial (vw)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta v \tag{29}$$

$$\frac{\partial w}{\partial t}+\frac{\partial (wu)}{\partial x}+\frac{\partial (wv)}{\partial y}+\frac{\partial (w^2)}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\Delta w \tag{30}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{31}$$

and impart to this inertial frame of reference fast oscillations along each of the coordinate axes

$$\Delta x=-\frac{\tilde{u}}{\omega}\sin\omega t,\ \ \Delta y=\frac{\tilde{v}}{\omega}\sin\omega t,\quad \Delta z=\frac{\tilde{w}}{\omega},\quad \omega>>\frac{1}{\tau},\quad i=1,2,3. \tag{32}$$

Here τ is the time scale upon which the changes of the velocities in the inertial system can be ignored, and $\tilde{u},\tilde{v},\tilde{w}$ are the components of the transport velocity generated by oscillation of the frame. Then each projection of velocity could be decomposed into relative and transport components

$$u = \bar{u} + \tilde{u}\cos\omega t \qquad (33)$$
$$v = \bar{v} + \tilde{v}\cos\omega t \qquad (34)$$
$$w = \bar{w} + \tilde{w}\cos\omega t \qquad (35)$$

It should be noticed that, as follows from (2.32), fast oscillating velocity of the frame practically does not change the metrics of the original frame

$$\Delta x, \Delta y, \Delta z \to 0 \quad if \quad \omega \to \infty, \qquad (36)$$

even if the amplitudes of oscillations $\tilde{u}, \tilde{v}, \tilde{w}$ depend upon the coordinates *x,y*, and z. Taking into account that

$$\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \bar{u}\,dt \cong \bar{u}, \quad \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \bar{v}\,dt \cong \bar{v} \quad \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \bar{w}\,dt \cong \bar{w} \qquad (37)$$

$$\int_0^{2\pi/\omega} \tilde{u}\cos\omega\,dt \cong 0 \quad \int_0^{2\pi/\omega} \tilde{v}\cos\omega\,dt \cong 0 \quad \int_0^{2\pi/\omega} \tilde{w}\cos\omega\,dt \cong 0 \qquad (38)$$

$$\int_0^{2\pi/\omega} \tilde{u}^2\cos^2\omega\,dt \cong \frac{1}{2}\tilde{u}^2, \quad \int_0^{2\pi/\omega} \tilde{v}^2\cos^2\omega\,dt \cong \frac{1}{2}\tilde{v}^2, \quad \int_0^{2\pi/\omega} \tilde{w}^2\cos^2\omega\,dt \cong \frac{1}{2}\tilde{w}^2, \qquad (39)$$

let us substitute the decomposed velocities (33)-(35) into Eqs. (28)-(31) and integrate these equations over time from 0 to $2\pi/\omega$.

Then one arrives at the following system of PDE

$$\frac{\partial\bar{u}}{\partial t} + \frac{\partial(\bar{u}^2)}{\partial x} + \frac{\partial(\bar{u}\bar{v})}{\partial y} + \frac{\partial(\bar{u}\bar{w})}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta\bar{u} - [\frac{\partial(\tilde{u}^2)}{\partial x} + \frac{\partial(\tilde{u}\tilde{v})}{\partial y} + \frac{\partial(\tilde{u}\tilde{w})}{\partial z}] \qquad (40)$$

$$\frac{\partial\bar{v}}{\partial t} + \frac{\partial(\bar{v}\bar{u})}{\partial x} + \frac{\partial(\bar{v}^2)}{\partial y} + \frac{\partial(\bar{v}\ \bar{w})}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta\bar{v} - [\frac{\partial(\tilde{v}\tilde{u})}{\partial x} + \frac{\partial(\tilde{v}^2)}{\partial y} + \frac{\partial(\tilde{v}\tilde{w})}{\partial z}] \qquad (41)$$

$$\frac{\partial\bar{w}}{\partial t} + \frac{\partial(\bar{w}\,\bar{u})}{\partial x} + \frac{\partial(\bar{w}\,\bar{v})}{\partial y} + \frac{\partial(\bar{w}^2)}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta\bar{w} - [\frac{\partial(\tilde{w}\tilde{u}}{\partial x} + \frac{\partial(\tilde{w}\tilde{v})}{\partial y} + \frac{\partial(\tilde{w}^2)}{\partial z}] \qquad (42)$$

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} + \frac{\partial\bar{w}}{\partial z} = 0 \qquad (43)$$

$$\frac{\partial\tilde{u}}{\partial x} + \frac{\partial\tilde{v}}{\partial y} + \frac{\partial\tilde{w}}{\partial z} = 0 \qquad (44)$$

A comparison of these equations with the Reynolds equations shows that they are identical. It means that the Reynolds equations can be interpreted as the Navier-Stokes equations referred to a non-inertial fast oscillating frame. Therefore the Reynolds stress tensor

$$\begin{pmatrix} \tilde{\sigma}_x & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{xy} & \tilde{\sigma}_y & \tilde{\tau}_{yz} \\ \tilde{\tau}_{xz} & \tilde{\tau}_{yz} & \tilde{\sigma}_z \end{pmatrix} = -\begin{pmatrix} \rho\tilde{u}^2 & \rho\tilde{u}\tilde{v} & \rho\tilde{u}\tilde{w} \\ \rho\tilde{u}\tilde{v} & \rho\tilde{v}^2 & \rho\tilde{v}\tilde{w} \\ \rho\tilde{u}\tilde{w} & \rho\tilde{v}\tilde{w} & \rho\tilde{w}^2 \end{pmatrix} \quad (45)$$

is composed of the inertia forces generated by a non-inertia fast oscillating frame of reference, while this frame can be chosen arbitrarily.

This interpretation completely disqualifies all the attempts to find an additional "constitutive equation " that would express the Reynolds stresses via the state variables of the Navier-Stokes equations and lead to the closure of the Reynolds equations. Such a hypothetical closure ***does not exist.*** However the interpretation introduced above delivers important information that leads to a closure through the Stabilization Principle to be formulated in the next section.

5. Formulation of the Stabilization Principle.

Let us return to the inverted pendulum discussed in the Section 3 and consider the inequality (27)

$$a^2\omega^2 > 2gL$$

It suggests that if the energy of the fast oscillating frame is sufficiently large, then the upper vertical position of the pendulum can be stabilized at

$$a^2\omega^2 = 2gL \quad (46)$$

Obviously Eq.(46) provide a neutral stability, and therefore, the underlying oscillating frame provides the "best view" of the inverted pendulum that is unstable in any inertial frame of reference.

Can we do the same with the Navier-Stokes equation, i.e. can we find such oscillating frame in which the Reynolds stresses stabilize the motion?

Before answering this question, let us discuss it from another angle:

Consider a dynamical model that in some domain of its parameters loses its space-time differentiability, i.e. it becomes unstable in the class of differentiable functions. As noticed earlier, this means that the corresponding physical phenomenon cannot be adequately described without an appropriate modification of the mathematical formalism representing the model. The modification should be based upon an enlarging the original class of functions in such a way that the instability is eliminated. The mathematical formulation of this statement can be expressed in the following *symbolic* form:

$$A_R \otimes X = A_0 \otimes X + \sigma_R \quad (47)$$

If the original dynamical model

$$A_0 \otimes X = 0 \quad (48)$$

is unstable, but the Reynolds model

$$A_0 \otimes X + \sigma_R = 0 \quad (49)$$

is stable, obviously the stabilization is performed by the Reynolds stresses σ_R that represent contribution of all the non-smooth components into the modified model. Indeed, driven by the mechanism of instability of the original model, they grow until the instability is suppressed down to a neutral stability. After that, there in no any mechanisms that would drive further drowth of the fluctuations, and that is the simplest justification of the stabilization process. From that viewpoint, the Prandil's closure can be considered as a feedback that stabilizes an originally unstable laminar flow. Turning, for instance, to a plane Poiseuille flow with a parabolic velocity profile, one arrives at its instability if the Reynolds number is larger than R ≅ 5772. Experiments show that the new steady turbulent profile is no longer parabolic: it is very flat near the center and is very steep near the walls, Fig. 6.

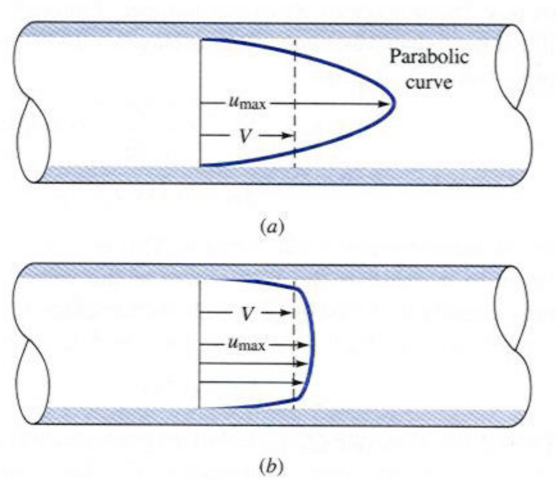


Figure 6. Laminar (a) and turbulent (b) profiles of the Poiseuille flow.

The same profile follows from the Prandtl solution. But since this profile can be experimentally observed, it must be stable, and this stabilization is carried out by the feedback

$$-\overline{v_i v_j} = l^2 \left| \frac{\partial \overline{v}_i}{\partial x_j} \right| \frac{\partial \overline{v}_i}{\partial x_j}$$

where a *non-local* parameter *l* (called mixing length) is supposed to be found from experiments. Mathematical justification of the *neutral* stability is discussed in [4]. Experimental verification of neutral stability of free turbulent jets was reported in [5].

Thus it turns out that the closure of the Reynolds equations looks more like a problem of control: given an unstable dynamical system (the Navier-Stokes equations) find such an input (in the form of the Reynolds stresses) that drives this system to the state of neutral stability.

This statement can be considered as the ***Stabilization Principle.***

It is clear that the Stabilization Principle is not a prerogative of fluids: it is valid for any dynamical system (see the pendulum example). The meaningfulness of the Stabilization Principle formulated above and its application to turbulence and chaos has been illustrated in [4]. However, a major limitation of this approach is a necessity to find a rate of instability of the original laminar flow prior to application of the Stabilization Principle, and this pre-condition is very complex and laborious despite the fact that the conditions of instability are well formulated. Nevertheless there are cases of practical importance that can be solved analytically, although such cases are rather exceptions than the rule. One of those cases will be consider in the next Section.

6. Closure in Poiseuille flow using Stabilization Principle.

a. Background. We start this section with preliminary information about the celebrated Orr-Sommerfeld equation that is a powerful tool for stability analysis of plain viscose flows.

The Orr–Sommerfeld equation, in fluid dynamics, is an eigenvalue equation describing the linear two-dimensional modes of disturbance to a viscous parallel flow. The solution of the Navier–Stokes equations for a parallel, laminar flow can become unstable at supercritical Raynolds numbers, and the Orr–Sommerfeld equation determines precisely what the conditions for hydrodynamic stability are.

The equation is derived by solving a linearized version of the Navier–Stokes equation for the perturbation velocity field

$$\mathbf{v} = [U(z) + u'(x,y,z), 0, w'(x,y,z)] \qquad (50)$$

where $[U(z),0,0)]$ is the unperturbed or basic flow. The perturbation velocity has the wave-like solution $u' \propto e^{i\alpha(x-ct)}$. Using the stream function representation for the flow, the following dimensionless form of the Orr–Sommerfeld equation is obtained:

$$\frac{1}{i\alpha \mathrm{R}}(D^2-\alpha^2)^2\varphi = (U-c)(D^2-\alpha^2)\varphi - U''\varphi \qquad (51)$$

where $\mathrm{R} = \frac{\rho U_0 h}{\mu}$ is the Reynolds number of the base flow, μ is the dynamic viscosity of the fluid, ρ is fluid density, φ is the stream function, U_0 is characteristic velocity, and *h* is characteristic length. The relevant boundary conditions are the no-slip boundary conditions. The eigenvalue parameter of the problem is c. If the imaginary part of the wave speed c is positive, then the base flow is unstable, and the small perturbation introduced to the system is amplified in time.

If we confine our study to flows whose instability can be found from linear analysis, (plane Poiseuille flow, boundary layers), then the closure problem can be formulated as follows: let the original laminar flow described by the Navier-Stokes equations be unstable, i.e some of the eigenvalues for the corresponding Orr-Sommerfeld equation have positive imaginary parts. Then the closure is found from the condition that all these positive imaginary parts vanish, and therefore, the solution possesses a neutral stability. However the closure can be written in the explicit form only if the criteria for the onset of instability are formulated explicitly. Since such a situation is an exception rather than a rule, one can apply a step-by-step strategy. This strategy is based upon the fact that the Reynolds stress disturbances grow much faster than the mean motion disturbances. Hence one can assume that these stresses will be large enough to stabilize the mean flow that is still sufficiently close to its original unperturbed state. But the Reynolds stresses being substituted in the Reynolds equations will change the mean velocity profile, and consequently, the conditions of instability. These new conditions, in turn, will change the Reynolds stresses, etc. By choosing the iteration steps to be sufficiently small, one can obtain acceptable accuracy. In this example, the first step approximation will be applied to a plane Poiseuille flow.

b. Formulation of the Problem. Let us consider a plane shear flow with a dimensionless velocity profile:

$$\bar{U} = \bar{U}(z), \quad 0 \le z \le 1 \tag{52}$$

with boundaries

$$z_1 = 0,\ z_2 = 1 \tag{53}$$

and the *x* coordinate being along the axis of symmetry. The stream function representing a single oscillation of the disturbance is assumed to be of the form

$$\psi(x,z,t) = \varphi(z)e^{i(\alpha x - \beta t)} \tag{54}$$

The function $\varphi(z)$ must satisfy the Orr-Sommerfeld equation,[6]

$$(U - C)(D^2 - \alpha^2)\varphi - U''\varphi = (ia\mathrm{R})^{-1}(D^2 - \alpha^2)^2\varphi \tag{55}$$

in which α and β are constants, *R* is the Reynolds number, and

$$C = \frac{\beta}{\alpha}, \qquad D\phi = \frac{d\phi}{dz} = \phi' \tag{56}$$

Equation (55) should be solved subject to the boundary conditions, that in case of a symmetric flow between rigid walls are

$$\varphi = D\varphi = 0 \quad at \quad z = z_2, \quad D\phi = D^3\phi = 0 \quad at \quad z = z_1 \tag{57}$$

We will start with the velocity profile characterized by the critical Reynolds number

$$R = R_{cr} \tag{58}$$

Any increase in the velocity when

$$R^* > R_{cr} \tag{59}$$

leads to instability of the laminar flow and to transition to a new turbulent flow.

We will concentrate our attention on the situation when the increase in the Reynolds number is sufficiently small

$$\frac{R^* - R_{cr}}{R_{cr}} << 1 \tag{60}$$

In this case we will be able to formulate a linearized version of the closure (51) explicitly based upon the conditions of instability of the Orr-Sommerfeld equation written for $R = R_{cr}$ and to obtain the mean velocity profile and Reynolds stress for the corresponding turbulent flow.

c. Generalized Orr-Sommerfeld Equation. In order to apply the stabilization principle and formulate the closure problem we have to incorporate the Reynolds stresses into the Orr-Sommerfeld equation. For this purpose let us start with the Reynolds equations for a plane shear flow expressed in terms of small perturbations:

$$\frac{\partial \tilde{U}}{\partial t} + U\frac{\partial \tilde{U}}{\partial x} + \tilde{V}\frac{d\bar{U}}{dz} + \frac{1}{\rho}\frac{\partial \tilde{P}}{\partial z} = \nu \nabla^2 \tilde{U} + \frac{\partial \tilde{\tau}}{\partial z} \qquad (61)$$

$$\frac{\partial \tilde{V}}{\partial t} + U\frac{\partial \tilde{V}}{\partial x} + \frac{1}{\rho}\frac{\partial \tilde{P}}{\partial z} = \nu \nabla^2 \tilde{V} + \frac{\partial \tilde{\tau}}{\partial x} \qquad (62)$$

$$\frac{\partial \tilde{U}}{\partial x} + \frac{\partial \tilde{V}}{\partial z} = 0 \qquad (63)$$

using the boundary layer approximation. Here $\bar{U}(z)$ is the mean velocity profile, $\tilde{U}, \tilde{V}, \tilde{P}$ are small velocity and pressure perturbations, ν is the kinematic viscosity and $\tilde{\tau}$ is the shear Reynolds stress which is sought in the form

$$\tilde{\tau} = \hat{\tau}(z)e^{i(\alpha x - \beta t)} \qquad (64)$$

Substituting Eq. (54) and Eq. (64) into Eqs.(61), (62), and (63) we obtain after the elimination of pressure, the generalized Orr-Sommerfeld equation in dimensionless form:

$$(\bar{U} - c)(D^2 - \alpha^2)\phi - \bar{U}''\phi - (iaR)^{-1}(D^2 - \alpha^2)^2\phi = -\frac{1}{\alpha}(D^2 + \alpha^2)\tau \qquad (65)$$

in which

$$\tau = \frac{\hat{\tau}}{\rho \bar{U}^2_{max}} \qquad (66)$$

Eq. (65) contains an additional term on the right hand side: the Reynolds stress disturbance, as yet unknown.

d. The Closure Problem. Returning to our problem, let us apply Eq. (65) to the case when

$$R = R^*, U = \bar{U}(z) \qquad (67)$$

Substituting the equalities (67) into Eq. (65), one obtains

$$(\bar{U} - C)(D^2 - \alpha^2)\phi - \bar{U}''\phi - (iaR^*)^{-1}(D^2 - \alpha^2)^2\phi = -\frac{1}{\alpha}(D^2 + \alpha^2)\tau \qquad (68)$$

With zero Reynolds stress $(\tau = 0)$, Eq. (68) would have eigenvalues with positive imaginary parts since $R^* > R_{cr}$. These positive imaginary parts of the eigenvalues would vanish if R^* is replaced by R_{cr}. Hence, according to the stabilization principle, the Reynolds stresses should be selected such that Eq. (68) is converted into Eq. (65) at $R = R_{cr}$, i.e.

$$(iaR^*)^{-1}(D^2-\alpha^2)^2\phi - \frac{1}{\alpha}(D^2+\alpha^2)\tau = (i\alpha R_{cr})^{-1}(D^2-\alpha^2)^2\phi \tag{69}$$

or

$$(D^2+\alpha^2)\tau = \left(\frac{1}{R_{cr}} - \frac{1}{R^*}\right)(D^2-\alpha^2)^2\phi \tag{70}$$

Eq. (70) relates the disturbance of the mean flow velocity and the Reynolds stress τ. With reference to Eqs. (54), and (64), Eq. (70) allows us to formulate a linearized version of closure of the Reynolds equations (61), (62) and (63)

$$\bar{\tau}'' + \alpha^2\bar{\tau} = \left(\frac{1}{R_{cr}} - \frac{1}{R^*}\right)(\bar{\psi}'''' - 2\alpha^2\bar{\psi}'' + \alpha^4\bar{\psi}) \tag{71}$$

in which $\bar{\psi}$ and $\bar{\tau}$ are the dimensionless stream function and Reynolds stresses characterizing the unperturbed flow (for instance, $\psi = -\partial U / \partial x$). Indeed, after perturbing Eq. (71) and substituting equations (54),(64), and (66), one returns to Eq. (70).

It is important to emphasize that Eq. (71) is not a universal closure: it contains two parameters (R_{cr} and α), that characterize a particular laminar flow. Here R_{cr} is the smallest value of the Reynolds number; below that number, all initially imparted disturbances decay, whereas above this number those disturbances that are characterized by α (see Eq. (54) and (64)) are amplified. Both of these numbers can be found from Eq. (55) as a result of classical analysis of hydrodynamics stability performed for a particular laminar flow. One should recall that the closure (71) implies a small increment of the Reynolds number over its critical value (see Eq. (60)). For large increments the procedure must be performed by steps: for each new mean velocity profile (that is sufficiently close to the previous one) the new R'_{cr} and α' are supposed to be found from the solution of the eigenvalue problem for the Orr-Sommerferd equation. Substituting R'_{cr} and α' into the closure Eq. (71) and solving it together with the corresponding Reynolds equation,

one finds the mean velocity profile and the Reynolds stress for the next increase of the Reynolds number, etc.

e. Plane Poiseuille Flow. In this sub-section we will apply the approach developed above to a plane Poiseuille flow with the velocity profile (see Fig. 6a)

$$\bar{U}^0(z) = 1 - z^2 \tag{72}$$

and

$$R_{cr} = 5772.2, \alpha = 1.021 \quad (73)$$

As a new (supercritical) Reynolds number we will take

$$R^* = 6000 \quad (74)$$

The closure (71) should be considered together with the governing equation for the unidirectional mean flow

$$\nu \bar{U}'' + \bar{\tau}' = C = const \quad (75)$$

or

$$\nu \bar{U}' + \bar{\tau} = \bar{C}_1 z + \bar{C}_2 = const \quad (76)$$

The constants $\bar{C}_1$ and $\bar{C}_2$ can be found from the condition

$$\bar{\tau} = 0 \quad at \quad z = 1 \quad and \quad z = 0 \quad (77)$$

expressing the fact that the Reynolds stress vanishes at the rigid wall and at the middle of the flow. Hence

$$\bar{C}_2 = 0 \quad (78)$$

since $\bar{U}' = 0 \quad at \quad z = 0$ and

$$\bar{C}_1 = \nu \bar{U}_1 \qquad (\bar{U}_1 = \bar{U} \qquad at \qquad z = 1) \quad (79)$$

Thus

$$\bar{\tau} = \nu (U_1' z - U') \quad (80)$$

or in dimensionless form,

$$\bar{\tau} = \frac{1}{R^*}(\bar{U}_1' z - \bar{U}') \quad (81)$$

Substituting Eq. (81) into the closure Eq. (71)), one obtains the governing equation for the mean velocity profile in terms of the stream function $\bar{\psi}$ while $\bar{U} = \partial \bar{\psi} / \partial z$

$$\frac{1}{R_{cr}} \psi'''' - \alpha^2 \left(\frac{2}{R_{cr}} - \frac{1}{R^*}\right) \bar{\psi}'' - \alpha^4 \left(\frac{1}{R_{cr}} - \frac{1}{R^*}\right) \bar{\psi} = \frac{\alpha^2}{R^*} \bar{\psi}_1'' z \quad (82)$$

in which $\bar{\psi}_1'' = \bar{\psi}'' \qquad at \qquad z = z_1$

Without loss of generality it can be set

$$\bar{\psi}|_{z=0} = 0 \quad (83)$$

Since at the rigid wall $\bar{U} = 0$, one obtains

$$\bar{\psi}_1' = 0 \quad (84)$$

In the middle of the flow due to symmetry

$$\bar{U}_0' = 0, \qquad i.e. \qquad \bar{\psi}_0'' = 0 \quad (85)$$

Finally, the flux of the turbulent flow should be the same as the flux of the original (unperturbed) laminar flow

$$\bar{\psi}_1 = \int_0^1 (1-z^2)\,dy = \frac{2}{3} \qquad (86)$$

These four non-homogeneous boundary conditions (83) – (86) allow one to find four arbitrary constants appearing as a result of integration of Eq. (82). After substituting the numerical values (73) and (74), one arrives at the following linear differential equation of the fourth order with respect to the dimensionless stream function

$$\bar{\psi}'''' - 1.08202\bar{\psi}'' - 0.04124\bar{\psi} = 1.044\bar{\psi}_1'' \qquad (87)$$

whence

$$\bar{\psi} = C_1 \sin 0.19199z + C_2 \cos 0.19199z + C_3 \sinh 0.19199z$$
$$C_4 \cosh 0.19199z - 25.3152\bar{\psi}_1'' z \qquad (88)$$

Applying the conditions (83) and (85), one finds that

$$C_2 = C_4 = 0 \qquad (89)$$

Taking into account that

$$\bar{\psi}_1'' = -0.00703C_1 + 0.00712C_3 \qquad (90)$$

one obtains

$$\bar{\psi} = C_1 \sin 0.19199z + C_3 \sinh 0.19199z + (0.17797C_1 - 0.18924C_3)z \qquad (91)$$

Now applying the conditions (84), (85) and (86) one arrives at the final form of the solution

$$\bar{\psi} = 11.278 \sin 0.19199z - 270.11 \sinh 0.19199z + 50.692 \qquad (92)$$

and therefore,

$$\bar{U} = 2.1653 \cos 0.19199z - 51.8584 \cosh 0.19199z + 50.692$$

Substituting the solution (92) into Eq. (81), one obtains the Reynolds stress profile

$$R^*\tau = 0.41572 \sin 0.19199z + 9.9563 \sinh 0.19199z - 2.00259z \qquad (94)$$

f. Analysis of the solution. We will start with the comparison of the original laminar velocity profile (72), Fig. 6a, and the mean velocity profile (92), Fig.6b. Both of them envelop the same area, i.e., the fluxes of the original laminar and post-instability turbulent flows are the same. However, the maximum turbulent mean velocity is smaller than the maximum velocity of the original laminar flow:

$$\bar{U}^T_{\max} = 0{,}9989 < \bar{U}^L_{\max} = 1 \qquad (95)$$

Also

$$|\bar{U}_0''|^T = 1.99132 < |\bar{U}_0''|^L = 2 \quad (96)$$

At the same time,

$$|\bar{U}_1'|^T = 1.99132 > |\bar{U}_1'|^L \quad (97)$$

Hence, the turbulent mean velocity profile is more flat at the center and it is steeper at the walls in comparison with the corresponding laminar flow. This property is typical for turbulent flows.

Turning to the Reynolds stress profile (94), one finds that the maximum of the stress module $|\tau|$ is shifted toward the wall:

$$z^* = 0.58 \quad (98)$$

that expresses the well-known wall effect.
Finally, the pressure gradient

$$\frac{\partial \bar{p}}{\partial x} = \frac{1}{R^*}\bar{U}_0'' + \bar{\tau}_0' \quad (99)$$

for the new turbulent flow is greater than for the original laminar flow:

$$\left|\frac{\partial \bar{p}}{\partial x}\right|^T = \frac{2.002586}{R^*} > \frac{2}{R^*} = \left|\frac{\partial p}{\partial x}\right|^L \quad (100)$$

Therefore, despite the fact that the Reynolds number R^* slightly exceeds the critical value R_{cr}, all the typical features of turbulent flows are clearly pronounced in the solution obtained above.

7. Conclusion.

Thus, it has been demonstrated that the closure in turbulence theory is based upon the principle of stabilization of the original laminar flow by fluctuation velocities. We will stress again the mathematical meaning of this procedure. It is well known that the concept of stability is related to a certain class of functions: a solution which is unstable in a class of smooth functions can be stable in an enlarged (non-smooth) class of functions. Reynolds enlarged the class of smooth functions by introducing the field of fluctuation velocities that generated additional (Reynolds) stresses in the Navier-Stokes equations. Now it is reasonable to extend this procedure by choosing these Reynolds stresses such that they eliminate the original instability, i.e. by applying the Stabilization Principle.

Another interpretation of the same effect was introduced in the previous Section: the Reynolds equations can be interpreted as the Navier-Stokes equations referred to a non-inertial fast oscillating frame; therefore the Reynolds stress tensor is composed of the inertia forces generated by that non-inertia frame, while this frame can be chosen arbitrarily. According to the Stabilization Principle, this arbitrariness must be used in such a way that the original instability is suppressed down to a neutral stability.

Obviously one cannot expect that the solution of the type of Eq. (93) would describe all the peculiarities of turbulent motion; it will rather extract the most essential properties of the motion, i.e., such properties that are reproducible, and therefore, have certain physical meaning.

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