Geometrodynamic Foundation of Classical Electrodynamics

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Abstract: We summarize how the Lorentz Force motion observed in classical electrodynamics may be understood as geodesic motion derived by minimizing the variation of the proper time along the worldline of test charges in external potentials, while the spacetime metric remains invariant under, and all other fields in spacetime remain independent of, any rescaling of the charge-to-mass ratio q/m. In order for this to occur, time is dilated or contracted due to attractive and repulsive electromagnetic interactions respectively, in very much the same way that time is dilated due to relative motion in special relativity, without contradicting the latter's well-corroborated experimental content. As such, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

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1. Motivation and Purpose

The equation of motion for a test particle along a geodesic line in curved spacetime specified by the metric interval $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ with metric tensor $g_{\mu\nu}$ was first obtained by Albert Einstein in §9 of his landmark 1915 paper [1] introducing the General Theory of Relativity. The infinitesimal linear element $d\tau = ds/c$ for the proper time is a scalar invariant which is independent of the chosen system of coordinates. Likewise the finite proper time $\tau = \int_A^B d\tau$ measured along the worldline of the test particle between two spacetime events *A* and *B* has an invariant meaning independent of the choice of coordinates. Specifically, the geodesic of motion is stationary, and results from a minimization of the variational equation

$$0 = \delta \int_A^B d\tau \, . \tag{1.1}$$

After carrying out the well-known calculation originally given by Einstein in [1], the particle's equation of geodesic motion is found to be:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{du^\beta}{d\tau} = -\Gamma^\beta{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu \, , \tag{1.2}$$

with the Christoffel connection defined by $-\Gamma^{\beta}{}_{\mu\nu} \equiv \frac{1}{2} g^{\beta\alpha}\left(\partial_{\alpha} g_{\mu\nu} - \partial_{\mu} g_{\nu\alpha} - \partial_{\nu} g_{\alpha\mu}\right)$ and the relativistic four-velocity given by $u^{\mu} \equiv dx^{\mu} / d\tau$.

The geodesic (1.2) can also be viewed at in alternative, yet equivalent way. In curved spacetime, $DB^{\beta} / D\tau \equiv \left(\partial x^{\nu} / \partial \tau\right)\partial_{;\nu} B^{\beta}$ defines the "derivative along the curve" for any contravariant vector B^{β}, using gravitationally-covariant derivatives $\partial_{;\nu} B^{\beta} = \partial_{\nu} B^{\beta} + \Gamma^{\beta}{}_{\sigma\nu} B^{\sigma}$ and the chain rule. So when $B^{\beta} = u^{\beta}$, then, in view of (1.2), we may also write:

$$\frac{Du^{\beta}}{D\tau} = \frac{\partial x^{\alpha}}{\partial \tau}\partial_{;\alpha} u^{\beta} = \frac{\partial x^{\alpha}}{\partial \tau}\left(\partial_{\alpha} u^{\beta} + \Gamma^{\beta}{}_{\sigma\alpha} u^{\sigma}\right) = \frac{\partial x^{\alpha}}{\partial \tau}\left(\frac{\partial}{\partial x^{\alpha}}\frac{dx^{\beta}}{d\tau} + \Gamma^{\beta}{}_{\sigma\alpha} u^{\sigma}\right) = \frac{du^{\beta}}{d\tau} + \Gamma^{\beta}{}_{\mu\nu} u^{\mu} u^{\nu} = 0. \quad (1.3)$$

This has exactly the same content as the geodesic equation (1.2). But given that $du^{\beta} / d\tau = 0$ describes Newtonian inertial motion when the gravitational connection $\Gamma^{\beta}{}_{\mu\nu} = 0$, we may think of $Du^{\beta} / D\tau = 0$ above as describing *covariantly-inertial* motion in the presence of gravitation. This is what gives gravitational geodesics their colloquial characterization as "straight lines," or more precisely, "inertial lines" in curved spacetime.

Just as ordinary derivatives $\partial_{\alpha} = \left(\partial / \partial t, \nabla\right)$ are replaced by gravitationally-covariant derivatives $\partial_{;\alpha}$ in curved spacetime, so too in gauge theory ordinary derivatives ∂_{α} are replaced by gauge-covariant or "canonical" derivatives $\mathcal{D}_{\alpha} \equiv \partial_{\alpha} - iqA_{\alpha}$, where q is the electric charge strength and A_{α} is the gauge field / vector potential, and where we use $\mathcal{D}_{\alpha}$ rather than the often-employed D_{α} to distinguish symbolically from the D of gravitational motion in (1.3). Motivated by the geodesic nature of gravitationally-covariant motion for which $Du^{\beta} / D\tau = 0$ rather than $du^{\beta} / d\tau = 0$ and how this motion stems directly from the replacement of ordinary with gravitationally-covariant derivatives, the purpose of this paper is to summarize how electrodynamic Lorentz Force motion is likewise geodesic motion which is *canonically-inertial* and which stems directly from the canonical derivatives of gauge theory. As will be shown, this comes about as a consequence of heretofore unrecognized time dilations and contractions which occur any time two material bodies are electromagnetically interacting.

2. Geometro-electrodynamics and Time Dilations and Contractions: An Overview

To begin, if the test particle, to which we now ascribe a mass $m > 0$, also has a non-zero net electrical charge $q \neq 0$ and the region of spacetime in which it subsists also has a nonzero electromagnetic field strength $F^{\beta\alpha} \neq 0$, then the equation of motion is no longer given by (1.2), but is supplemented by an additional term which contains the Lorentz Force law, namely:

$$\frac{d^{2}x^{\beta}}{d\tau^{2}} = \frac{du^{\beta}}{d\tau} = -\Gamma^{\beta}{}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha}\frac{dx^{\sigma}}{cd\tau} = -\Gamma^{\beta}{}_{\mu\nu} u^{\mu} u^{\nu} + \frac{q}{m} g_{\sigma\alpha} F^{\beta\alpha}\frac{u^{\sigma}}{c}. \quad (2.1)$$

In the above, the field strength $F^{\beta\alpha}$ containing the electric and magnetic field bivectors **E** and **B** is defined as usual by $F^{\beta\alpha} \equiv \partial^\beta A^\alpha - \partial^\alpha A^\beta$ in relation to the gauge potential four-vector A^α. The above force law is of course a well-known, well-corroborated, well-established law of physics.

Given that the gravitational geodesic (1.2) specifies a path of minimized proper time (1.1), the question arises whether there is a way to obtain (2.1) from the same variation as in (1.1), thus revealing the electrodynamic motion to also entail particles moving through spacetime along paths of minimized proper time in four spacetime dimensions. Conceptually, it cannot be argued other than that this would be a desirable state of affairs. But physically the difficulty rests in how to accomplish this without ruining the integrity of the metric and the background fields in spacetime by making them a function of the charge-to-mass ratio q/m. This ratio is and must remain a characteristic of the test particle alone. It is not and cannot be a characteristic of the line element $d\tau$, or the metric tensor $g_{\mu\nu}$, or the gauge field A^α, or the field strength $F^{\beta\alpha}$ which define the field-theoretical spacetime background through which the test particle is moving. And, at bottom, this difficulty springs from the *inequivalence* of the "electrical mass" (a.k.a. charge) *q* and the inertial mass *m*, versus the Newtonian equivalence of gravitational and inertial mass. In (2.1), this is captured by the fact that *m* does *not* appear in the gravitational term $-\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu$, while the q/m ratio *does* appear in the electrodynamic Lorentz Force term that we rewrite as $(q/m)F^\beta{}_\sigma u^\sigma$ in natural units with $c=1$.

This may also be seen very simply if we compare Newton's law with Coulomb's law. In the former case we start with a force $F=-GMm/r^2$ (with the minus sign indicating that gravitation is attractive) and in the latter $F=-k_e Qq/r^2$ (for which we choose an attractive interaction to provide a direct comparison to gravitation), where *G* is Newton's gravitational constant and the analogous $k_e = 1/4\pi\varepsilon_0 = c^2\mu_0/4\pi$ is Coulomb's constant. If the gravitational field is taken to stem from *M* and the electrical field from *Q*, then the test particle in those fields has gravitational mass *m* and electrical mass *q*. But the Newtonian force $F=ma$ always contains the inertial mass *m*. So in the former case, because the gravitational and inertial mass are equivalent, the acceleration $a=F/m=-GMm/mr^2=-GM/r^2$ and these two masses cancel, giving $-\Gamma^\beta{}_{\mu\nu} u^\mu u^\nu$ without any mass in (2.1). But in the latter case the acceleration $a=F/m=-k_e Qq/mr^2=-(q/m)k_e Q/r^2$ because the electrical and inertial masses are not equivalent, hence $(q/m)F^\beta{}_\sigma u^\sigma$ containing this same ratio in (2.1). Here, the motion is distinctly dependent on the electrical and inertial masses *q* and *m* of the test particle. And as a result, different charges *q* with different masses *m* may all be moving through the exact same background fields and yet have different observable motions.

So, were we to pursue the conceptually-attractive goal of understanding electrodynamic motion as the result of particles moving through spacetime along paths of minimized proper time, with (1.1) applying to electrodynamic motion just as it does to gravitational motion, the line element $d\tau$ would inescapably have to be a function $d\tau(q/m)$ of q/m. And this in turn would *appear* to violate the integrity of the line element $d\tau$ as well as the metric tensor $g_{\mu\nu}$ in

$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, because these would all *seem to be* dependent upon the attributes q and m of the test particles that are moving through the spacetime background. Were this to be reality and not just seeming appearance, this would be physically impermissible.

Consequently, despite there being many known derivations of the Lorentz Force law, there does not, to date, appear to be an acceptable rooting of the Lorentz Force law in the variational equation $0 = \delta \int_A^B d\tau$ which would reveal electrodynamic motion to be geodesic motion just like the familiar gravitational motion. And this is because it has not been understood how to obtain electrodynamic motion from a minimized variation while simultaneously maintaining the integrity of field theory such that the metric and the background fields do not depend upon the attributes of the test particles which may move through these fields. This, in turn, is because electrical mass is not equivalent to the inertial mass, which causes different test particles to move differently even when in the exact same background fields, in contrast to the Newtonian equivalence of the gravitational and inertial masses from which all particles respond alike in the same background.

So, when a first test particle with electrical mass q and inertial mass m is placed in a field $F^{\beta\alpha}$, and a second test particle with electrical mass q' and inertial mass m' of a different ratio $q'/m' \neq q/m$ is placed at equipotential in the same field $F^{\beta\alpha}$, there are observably-different Lorentz Force motions for these two different test particles even though they are at equipotential. As a result, having the line element $d\tau$ be a mathematical function of q/m yet be physically independent of q/m may seem paradoxical. Nevertheless, it is possible to have a line element $d\tau(q/m)$ which is a function of the electrical-to-inertial mass ratio q/m, from which the variational equation $0 = \delta \int_A^B d\tau$ does yield the combined gravitational and electrodynamic equation of motion (2.1), yet for which the line element $d\tau$, the metric tensor $g_{\mu\nu}$, the gauge field A^α, and the electromagnetic field strength $F^{\beta\alpha}$ are all independent of this q/m ratio. Specifically, close study reveals that this paradox may be resolved by recognizing that *time does not flow at the same rate for these two test particles in very much the same way that time does not flow at the same rate for two reference frames in special relativity which are in motion relative to one another*.

In particular, in the absence of gravitation with $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\beta{}_{\mu\nu} = 0$, the first test particle will have a Lorentz motion given by:

$$\frac{d^2 x^\beta}{d\tau^2} = \frac{q}{m} \eta_{\sigma\alpha} F^{\beta\alpha} \frac{dx^\sigma}{c d\tau}. \tag{2.2}$$

Note that this Lorentz motion also contains a set of coordinates x^μ. Now usually it is assumed that for the second test particle the motion is given by this same equation (2.2), merely with the substitution of $q \to q'$ and $m \to m'$; that is, by:

$$\frac{d^2x^\beta}{d\tau^2}=\frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx^\sigma}{cd\tau}. \tag{2.3}$$

The particular assumption here is that there is no change in the rate at which time flows when (2.2) is replaced with (2.3); and more generally the assumption is that the coordinate interval dx^σ in (2.3) is identical to the dx^σ in (2.2). Yet, it is impossible to have both (2.2) and (2.3) emerge through the variation $0=\delta\int_A^B d\tau$ from the same metric element $d\tau$, and simultaneously maintain the integrity of the field theory, unless the coordinates are different, wherein dx^σ in (2.2) is *not identical* to what must now be $dx^\sigma \to dx'^\sigma \neq dx^\sigma$ in (2.3).

In fact, the very physics of having electric charges in electromagnetic fields induces a change in coordinates as between these two test charges with different $q'/m' \neq q/m$, very similar to the coordinate change via Lorentz transformations induced by relative motion. As a result, the electrodynamic motion of the second test charge is given, not by (2.3), but by:

$$\frac{d^2x'^\beta}{d\tau^2}=\frac{q'}{m'}\eta_{\sigma\alpha}F^{\beta\alpha}\frac{dx'^\sigma}{cd\tau}. \tag{2.4}$$

Here, x^β in (2.2) and $x'^\beta \neq x^\beta$ in (2.4), respectively, are two different sets of coordinates. Yet, they are interrelated by a definite transformation. Most importantly, this results in *time itself* being induced to flow differently as between these two sets of coordinates, making time dilation and contraction as fundamental an aspect of electrodynamics, as it already is of the special relativistic theory of motion and the general relativistic theory of gravitation. In fact, what is really happening – physically – is that the placement of a charge in an electromagnetic field is *inducing a physically-observable change of coordinates* $x^\beta(q/m)\to x'^\beta(q'/m')$ in the very same way that relative motion between the coordinate systems $x^\beta(v)$ and $x'^\beta(v')$ of two different inertial reference frames with velocities *v* and v' induces a Lorentz transformation $x^\beta(v)\to x'^\beta(v')$ that relates the two coordinate systems to one another via $c^2d\tau^2=\eta_{\mu\nu}dx^\mu(v)dx^\nu(v)=\eta_{\mu\nu}dx'^\mu(v')dx'^\nu(v')$, with an invariant line element $d\tau^2=d\tau'^2$ and the same metric tensor $\eta_{\mu\nu}=\eta'_{\mu\nu}$ in either reference frame.

As it turns out, the line element that yields (2.1) from (1.1), including electrodynamic motion, which is quadratic in $d\tau$, is:

$$c^2d\tau^2=g_{\mu\nu}\left(dx^\mu+\frac{q}{mc}d\tau A^\mu\right)\left(dx^\nu+\frac{q}{mc}d\tau A^\nu\right)=g_{\mu\nu}\mathfrak{D}x^\mu\mathfrak{D}x^\nu. \tag{2.5}$$

Above, we have defined a gauge-covariant coordinate interval $\mathfrak{D}x^\mu \equiv dx^\mu+\left(q/mc\right)d\tau A^\mu$, again with a canonical $\mathfrak{D}$ to distinguish from the gravitational *D* in (1.3). And it will be seen that upon multiplying through by m^2 and dividing through by $d\tau^2$ this becomes:

$$m^2c^2 = g_{\mu\nu}\left(m\frac{dx^\mu}{d\tau}+\frac{q}{c}A^\mu\right)\left(m\frac{dx^\nu}{d\tau}+\frac{q}{c}A^\nu\right) = g_{\mu\nu}\left(p^\mu+\frac{q}{c}A^\mu\right)\left(p^\nu+\frac{q}{c}A^\nu\right) = g_{\mu\nu}\pi^\mu\pi^\nu . \tag{2.6}$$

This, it will be recognized, is the usual relationship between the rest mass m and the canonical energy-momentum $\pi^\mu \equiv mdx^\mu / d\tau + qA^\mu / c = p^\mu + qA^\mu / c$, where ordinary mechanical / kinetic energy-momentum is $p^\mu = mdx^\mu / d\tau$. Some authors continue to use p^μ to denote the canonical momentum when there are charges and gauge field present; we find it preferable to employ the different symbol π^μ to avert confusion. The gauge interval $\mathfrak{D}x^\mu \equiv dx^\mu + (q/mc)d\tau A^\mu$ defined in (2.5) is then seen to be merely a restatement of the gauge-covariant derivatives $\mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and canonical momenta $\pi^\mu \equiv p^\mu + qA^\mu / c$ which emerge from gauge theory and relate to one another via $i\partial_\sigma \Leftrightarrow p_\sigma$ and $i\mathfrak{D}_\sigma \Leftrightarrow \pi_\sigma$, and in particular from the mandate for local gauge (really, phase) symmetry.

Now, the line element (2.5) is clearly a function of q/m and so has the *appearance* of depending on the ratio q/m. But this is only appearance. For, when we now place the second test charge with the second ratio $q'/m' \neq q/m$ in the exact same metric measured by the invariant line element $d\tau$ and moving through the exact same fields $g_{\mu\nu}$ and A^μ, this metric gives:

$$c^2d\tau'^2 = c^2d\tau^2 = g_{\mu\nu}\left(dx'^\mu+\frac{q'}{m'c}d\tau A^\mu\right)\left(dx'^\nu+\frac{q'}{m'c}d\tau A^\nu\right) = g_{\mu\nu}\mathfrak{D}x'^\mu\mathfrak{D}x'^\nu , \tag{2.7}$$

with $\mathfrak{D}x'^\mu = dx'^\mu + (q'/m'c)d\tau A^\mu$. So despite $d\tau$ being a function of the q/m ratio, this $d\tau = d\tau'$ as a measured proper time element is actually *invariant* with respect to the q/m ratio because *the differences between different q/m and q'/m' are entirely absorbed into the coordinate transformation $x^\mu \to x'^\mu$, which is quite analogous to the Lorentz transformation of special relativity*. The counterpart to (2.6) now becomes:

$$m'^2c^2 = g_{\mu\nu}\left(m'\frac{dx'^\mu}{d\tau}+\frac{q'}{c}A^\mu\right)\left(m'\frac{dx'^\nu}{d\tau}+\frac{q'}{c}A^\nu\right) = g_{\mu\nu}\pi'^\mu\pi'^\nu , \tag{2.8}$$

with an invariant $d\tau$ and unchanged background fields $g_{\mu\nu}$ and A^μ.

In fact, this transformation $x^\mu \to x'^\mu$ is *defined* so as to keep $d\tau = d\tau'$ invariant, and $g_{\mu\nu} = g'_{\mu\nu}$ and $A^\mu = A'^\mu$ and by implication the field strength bivector $F^{\beta\alpha} = F'^{\beta\alpha}$ all unchanged, just as Lorentz transformations are defined so as to maintain a constant speed of light for all inertial reference frames independently of their state of motion. That is, combining (2.5) and (2.7), this transformation $x^\mu \to x'^\mu$ which results in time dilations and contractions, is *defined* by:

$$c^2 d\tau^2 = g_{\mu\nu}\left(dx^\mu + \frac{q}{mc}d\tau A^\mu\right)\left(dx^\nu + \frac{q}{mc}d\tau A^\nu\right) \equiv g_{\mu\nu}\left(dx'^\mu + \frac{q'}{m'c}d\tau A^\mu\right)\left(dx'^\nu + \frac{q'}{m'c}d\tau A^\nu\right). \quad (2.9)$$

Consequently, $d\tau = d\tau'$ is a function of charge q and mass m yet is invariant with respect to the same, and there is no inconsistency in having $d\tau = d\tau'$ be a function of, yet be invariant under, a rescaling of the q/m ratio. Likewise, the fields $g_{\mu\nu} = g'_{\mu\nu}$ and $A^\mu = A'^\mu$ are independent of the charge and the mass of the test particle, because again, everything stemming from the different ratios q/m and q'/m' is absorbed into a coordinate transformation $x^\mu \to x'^\mu$. Thus, while "gauge" is a historical misnomer for what is really invariance under local *phase* transformations $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ applied to a wavefunction ψ, we see in (2.9) that the line element $d\tau$ truly is invariant under what can be genuinely called a *re-gauging* of the q/m ratio. And from (2.6) and (2.8), we see that this symmetry is really not new. It is merely a restatement of the usual relationship $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ between rest mass and canonical momentum.

As a result, each and every different test particle carries its own coordinates, all interrelated so as to keep $d\tau$ invariant, and $g_{\mu\nu}$, A^μ and $F^{\beta\alpha}$ unchanged. The coordinate transformation interrelating all the test particles causes time $x^0 = t$ to dilate for electrical attraction and to contact for repulsion, with a dimensionless ratio $dt/d\tau \equiv \gamma_{em}$ that integrally depends upon the magnitude of the likewise-dimensionless ratio qA^μ/mc^2 of electromagnetic interaction energy qA^μ to the test particle's rest energy mc^2. This in turn supplements the ratio $dt/d\tau = \gamma_v = 1/\sqrt{1-v^2/c^2}$ for motion in special relativity and $dt/d\tau = \gamma_g = 1/\sqrt{g_{00}}$ for a clock at rest in a gravitational field, and assembles them into the overall product combination $dt/d\tau = \gamma_{em}\gamma_g\gamma_v$ governing time dilation and contraction when all of motion and gravitational and electromagnetic interactions are present.

Operationally, the electromagnetic contribution γ_{em} to this time dilation or contraction would be measured in principle by comparing the rate at which time is kept by otherwise identical, synchronized geometrodynamic clocks or oscillators which are then electrically charged with different q/m ratios, and then placed at rest into a background potential $A^\mu = (\phi, \mathbf{A}) = (\phi_0, \mathbf{0})$ at equipotential, where φ_0 is the proper potential. Or more generally, this would be measured by electrically charging otherwise identical clocks and then placing them into the potential to have differing dimensionless $q\phi_0/mc^2$ ratios.

Empirically, for $q\phi_0/mc^2 << 1$, and for an attractive Coulomb force $F = -k_e Qq/r^2$, the interaction energies $E_{em} = \int F dr = +k_e Qq/r$ plus integration constant are related to these electromagnetic time dilations in a manner identical to how the kinetic energy $E_v = \frac{1}{2}mv^2$ is contained in $mc^2\gamma_v = mc^2/\sqrt{1-v^2/c^2} \cong mc^2 + \frac{1}{2}mv^2$ for nonrelativistic velocities $v << c$ in special relativity. In fact, the actual expression for the electromagnetic contribution to the time dilation for $q\phi_0/mc^2 << 1$ interactions is $\gamma_{em} = 1 - q\phi_0/mc^2$. And for a Coulomb proper potential

$\phi_0 = -k_e Q / r$ for an electrical interaction chosen to be attractive like gravitation, this is $\gamma_{em} = 1 + k_e Qq / mc^2 r$. So the combined time dilation $dt / d\tau = \gamma_{em} \gamma_g \gamma_v$ mentioned earlier, employing the gravitational factor $\gamma_g = 1 / \sqrt{g_{00}(r)} \cong 1 + GM / c^2 r$ in the weak field Newtonian limit (where the Reissner–Nordström metric term $Gk_e Q^2 / c^4 r^2$ may clearly be neglected), produces an overall energy which, in the low velocity, weak-gravitational and weak-electromagnetic interaction limit, is given by:

$$\begin{aligned} E &= mc^2 \frac{dt}{d\tau} = mc^2 \gamma_{em} \gamma_g \gamma_v = mc^2 \frac{1 + k_e Qq / mc^2 r}{\sqrt{g_{00}} \sqrt{1 - v^2 / c^2}} \cong mc^2 \left(1 + \frac{GM}{c^2 r}\right)\left(1 + \frac{k_e Qq}{mc^2 r}\right)\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) \\ &= mc^2 + \frac{1}{2} mv^2 + \frac{k_e Qq}{r} + \frac{1}{2}\frac{k_e Qq}{c^2 r} v^2 + \frac{GMm}{r} + \frac{1}{2}\frac{GMm}{c^2 r} v^2 + \frac{GM}{r}\frac{k_e Qq}{c^2 r} + \frac{1}{2}\frac{GM}{c^2 r}\frac{k_e Qq}{c^2 r} v^2 \end{aligned} . \quad (2.10)$$

What we see here, in succession, are 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the Coulomb interaction energy of the charged mass, 4) the kinetic energy of the Coulomb energy, 5) the gravitational interaction energy of the mass, 6) the kinetic energy of the gravitational energy, 7) the gravitational energy of the Coulomb energy and 8) the kinetic energy of the gravitational energy of the Coulomb energy. It is clear that this accords entirely with empirical observations of the linear limits of these same energies.

Importantly, unlike gravitational redshifts or blueshifts which are a consequence of spacetime curvatures, these electromagnetic time dilations *do not stem directly from curvature*. They only affect curvature indirectly through any changes in energy to which they give rise because gravitation still "sees" all energy. Hermann Weyl's ill-fated attempt from 1918 until 1929 in [2], [3], [4] to base electrodynamics on *real* gravitational curvature foreclosed any such real curvature explanation. This is because Weyl's initial attempt was rooted in invariance under a non-unitary local transformation $\psi \to \psi' = e^{\Lambda(t,\mathbf{x})} \psi$ which re-gauges the magnitude of a wavefunction, rather than under the correct transformation $\psi \to \psi' = U\psi = e^{i\Lambda(t,\mathbf{x})}\psi$ with an imaginary exponent that simply redirects the phase. Specifically, the latter correct phase transformation is associated with an *imaginary*, not real, curvature that places a factor $i = \sqrt{-1}$ into the geodesic deviation $D^2 \xi^\mu / D\tau^2$ when expressed in terms of the commutativity $\left[\partial_{;\mu}, \partial_{;\nu}\right]$ of spacetime derivatives. So at best, electrodynamics can be understood on the basis of a mathematically-imaginary spacetime curvature. The alteration of time flow in electrodynamics that we suggest here, is therefore much more akin to the time dilation of special relativity than it is to the gravitational redshifts and blueshifts of general relativity. It may transpire entirely in flat spacetime, and real spacetime curvature only becomes implicated when the energies added to mc^2 reach sufficient magnitude beyond their linear limits shown in (2.10) to curve the nearby spacetime.

Also importantly, the similarity of the ratios $q\phi_0 / mc^2$ and v^2 / c^2 as the driving number in $\gamma_{em} = 1 - q\phi_0 / mc^2$ and $\gamma_v = 1 / \sqrt{1 - v^2 / c^2}$, respectively, is more than just an analogy. Just as $v < c$ (a.k.a. $mv^2 < mc^2$) is a fundamental limit on the motion of material subluminal particles, so too, it turns out that $q\phi_0 < mc^2$ is a material limit on the strength of the interaction energy between a test charge *q* with mass *m* interacting with the sources of the proper potential ϕ_0. This transpires by requiring particle and antiparticle energies to always be positive and time to always flow forward in accordance with Feynman-Stueckelberg, and by maintaining the speed of light as the material limit which it is known to be. Further, it turns out that when $\phi_0 = k_e Q / r$ is the Coulomb potential whereby this limit becomes $k_e Qq / r < mc^2$ (a.k.a. $r > k_e Qq / mc^2$), we find that there is a lower physical limit on how close two interacting charges can get to one another, thereby solving the long-standing problem of how to circumvent the $r = 0$ singularity in Coulomb's law.

To be sure, these electromagnetic time dilations are miniscule for everyday electromagnetic interactions, as are special relativistic time dilations for everyday motion. So testing of $dt / d\tau$ changes for electrodynamics may perhaps be best pursued with experimental approaches similar to those used to test relativistic time dilations. As a very simple example to establish a numeric benchmark, consider two bodies with charges $Q = q = 1\,\mathrm{C}$ (Coulomb) separated by $r = 1\,\mathrm{m}$ (meter). In this event, the Coulomb interaction energy has a magnitude $k_e Qq / r = k_e = 1/4\pi\varepsilon_0 = 8.897 \times 10^9\ \mathrm{J}$ (Joules). Yet, if the test particle which we take to have the charge q has a rest mass $m = 1\,\mathrm{kg}$ (kilogram), then the electrodynamic time dilation factor contained in (2.10) is $\gamma_{em} = 1 + k_e / c^2 = 1 + \mu_0 / 4\pi = 1 + 10^{-7} = 1.0000001$. This is a very tiny time dilation for a tremendously energetic interaction. The release of this much energy per second would yield a power of approximately 8.99 GW (gigawatts), which roughly approximates seven or eight nuclear power plants, or roughly four times the power of the Hoover Dam, or the power output of a single space shuttle launch, or the power of about seventy five jet engines, or that of a single lightning bolt. For a special relativistic comparison, consider an airplane which flies one mile in six seconds, versus light which travels a bit over one million miles in six seconds. Here, $v / c \cong 10^{-6}$ and the time dilation is $\gamma_v = 1/\sqrt{1 - v^2 / c^2} \cong 1.0000000000005$. So in fact the exemplary electrodynamic time dilation is substantially less miniscule than this exemplary special relativistic dilation. However in daily experience where one encounters watts and kilowatts not gigawatts, these time dilations would be of similar magnitude.

In short, in order to be able to obtain equation (2.1) for gravitational and electrodynamic motion from the minimized proper time variation (1.1) in a way that preserves the integrity of the metric and the background fields independently of the q / m ratio for a given test charge and thereby achieves the conceptually-attractive goal of understanding electrodynamic motion to be geodesic motion just like gravitational motion, we are forced to recognize that attractive electrodynamic interactions inherently dilate and repulsive interactions inherently contract time itself, *as an observable physical effect*. This is identical to how relative motion dilates time, and to how gravitational fields dilate (redshift) or contract (blueshift) time. In this way, it becomes possible to have a spacetime metric which – although a function of the electrical charge and inertial mass of test particles – also remains invariant with respect to those charges and masses and particularly with respect to a re-gauging of the charge-to-mass ratio. This preserves the integrity of the field theory, and establishes that electrodynamic motion is in fact geodesic motion which satisfies the minimized proper time variation $0 = \delta \int_A^B d\tau$ from (1.1). As a result, it becomes possible to lay an entirely geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

In the next section we shall review in detail exactly how (2.1), which includes gravitational and electrodynamic motion, is deductively derived from minimizing the action (1.1) using the line element (2.5) and the related equation (2.6) for the canonical energy-momentum. As we shall see in (3.4), this derivation produces an additional term in the Lorentz force that is not gauge-invariant, and thus leaves an unobservable ambiguity in the physical motion. To address this, as reviewed in section 4, it is necessary to impose two conditions on the gauge field. The first condition fixes the gauge field to the Maxwell Lagrangian in lieu of the often-imposed Lorenz gauge, but still leaves some residual ambiguity in the gauge field. The second condition fixes the additional Lorentz force term to zero, thereby removing the remaining gauge ambiguity. Then, in section 5, we reformulate the former Lagrangian-based gauge condition in terms of the Maxwell action. In sections 6 and 7, respectively, we use these gauge conditions to uncover a covariant scalar equation for power, and a scalar field equation for energy flux, in the presence of both gravitational and electrodynamic interactions and sources. In essence, sections 3 through 7 directly explicate the derivation of the Lorentz force (2.1) from the minimized variation (1.1) and the immediate consequences of this in terms of required gauge fixing conditions and resulting power and energy flux equations. Section 8 then shows precisely how the time dilation and contraction summarized above, as well as the time flow / energy relation (2.10), are derived by simply requiring that the metric line element must remain invariant and the background fields in spacetime must remain unchanged, under a re-gauging of the electrodynamic charge-to-mass ratio q/m. Finally, section 9 contains concluding remarks.

3. Derivation of Lorentz Force Geodesic Motion from Variation Minimization

The foundational calculation to derive (2.1) including the Lorentz force from the minimized variation (1.1) begins with the spacetime metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ which is multiplied through by m^2 and turned into the free particle energy-momentum relation $m^2c^2 = g_{\mu\nu} p^\mu p^\nu$ containing the mechanical momentum $p^\mu = m dx^\mu / d\tau$. This in turn is readily turned into Dirac's $\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0$ for a free electron in flat spacetime making use of $\eta^{\mu\nu} = \tfrac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\}$. Then, we simply use Weyl's well-known gauge prescription [4] which transforms the mechanical momentum to the canonical momentum $p^\mu \to \pi^\mu \equiv p^\mu + qA^\mu / c$ thus the energy-momentum relation to $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ in (2.6), and the ordinary derivatives to gauge-covariant derivatives $\partial_\sigma \to \mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and thus Dirac's equation to $\left(i\gamma^\mu \mathfrak{D}_\mu - m\right)\psi = 0$ for interacting particles. All of this emerges by requiring "gauge" symmetry under the local phase transformation $\varphi \to \varphi' = U\varphi = e^{i\Lambda(t,\mathbf{x})}\varphi$ acting generally on the scalar fields $\varphi = \phi$ of the Klein-Gordon equation and the fermion fields $\varphi = \psi$ of Dirac's equation, redirecting phase but preserving magnitude. This is all well-known, so it is not necessary to detail this further. The point is that the relation $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ in (2.6) is easily derived from the metric $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ using local gauge symmetry, and that nothing more is needed to furnish the starting point to minimize the variation and arrive at the combined gravitational and electrodynamic motion (2.1).

Starting with (2.6) and dividing through by m^2c^2, we form the number 1 as such:

$$1 = g_{\mu\nu}\left(\frac{dx^\mu}{cd\tau} + \frac{q}{mc^2}A^\mu\right)\left(\frac{dx^\nu}{cd\tau} + \frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\left(\frac{u^\mu}{c} + \frac{q}{mc^2}A^\mu\right)\left(\frac{u^\nu}{c} + \frac{q}{mc^2}A^\nu\right) = g_{\mu\nu}\frac{U^\mu}{c}\frac{U^\nu}{c}. \quad (3.1)$$

This will be useful in a variety of circumstances. The above includes the mechanical four-velocity $u^\mu \equiv dx^\mu / d\tau$ and a canonical four-velocity defined by $U^\mu \equiv u^\mu + qA^\mu / mc$. From here, we shall work in natural units $c = 1$ and use dimensional rebalancing to restore *c* only after a final result.

The first place that "1" above will be useful is in (1.1), where, distributing the expression after the first equality while absorbing $g_{\mu\nu}$ into the electrodynamic term indices, we write:

$$0 = \delta\int_A^B d\tau(1) = \delta\int_A^B d\tau\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\frac{q}{m}A_\sigma\frac{dx^\sigma}{d\tau} + \frac{q^2}{m^2}A_\sigma A^\sigma\right)^{.5}. \quad (3.2)$$

From here, we carry out the variational calculation, which deductively culminates in:

$$0 = \delta\int_A^B d\tau = \int_A^B \delta x^\alpha d\tau\left(\begin{array}{l} -g_{\alpha\nu}\dfrac{d^2x^\nu}{d\tau^2} + \dfrac{1}{2}\left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right)\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} \\ +\dfrac{q}{m}\left(\partial_\alpha A_\sigma - \partial_\sigma A_\alpha\right)\dfrac{dx^\sigma}{d\tau} + \dfrac{1}{2}\dfrac{q^2}{m^2}\partial_\alpha\left(A_\sigma A^\sigma\right) \end{array}\right). \quad (3.3)$$

Going from (3.2) to (3.3) is straightforward. The top line contains the same result usually obtained for gravitational geodesics, which is the result of setting $q = 0$ in (3.2). This is the calculation Einstein first presented in §9 of [1], and does not need to be reviewed further. The terms on the bottom line emerge as a direct and immediate consequence of starting with the canonical $m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ rather than the ordinary mechanical $m^2c^2 = g_{\mu\nu}p^\mu p^\nu$ energy-momentum relation, which is to say, the bottom line is a result merely of mandating local gauge symmetry. Some specific guides to note when performing the detailed calculation include: a) we assume no variation in the charge-to-mass ratio, i.e., that $\delta(e/m) = 0$, over the path from *A* to *B*; b) applied to gauge field terms, the variations are $\delta A_\sigma = \delta x^\alpha \partial_\alpha A_\sigma$ and $\delta\left(A_\sigma A^\sigma\right) = \delta x^\alpha \partial_\alpha\left(A_\sigma A^\sigma\right)$; c) we also use $dA_\sigma / d\tau = \partial_\alpha A_\sigma dx^\alpha / d\tau$; and d) there is an integration-by-parts in the calculation. This integration-by-parts produces a boundary term $\int_A^B d\left(A_\sigma \delta x^\sigma\right) = \left(A_\sigma \delta x^\sigma\right)\Big|_A^B = 0$ that can be eliminated, and for the remaining term causes the sign reversal appearing in $\partial_\alpha A_\sigma - \partial_\sigma A_\alpha$.

The proper time $d\tau \neq 0$ for material worldlines, and between the boundaries at *A* and *B* the variation $\delta x^\sigma \neq 0$. So the large parenthetical expression in (3.3) must be zero. The connection

$-\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right)$ and field strength $F_{\alpha\sigma} = \partial_\alpha A_\sigma - \partial_\sigma A_\alpha = \partial_{;\alpha} A_\sigma - \partial_{;\sigma} A_\alpha$. So with *c* restored, this enables us to extract:

$$\frac{d^2 x^\beta}{d\tau^2} = -\Gamma^{\beta}{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{q}{m}F^{\beta}{}_{\sigma}\frac{dx^\sigma}{cd\tau} + \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right). \tag{3.4}$$

This clearly reproduces (2.1) and includes the Lorentz force motion alongside the gravitational geodesic, all obtained from the minimized variation (3.2). Therefore, (3.4) does represent geodesic motion, although when contrasted to the Lorentz motion it contains an additional term $\partial^\beta\left(A_\sigma A^\sigma\right)$ that we shall shortly review in depth.

As with (1.3), we may view (3.4) in an alternative albeit equivalent way that highlights how Lorentz motion plus the extra term is now merely a consequence of local gauge symmetry: It is well-known how imposing gauge symmetry spawns the heuristic rules $\partial_\sigma \rightarrow \mathfrak{D}_\sigma \equiv \partial_\sigma - iqA_\sigma$ and $p^\mu \rightarrow \pi^\mu \equiv p^\mu + qA^\mu / c$ for gauge-covariant derivatives and canonical momentum, and $m^2c^2 = g_{\mu\nu} p^\mu p^\nu \rightarrow m^2c^2 = g_{\mu\nu}\pi^\mu\pi^\nu$ for the energy momentum relation. Here, referring to (1.3), we see another heuristic rule which emerges in lockstep with these others, namely:

$$\frac{Du^\beta}{D\tau} = \frac{du^\beta}{d\tau} + \Gamma^{\beta}{}_{\mu\nu}u^\mu u^\nu \rightarrow \mathrm{A}^\beta \equiv \frac{\mathfrak{D}u^\beta}{\mathfrak{D}\tau} \equiv \frac{Du^\beta}{D\tau} - \frac{q}{mc}F^{\beta}{}_{\sigma}u^\sigma - \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right) = 0, \tag{3.5}$$

which in the absence of gravitation we may write as:

$$\frac{du^\beta}{d\tau} \rightarrow \frac{\mathfrak{D}u^\beta}{\mathfrak{D}\tau} \equiv \frac{du^\beta}{d\tau} - \frac{q}{mc}F^{\beta}{}_{\sigma}u^\sigma - \frac{1}{2}\frac{q^2}{m^2c^2}\partial^\beta\left(A_\sigma A^\sigma\right) = 0. \tag{3.6}$$

In the above, $\mathfrak{D}u^\beta / \mathfrak{D}\tau$ symbolizes the gauge-covariant or *canonical acceleration*, which is rooted in the further heuristic $dx^\mu \rightarrow \mathfrak{D}x^\mu \equiv dx^\mu + \left(q / mc\right) d\tau A^\mu$ defined in (2.5). And more generally, using the boldface $\boldsymbol{\mathfrak{D}}$ notation whenever there are both gravitational and electrodynamic fields, we have used $\mathrm{A}^\beta \equiv \boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau = 0$ to denote the *gravitationally-* *<u>and</u>* *gauge-covariant* acceleration. The equation $\boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau = 0$ in (3.5) states that *covariant canonical acceleration* is gravitationally-covariant and gauge-covariant, which we shall refer to generally as "canonical covariance." Yet, when shown in terms of *mechanical* four-velocities $u^\mu = dx^\mu / d\tau$, the *mechanical acceleration* contains the geodesic motion of gravitation and the Lorentz force motion of electrodynamics. In the absence of any charge or electromagnetic potential / field the above reverts back to $Du^\beta / D\tau = du^\beta / d\tau + \Gamma^{\beta}{}_{\mu\nu}u^\mu u^\nu = 0$ for gravitationally-covariant motion (1.3). In the absence of gravitation we reduce to (3.6) for the canonically-covariant Lorentz force alone. And in the absence of both gravitation and electromagnetism what remains is merely $du^\beta / d\tau = 0$ for the Newtonian inertial motion governed by special relativity alone. From this view, all physical motion is inertial and geodesic because $\boldsymbol{\mathfrak{D}}u^\beta / \boldsymbol{\mathfrak{D}}\tau \equiv 0$; the motion is simply *covariantly and* *canonically-inertial* with any gravitational curvature and any canonical gauge elements. What we observe physically are the mechanical counterparts to the covariant canonical motion.

All of the above provides a conceptually-compelling view of classical physical motion. However, (3.4) yields a term $\partial^\beta\left(A_\sigma A^\sigma\right)$ which is not ordinarily a part of the Lorentz force law. And in fact, this term needs to be removed for one empirical reason and two theoretical reasons: The empirical reason is that this term is not part of the well-established, well-corroborated. universally-observed Lorentz Force law (2.1). The first theoretical reason is that the motion cannot depend upon a term $\partial_\beta\left(A_\sigma A^\sigma\right)$ which in turn depends upon and changes as a function of the unobservable local phase $\Lambda(t,\mathbf{x})$. Specifically, the gauge transformation $qA_\sigma \to qA'_\sigma = qA_\sigma - \partial_\sigma\Lambda$ would introduce the phase into (3.4) and thus leave the observable motion ambiguous and in violation of gauge symmetry. The second theoretical reason is that by removing this term, (3.4) now does fully describe the Lorentz motion as geodesic motion, which is conceptually attractive. So the question arises whether there is some clear natural basis upon which this term does in fact get removed in the physical world.

A simple fix would be to modify the metric (2.5) by subtracting out the second-order term with $A_\sigma A^\sigma$, and to then start the variation of (3.2) on the basis of:

$$c^2 d\tau^2 = \mathfrak{D}x_\sigma \mathfrak{D}x^\sigma - \frac{q^2}{m^2c^2}d\tau^2 A_\sigma A^\sigma = \left(dx_\sigma + \frac{q}{mc}d\tau A_\sigma\right)\left(dx^\sigma + \frac{q}{mc}d\tau A^\sigma\right) - \frac{q^2}{m^2c^2}d\tau^2 A_\sigma A^\sigma . \qquad (3.7)$$

When turned into the number "1" as in (3.1) and then used in the variation as in (3.2), it is clear that this will result in (3.4) but without the extra term $\partial^\beta\left(A_\sigma A^\sigma\right)$ because the source of that term is subtracted out of (3.7). So the result is the Lorentz force plus gravitational motion, precisely, as desired. However, this approach loses some conceptual strength, because the Lorentz force does not emerge simply from applying local gauge symmetry and the heuristic rules which emerge from this symmetry as reviewed in equations (3.5) and (3.6). Now the rule becomes: apply gauge symmetry, *and then take the extra step* of subtracting off the $A_\sigma A^\sigma$ term to get a desired result. Occam's razor would in this circumstance compel us to see if this second step can be eliminated, and whether the term $\partial^\beta\left(A_\sigma A^\sigma\right)$ can be removed from (3.4) in some other, more natural way.

As we shall now see in sections 4 through 7, this extra term in (3.4), and the process for its prospective removal from (3.4), is intimately connected with gauge fixing, Maxwell's electric charge equation, the electrodynamic Lagrangian and action, electrodynamic and gravitational power, and the sources $T^{\mu\nu}$ in Einstein's field equation for gravitation.

4. The Lagrangian Gauge and the Geodesic Gauge, and Canonically-Inertial Motion

To study the extra term $\partial^\beta\left(A_\sigma A^\sigma\right)$ in (3.4), we start with Maxwell's equation $J^\beta = \partial_{;\alpha}F^{\alpha\beta}$ for the electric charge density. Via the usual expression $F^{\alpha\beta} = \partial^{;\alpha}A^\beta - \partial^{;\beta}A^\alpha = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ for the field strength we write this in terms of the gauge fields as $J^{\beta} - \partial_{\alpha;}\partial^{\alpha}A^{\beta} + \partial^{\beta}\partial_{;\alpha}A^{\alpha} = 0$. But we do *not* use the Lorenz condition $\partial_{;\alpha}A^{\alpha} = 0$ to fix the gauge; rather for now we leave this term as is. We then multiply this Maxwell equation through by A_{β}, thus writing the scalar equation:

$$A_{\beta}J^{\beta} - A_{\beta}\partial_{;\alpha}\partial^{\alpha}A^{\beta} + A_{\beta}\partial^{\beta}\partial_{;\alpha}A^{\alpha} = 0 . \tag{4.1}$$

For the second term above we have $-A_{\beta}\partial_{;\alpha}\partial^{\alpha}A^{\beta} = \partial_{;\alpha}A_{\beta}\partial^{\alpha}A^{\beta} - \partial_{;\alpha}\left(A_{\beta}\partial^{\alpha}A^{\beta}\right)$, using the product rule. We may also form the identity $A_{\beta}\partial^{\alpha}A^{\beta} = \frac{1}{2}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right)$. Using both of these in (4.1) yields:

$$A_{\beta}J^{\beta} + \partial_{;\alpha}A_{\beta}\partial^{\alpha}A^{\beta} - \tfrac{1}{2}\partial_{;\alpha}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right) + A_{\beta}\partial^{\beta}\partial_{;\alpha}A^{\alpha} = 0 . \tag{4.2}$$

The second term $\partial_{;\alpha}A_{\beta}\partial^{\alpha}A^{\beta} = \partial_{\alpha}A_{\beta}\partial^{\alpha}A^{\beta} = \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$, and with this, the first two terms are equivalent to minus the electrodynamic Lagrangian density, $A_{\beta}J^{\beta} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} = -\mathcal{L}_{em}$. Therefore, (4.2) is simply:

$$-\tfrac{1}{2}\partial_{;\alpha}\partial^{\alpha}\left(A_{\beta}A^{\beta}\right) + A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha} = \mathcal{L}_{em} . \tag{4.3}$$

Again, this is an alternative way of saying that $A_{\beta}J^{\beta} = A_{\beta}\partial_{;\alpha}F^{\alpha\beta}$, which is a four-dimensional scalar product of Maxwell's charge equation with the gauge field. Note that $\partial_{\beta}\partial_{;\alpha}A^{\alpha} = \partial_{;\beta}\partial_{;\alpha}A^{\alpha}$ because the gravitationally-covariant derivative of any scalar is equal to the ordinary derivative of the same. As is easily seen, the first term above contains the extra term $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$ that appeared in (3.4). And the second term contains $\partial_{;\alpha}A^{\alpha}$ which in the Lorenz gauge is fixed to $\partial_{;\alpha}A^{\alpha} = 0$. The latter is a covariant scalar condition which removes one degree of freedom from the gauge field A^{α}.

Now, because photons which comprise the gauge field are massless, we are not *required* to use $\partial_{;\alpha}A^{\alpha} = 0$ as we would be if photons were massive. Instead, we are permitted to fix the gauge directly to the physical Maxwell Lagrangian by setting:

$$A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha} \equiv \mathcal{L}_{em} . \tag{4.4}$$

This is also a covariant scalar gauge condition which removes one degree of freedom, so it would be a suitable replacement for the Lorenz gauge. For obvious reasons we shall refer to this as the "*Lagrangian gauge*." If we were to impose this condition, then as a consequence of combining (4.4) with Maxwell's equation represented via (4.3), we would also find (renaming indexes) that:

$$\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right) = 0 . \tag{4.5}$$

Therefore, at the very least, the *four-gradient* $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)$ of the term $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$ would become zero. The question now is: may we and should we adopt the Lagrangian gauge (4.4), and also, the stronger condition that $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ itself?

Were we to impose the condition $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ and thus add further constraint beyond the covariant scalar relation (4.4), then (4.5) would still remain true and thus be compatible with the Lagrangian gauge condition (4.4). And all of this would remain compatible with the scalar representation (4.3) of Maxwell's equation in $A_{\beta}J^{\beta}=A_{\beta}\partial_{;\alpha}F^{\alpha\beta}$. So there is no apparent conflict or contradiction that arises from setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$. But it is also well-known that a covariant scalar gauge condition such as the Lorenz gauge $\partial_{;\alpha}A^{\alpha}=0$ or the Lagrangian gauge of (4.4) still leaves some residual ambiguity in the gauge field, which ambiguity still needs to be removed. The question is how we do so. Because setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$ would be an even stronger constraint than (4.5), clearly this would squeeze out some further ambiguity. The question now is whether this would remove just enough ambiguity to eliminate *all* residual ambiguity, and at the same time not over-determine the results by imposing too much constraint.

This brings us back to (3.4). As noted in the paragraph prior to (3.7), a gauge transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$ applied to (3.4) would leave the physical motion ambiguous because of the extra term $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)$. Further, there is no way to completely remove this ambiguity without removing this term entirely. The weaker condition (4.5) which via (4.3) is a proxy for the Lagrangian gauge (4.4), which in turn is a substitute for the Lorenz gauge, would remove all traces of this extra term from the *third-derivative* expression that would result were we to take $\partial_{;\beta}d^{2}x^{\beta}/d\tau^{2}$ by applying $\partial_{;\beta}$ to (3.4). But there would still remain some ambiguity at the second derivative which is (3.4) because of what happens when we apply the transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$. Therefore, to remove *all* ambiguity from the physical motion, we do need to apply the stronger condition $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$. Once we do so, all of the remaining ambiguity is removed from the physical motion of (3.4), and the result is no more and no less than the Lorentz force law. And because the Lorentz force law is entirely symmetric under the gauge transformation $qA_{\sigma}\rightarrow qA'_{\sigma}=qA_{\sigma}-\partial_{\sigma}\Lambda$, we are assured that not only have we removed all physical ambiguity by setting $\partial^{\beta}\left(A_{\sigma}A^{\sigma}\right)=0$, but also that we have not removed too much ambiguity so as to over-determine the physical result. Rather, we have precisely determined the physical result. And, we are assured from the derivation (4.1) through (4.5) that there is no contradiction whatsoever with Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$.

Therefore, we shall now formally take the following two steps: First, to covariantly remove one degree of freedom from the gauge field, we shall fix the gauge using the Lagrangian gauge condition $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\mathcal{L}_{em}$ of (4.4). This is in lieu of applying the Lorenz gauge condition

$\partial_{;\alpha}A^{\alpha}=0$. Second, to remove any additional ambiguity from the gauge field, we shall impose the condition:

$$\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0 \tag{4.6}$$

on the four-gradient of the scalar quantity $A_{\alpha}A^{\alpha}$. The d'Alembertian of this scalar will then also be zero as shown in (4.5), which is fully compatible with Maxwell's electric charge equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$. By imposing both conditions (4.4) and (4.6), the result in (3.4) now reduces to:

$$\frac{d^{2}x^{\beta}}{d\tau^{2}}=-\Gamma^{\beta}{}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+\frac{q}{m}F^{\beta}{}_{\sigma}\frac{dx^{\sigma}}{cd\tau}. \tag{4.7}$$

Note, because we now have $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\mathcal{L}_{em}$, that the additional use of the Lorenz gauge $\partial_{;\alpha}A^{\alpha}=0$ is *not permitted*: imposing this condition would cause $\mathcal{L}_{em}=0$ and thereby over-determine the physical results.

Now, the Lorentz force law has been derived from the minimized variation $0=\delta\int_{A}^{B}d\tau$ of (1.1) starting at (3.2) by merely requiring local gauge symmetry and, true to Occam's razor, nothing more. The extra term $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)$ has been removed not by the unnatural fix of (3.7), but rather by the natural solution of fixing the gauge to entirely remove any ambiguity from the physical motion without over-determination. Following all of this, (3.5) reduces to:

$$\mathrm{A}^{\beta}=\frac{\mathfrak{D}u^{\beta}}{\mathfrak{D}\tau}\equiv\frac{Du^{\beta}}{D\tau}-\frac{q}{m}F^{\beta}{}_{\sigma}u^{\sigma}=\frac{du^{\beta}}{d\tau}+\Gamma^{\beta}{}_{\mu\nu}u^{\mu}u^{\nu}-\frac{q}{m}F^{\beta}{}_{\sigma}u^{\sigma}=0, \tag{4.8}$$

and the combined Lorentz and gravitational acceleration truly is geodesic motion. Specifically, the motion (4.8) is inertial in both a gravitationally- and canonically-covariant manner. As a shorthand, we shall refer to this simply as "*canonically-inertial motion*." This is a generalization of Newtonian inertial motion $du^{\beta}/d\tau=0$ to the circumstance where gravitational and electromagnetic fields are present and the test particle has a charge q that interacts with the electromagnetic fields. Here, the canonical $\mathfrak{D}u^{\beta}/\mathfrak{D}\tau=0$ instead, while the mechanical motion $du^{\beta}/d\tau\neq 0$, which is not inertial, describes what is observed when the motion is referred to the coordinates x^{β} of $u^{\beta}=dx^{\beta}/d\tau$ and then clocked in relation to the proper time linear metric element $d\tau$. Given all of this, we shall refer to (4.6) as the "*geodesic gauge*" condition.

The foregoing is yet another example of the general heuristic rule that when gauge fields and charges are present, canonical quantities behave in the same way that their counterpart mechanical quantities behave in the absence of the gauge fields and charges. Thus, for example, the mechanical $\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=0$ is inherited by Dirac's canonical $\left(i\gamma^{\mu}\mathfrak{D}_{\mu}-m\right)\psi=0$; while the mechanical energy relation $m^{2}c^{2}=g_{\mu\nu}p^{\mu}p^{\nu}$ is inherited by the canonical $m^{2}c^{2}=g_{\mu\nu}\pi^{\mu}\pi^{\nu}$ of

(2.6); and the mechanical $du^{\beta}/d\tau = 0$ is inherited (absent gravitation) by the canonical $\mathfrak{D}u^{\beta}/\mathfrak{D}\tau = 0$ for the Lorentz force, see (3.6) without the extra term. And for yet another example, absent gauge fields and charges, the momenta along different space axes are compatible, $\left[p^{i}, p^{j}\right] = 0$. But once gauge fields and charges are added, then it is the canonical $\left[\pi^{i}, \pi^{j}\right] = 0$ which inherit this compatibility. Using $\pi^{\mu} \equiv p^{\mu} + qA^{\mu}$ and the canonical commutativity relation $\left[p^{i}, A^{j}\right] = -i\hbar\partial^{i}A^{j}$, it is then straightforward to show that the mechanical momenta $\left[p^{i}, p^{j}\right] = iq\hbar F^{ij}$ now become incompatible, where $F^{ij} = \partial^{i}A^{j} - \partial^{j}A^{i} - iq\left[A^{i}, A^{j}\right]/\hbar$ are the space components of a non-abelian field strength. For the electromagnetic field which is abelian, this becomes $\left[p^{i}, p^{j}\right] = iq\hbar\varepsilon^{ijk}B^{k}$ where $B^{k} = \mathbf{B}$ is the magnetic field bivector and the Levi-Civita tensor $\varepsilon^{123} = -1$ given that the lower-indexed $\varepsilon_{123} \equiv +1$. So the magnetic fields measure the incompatibility of the mechanical momentum components.

This last example, via the heuristic interchange $i\partial_{\mu} \Leftrightarrow p_{\mu}$, is simply a variant of the fundamental premise that in gauge theory, the field strength is an *imaginary* measure $\left[\mathfrak{D}^{\mu}, \mathfrak{D}^{\nu}\right]\phi = -iqF^{\mu\nu}\phi$ of the extent to which the gauge-covariant (canonical) derivatives do not commute when acting on a generalized field ϕ. This is *why* Hermann Weyl pursued gauge theory to begin with, as an effort to generalize into electrodynamics, general relativistic curvatures for which $R^{\alpha}{}_{\beta\mu\nu}A_{\alpha} = \left[\partial_{;\nu}, \partial_{;\mu}\right]A_{\beta}$. And this is why $F^{\mu\nu}$ is often referred to as the "curvature" tensor. However, as discussed after (2.10), it took Weyl just over a decade [2], [3], [4] to finally realize that $qF^{\mu\nu}\phi = i\left[\mathfrak{D}^{\mu}, \mathfrak{D}^{\nu}\right]\phi$ must bear an imaginary, not real, relation to curvature, and that the root symmetry was not under a re-gauging, but rather under a re-phasing, of electron wavefunctions.

Now, let us explore some further significant results which arise from the Lagrangian gauge (4.4) and the geodesic gauge (4.6). As noted at the end of the previous section, these result relate to the electrodynamic Lagrangian and action (the former already seen in the Lagrangian gauge $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha} = \mathcal{L}_{em}$ of (4.4)), electrodynamic and gravitational power, and the sources $T^{\mu\nu}$ in Einstein's equation.

5. The Electrodynamic Action in Lagrangian Gauge

It is very illustrative to rewrite the Lagrangian gauge (4.4) using the product rule as

$$\mathcal{L}_{em} = A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha} = \partial_{\beta}\left(A^{\beta}\partial_{;\alpha}A^{\alpha}\right) - \partial_{\beta}A^{\beta}\partial_{;\alpha}A^{\alpha}, \tag{5.1}$$

and then obtain the electrodynamic action $S_{em} = \int d^{4}x\mathcal{L}_{em}$. Once inside the action integral, we may set $\int d^{4}x\partial_{\beta}\left(A^{\beta}\partial_{;\alpha}A^{\alpha}\right) = 0$ via the boundary condition $A_{\beta}(t, \mathbf{x}) = 0$ at the extremum $t, \mathbf{x} = \pm\infty$. What we then end up with is an action:

$$S_{em} = \int d^4 x \mathcal{L}_{em} = -\int d^4 x \partial_\beta A^\beta \partial_{;\alpha} A^\alpha = -\int d^4 x \left(\partial_\beta A^\beta \partial_\alpha A^\alpha + \Gamma^\alpha{}_{\sigma\alpha} A^\sigma \partial_\beta A^\beta \right), \tag{5.2}$$

noting also that $\Gamma^\alpha{}_{\sigma\alpha} = \partial_\sigma \sqrt{-g} \big/ \sqrt{-g}$ where g is the metric tensor determinant. In flat spacetime, with $\partial_\sigma \sqrt{-g} = 0$, this becomes the very simple action:

$$S_{em} = \int d^4 x \mathcal{L}_{em} = -\int d^4 x \left(\partial_\alpha A^\alpha \right)^2 . \tag{5.3}$$

It will be seen that (5.3) is analogous to the R_ξ gauge conditions, which are ordinarily written as $\delta \mathcal{L} = -\left(\partial_\alpha A^\alpha \right)^2 / 2\xi$. However, (5.2) and (5.3) are not local conditions; they are global because they represent an integral over the entire volume of the four-dimensional spacetime.

Once we are working with the action, we are but a step away from Quantum Electrodynamics, which is generated through the path integration $Z_{em} = \int DA^\alpha \exp\left(iS_{em} / \hbar \right)$. As usual, we may start with $A_\beta J^\beta + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = -\mathcal{L}_{em}$ to obtain the electrodynamic action $S_{em} = \int d^4 x \left(\frac{1}{2} A_\mu \left(g^{\mu\nu} \partial_\sigma \partial^\sigma - \partial^\mu \partial^\nu \right) A_\nu - J_\mu A^\mu \right)$. Note that this has no expressly-appearing gravitationally-covariant derivatives, because of the cancellations that occur via $F^{\alpha\beta} = \partial^{;\alpha} A^\beta - \partial^{;\beta} A^\alpha = \partial^\alpha A^\beta - \partial^\beta A^\alpha$. However, there is an implicit gravitational term, because $J^\beta = \partial_{;\alpha} F^{\alpha\beta}$. This is the exact origin starting at (4.1) of the $\partial_{;\alpha}$ appearing in (5.1) and (5.2). Then we use Gaussian integration to path integrate as usual. But the upshot of (5.2) is to tell us that:

$$S_{em} = \int d^4 x \left(\frac{1}{2} A_\mu \left(g^{\mu\nu} \partial_\alpha \partial^\alpha - \partial^\mu \partial^\nu \right) A_\nu - J_\alpha A^\alpha \right) = -\int d^4 x \left(\left(\partial_\alpha A^\alpha \right)^2 + \Gamma^\alpha{}_{\sigma\alpha} A^\sigma \partial_\beta A^\beta \right). \tag{5.4}$$

This provides a second expression for the action based on employing the Lagrangian gauge (4.4) in the process of deducing the combined gravitational and Lorentz force motion of (4.7) and (4.8) from the minimized variation (1.1) as applied in (3.2). The other constraint, of course, is the geodesic gauge condition $\partial^\beta \left(A_\alpha A^\alpha \right) = 0$ of (4.6) to which we now turn. This constraint leads to a relation for electrodynamic and gravitational power, and leads to a direct connection with the sources $T^{\mu\nu}$ in Einstein's equation.

6. The Geodesic Gauge and the Electro-Gravitational Power Equation

Next, we study the effect of the geodesic gauge condition (4.6) on the canonical energy-momentum relation (2.6). We first return to (2.6), which, with indices summed and with $c = 1$, we expand without commuting the left-right ordering of the momenta and the gauge fields, to obtain $m^2 = p_\sigma p^\sigma + qA_\sigma p^\sigma + qp_\sigma A^\sigma + q^2 A_\sigma A^\sigma$. The reason we refrain from commuting is to highlight that were we to combine the two middle terms into $qA_\sigma p^\sigma + qp_\sigma A^\sigma = 2qA_\sigma p^\sigma$ we would need to commute p_σ and A^σ which needs to be done with care given the Heisenberg commutation relation $\left[p_j, B\right] = -i\hbar\partial_j B$ for any field $B(t,\mathbf{x})$ which is a function of the spacetime coordinates. And as to the time component, we would also want to be mindful of the Heisenberg equation of motion $\left[H_0, O\right] = -i\hbar\partial_0 O$ for an operator O with no explicit time dependence, together with relationship $H_0|\psi\rangle = p_0|\psi\rangle$ between the Hamiltonian H_0 operator and the observable energy $p_0 = E$ which contains its eigenvalues. Therefore, even if we were to commute the energy with the time component of the potential $A^0 = \phi$ thus setting $\left[p_0, A^0\right] = 0$, we would still have to recognize that $p_j A^j = A_j p^j - i\hbar\partial_j A^j$ and thus include a term of the form $-i\hbar\partial_j A^j$ if not $-i\hbar\partial_\sigma A^\sigma$, if it was our desire to move beyond classical physics and account for the quantum mechanical non-commutativity.

For present purposes, to be completely general, let us use the relationship $\left[p_\sigma, A^\sigma\right] = -i\hbar\partial_\sigma A^\sigma$ a.k.a. $p_\sigma A^\sigma = A_\sigma p^\sigma - i\hbar\partial_\sigma A^\sigma$ covariantly extended into the time dimension, recognizing that we may always restrict this to the space components by setting $\left[p_0, A^0\right] = 0$, thus $\partial_0 A^0 = 0$, and may additionally ignore quantum effects entirely by setting $\left[p_j, A^j\right] = 0$, thus the space divergence $\partial_j A^j = \nabla \cdot \mathbf{A} = 0$. Therefore, we start by writing (2.6), with $\hbar = c = 1$, as:

$$m^2 = p_\sigma p^\sigma + 2qA_\sigma p^\sigma + q^2 A_\sigma A^\sigma - iq\partial_\sigma A^\sigma . \tag{6.1}$$

The final term $\partial_\sigma A^\sigma$ arises from the commutativity just discussed, and may be removed or ignored under the circumstances just discussed.

Now, let us take the covariant spacetime gradient $\partial_{;\beta}$ of the above. The rest mass is invariant, and so its four-gradient $\partial_{;\beta} m = \partial_\beta m = 0$. Therefore, after reduction we obtain:

$$0 = p_\sigma \partial_{;\beta} p^\sigma + q\partial_{;\beta} A_\sigma p^\sigma + qA_\sigma \partial_{;\beta} p^\sigma + \tfrac{1}{2} q^2 \partial_{;\beta}\left(A_\sigma A^\sigma\right) - \tfrac{1}{2} iq\partial_{;\beta}\partial_\sigma A^\sigma . \tag{6.2}$$

Now we apply the geodesic gauge (4.6), so the term $\partial_{;\beta}\left(A_\sigma A^\sigma\right) = \partial_\beta\left(A_\sigma A^\sigma\right) = 0$ is removed. We may also use the field strength to replace $\partial_{;\beta} A_\sigma = F_{\beta\sigma} + \partial_{;\sigma} A_\beta$. Additionally, $p^\sigma = mu^\sigma$ is the ordinary mechanical momentum, so we can divide out *m*, whereby $p^\sigma \to u^\sigma$ throughout the contravariant momentum terms in the above. Thus, segregating the field strength term on the left, (6.2) becomes:

$$qF_{\beta\sigma} u^\sigma = -p_\sigma \partial_{;\beta} u^\sigma - qA_\sigma \partial_{;\beta} u^\sigma - q\partial_{;\sigma} A_\beta u^\sigma + \tfrac{1}{2} i\left(q/m\right)\partial_{;\beta}\partial_\sigma A^\sigma . \tag{6.3}$$

We of course recognize $qF_{\beta\sigma} u^\sigma$ as a variant of the Lorentz force term in (2.1).

Now, we wish to express the terms on the right in relation to the passage of proper time, that is, as derivatives along the curve, see (3.5) and (3.7). For the next-to-last term in (6.3) we may substitute $\partial_{;\sigma} A_\beta u^\sigma = dA_\beta / d\tau - \Gamma^\tau{}_{\sigma\beta} A_\tau u^\sigma$ derived using the gravitationally-covariant derivative and the chain rule. So also with $\partial_{;\beta}\partial_\sigma A^\sigma = \partial_\beta \partial_\sigma A^\sigma$, (6.3) advances to:

$$qF_{\beta\sigma}u^\sigma = -p_\sigma \partial_{;\beta} u^\sigma - qA_\sigma \partial_{;\beta} u^\sigma - q\frac{dA_\beta}{d\tau} + q\Gamma^\tau{}_{\sigma\beta} A_\tau u^\sigma + \frac{1}{2} i (q/m) \partial_\beta \partial_\sigma A^\sigma . \tag{6.4}$$

As to the remaining terms, we now multiply by $u^\beta = dx^\beta / d\tau$ throughout, giving us a $u^\beta \partial_{;\beta} u^\sigma$ in the first two terms after the equality. Then we may similarly derive and then substitute $u^\beta \partial_{;\beta} u^\sigma = du^\sigma / d\tau + \Gamma^\sigma{}_{\beta\tau} u^\beta u^\tau$. Also writing $p_\sigma = mu_\sigma$ for the remaining mechanical momentum, and seeing that the terms with $\Gamma^\tau{}_{\sigma\beta} A_\tau u^\beta u^\sigma$ cancel identically, and using the chain rule in the final term $u^\beta \partial_\beta \partial_\sigma A^\sigma = (d/d\tau) \partial_\sigma A^\sigma / = \partial_\sigma dA^\sigma / d\tau$, with renamed indices and $\hbar = c = 1$ restored, we now have:

$$\frac{q}{c} F_{\mu\nu} u^\mu u^\nu = -\left(mu_\sigma + \frac{q}{c} A_\sigma \right) \frac{du^\sigma}{d\tau} - \frac{q}{c} u^\sigma \frac{dA_\sigma}{d\tau} - m\Gamma^\sigma{}_{\mu\nu} u_\sigma u^\mu u^\nu + \frac{1}{2} i\hbar \frac{q}{mc} \partial_\sigma \frac{dA^\sigma}{d\tau} . \tag{6.5}$$

This $(q/c) F_{\mu\nu} u^\mu u^\nu$ term on the left is a scalar number, and it has dimensions of power. So this is an expression for electrodynamic and gravitational power. However, because $F_{\mu\nu}$ is an antisymmetric tensor, this term vanishes identically. Therefore, moving all of the mechanical and gravitational terms to the left and keeping the electrodynamic terms on the right, we may consolidate to:

$$mu_\sigma \left(\frac{du^\sigma}{d\tau} + \Gamma^\sigma{}_{\mu\nu} u^\mu u^\nu \right) = -\frac{q}{c} \frac{d}{d\tau} \left(A_\sigma u^\sigma \right) + \frac{1}{2} i\hbar \frac{q}{mc} \partial_\sigma \frac{dA^\sigma}{d\tau} . \tag{6.6}$$

It is easily seen that when the right hand side becomes zero in the absence of electrodynamics, the left hand side contains the gravitational geodesic motion (1.1). The final term may also be vanished by setting $\hbar = 0$, i.e., in the classical limit. In terms of spacetime coordinates with all terms expanded, and isolating all the acceleration terms on the left, another way to express this is:

$$\left(m\frac{dx_\sigma}{d\tau} + \frac{q}{c} A_\sigma \right) \frac{d^2 x^\sigma}{d\tau^2} = -\left(m\Gamma^\sigma{}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{q}{c} \frac{dA^\sigma}{d\tau} \right) \frac{dx_\sigma}{d\tau} + \frac{1}{2} i\hbar \frac{q}{mc} \partial_\sigma \frac{dA^\sigma}{d\tau} . \tag{6.7}$$

In the absence of gravitation, we merely set $\Gamma^\sigma{}_{\mu\nu} = 0$. And if we neglect the non-commutativity discussed in the first paragraph of this section, then we may set $\hbar = 0$ to vanish the final term. The effect of the geodesic gauge (4.6) in all of the foregoing, starting at (6.3), is to have removed the terms $A_\sigma A^\sigma$ which are of second order in the gauge field.

Now let us see how this connects to Einstein's equation and gravitational curvature.

7. The Electro-Gravitational Energy Flux Field Equation

As already reviewed, by fixing to the Lagrangian gauge $A^\beta \partial_\beta \partial_{;\alpha} A^\alpha \equiv \mathcal{L}_{em}$ of (4.4) in lieu of the Lorenz gauge $\partial_{;\alpha} A^\alpha = 0$, Maxwell's equation $J^\beta = \partial_{;\alpha} F^{\alpha\beta}$ also constrains us to require the relation $\partial_{;\beta}\partial^\beta \left(A_\alpha A^\alpha\right) = 0$ of (4.5). The stronger geodesic gauge $\partial^\beta \left(A_\alpha A^\alpha\right) \equiv 0$ of (4.6) was used to remove the remaining gauge ambiguity from the equation of motion (3.4), or (3.5), thereby producing the combined gravitational and Lorentz force law of motion (4.7). This raises an interesting question: if we want to explore the impact on the equation of motion of the weaker condition $\partial_{;\beta}\partial^\beta \left(A_\alpha A^\alpha\right) = 0$ which is required for compatibility with Maxwell's equation, then it is clear that this impact can be seen by taking the covariant gradient $\partial_{;\beta}$ of the original equation of motion (3.4) from before we imposed the stronger condition of (4.6). What makes this interesting is that this ties together the sources in both the Einstein equation for gravitation and Maxwell's equation for electric charges, as we shall now see.

Mindful that $A_\beta J^\beta + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} = -\mathcal{L}_{em}$, we start by taking the covariant gradient $\partial_{;\beta}$ of (3.5), and then applying (4.3) which stems from Maxwell's charge equation, to obtain:

$$\partial_{;\beta} \mathrm{A}^\beta = \partial_{;\beta} \frac{\mathfrak{D} u^\beta}{\mathfrak{D}\tau} = \partial_{;\beta} \frac{Du^\beta}{D\tau} - \frac{q}{m} \partial_{;\beta} \left(F^\beta{}_\sigma u^\sigma\right) + \frac{q^2}{m^2} \left(\mathcal{L}_{em} - A^\beta \partial_\beta \partial_{;\alpha} A^\alpha\right) = 0 . \tag{7.1}$$

To be clear, the above via the development laid out from (3.2) to (3.5) is a direct deductive consequence of taking the variation $0 = \delta \int_A^B d\tau$ based on the canonical mass-energy-momentum relation $m^2 c^2 = g_{\mu\nu} \pi^\mu \pi^\nu$ of (2.6) in combination with Maxwell's charge equation $J^\beta = \partial_{;\alpha} F^{\alpha\beta}$. No additional assumptions are used to obtain (7.1), and in particular, no gauge conditions have yet been imposed on (7.1).

First, let us focus on the term $\partial_{;\beta} Du^\beta / D\tau$. Using the expression $R^\alpha{}_{\beta\mu\nu} B_\alpha = \left[\partial_{;\nu}, \partial_{;\mu}\right] B_\beta$ which relates the Riemann tensor to the degree to which gravitationally-covariant derivatives do not commute when operating on an arbitrary vector B_α, from which we deduce $R^\alpha{}_\nu u_\alpha = R^{\alpha\beta}{}_{\beta\nu} u_\alpha = \left[\partial_{;\nu}, \partial_{;\beta}\right] u^\beta$ for the velocity four-vector u^β, it is easily seen that:

$$\partial_{;\beta} \frac{Du^\beta}{D\tau} = \partial_{;\beta} \left(\frac{\partial x^\nu}{\partial \tau} \partial_{;\nu} u^\beta \right) = \partial_{;\beta} \frac{\partial x^\nu}{\partial \tau} \partial_{;\nu} u^\beta + \frac{\partial x^\nu}{\partial \tau} \partial_{;\beta} \partial_{;\nu} u^\beta = \partial_{;\beta} u^\nu \partial_{;\nu} u^\beta + u^\nu \partial_{;\nu} \partial_{;\beta} u^\beta - R_{\mu\nu} u^\mu u^\nu . \tag{7.2}$$

So the Ricci tensor which is part of the Einstein equation $-\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ and thus related to the energy tensor $T_{\mu\nu}$ which is the source of gravitation, is seen to be contained in (7.1). This is especially direct using the inverse form $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$.

Next let us insert (7.2) into (7.1) and also expand terms while applying Maxwell's $J_\sigma = \partial_{;\beta} F^\beta{}_\sigma$. With some index renaming, this now yields a scalar equation:

$$\begin{aligned}\partial_{;\beta} \mathrm{A}^\beta = \partial_{;\beta} \frac{\mathfrak{D} u^\beta}{\mathfrak{D}\tau} = -R_{\mu\nu} u^\mu u^\nu - \frac{q}{m} J^\beta{}_\sigma u^\sigma + \partial_{;\beta} u^\nu \partial_{;\nu} u^\beta + u^\nu \partial_{;\nu} \partial_{;\beta} u^\beta - \frac{q}{m} F^\beta{}_\sigma \partial_{;\beta} u^\sigma \\ + \frac{q^2}{m^2}\left(\mathcal{L}_{em} - A^\beta \partial_\beta \partial_{;\alpha} A^\alpha\right) = 0\end{aligned} \quad . \tag{7.3}$$

We now find both gravitational sources in $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$ and electric charge sources $\mu_0 J_\sigma = \partial_{;\beta} F^\beta{}_\sigma$ (with $\mu_0 = 1/\varepsilon_0 c^2$ balancing dimensionality) all as part of the same dynamical equation. Now, to eliminate the entire second line of (7.3), we impose the Lagrangian gauge condition $A^\beta \partial_\beta \partial_{;\alpha} A^\alpha \equiv \mathcal{L}_{em}$ of (4.4) which covariantly removes just as much freedom from this equation as does the Lorenz gauge $\partial_{;\alpha} A^\alpha = 0$. We may also write $\partial_{;\mu}\partial_{;\nu} u^\nu = \partial_\mu \partial_{;\nu} u^\nu$ because $\partial_{;\nu} u^\nu$ is a scalar. We also multiply the above through by *m*, while noting that $mR_{\mu\nu} u^\mu u^\nu$ has dimensions of energy per area i.e. energy flux. We then restore *c* so as to give all terms this same dimensionality, while mindful that $\kappa = 8\pi G / c^4$ and $\mu_0 \varepsilon_0 c^2 = 1$. And, we make explicit use of $R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$ while isolating all sources on the left. With all of this, these sources are now seen to bring about motion via the differential equation:

$$-\kappa T_{\mu\nu} m u^\mu u^\nu + \tfrac{1}{2}\kappa T m u_\sigma u^\sigma + \mu_0 q J_\sigma u^\sigma = m \partial_{;\nu} u^\mu \partial_{;\mu} u^\nu + m u^\mu \partial_\mu \partial_{;\nu} u^\nu - \left(q/c\right) F^\tau{}_\sigma \partial_{;\tau} u^\sigma . \tag{7.4}$$

This is a combined differential equation for the gravitational and electrodynamic motion of material bodies with a four-velocity u^ν and its first and second covariant derivatives contained in the foregoing. Because all terms have dimensions of energy per area, i.e. energy flux, we recognize this to be a scalar energy flux equation.

In general one may find it helpful to keep this equation in the form of (7.4). To the extent one wishes to be more explicit about the derivatives involved in (7.4), we may expand using $\partial_{;\nu} u^\mu = \partial_\nu u^\mu + \Gamma^\mu{}_{\sigma\nu} u^\sigma$ and the like. So the first term after the equality is:

$$m \partial_{;\nu} u^\mu \partial_{;\mu} u^\nu = m \partial_\nu u^\mu \partial_\mu u^\nu + 2\Gamma^\mu{}_{\sigma\nu} m u^\sigma \partial_\mu u^\nu + \Gamma^\nu{}_{\tau\mu} \Gamma^\mu{}_{\sigma\nu} m u^\sigma u^\tau . \tag{7.5}$$

Because $\partial_\nu u^\nu = 0$ by the chain rule, we have $\partial_{;\nu} u^\nu = \Gamma^\nu{}_{\sigma\nu} u^\sigma$. Noting as well that $\Gamma^\nu{}_{\sigma\nu} = \partial_\sigma \sqrt{-g} / \sqrt{-g} = \frac{1}{2}\left(1/g\right)\partial_\sigma g$, with further use of the chain rule the next term in (7.4) is:

$$\begin{aligned} mu^\mu \partial_\mu \partial_{;\nu} u^\nu &= mu^\mu \partial_\mu \left(\Gamma^\nu{}_{\sigma\nu} u^\sigma \right) = m \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} \left(\frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial x^\sigma} \frac{dx^\sigma}{d\tau} \right) \\ &= \frac{1}{2} m \left(-\frac{1}{g^2} \left(\frac{dg}{d\tau} \right)^2 + \frac{1}{g} \frac{d^2 g}{d\tau^2} + \frac{1}{g} \frac{\partial g}{\partial x^\sigma} \frac{d^2 x^\sigma}{d\tau^2} \right) \end{aligned} . \tag{7.6}$$

Placing (7.5) and (7.6) into (7.4) and also expanding the $F^\tau{}_\sigma \partial_{;\tau} u^\sigma$ term, we then obtain the final expanded form of the energy flux equation:

$$\begin{aligned} &-\kappa T_{\mu\nu} mu^\mu u^\nu + \tfrac{1}{2} \kappa T m u_\sigma u^\sigma + \mu_0 q J_\sigma u^\sigma \\ &= m \partial_\nu u^\mu \partial_\mu u^\nu + 2\Gamma^\mu{}_{\sigma\nu} m u^\sigma \partial_\mu u^\nu + \Gamma^\alpha{}_{\mu\beta} \Gamma^\beta{}_{\nu\alpha} m u^\mu u^\nu \\ &-\frac{1}{2} m \frac{1}{g^2} \left(\frac{dg}{d\tau} \right)^2 + \frac{1}{2} m \frac{1}{g} \frac{d^2 g}{d\tau^2} + \frac{1}{2} m \frac{1}{g} \frac{\partial g}{\partial x^\sigma} \frac{d^2 x^\sigma}{d\tau^2} - \frac{q}{c} F^\tau{}_\sigma \partial_\tau u^\sigma - \frac{q}{c} \Gamma^\sigma{}_{\alpha\tau} F^\tau{}_\sigma u^\alpha \end{aligned} . \tag{7.7}$$

In regions of spacetime where there is no gravitating matter, i.e., *in vacuo*, we set $T_{\mu\nu} = 0$ and $T = 0$ above, and then solve for the motion, given only the probability density contained in the time component of $J_\sigma = \rho_0 u_\sigma = \overline{\psi} \gamma_\sigma \psi$. In the further absence of electrodynamic sources we set $J_\sigma = 0$ so the entire top line of the above equation becomes zero.

One interesting way to use (7.7) is to remove all energy sources except for the Maxwell-Poynting electromagnetic field tensor which is $4\pi \mu_0 c^2 T_{\mu\nu} = -F_{\sigma\mu} F^\sigma{}_\nu + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$ with dimensional balancing, with $\mu_0 \varepsilon_0 c^2 = 1$. This tensor of course has no trace, which is related to why electromagnetic fields travel at the speed of light and photons are massless. So when this is the only energy present – and recognizing that this energy still gravitates and thus affects the metric and the spacetime curvature – then, with the source term $\mu_0 c q J_\sigma u^\sigma$ isolated on the left, and with the constants reorganized via $\kappa / 4\pi \mu_0 c^2 = G / 2\pi c^4 k_e$ to display the embedded ratio G / k_e of Newton's to Coulomb's constant, (7.7) becomes:

$$\begin{aligned} &\mu_0 q J_\sigma u^\sigma \\ &= m \partial_\nu u^\mu \partial_\mu u^\nu + 2\Gamma^\mu{}_{\sigma\nu} m u^\sigma \partial_\mu u^\nu + \left(\Gamma^\alpha{}_{\mu\beta} \Gamma^\beta{}_{\nu\alpha} - \frac{G}{2\pi c^4 k_e} F_{\sigma\mu} F^\sigma{}_\nu \right) m u^\mu u^\nu + \frac{G}{8\pi c^4 k_e} F_{\alpha\beta} F^{\alpha\beta} m u_\sigma u^\sigma \\ &-\frac{1}{2} m \frac{1}{g^2} \left(\frac{dg}{d\tau} \right)^2 + \frac{1}{2} m \frac{1}{g} \frac{d^2 g}{d\tau^2} + \frac{1}{2} m \frac{1}{g} \frac{\partial g}{\partial x^\sigma} \frac{d^2 x^\sigma}{d\tau^2} - \frac{q}{c} F^\tau{}_\sigma \partial_\tau u^\sigma - \frac{q}{c} \Gamma^\sigma{}_{\alpha\tau} F^\tau{}_\sigma u^\alpha \end{aligned} . \tag{7.8}$$

An equation free of electrodynamic source charges then results from setting $J_\sigma = 0$ in the above.

It is important to keep in mind that (7.7) may be derived directly from the known Lorentz force law (2.1) as represented in (4.8), even had we not obtained this from the minimization of the action (1.1). This is because (7.7) is simply the spacetime gradient $\partial_{;\beta}$ applied to (4.8) as starting at (7.1), and (4.8) is true whether or not we obtain it from a variation. But the motivation to operate on the Lorentz force law in this way comes from the fact that when we do obtain the Lorentz force from a variation, Maxwell's equation $J^{\beta}=\partial_{;\alpha}F^{\alpha\beta}$ together with the Lagrangian gauge $A^{\beta}\partial_{\beta}\partial_{;\alpha}A^{\alpha}=\mathcal{L}_{em}$ of (4.4) mandate the gauge condition $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$, which is a weaker condition than the geodesic gauge $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0$ of (4.6). So when we study the impact of this weaker condition $\partial_{;\beta}\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)=0$ on the Lorentz force, the result is the energy flux field equation (7.7). When we impose the stronger condition $\partial^{\beta}\left(A_{\alpha}A^{\alpha}\right)\equiv 0$, the result is the Lorentz force itself. What is important about (7.7) and (7.8) is that they put the energy source tensor $T_{\mu\nu}$ or the spacetime curvature $R_{\mu\nu}$ (as chosen for best convenience in any given calculation), directly into the dynamical equation for energy flux.

Having now reviewed how the combined gravitational and Lorentz motion (2.1) is derived from the variational equation (1.1), and the required gauge conditions and the immediately-consequent power and energy flux equations, we now show how to derive the electrodynamic time dilation and contraction summarized in section 2. Again, this is premised on requiring the line element to remain invariant and the background fields in spacetime to remain unchanged, under a re-gauging of the electrodynamic charge-to-mass ratio q/m.

8. Electrodynamic Time Dilation and Contraction

As noted earlier, the number "1" constructed in (3.1) is useful in a variety of circumstances. Another such circumstance is to explicitly introduce the Lorentz contraction factor $\gamma_{v}=1/\sqrt{1-v^{2}/c^{2}}$ and the ordinary four-velocity $v^{\mu}/c=\left(1,\mathbf{v}/c\right)$. With $g_{\mu\nu}=\eta_{\mu\nu}$, it is easily shown and well-known that $\eta_{\mu\nu}\left(\gamma_{v}v^{\mu}\right)\left(\gamma_{v}v^{\nu}\right)/c^{2}=1$, which is another "1." So if we write (3.1) in flat spacetime as $\eta_{\mu\nu}U^{\mu}U^{\nu}/c^{2}=1$, we see that the canonical velocity U^{μ}, not the mechanical velocity u^{μ}, is related expressly to γ_{v} and v^{μ} by:

$$U^{\mu}=\gamma_{v}v^{\mu}. \tag{8.1}$$

This may then be generalized into curved spacetime. Additionally, we may ascertain from the final equality in (3.1), when combined with (8.1), that:

$$U^{\mu}=u^{\mu}+\frac{q}{mc}A^{\mu}=\frac{dx^{\mu}}{d\tau}+\frac{q}{mc}A^{\mu}=\gamma_{v}v^{\mu}. \tag{8.2}$$

This may be conversely rewritten in terms of the ordinary mechanical velocity as:

$$u^{\mu}=\frac{dx^{\mu}}{d\tau}=U^{\mu}-\frac{q}{mc}A^{\mu}=\gamma_{v}v^{\mu}-\frac{q}{mc}A^{\mu}. \tag{8.3}$$

With these relationships, we return to (2.9), which states that the line element $d\tau$ must be invariant, and the metric tensor $g_{\mu\nu}$ and the gauge field A^{μ} [the latter now subject to the Lagrangian and geodesic gauge conditions (4.4) and (4.6)] must be unchanged under a rescaling of $q/m \rightarrow q'/m'$. Thus, it is (2.9) which *defines* the coordinate transformation $x^{\mu} \rightarrow x'^{\mu}$ leading to electrodynamic time dilation and contraction. Now we show exactly how this occurs.

Generally, we will wish to compare the rate at which time flows for a massive body which has a net charge of zero and so is neutral, in relation to a material body with a nonzero net charge. We assume for now that there is no gravitation. Via (2.9), this means that we shall set $q=0$ (neutrality) and leave q' as it is (i.e., charged). Therefore, (2.9) becomes:

$$c^{2}d\tau^{2}=g_{\mu\nu}dx^{\mu}dx^{\nu}=g_{\mu\nu}\left(dx'^{\mu}+\frac{q'}{m'c}d\tau A^{\mu}\right)\left(dx'^{\nu}+\frac{q'}{m'c}d\tau A^{\nu}\right). \tag{8.4}$$

From this, we can immediately extract the coordinate transformation:

$$dx'^{\mu}=dx^{\mu}-\frac{q'}{m'c}d\tau A^{\mu}. \tag{8.5}$$

Because the coordinates x^{μ} are associated with a neutral net charge, as a notational convenience we shall drop the primes from the mass and charge and write this as $dx'^{\mu}=dx^{\mu}-(q/mc)d\tau A^{\mu}$. Thus, dx'^{μ} represents the coordinates of the body with q/m, and dx^{μ} the coordinates of the neutral body. With this notational adjustment, and dividing through by $d\tau$, we obtain the relation:

$$u'^{\mu}=\frac{dx'^{\mu}}{d\tau}=\frac{dx^{\mu}}{d\tau}-\frac{q}{mc}A^{\mu}=u^{\mu}-\frac{q}{mc}A^{\mu}. \tag{8.6}$$

The time component of this with $x^{\mu}=(ct,\mathbf{x})$ and $A^{\mu}=(\phi,\mathbf{A})$ is easily seen to be:

$$\frac{dt'}{d\tau}=\frac{dt}{d\tau}-\frac{q\phi}{mc^{2}}. \tag{8.7}$$

So in the rest frame where $dt/d\tau=1$ for the neutral body (because we have posited no gravitation for now) and $A^{\mu}=(\phi_{0},0)$ with ϕ_{0} being the proper scalar potential, this becomes:

$$\gamma_{em}\equiv\frac{dt'}{d\tau}=1-\frac{q\phi_{0}}{mc^{2}}. \tag{8.8}$$

This is where we define the factor γ_{em}, first introduced between (2.9) and (2.10), to be the rate of time flow for a net-charged body q in a proper potential ϕ_0, in relation to the rate of time flow for a net-neutral body, all at relative rest. And this is where the dimensionless ratio $q\phi_0 / mc^2$ which is central to this variable time flow first arises. As obtained from (8.4), the above (8.8) is what allows the Lorentz force motion (2.1) to be deduced from the minimized variation (1.1) without compromising the integrity of the background fields.

Now, because $A^\mu = (\phi_0, \mathbf{0})$ at rest, the question also arises how to specify A^μ generally when there is motion. Specifically, the choice would be between $A^\mu = \phi_0 U^\mu / c$ using the canonical velocity, or $A^\mu = \phi_0 u^\mu / c$ using the mechanical velocity. But we see from $U^\mu = \gamma_v v^\mu$ in (8.1) that $A^\mu = \phi_0 U^\mu / c$ is the proper choice, that is:

$$A^\mu = \phi_0 U^\mu / c = \phi_0 \gamma_v v^\mu / c , \tag{8.9}$$

because at rest $\gamma_v v^\mu / c = (1, \mathbf{0})$, and this yields the correct result that $A^\mu = (\phi_0, \mathbf{0})$ at rest.

With (8.9) we may now obtain several other important results. Using this in (8.3) yields:

$$u^\mu = \frac{dx^\mu}{d\tau} = \left(1 - \frac{q\phi_0}{mc^2}\right) \gamma_v v^\mu = \gamma_{em} \gamma_v v^\mu = \gamma_{em} U^\mu . \tag{8.10}$$

So we see that the mechanical velocity u^μ is related to the canonical velocity U^μ through a multiplicative factor given by γ_{em}. The inverse result $U^\mu = u^\mu / \gamma_{em}$ can be combined with (8.2) with everything multiplied through by m to also obtain:

$$mU^\mu = \frac{1}{\gamma_{em}} mu^\mu = \frac{1}{\gamma_{em}} p^\mu = mu^\mu + \frac{q}{c} A^\mu = \pi^\mu . \tag{8.11}$$

This contains the relationship $p^\mu = \gamma_{em} \pi^\mu$ between the mechanical and canonical momentum, mirroring $u^\mu = \gamma_{em} U^\mu$ in (8.10). For the gauge field itself, we may combine (8.9) and (8.10) thus:

$$A^\mu = \frac{\phi_0 U^\mu}{c} = \frac{\phi_0 \gamma_v v^\mu}{c} = \frac{1}{\gamma_{em}} \frac{\phi_0 u^\mu}{c} = \frac{1}{\gamma_{em}} \frac{\phi_0}{c} \left(1 - \frac{q\phi_0}{mc^2}\right) \gamma_v v^\mu . \tag{8.12}$$

Then, we may multiply (8.10) through by mc to obtain the energy-dimensioned four vector, and also use (8.11), to write:

$$cp^\mu = mcu^\mu = mc\frac{dx^\mu}{d\tau} = mc\gamma_{em} \gamma_v v^\mu = mc\gamma_{em} U^\mu = c\gamma_{em} \pi^\mu . \tag{8.13}$$

All of this finally leads us to take the time component of (8.13) in the non-relativistic limit, namely:

$$E = cp^0 = mc^2\gamma_{em}\gamma_v = mc^2\frac{1-\dfrac{q\phi_0}{mc^2}}{\sqrt{1-\dfrac{v^2}{c^2}}} \cong mc^2\left(1-\frac{q\phi_0}{mc^2}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) = mc^2 + \frac{1}{2}mv^2 - q\phi_0 - \frac{1}{2}\frac{q\phi_0}{c^2}v^2 \,. \tag{8.14}$$

This is how the key energy relationship (2.10) originates. Here, in succession, we see 1) the rest energy mc^2, 2) the kinetic energy of the mass *m*, 3) the electrical interaction energy of the charged mass, and 4) the kinetic energy of the electrical energy. If we then choose a Coulomb proper potential $\phi_0 = -k_e Q / r$ so that the charges have *opposite signs* and so are attracting in the same way that gravitation attracts, then we arrive precisely at the first four terms of (2.10).

Then to add gravitation, it is convenient to start with the metric (2.5) in the form $c^2 d\tau^2 = g_{\mu\nu} Dx'^{\mu} Dx'^{\nu}$ for the charged mass that has the x'^{μ} coordinates. We take this mass to be at rest in the gravitational field so that $d\tau^2 = g_{00} Dt'^2$, a.k.a. $Dt'/d\tau = 1/\sqrt{g_{00}}$. Earlier, we set $dt/d\tau = 1$ to arrive at (8.8), which was appropriate for a neutral body because we assumed an absence of gravitation. But when gravitation is present, then even for a neutral body, we must use (8.8) in the form $dt'/d\tau = dt/d\tau - q\phi_0/mc^2$, because time dilation and contraction in the gravitational field will cause $dt/d\tau$ to be some number that is not precisely equal to 1. That is, $dt/d\tau$ cannot be summarily set to 1 once there is gravitation. So, if we were to write out $c^2 d\tau^2 = g_{\mu\nu} Dx'^{\mu} Dx'^{\nu}$ using $Dx'^{\mu} = dx'^{\mu} + (q/mc) d\tau A^{\mu}$, and also use (8.8) in the form $dt'/d\tau = dt/d\tau - q\phi_0/mc^2$ for the reasons just mentioned, we obtain:

$$\frac{1}{\sqrt{g_{00}}} = \frac{Dt'}{d\tau} = \frac{dt'}{d\tau} + \frac{q\phi_0}{mc^2} = \frac{dt}{d\tau} - \frac{q\phi_0}{mc^2} + \frac{q\phi_0}{mc^2} = \frac{dt}{d\tau} \,. \tag{8.15}$$

The electrodynamic terms cancel, leaving the usual relationship $dt/d\tau = 1/\sqrt{g_{00}} \equiv \gamma_g$ for time dilation or contraction for a particle at rest in a gravitational field. This then supplements $\gamma_{em}\gamma_v \to \gamma_g\gamma_{em}\gamma_v$ in (8.10), (8.13) and (8.14). Particularly, (8.14) now becomes $E = cp^0 = mc^2\gamma_g\gamma_{em}\gamma_v$, which is synonymous with (2.10), and it then becomes possible to simultaneously represent the combined effects of gravitation, electrodynamics and motion, upon time and energy. The widely-corroborated, well-established energy relation $E = cp^0 = mc^2\gamma_g\gamma_{em}\gamma_v$ shown in (2.10) , then results directly from merging (8.14) and (8.15).

9. Conclusion

The fact that (2.10) correctly reproduces widely-corroborated, well-established energy relations, is an important point of validation that the geometro-electrodynamic viewpoint which has been presented here is empirically correct. However, the mainspring which enables everything to fit together without contradiction is the time flow relationship

$$\frac{dt}{d\tau}=\gamma_g\gamma_{em}\gamma_v=\frac{1+q\phi_0/mc^2}{\sqrt{g_{00}}\sqrt{1-v^2/c^2}}\cong\left(1+\frac{GM}{c^2r}\right)\left(1+\frac{q\phi_0}{mc^2}\right)\left(1+\frac{1}{2}\frac{v^2}{c^2}\right) \tag{9.1}$$

contained within (8.14) when supplemented by (8.15) and applied to gravitation in the Newtonian limit. This is the time-component of the four-velocity, by which (8.10) becomes extended to:

$$u^\mu=\frac{dx^\mu}{d\tau}=\gamma_g\gamma_{em}\gamma_v v^\mu=\gamma_g\gamma_{em}U^\mu. \tag{9.2}$$

This is turn is the four velocity that appears throughout the key dynamical equations developed here. For example, this four velocity (9.2) with the time component (9.1) appears in the equation of motion (4.8) itself, in the power equations (6.6) and (6.7), and in the energy flux equations (7.7) and (7.8). Given the direct relation between (9.1) and the energy relation (2.10), it should be clear that the energies and powers governed by these dynamical equations are the energies of motion, and of gravitational and electrodynamic interaction, all taken together.

Consequently, it becomes most important to perform experimental tests of these predicted time flow changes for charged bodies in electromagnetic fields. Although these time flow relations (9.1) go hand-in-hand with the energy relations (2.10), it is (9.1) which nevertheless is the theoretical foundation of the energy relations (2.10). That is, the widely-corroborated energy relations (2.10) are seen in the present analysis to be rooted in geometrodynamic measurement of the flow rates of time. Experimental observation of a change in the rate at which time flows for charged bodies in electromagnetic fields in accordance with (9.1) – or possibly the explanation of additional known physics on the basis of these time flow rates – would therefore support the validity of this geometrodynamic foundation for classical electrodynamics in four spacetime dimensions.

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