# Collection of formulas for pi : Part 1

Edgar Valdebenito

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**Abstract**

In this note we show some formulas related with the constant pi

**Resumen**

En esta nota mostramos una colección de fórmulas relacionadas con la constante pi:

$\pi = 3.14159265358979 \ldots.$

Keywords: número pi,series,productos infinitos,radicales continuos,función zeta de Riemann,fracciones continuas,funciones hipergeométricas.

# I. Introducción

Notación:

$$\mathbb{N} = \{1,2,3,\ldots\}, \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$$

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1)\,, n \in \mathbb{N}\,;\ (a)_0 = 1$$

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\ldots$$

$$F_{n+2} = F_{n+1} + F_n\,, F_1 = F_2 = 1\ \ , Sucesión\ de\ Fibonacci$$

$$\zeta(x) = \sum_{n=1}^{\infty} n^{-x}\ \ , x > 1\ , función\ zeta\ de\ Riemann$$

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n\ \ , -1 < x < 1\,, función\ hipergeométrica\ de\ gauss$$

$$B_n = \left\{\frac{1}{6},\frac{1}{30},\frac{1}{42},\frac{1}{30},\frac{5}{66},\ldots\right\}\ \ , números\ de\ Bernoulli$$

$$B_n = (-1)^{n-1} \sum_{k=0}^{2n} \frac{1}{k+1} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^{2n}\,, n \in \mathbb{N}$$

$$\{E_k : k \in \mathbb{N}_0\} = \{1,1,5,61,1385,\ldots\}\,, números\ de\ Euler$$

$$\gamma\ = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.57721\ldots$$

$$H_n = \sum_{k=1}^{n} k^{-n}$$

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt \quad , x > 0 \;\; , función\ gamma$$

$$\Gamma(x) = \frac{1}{x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{x} \quad , x > 0$$

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s} \quad , |z| > 1, s > 0, v \neq 0, -1, -2, \dots, función\ de\ Lerch$$

$$F(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^n}{n!} \quad , |z| < 1$$

Recordamos algunas fórmulas clásicas:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \cdots$$

$$\pi = \sum_{k=1}^{\infty} \frac{3^k - 1}{2^{2k}} \zeta(k+1)$$

$$\pi = 4 \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2k+1}}\right)$$

$$\pi = 4 \tan^{-1}\left(\frac{1}{2}\right) + 4 \tan^{-1}\left(\frac{1}{5}\right) + 4 \tan^{-1}\left(\frac{1}{8}\right)$$

$$\pi = \sum_{k=0}^{\infty} 2^{-4k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \binom{2k}{k}^3 \frac{(42k+5)}{2^{12k+4}}$$

$$\frac{\pi\sqrt{3}}{9} = \sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!\, k}$$

## II. Fórmulas

(1) $$a\pi = \sum_{n=1}^{\infty} \frac{\sin(na\pi)}{n\, 2^n\, (\cos(a\pi))^n} \quad , |a| > 1/3$$

(2) $$a\pi = 2\sum_{n=1}^{\infty} \frac{\sin\big((2n-1)a\pi\big)}{(2n-1)\left(\cos(a\pi) + \sqrt{1 + (\cos(a\pi))^2}\right)^{2n-1}} \quad , |a| > 1/2$$

(3) $$a\pi = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cos\big((2n-1)a\pi\big)}{2n-1}\left(\frac{\sin(a\pi)}{(\cos(a\pi))^2 + \sqrt{(\cos(a\pi))^4 + (\sin(a\pi))^2}}\right)^{2n-1}$$

$$|a| < 1/2$$

Ejemplos:

(4) $$\pi = 4\sum_{n=1}^{\infty} \frac{1}{16^n}\left(\frac{8}{8n-7} + \frac{8}{8n-6} + \frac{4}{8n-5} - \frac{2}{8n-3} - \frac{2}{8n-2} - \frac{1}{8n-1}\right)$$

(5) $$\pi = 4\sqrt{2}\sum_{n=1}^{\infty}\left(\frac{x^{8n-7}}{8n-7} + \frac{x^{8n-5}}{8n-5} - \frac{x^{8n-3}}{8n-3} - \frac{x^{8n-1}}{8n-1}\right) \quad , x = \frac{\sqrt{2}}{1+\sqrt{3}}$$

(6) $$\pi = 6\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{x^{6n-5}}{2(6n-5)} + \frac{x^{6n-3}}{6n-3} + \frac{x^{6n-1}}{2(6n-1)}\right) \quad , x = \frac{2\sqrt{3}}{1+\sqrt{13}}$$

(7) $$\frac{e^{\pi 2^{-k-1}} + e^{-\pi 2^{-k-1}} + r_k}{e^{\pi 2^{-k-1}} + e^{-\pi 2^{-k-1}} - r_k} = \prod_{n=0}^{\infty}\left(\frac{1 + (2^k + 1 + n2^{k+1})^2}{1 + (2^k - 1 + n2^{k+1})^2}\right)^{(-1)^n} \quad , k \in \mathbb{N}$$

donde

(8) $$r_k = \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{k-radicales} \quad , \qquad k \in \mathbb{N}$$

(9) $$exp\left(\frac{\pi}{2^{k+1}}\right) = R_k - \cfrac{1}{R_k - \cfrac{1}{R_k - \cfrac{1}{R_k - \cdots}}} \quad , k \in \mathbb{N}$$

donde

(10) $$R_k = \frac{P_k + 1}{P_k - 1} r_k \quad , \quad P_k = \prod_{n=0}^{\infty}\left(\frac{1 + (2^k + 1 + n2^{k+1})^2}{1 + (2^k - 1 + n2^{k+1})^2}\right)^{(-1)^n}$$

(11) $$\frac{1}{\pi} = \frac{2}{3\sqrt{3}}\prod_{n=1}^{\infty}\left(\frac{4x_n^2 - 1}{3}\right)$$

donde

(12) $$x_1 = \sqrt[3]{\frac{1}{8}+\frac{3}{4}\sqrt[3]{\frac{1}{8}+\frac{3}{4}\sqrt[3]{\frac{1}{8}+\cdots}}}$$

(13) $$x_{n+1} = \sqrt[3]{\frac{x_n}{4}+\frac{3}{4}\sqrt[3]{\frac{x_n}{4}+\frac{3}{4}\sqrt[3]{\frac{x_n}{4}+\cdots}}}\quad , \qquad n \in \mathbb{N}$$

(14) $$x_n = \cos\left(\frac{\pi}{3^{n+1}}\right)\ , 0 < x_n < x_{n+1} < 1\ , n \in \mathbb{N}$$

(15) $$\pi^2\left(\frac{1}{6}-\frac{2^{k+1}-1}{2^{2k+2}}\right)+\frac{1}{3\cdot 2^{2k-1}}-\sum_{n=1}^{k-1}\frac{1}{2^{2n-2}}\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}_{(k-n)-radicales}$$

$$= 4\sum_{n\in A}\frac{1}{n^2}\cos\left(\frac{n\pi}{2^{k+1}}\right)\quad , k \in \mathbb{N}$$

donde

(16) $$A = \{2^n(2m+1) : n, m \in \mathbb{N}\} = \{6,10,12,14,18,\dots\}$$

Ejemplos:

(17) $$\frac{\pi^2}{48} = \frac{1}{6} - 4\sum_{n\in A}\frac{1}{n^2}\cos\left(\frac{n\pi}{4}\right) = \frac{1}{6} + 4\left(\frac{1}{12^2}+\frac{1}{20^2}-\frac{1}{24^2}+\frac{1}{28^2}+\frac{1}{36^2}-\cdots\right)$$

(18) $$\frac{\pi^2}{192}+\frac{1}{24}-\sqrt{\frac{1}{2}} = 4\sum_{n\in A}\frac{1}{n^2}\cos\left(\frac{n\pi}{8}\right) = 4\left(-\frac{1}{6^2\sqrt{2}}-\frac{1}{10^2\sqrt{2}}+\frac{1}{14^2\sqrt{2}}+\cdots\right)$$

(19) $$\frac{83\pi^2}{768}+\frac{1}{96}-\sqrt{\frac{1}{2}}-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}} = 4\sum_{n\in A}\frac{1}{n^2}\cos\left(\frac{n\pi}{16}\right)$$

(20) $$\frac{\pi}{3\mu^2-1}\left(\frac{1}{\sqrt{\mu}}-\frac{\sqrt{2\sqrt{\mu^2-1}-\mu}}{2\sqrt{\mu^2-1}}+\frac{3\mu\sqrt{2\sqrt{\mu^2-1}+\mu}}{2\sqrt{\mu^2-1}\sqrt{3\mu^2-4}}\right)$$

$$= \sum_{n=0}^{\infty} n!\left(\frac{1}{\left(2n+\frac{5}{2}\right)_{n+1}}+\frac{1}{2\left(\frac{2n+1}{4}\right)_{n+1}}\right)$$

donde

(21) $$\mu = \sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}} = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}$$

(22) $$\mu^3 - \mu - 1 = 0$$

(23) $$\frac{1}{\mu} = -\frac{1}{3} + \sqrt[3]{\frac{25}{54}+\frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{25}{54}-\frac{1}{6}\sqrt{\frac{23}{3}}}$$

(24) $$\mu = \sqrt[3]{4}\sum_{n=0}^{\infty}\frac{(-1/3)_{2n}}{(2n)!}\left(\frac{23}{27}\right)^n$$

(25) $$\frac{1}{\mu} = -\frac{1}{3} + \frac{10^{2/3}}{3}\sum_{n=0}^{\infty}\frac{(-1/3)_{2n}}{(2n)!}\left(\frac{621}{625}\right)^n$$

(26) $$\pi = 4\tan^{-1}(\mu^{-1}) + 4\tan^{-1}(\mu^{-7})$$

(27) $$\frac{1}{\pi^2}\big(c(n)-c(m)\big) = \frac{1}{8}\left(\frac{1}{2^{2m}}-\frac{1}{2^{2n}}\right)\prod_{k=1}^{\infty}\left(1-\frac{(2^n+2^m)^2}{2^{2n+2m+4}\,k^2}\right)\left(1-\frac{(2^n-2^m)^2}{2^{2n+2m+4}\,k^2}\right)$$

$$n, m \in \mathbb{N}, n > m$$

(28) $$\frac{1}{\pi}s(n)\big(1+2c(n-1)\big) = \frac{3}{2^{n+1}}\prod_{k=1}^{\infty}\left(1-\frac{9}{2^{2n+2}\,k^2}\right) \quad, n \in \mathbb{N}-\{1\}$$

(29) $$\frac{1}{\pi}\big(s(n)\cos(\alpha\pi)+c(n)\sin(\alpha\pi)\big) = \left(\alpha+\frac{1}{2^{n+1}}\right)\prod_{k=1}^{\infty}\left(1-\frac{1}{k^2}\left(\alpha+\frac{1}{2^{n+1}}\right)^2\right)$$

$$n \in \mathbb{N}, \alpha \in \mathbb{R}$$

(30) $$\frac{1}{\pi}\big(s(n)\cos(\alpha\pi)-c(n)\sin(\alpha\pi)\big) = \left(\frac{1}{2^{n+1}}-\alpha\right)\prod_{k=1}^{\infty}\left(1-\frac{1}{k^2}\left(\frac{1}{2^{n+1}}-\alpha\right)^2\right)$$

$$n \in \mathbb{N}, \alpha \in \mathbb{R}$$

(31) $$\frac{\pi}{2}\left(1+\frac{1}{2^n}\right)(1-s(n)) = \prod_{k=1}^{\infty}\frac{\big(8k(4^n(2k-1)^2-1)\big)^2}{(2k-1)^2(4^{n+2}k^2-(2^n+1)^2)(4^{n+1}(2k-1)^2-(2^n-1)^2)}$$

$$n \in \mathbb{N}$$

(32) $$\frac{\pi}{2}\left(1-\frac{1}{2^n}\right)(1+s(n)) = \prod_{k=1}^{\infty}\frac{\big(8k(4^n(2k-1)^2-1)\big)^2}{(2k-1)^2(4^{n+2}k^2-(2^n-1)^2)(4^{n+1}(2k-1)^2-(2^n+1)^2)}$$

$$n \in \mathbb{N}$$

En las fórmulas (27),(28),(29),(30),(31),(32), se tiene:

(33) $$c(n)=\underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}_{n-radicales}\quad,\qquad s(n)=\underbrace{\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}+\cdots+\frac{1}{2}\sqrt{\frac{1}{2}}}}}_{n-radicales}\quad,n\in\mathbb{N}$$

(34) $$1-\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}}}-\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}+\cdots=\pi\sum_{n=0}^{\infty}\frac{p(n)}{2^{2n+2}}$$

donde

(35) $$p(n)=\prod_{k=1}^{\infty}\left(1-\frac{1}{2^{4n+6}k^2}\right)\left(1-\frac{9}{2^{4n+4}(2k-1)^2}\right)$$

(36) $$\pi=\frac{8}{3}+\frac{8}{35}\varphi^{-1}+8\sum_{n=1}^{\infty}\frac{\varphi^{-n-1}\,v_n}{(4n+1)(4n+3)(4n+5)(4n+7)}$$

donde

(37) $$\varphi=\frac{1+\sqrt{5}}{2}\quad,v_n=2(16n^2+32n+19)F_n+(4n+1)(4n+3)F_{n+1}$$

donde $F_n$ es la sucesión de Fibonacci.

(38) $$\frac{\pi^2}{e^{\pi}-e^{-\pi}-2c(n)}=a(n)\prod_{k=2}^{\infty}\frac{2^{4n+8}\,k^4}{2^{4n+8}\,k^4+2^{2n+5}(2^{2n+2}-1)k^2+(2^{2n+2}+1)^2}$$

donde $c(n)$ se define por (33) , y $a(n)$ se define por:

(39) $$a(n)=\frac{2^{6n+10}}{(2^{2n+2}+1)^2(9\cdot 2^{2n+2}+1)+2^{2n+6}(2^{4n+4}-1)}$$

$$=\frac{2^{6n+10}}{(2^{2n+2}+1)(2^{2n+4}+2^{2n+2}+2^{n+3}-1)(2^{2n+4}+2^{2n+2}-2^{n+3}+1)}$$

Ejemplo:

(40) $$\frac{\pi^2}{e^{\pi}-e^{-\pi}-2\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}=\frac{2^{22}}{65\cdot 289\cdot 353}\prod_{k=2}^{\infty}\frac{2^{16}\,k^4}{2^{16}\,k^4+2^9(2^6-1)k^2+(2^6+1)^2}$$

(41) $$\frac{\pi}{\sqrt{4ac-b^2}} = \frac{1}{c}\sum_{k=0}^{\infty}\frac{\alpha_{2k}}{2k+1}\left(\sqrt{\frac{c}{a}}-\varepsilon\right)^{2k+1} + \frac{1}{a}\sum_{k=0}^{\infty}\frac{\beta_{2k}}{2k+1}\left(\sqrt{\frac{c}{a}}+\varepsilon\right)^{-(2k+1)} + \frac{1}{\sqrt{4ac-b^2}}\left(\tan^{-1}\left(\frac{2\sqrt{ac}+2a\varepsilon+b}{\sqrt{4ac-b^2}}\right)-\tan^{-1}\left(\frac{2\sqrt{ac}-2a\varepsilon+b}{\sqrt{4ac-b^2}}\right)\right) + \frac{1}{\sqrt{4ac-b^2}}\left(\tan^{-1}\left(\frac{2\sqrt{ac}+2a\varepsilon-b}{\sqrt{4ac-b^2}}\right)-\tan^{-1}\left(\frac{2\sqrt{ac}-2a\varepsilon-b}{\sqrt{4ac-b^2}}\right)\right)$$

donde

(42) $$a>0\,, c>0\,, 4ac-b^2>0\,, b\in\mathbb{R}\,, 0<\varepsilon<\sqrt{\frac{c}{a}}$$

(43) $$\alpha_{n+2} = -\frac{b}{c}\,\alpha_{n+1} - \frac{a}{c}\,\alpha_n \;\;, \alpha_0 = 1\,, \alpha_1 = -b/c$$

(44) $$\beta_{n+2} = -\frac{b}{a}\,\beta_{n+1} - \frac{c}{a}\,\beta_n \;\;, \beta_0 = 1\,, \beta_1 = -b/a$$

(45) $$\frac{\pi}{\sin\left(\frac{a\pi}{b}\right)}\left(\frac{a}{b}\right) = \left[\prod_{k=1}^{n}\frac{(bk)^2}{(bk)^2-a^2}\right]\left(1-\left(\frac{a}{bn}\right)^2\right)^{n+\frac{1}{2}}\left(\frac{bn+a}{bn-a}\right)^{a/b} exp\big(y(a,b,n,\theta)\big)$$

donde

(46) $$a,b\in\mathbb{N}\,, 0<a<b\,, n\in\mathbb{N}\,, n\gg 1$$

(47) $$y(a,b,n,\theta) = \frac{a^2}{6(b^2n^3-a^2n)} + \frac{\theta a^2(3b^2n^2-a^2)}{60(b^2n^3-a^2n)^2} \;\;, -1<\theta<1$$

Ejemplo:

(48) $$\frac{\pi}{3\left(\sqrt{6}-\sqrt{2}\right)} = \left[\prod_{k=1}^{n}\frac{(12k)^2}{(12k)^2-1}\right]\left(1-\left(\frac{1}{12n}\right)^2\right)^{n+\frac{1}{2}}\left(\frac{12n+1}{12n-1}\right)^{1/12} exp\big(y(1,12,n,\theta)\big)$$

(49) $$y(1,12,n,\theta) = \frac{1}{6(144n^3-n)} + \frac{\theta(432n^2-1)}{60(144n^3-n)^2} \;\;, -1<\theta<1$$

(50) $$\frac{\pi}{3} = \left[\prod_{k=1}^{n}\frac{(6k)^2}{(6k)^2-1}\right]\left(1-\left(\frac{1}{6n}\right)^2\right)^{n+\frac{1}{2}}\left(\frac{6n+1}{6n-1}\right)^{1/6} exp\big(y(1,6,n,\theta)\big)$$

(51) $$y(1,6,n,\theta) = \frac{1}{6(36n^3-n)} + \frac{\theta(108n^2-1)}{60(36n^3-n)^2} \;\;, -1<\theta<1$$

(52) $$\frac{\pi\sqrt{2}}{4} = \left[\prod_{k=1}^{n}\frac{(12k)^2}{(12k)^2-1}\right]\left(1-\left(\frac{1}{4n}\right)^2\right)^{n+\frac{1}{2}}\left(\frac{4n+1}{4n-1}\right)^{1/12} exp\big(y(1,4,n,\theta)\big)$$

(53) $$y(1,4,n,\theta) = \frac{1}{6(16n^3 - n)} + \frac{\theta(48n^2 - 1)}{60(16n^3 - n)^2} \ , -1 < \theta < 1$$

(54) $$\pi = 8 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \cosh\big((2n+1)x\big)$$

donde

(55) $$0 < a < \sqrt{2} - 1 \, , x > 0 \, , 0 < a < e^{-x}$$

(56) $$\cosh x = \frac{1 - a^2}{2a} \quad , \qquad a = \sqrt{1 + (\cosh x)^2} - \cosh x$$

(57) $$\pi = 4 \tan^{-1}\left(\frac{1 - a^2 - 2a\cosh x}{1 - a^2 + 2a\cosh x}\right) + 8 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \cosh\big((2n+1)x\big)$$

donde

(58) $$0 < a < \sqrt{2} - 1 \ , 0 < x < \cosh^{-1}\left(\frac{1 - a^2}{2a}\right)$$

(59) $$\pi = 2 \tan^{-1}\left(\frac{1 - a^2}{2a\cosh x}\right) + 4 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \cosh\big((2n+1)x\big)$$

donde

(60) $$0 < a < 1 \, , x > 0 \, , 0 < a < e^{-x} \, , \cosh x > \frac{1 - a^2}{2a}$$

(61) $$\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \sinh\big((2n+1)x\big)$$

donde

(62) $$0 < a < \frac{\sqrt{6} - \sqrt{2}}{2} \, , 0 < a < e^{-x} \, , \sinh x = \frac{1 + a^2}{2a\sqrt{3}}$$

(63) $$\pi = 4 \tan^{-1}\left(\frac{1 + a^2 - 2a\sinh x}{1 + a^2 + 2a\sinh x}\right) + 8 \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{2n+1} \sinh\big((2n+1)x\big)$$

donde

(64) $$0 < a < 1 \, , x > 0 \, , 0 < a < e^{-x}$$

(65) $$\pi \frac{\sin\left(\frac{a\pi}{c}\right)}{\cos\left(\frac{a\pi}{c}\right) + \cos\left(\frac{b\pi}{c}\right)} = 4ac \sum_{n=0}^{\infty} \frac{(2n+1)^2 c^2 - (a^2 - b^2)}{\big((2n+1)^2 c^2 - (a+b)^2\big)\big((2n+1)^2 c^2 - (a-b)^2\big)}$$

$$c > |a| + |b|$$

(66) $$\pi\sqrt{3} = 24 \sum_{n=0}^{\infty} \frac{1}{9(2n+1)^2 - 4}$$

(67) $$\pi = 16\sum_{n=0}^{\infty}\frac{(8n+4)^2+3}{((8n+4)^2-9)((8n+4)^2-1)}$$

(68) $$\pi\,\frac{r(m)}{\sqrt{2}+s(m)} = 2^{m+3}\sum_{n=0}^{\infty}\frac{2^{2m+2}(2n+1)^2+2^{2m-2}-1}{(2^{2m+2}(2n+1)^2-(2^{m-1}+1)^2)(2^{2m+2}(2n+1)^2-(2^{m-1}-1)^2)}$$

donde

(69) $$m\in\mathbb{N}\,, r(m) = \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicales}\,, s(m) = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{m-radicales}$$

(70) $$\sin\left(\pi\sqrt{\frac{\sqrt{5}-1}{2}}\right)\sinh\left(\pi\sqrt{\frac{\sqrt{5}+1}{2}}\right) = \pi^2\prod_{n=1}^{\infty}(1+n^{-2}-n^{-4})$$

(71) $$\left(\sinh\frac{\pi}{\sqrt{2}}\right)^2+\left(\sin\frac{\pi}{\sqrt{2}}\right)^2 = \left(\cosh\frac{\pi}{\sqrt{2}}\right)^2-\left(\cos\frac{\pi}{\sqrt{2}}\right)^2 = \pi^2\prod_{n=1}^{\infty}(1+n^{-4})$$

(72) $$\left(\cosh\frac{\pi\sqrt{3}}{2}\right)^2 = \pi^2\prod_{n=1}^{\infty}(1+n^{-2}+n^{-4})$$

(73) $$\sin\left(\pi\sqrt{\frac{\sqrt{5}+1}{2}}\right)\sinh\left(\pi\sqrt{\frac{\sqrt{5}-1}{2}}\right) = \pi^2\prod_{n=1}^{\infty}(1-n^{-2}-n^{-4})$$

(74) $$\left(\sinh\frac{\pi}{2}\right)^2+\left(\sin\frac{\pi\sqrt{3}}{2}\right)^2 = \left(\cosh\frac{\pi}{2}\right)^2-\left(\cos\frac{\pi\sqrt{3}}{2}\right)^2 = \pi^2\prod_{n=1}^{\infty}(1-n^{-2}+n^{-4})$$

(75) $$\frac{\pi^3}{8b^3}\,\frac{\sin\left(\frac{a\pi}{2b}\right)}{\left(\cos\left(\frac{a\pi}{2b}\right)\right)^3} = \sum_{n=0}^{\infty}\left[\left((2n+1)b-a\right)^{-3}-\left((2n+1)b+a\right)^{-3}\right]$$

$$0<a<b$$

Ejemplo:

(76) $$\frac{\pi^3}{32} = \sum_{n=0}^{\infty}[(4n+1)^{-3}-(4n+3)^{-3}]$$

(77) $$\frac{\pi^3}{162\sqrt{3}} = \sum_{n=0}^{\infty}[(6n+2)^{-3}-(6n+4)^{-3}]$$

(78) $$\frac{\pi^3}{2^{3k+1}}\underbrace{\sqrt{2-\sqrt{2+\cdots+\sqrt{2}}}}_{k-radicales}\left(\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{k-radicales}\right)^{-3}$$
$$=\sum_{n=0}^{\infty}[(2^{k+1}n+2^k-1)^{-3}-(2^{k+1}n+2^k+1)^{-3}] \quad ,k\in\mathbb{N}$$

(79) $$\frac{\pi}{7\cdot 5^3}\prod_{n=1}^{\infty}\frac{(14n)^4((14n)^2-4)}{((14n)^2-25)^3}=\sum_{n=0}^{\infty}[(14n+5)^{-3}-(14n+9)^{-3}]$$

(80) $$\frac{1}{\pi}=\frac{(2m+1)}{2^{4m+1}}\binom{2m}{m}^2 e^{z(m)}$$

(81) $$\pi=\frac{2^{4n+2}}{m+1}\binom{2m+1}{m}^{-2} e^{z(m-\frac{1}{2})}$$

donde

(82) $$z(m)=-\sum_{k=1}^{\infty}\frac{1}{k\,2^{2k}}\Phi(1,2k,m+1)$$

(83) $$z(m)=-\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{(2k-2n+2m+2)^{-2n}}{n}=-\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(2k+2m)^{-2n+2k-2}}{n-k+1}$$

$\Phi(x,y,w)$ : Lerch Function

(84) $$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(1/2)_n^{p+1}\,a^p}{(n!)^{q+2}}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(1/2)_m^p(1/2)_{n-m}a^m}{(m!)^{q+1}(n-m)!\,(2n+1)}$$

donde

(85) $$p<q+1\,,|a|\le 1\,,Si\ p=q+1\ entonces\ |a|<1$$

(86) $$\pi\sum_{k=0}^{n}\binom{n}{k}^2 a^{2k}=\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!}\sum_{m=0}^{n}\binom{n}{m}\frac{(-2a)^m(1+a^2)^{n-m}(1+(-1)^m)}{2k+m+1}$$

donde

(87) $$0<a<1\,,n\in\mathbb{N}_0$$

Ejemplo:

(88) $$\pi\sum_{k=0}^{2N}\binom{2N}{k}^2 a^{2k}=2\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!}\sum_{m=0}^{n}\binom{2N}{2m}\frac{(2a)^{2m}(1+a^2)^{2N-2m}}{2k+2m+1}$$

$$0<a<1\,,N\in\mathbb{N}_0$$

(89) $$\pi\sum_{k=0}^{2N+1}\binom{2N+1}{k}^2 a^{2k}=2\sum_{k=0}^{\infty}\frac{(1/2)_k}{k!}\sum_{m=0}^{n}\binom{2N+1}{2m}\frac{(2a)^{2m}(1+a^2)^{2N-2m+1}}{2k+2m+1}$$

$$0<a<1\,,N\in\mathbb{N}_0$$

(90) $$\pi=A\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{m}\frac{b_k}{a_k^{2n}}\right)\left(\sum_{k=1}^{m}\frac{c_k}{2n+2k-1}\right)$$

donde

(91) $$m \in \mathbb{N}\,;A, a_k, b_k, c_k \in \mathbb{R}$$

Ejemplos:

$$m = 1 \leftrightarrow \begin{cases} A & a_1 & b_1 & c_1 \\ 4 & 1 & 1 & 1 \\ 2\sqrt{3} & \sqrt{3} & 1 & 1 \end{cases}$$

$$m = 2$$

| A | $a_1$ | $a_2$ | $b_1$ | $b_2$ | $c_1$ | $c_2$ |
|---|---|---|---|---|---|---|
| $\frac{1}{9}$ | 2 | 3 | 9 | $-4$ | 6 | 1 |
| $\frac{1}{4410}$ | 3 | 7 | 49 | $-9$ | 357 | 13 |
| $\frac{81}{443423050}$ | 7 | $\frac{79}{3}$ | $\frac{6241}{9}$ | $-49$ | $\frac{241661}{9}$ | $\frac{263}{3}$ |
| $\frac{\sqrt{2}}{6}$ | $\sqrt{2}$ | $2\sqrt{2}$ | 4 | $-1$ | 5 | 1 |
| $\frac{\sqrt{3}}{756}$ | $3\sqrt{3}$ | $4\sqrt{3}$ | 16 | $-9$ | 198 | 5 |

$$m = 3 \leftrightarrow \begin{cases} A = \frac{1}{960} \quad a_1 = 2 \quad a_2 = 5 \quad a_3 = 8 \\ b_1 = \frac{80}{7} \quad b_2 = -\frac{256}{91} \quad b_3 = \frac{5}{13} \quad c_1 = 352 \quad c_2 = 50 \quad c_3 = 1 \end{cases}$$

$$m = 3 \leftrightarrow \begin{cases} A = \left(\frac{2}{1648353}\right)^2 \quad a_1 = 7 \quad a_2 = 53 \quad a_3 = 4443 \\ b_1 = \frac{18483453147}{18160984000} \quad b_2 = -\frac{322424067}{18158444800} \quad b_3 = \frac{137641}{389621013088000} \\ c_1 = 2147047364973 \quad c_2 = 4901346394 \quad c_3 = 9133 \end{cases}$$

(92) $$\pi = 4 \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1}\right)$$

(93) $$\pi = \frac{1}{9} \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{2^{2n}} - \frac{4}{3^{2n}}\right)\left(\frac{6}{2n+1} + \frac{1}{2n+3}\right)$$

(94) $$\pi = \frac{1}{960}\sum_{n=0}^{\infty}(-1)^n\left(\frac{80/7}{2^{2n}} - \frac{256/91}{5^{2n}} + \frac{5/13}{8^{2n}}\right)\left(\frac{352}{2n+1} + \frac{50}{2n+3} + \frac{1}{2n+5}\right)$$

(95) $$\frac{\pi x - 2\ln x}{2(1+x^2)} - i\,\frac{\pi + 2x\ln x}{2(1+x^2)} = \prod_{n=1}^{\infty}\frac{4}{2 + x^{2^{-n}}(s(n) + i\,r(n))} \quad ,x > 0$$

(96) $$\frac{\pi x - 2\ln x}{2(1+x^2)} + i\,\frac{\pi + 2x\ln x}{2(1+x^2)} = \prod_{n=1}^{\infty}\frac{4}{2 + x^{2^{-n}}(s(n) - i\,r(n))} \quad ,x > 0$$

(97) $$\frac{\pi(1-i)}{4} = \prod_{n=1}^{\infty}\frac{4}{2 + s(n) + i\,r(n)}$$

(98) $$\frac{\pi(1+i)}{4} = \prod_{n=1}^{\infty}\frac{4}{2 + s(n) - i\,r(n)}$$

(99) $$\frac{\pi x - 2\ln x}{1+x^2} = \prod_{n=1}^{\infty}\frac{4}{2 + x^{2^{-n}}(s(n) + i\,r(n))} + \prod_{n=1}^{\infty}\frac{4}{2 + x^{2^{-n}}(s(n) - i\,r(n))} \quad ,x > 0$$

En las formulas (95)-(99), $s(n), r(n)$ ,se definen como en (69).

(100) $$\frac{X_1}{\pi} = \frac{3}{14}\prod_{n=1}^{\infty}\left(1 - \left(\frac{3}{14n}\right)^2\right)$$

(101) $$\frac{X_2}{\pi} = -\frac{1}{14}\prod_{n=1}^{\infty}\left(1 - \left(\frac{1}{14n}\right)^2\right)$$

(102) $$\frac{X_3}{\pi} = -\frac{5}{14}\prod_{n=1}^{\infty}\left(1 - \left(\frac{5}{14n}\right)^2\right)$$

(103) $$\frac{\pi}{X_1} = 196\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(14n+7)^2 - 16}$$

(104) $$\frac{\pi}{X_2} = 196\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(14n+7)^2 - 64}$$

(105) $$\frac{\pi}{X_3} = 196\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{(14n+7)^2 - 144}$$

donde

(106) $$X_1 = -\frac{1}{6} + \frac{1}{6}\sqrt[3]{7 + 21\sqrt[3]{7 + 21\sqrt[3]{7 + \cdots}}}$$

(107) $$X_2 = -\frac{1}{4} - \frac{1}{2}X_1 + \frac{1}{4}\sqrt{3 + 2X_1 - \frac{3}{2X_1}}$$

(108) $$X_3 = -\frac{1}{4} - \frac{1}{2}X_1 - \frac{1}{4}\sqrt{3 + 2X_1 - \frac{3}{2X_1}}$$

(109) $$\frac{1}{X_3} = -\frac{4}{3} + \frac{1}{3}\left[\frac{2}{3} + \frac{1}{84}\left(\frac{2}{3} + \frac{1}{84}\left(\frac{2}{3} + \cdots\right)^3\right)^3\right]$$

(110) $$\frac{1}{X_2} = -2 - \frac{1}{2X_3} - \sqrt{5 + \frac{1}{X_3} - 6X_3}$$

(111) $$\frac{1}{X_1} = -2 - \frac{1}{2X_3} + \sqrt{5 + \frac{1}{X_3} - 6X_3}$$

(112) $$\frac{1 + 2\sqrt[3]{7 + 21\sqrt[3]{7 + 21\sqrt[3]{7 + \cdots}}}}{2 + \sqrt[3]{7 + 21\sqrt[3]{7 + 21\sqrt[3]{7 + \cdots}}}} = \pi^2 \frac{9}{49}\prod_{n=1}^{\infty}\left(1 - \left(\frac{3}{14n}\right)^2\right)^2$$

(113) $$\frac{1}{\pi^3} = \frac{15}{343}\prod_{n=1}^{\infty}\left(1 - \left(\frac{1}{14n}\right)^2\right)\left(1 - \left(\frac{3}{14n}\right)^2\right)\left(1 - \left(\frac{5}{14n}\right)^2\right)$$

(114) $$\pi = 4\tan^{-1}\left(\frac{1}{x}\right) + 4\tan^{-1}\big((1 + x)(2 - x)\big)$$

donde

(115) $$x = \sqrt[3]{3 + 2\sqrt[3]{3 + 2\sqrt[3]{3 + \cdots}}}$$

(116) $$\pi = 4\tan^{-1}\left(\frac{1}{x}\right) + 4\tan^{-1}\left(\left(\frac{1}{2} + x\right)\left(\frac{3}{2} - x\right)\right)$$

donde

(117) $$x = \frac{1}{2}\sqrt[3]{14 + 3\sqrt[3]{14 + 3\sqrt[3]{14 + \cdots}}}$$

(118) $$\pi = 4\tan^{-1}\left(\frac{1}{x}\right) + 4\tan^{-1}\big((3 + x)(4 - x)\big)$$

donde

(119) $$x = \sqrt[3]{13 + 12\sqrt[3]{13 + 12\sqrt[3]{13 + \cdots}}}$$

(120) $$\pi = 4\tan^{-1}\left(\frac{1}{x}\right) + 4\tan^{-1}\left(\left(\frac{11}{3} + x\right)\left(\frac{14}{3} - x\right)\right)$$

donde

(121) $$x = \frac{1}{3}\sqrt[3]{489 + 154\sqrt[3]{489 + 154\sqrt[3]{489 + \cdots}}}$$

(122) $$\pi = 4\tan^{-1}(x) + 4\tan^{-1}\left((1+x)\left(x+\frac{\sqrt{13}-1}{6}\right)\left(x-\frac{\sqrt{13}+1}{6}\right)\right)$$

donde

(123) $$x = \frac{2}{\sqrt[4]{12+10\sqrt[4]{12+10\sqrt[4]{12+\cdots}}}}$$

(124) $$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{x}}\right) + 4\tan^{-1}\left(\left(\left(\sqrt[3]{x}-\frac{\sqrt{13}-1}{6}\right)\left(\sqrt[3]{x}+\frac{\sqrt{13}+1}{6}\right)\left(\sqrt[3]{x}-1\right)\right)^{-1}\right)$$

donde

(125) $$x = \left(\frac{4}{3}+\frac{5}{3}\left(\frac{4}{3}+\frac{5}{3}\left(\frac{4}{3}+\cdots\right)^{3/4}\right)^{3/4}\right)^{3/4}$$

(126) $$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt{x}}\right) + 4\tan^{-1}\left(\frac{1}{\sqrt{1+x}}\right)$$

donde

(127) $$x = \frac{2}{3}+\frac{1}{3}\sqrt[3]{359+12\sqrt{78}}+\frac{1}{3}\sqrt[3]{359-12\sqrt{78}}$$

(128) $$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} x_n^3 - \sum_{n=1}^{\infty}\frac{x_n}{n^2} = \frac{5}{8}+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-2)^{n-k+1}x_{k+2}^{n-k+3}$$
$$= \frac{3}{4}+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{n+1}2^{n-k+1}x_{k+2}^{n-k+3}$$

donde la sucesión $x_n, n \in \mathbb{N}$ , se define por:

(129) $$x_n = \sqrt[3]{\frac{1}{4n^2}+\frac{1}{n^2}\sqrt{\frac{1}{16}-\frac{1}{27n^2}}}+\sqrt[3]{\frac{1}{4n^2}-\frac{1}{n^2}\sqrt{\frac{1}{16}-\frac{1}{27n^2}}}$$

(130) $$x_n = \sqrt[3]{\frac{1}{2n^2}+\frac{1}{n^2}\sqrt[3]{\frac{1}{2n^2}+\cdots}} = \frac{1}{2n}\sqrt[3]{4n+4\sqrt[3]{4n+\cdots}}$$

(131) $$0 < x_{n+1} < x_n < x_1 \,, n \geq 2 \,, \lim_{n\to\infty} x_n = 0$$

(132) $$\pi^{2m+2}\,V(m) = \sum_{k=1}^{m-1}\zeta(2k+1)\,\zeta(2m-2k+1) + 2\sum_{n=1}^{\infty}\frac{H_n}{n^{2m+1}} \quad , m \in \mathbb{N}$$

donde

(133) $$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}\ ,Re(s)>1\ , \qquad H_n=\sum_{k=1}^{n}\frac{1}{k}\ \ ,n\in\mathbb{N}$$

(134) $$V(m)=2^{2m}\sum_{k=1}^{m}\frac{B_k B_{m-k+1}}{(2k)!\,(2m-2k+2)!}\ \ ,m\in\mathbb{N}$$

(135) $$V(m)=(-1)^{m-1}2^{2m}\sum_{k=1}^{m}\sum_{i=0}^{2k}\sum_{j=0}^{2m-2k+2}\sum_{r=0}^{i}\sum_{s=0}^{j}\binom{i}{r}\binom{j}{s}\frac{(-1)^{r+s}r^{2k}s^{2m-2k+2}}{(i+1)(j+1)}$$

(136) $$V(m)=\left\{\frac{1}{36},\frac{1}{270},\frac{1}{2100},\frac{1}{17010},\frac{691}{98232750},\frac{1}{1216215},\dots\right\}$$

(137) $$\ln\left(\frac{\pi}{2}\right)=2\sum_{n=1}^{\infty}\tanh^{-1}\left(\frac{1}{8n^2-1}\right)$$

(138) $$\ln\pi=2\tanh^{-1}\frac{1}{3}+2\sum_{n=1}^{\infty}\tanh^{-1}\left(\frac{1}{8n^2-1}\right)$$

(139) $$\ln\left(\frac{\pi x}{\sin(\pi x)}\right)=2\sum_{n=1}^{\infty}\tanh^{-1}\left(\frac{x^2}{2n^2-x^2}\right)\ \ ,0<x<1/2$$

(140) $$\sigma=\frac{2-\sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\cdots}}}}{2+\sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\cdots}}}}=\frac{\pi^2}{324}\prod_{n=1}^{\infty}\left(\frac{2n-1}{2n}\right)^4\left(\frac{(18n)^2-1}{4(9n-5)(9n-4)}\right)^2$$

(141) $$\sigma=\sum_{n=1}^{\infty}\frac{\pi^{2n}}{3^{4n}}\sum_{k=1}^{n}\frac{(2^{2k}-1)(2^{2n-2k}-1)B_k B_{n-k}}{(2k)!\,(2n-2k)!}$$

$B_k$ , números de Bernoulli

(142) $$\sigma=-1+\frac{81}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{(9n-5)^2}+\frac{1}{(9n-4)^2}\right)$$

(143) $$\sigma=\frac{81}{\pi^2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{(9k-5)(9k-4)(9n-9k-5)(9n-9k-4)}$$

(144) $$\sigma=\frac{1}{11+2\sqrt[3]{296+84\sqrt[3]{296+84\sqrt[3]{296+\cdots}}}}$$

(145) $$3\sigma^3-27\sigma^2+33\sigma-1=0$$

(146) $$\pi=3\sin^{-1}\left(\frac{1}{2}\sqrt[3]{2x+3\sqrt[3]{2x+3\sqrt[3]{2x+\cdots}}}\right)-\sin^{-1}x\quad ,0<x<1$$

(147) $$\pi=4\sin^{-1}r-2\sin^{-1}s$$

donde

(148) $$r=\sqrt[4]{\frac{3}{8}+\frac{1}{2}\sqrt[4]{\frac{3}{8}+\cdots}}=\frac{1}{2}\sqrt[4]{6+4\sqrt[4]{6+\cdots}}$$

(149) $$s=\frac{1}{2}\left(-2+\sqrt{2+2\sqrt[3]{2}+\sqrt[3]{4}}\right.$$
$$\left.+\sqrt{4-2\sqrt[3]{2}-\sqrt[3]{4}-4\sqrt{2+2\sqrt[3]{2}+\sqrt[3]{4}}+2\sqrt{16+6\sqrt[3]{2}+6\sqrt[3]{4}}}\right)$$

(150) $$\pi=5\sin^{-1}r-\sin^{-1}s$$

donde

(151) $$r=\frac{1}{6}\left(\sqrt[3]{27+3\sqrt{78}}+\sqrt[3]{27-3\sqrt{78}}\right)$$

(152) $$s=\frac{10}{3}-\frac{1}{6}\left(\sqrt[3]{27+3\sqrt{78}}+\sqrt[3]{27-3\sqrt{78}}\right)-\frac{1}{3}\left(\sqrt[3]{53+6\sqrt{78}}+\sqrt[3]{53-6\sqrt{78}}\right)$$

(153) $$\pi=2\sin^{-1}r+2\sin^{-1}s$$

donde

(154) $$r=\sqrt[5]{a+b\sqrt[5]{a+b\sqrt[5]{a+\cdots}}}$$

(155) $$s=\sqrt{1-\sqrt{b+a\sqrt[4]{\frac{1}{b+a\sqrt[4]{\frac{1}{b+\cdots}}}}}}$$

(156) $$0<a<1\,,0<b<1\,,a+b<1$$

(157) $$\pi=5\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{5-\sqrt{7}}{2}}\right)+\sin^{-1}\left(\frac{1}{4}\sqrt{\frac{5-\sqrt{7}}{2}}\right)$$

(158) $$\pi=5\sin^{-1}\left(\frac{1}{2\sqrt{2}}\right)+\sin^{-1}\left(\frac{11}{8\sqrt{2}}\right)$$

(159) $$\pi=\frac{8}{3}\sin^{-1}\left(\sqrt{\frac{3+\sqrt{6}}{6}}\right)-\frac{2}{3}\sin^{-1}\left(\frac{1}{3}\right)$$

(160) $$\pi=3\sin^{-1}\left(\frac{22}{25}\right)-\sin^{-1}\left(\frac{1342}{15625}\right)$$

(161) $$\pi=3\sin^{-1}\left(\frac{9}{10}\right)-\sin^{-1}\left(\frac{27}{125}\right)$$

(162) $$\pi=8\sin^{-1}\left(\frac{1}{3}\right)+2\sin^{-1}\left(\frac{17}{81}\right)$$

(163) $$\pi = \frac{2^{n+1}}{2^n - 1}\sin^{-1}\big(R(n,x)\big) - \frac{2}{2^n - 1}\sin^{-1} x$$

donde

(164) $$R(n,x) = \frac{1}{2}\underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + 2x}}}}_{n-radicales} \quad ,0 < x < 1\,, n \in \mathbb{N}$$

(165) $$\pi = \frac{5}{2}\sin^{-1}\left(\frac{1}{r}\right) + \frac{1}{2}\sin^{-1}\left(\frac{16}{r^4} + \frac{20}{r^3} - \frac{5}{r} - 16\right) \;, r = \sqrt[5]{1 + \sqrt[5]{1 + \sqrt[5]{1 + \cdots}}}$$

(166) $$\pi = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) + \sin^{-1}\left(\frac{3\sqrt{2} - \sqrt{3}}{6}\right)$$

(167) $$\pi = \sin^{-1}\left(\frac{3\sqrt{2} + \sqrt{3}}{6}\right) - \sin^{-1}\left(\frac{3\sqrt{2} - \sqrt{3}}{6}\right)$$

(168) $$\pi = 3\sin^{-1}\big(\sqrt{3} - \sqrt{2}\big) + 3\sin^{-1}\left(\frac{\sqrt{6\sqrt{6} - 12} - \sqrt{3} + \sqrt{2}}{2}\right)$$

(169) $$\pi = 3\sin^{-1}\left(\frac{\sqrt{3} - \sqrt{2} + \sqrt{6\sqrt{6} - 12}}{2}\right) - 3\sin^{-1}\big(\sqrt{3} - \sqrt{2}\big)$$

(170) $$\pi = \frac{6}{5}(\sin^{-1} r + \sin^{-1} s)$$

donde

(171) $$r = \frac{1}{2}\sqrt[3]{\frac{6}{5} + 3\sqrt[3]{\frac{6}{5} + 3\sqrt[3]{\frac{6}{5} + \cdots}}} \quad , \qquad s = \frac{1}{2}\sqrt[3]{\frac{8}{5} + 3\sqrt[3]{\frac{8}{5} + 3\sqrt[3]{\frac{8}{5} + \cdots}}}$$

(172) $$\pi = \frac{8}{3}\sin^{-1}\left(\sqrt[4]{\frac{x+3}{8} + \frac{1}{2}\sqrt{\frac{x+1}{2}}}\right) - \frac{2}{3}\sin^{-1} x \quad ,0 \le x \le 1$$

(173) $$\pi = \frac{8}{3}\sin^{-1}\left(\sqrt[4]{\frac{3n+1}{8n} + \frac{\sqrt{2n(n+1)}}{4n}}\right) - \frac{2}{3}\sin^{-1}\left(\frac{1}{n}\right) \quad ,n \in \mathbb{N}$$

(174) $$\pi = 3\sin^{-1} x - \sin^{-1}(4x^3 - 3x) \quad ,\frac{\sqrt{3}}{2} < x < 1$$

(175) $$\pi = \frac{8}{3}\sin^{-1} x - \frac{2}{3}\sin^{-1}(8x^4 - 8x^2 + 1) \;, \qquad \frac{\sqrt{2 + \sqrt{2}}}{2} < x < 1$$

(176) $$\pi = 5\sin^{-1} z + \sin^{-1}\big((16 - x)z^5\big)$$

donde

(177) $$0 < x < 20 \ , z = \sqrt{\frac{10 - \sqrt{100 - 5x}}{x}}$$

Ejemplos:

(178) $$x = 2\,, z = \sqrt{\frac{10 - 3\sqrt{10}}{2}}\ , \pi = 5\sin^{-1} z + \sin^{-1}(14z^5)$$

(179) $$x = 7\,, z = \sqrt{\frac{10 - \sqrt{65}}{7}}\ , \pi = 5\sin^{-1} z + \sin^{-1}(9z^5)$$

(180) $$x = 8\,, z = \sqrt{\frac{5 - \sqrt{15}}{4}}\ , \pi = 5\sin^{-1} z + \sin^{-1}(8z^5)$$

(181) $$x = 15\,, z = \frac{1}{\sqrt{3}}\ , \pi = 5\sin^{-1} z + \sin^{-1}(z^5)$$

(182) $$\pi = 5\sin^{-1} z + \sin^{-1}\left(\left(\frac{m^2 - 20}{5}\right)z^5\right) \quad , 0 < m < 10\,, z = \sqrt{\frac{5}{10 + m}}$$

(183) $$\pi = 5\sin^{-1}(z) - \sin^{-1}\left(\frac{19}{5}z^5\right) \quad , z = \sqrt{\frac{5}{11}}$$

(184) $$\pi = \frac{5}{2}\sin^{-1} z - \frac{1}{2}\sin^{-1}\left(\frac{z^5}{n}\right) \quad , \qquad n \in \mathbb{N}\,, z = \sqrt{\frac{10n + \sqrt{n(20n + 5)}}{16n - 1}}$$

(185) $$\pi = 2u \prod_{n=1}^{\infty}\left(\frac{\tanh(nv)}{\tanh(nu)}\right)^{2(-1)^{n-1}} \quad , u > 0, v > 0, uv = \frac{\pi^2}{4}$$

(186) $$\pi = \frac{ue^v}{2} \prod_{n=1}^{\infty}(\tanh(nu))^2 \left(\frac{1 + \coth(4nv)}{1 + \coth\big((4n - 2)v\big)}\right)^2 \quad , u > 0, v > 0, uv = \frac{\pi^2}{4}$$

(187) $$\pi = v\, e^{(u-v)/6} \prod_{n=1}^{\infty}\left(\frac{1 + \coth(nu)}{1 + \coth(nv)}\right)^2 \quad , u > 0, v > 0, uv = \pi^2$$

(188) $$\frac{7\pi}{12\sqrt{3}} + \ln\left(\left(\frac{\left(2\sqrt{3} + 1\right)^2\left(3\sqrt{3} + 1\right)^2}{50}\right)^{1/6}\right)$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{3n + 1}\left(\left(\frac{2 + \sqrt{3}}{4}\right)^{3n+1} + \left(\frac{3 + \sqrt{3}}{6}\right)^{3n+1}\right)$$

(189) $$\frac{\pi\sqrt{3}}{4}+\ln\left(\left(\frac{3\sqrt{3}+1}{10}\right)^{1/3}\left(\frac{\left(7\sqrt{3}+1\right)^2}{50}\right)^{1/6}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(2\left(\frac{3+\sqrt{3}}{6}\right)^{3n+1}+\left(\frac{7+\sqrt{3}}{14}\right)^{3n+1}\right)$$

(190) $$\frac{\pi}{2\sqrt{3}}+\ln\left(2\cdot 3^{1/6}\left(\frac{5}{19}\right)^{1/3}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\left(\frac{3}{5}\right)^{3n+1}+\left(\frac{7}{8}\right)^{3n+1}\right)$$

(191) $$\frac{\pi}{3\sqrt{3}}+\ln\left(\left(\frac{11\cdot 17}{89}\right)^{1/3}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\left(\frac{5}{6}\right)^{3n+1}+\left(\frac{7}{10}\right)^{3n+1}-\left(\frac{32}{57}\right)^{3n+1}\right)$$

(192) $$\frac{\pi}{2\sqrt{3}}+\ln\left(\left(\frac{7}{2\cdot 13}\right)^{1/3}\left(\frac{1}{3}\right)^{1/6}5^{2/3}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\left(\frac{3}{4}\right)^{3n+1}+\left(\frac{5}{7}\right)^{3n+1}\right)$$

(193) $$\frac{11\pi}{4\sqrt{3}}+\ln\left(\left(\frac{\left(18\sqrt{3}+1\right)^2}{325}\right)^{2}\left(\frac{\left(57\sqrt{3}+1\right)^2}{3250}\right)^{4/3}\left(\frac{\left(239\sqrt{3}+1\right)^2}{57122}\right)^{-5/6}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(12\left(\frac{18+\sqrt{3}}{36}\right)^{3n+1}+8\left(\frac{57+\sqrt{3}}{114}\right)^{3n+1}\right.$$
$$\left.-5\left(\frac{239+\sqrt{3}}{478}\right)^{3n+1}\right)$$

(194) $$\frac{2\pi}{3\sqrt{3}}+\ln\left(\left(13\left(\frac{5}{7}\right)^2\right)^{1/3}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(2\left(\frac{2}{3}\right)^{3n+1}+\left(\frac{5}{8}\right)^{3n+1}\right)$$

(195) $$\frac{\pi\sqrt{3}}{4}+\ln\left(\left(\frac{\left(2\sqrt{3}+1\right)^2\left(5\sqrt{3}+1\right)^2\left(8\sqrt{3}+1\right)^2}{5\cdot 26\cdot 65}\right)^{1/6}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\left(\frac{2+\sqrt{3}}{4}\right)^{3n+1}+\left(\frac{5+\sqrt{3}}{10}\right)^{3n+1}+\left(\frac{8+\sqrt{3}}{16}\right)^{3n+1}\right)$$

(196) $$\frac{11\pi}{12\sqrt{3}}+\ln\left(\left(\frac{\left(5\sqrt{3}+1\right)^2}{26}\right)^{2/3}\left(\frac{\left(70\sqrt{3}+1\right)^2}{4901}\right)^{-1/6}\left(\frac{\left(99\sqrt{3}+1\right)^2}{9802}\right)^{1/6}\right)$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(4\left(\frac{5+\sqrt{3}}{10}\right)^{3n+1}-\left(\frac{70+\sqrt{3}}{140}\right)^{3n+1}+\left(\frac{99+\sqrt{3}}{198}\right)^{3n+1}\right)$$

(197) $$\pi=2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\left(\frac{1+m^{2n+1}+m^{4n+2}}{2^{2n}m^{4n+2}}\right)$$

donde

(198) $$m\in\mathbb{N}-\{1\},\, x=(1+m)^2-\sqrt{(1+m)^4-4m^2}$$

(199) $$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(9-\sqrt{65}\right)^{2n+1} \left(\frac{1+2^{2n+1}+2^{4n+2}}{2^{6n+1}}\right)$$

(200) $$\ln \pi = 2\ln 2 - \sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1}\left(\frac{\tan(1)-1}{\tan(1)+1}\right)\right)^n$$

(201) $$\pi = \frac{b}{a}\left(\frac{1}{2}c - \frac{1}{2\cdot 4}c^2 + \frac{1\cdot 3}{2\cdot 4\cdot 6}c^3 - \cdots\right)$$

(202) $$\frac{1}{\pi} = \frac{1}{c}\left(\frac{2a}{b}\right) + \frac{a}{b}\left(\frac{1}{2} - \frac{1}{2\cdot 4}c + \frac{1\cdot 3}{2\cdot 4\cdot 6}c^2 - \cdots\right)$$

donde

(203) $$a, b \in \mathbb{N}, 0 < \frac{a}{b} < 0.131848\ldots, c = \frac{8a}{b^2}\sum_{n=0}^{\infty}\left(\frac{4bn+a+b}{(4n+1)^2} - \frac{4bn+3b-a}{(4n+3)^2}\right)$$

(204) $$\sqrt[3]{\frac{p}{4}+\frac{3}{4}\sqrt[3]{\frac{p}{4}+\cdots}} - \sqrt[3]{\frac{q}{4}+\frac{3}{4}\sqrt[3]{\frac{q}{4}+\cdots}}$$
$$= \pi^2\left(\frac{b^2-a^2}{18}\right)\prod_{n=1}^{\infty}\left(1-\frac{(a+b)^2}{36n^2}\right)\left(1-\frac{(a-b)^2}{36n^2}\right)$$

donde

(205) $$0 < a < b < \frac{1}{2}, \cos(a\pi) = p, \cos(b\pi) = q$$

(206) $$\frac{\pi}{12} + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2-\sqrt{3}}{2^n}\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{\left(4-2\sqrt{3}\right)^{4n+1}}{(4n+1)(2^{4n+1}-1)} + \frac{\left(4-2\sqrt{3}\right)^{4n+3}}{(4n+3)(2^{4n+3}-1)}\right)$$

(207) $$\frac{\pi}{6} + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2^n\sqrt{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\left(\frac{2^{4n+1}}{3^{2n}(4n+1)(2^{4n+1}-1)} + \frac{2^{4n+3}}{3^{2n+1}(4n+3)(2^{4n+3}-1)}\right)$$

(208) $$\frac{\pi}{6} + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{35}-3\sqrt{3}}{2^{n+2}}\right)$$
$$= \sum_{n=0}^{\infty}\left(\frac{\left(\sqrt{35}-3\sqrt{3}\right)^{4n+1}}{(4n+1)(2^{4n+1}-1)} + \frac{\left(\sqrt{35}-3\sqrt{3}\right)^{4n+3}}{(4n+3)(2^{4n+3}-1)}\right)$$

(209) $$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n F_{2n+1}}{5^n(2n+1)}\left(\left(\frac{2a}{\sqrt{4a^2+5}+\sqrt{5}}\right)^{2n+1}+\left(\frac{2(1-a)}{\sqrt{4(1-a)^2+5(1+a)^2}+(1+a)\sqrt{5}}\right)^{2n+1}\right)$$

donde $F_n$ es la sucesión de Fibonacci.

(210) $$\frac{\pi}{4\sqrt{3}}+\frac{1}{6}\ln P+\frac{1}{6}\ln Q=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(b^{3n+1}-a^{3n+1}+d^{3n+1}-c^{3n+1}\right)$$

donde

(211) $$P=\left(\frac{1+b}{1+a}\right)\left(\frac{a^2-a+1}{b^2-b+1}\right)\ ,Q=\left(\frac{1+d}{1+c}\right)\left(\frac{c^2-c+1}{d^2-d+1}\right)$$

(212) $$-1<a<\frac{2\sqrt{3}-1}{2\sqrt{3}+1}\ ,b=\frac{2+\left(2\sqrt{3}-1\right)a}{\left(2\sqrt{3}+1\right)-2a}$$

(213) $$-1<c<\frac{3\sqrt{3}-1}{3\sqrt{3}+1}\ ,d=\frac{2+\left(3\sqrt{3}-1\right)c}{\left(3\sqrt{3}+1\right)-2c}$$

(214) $$\frac{\pi}{4\sqrt{3}}+\frac{1}{6}(\ln P+\ln Q+\ln R)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(a_2^{3n+1}-a_1^{3n+1}+b_2^{3n+1}-b_1^{3n+1}+c_2^{3n+1}-c_1^{3n+1}\right)$$

donde

(215) $$P=\left(\frac{1+a_2}{1+a_1}\right)\left(\frac{a_1^2-a_1+1}{a_2^2-a_2+1}\right),-1<a_1<\frac{2\sqrt{3}-1}{2\sqrt{3}+1}\ ,a_2=\frac{2+\left(2\sqrt{3}-1\right)a_1}{2\sqrt{3}+1-2a_1}$$

(216) $$Q=\left(\frac{1+b_2}{1+b_1}\right)\left(\frac{b_1^2-b_1+1}{b_2^2-b_2+1}\right),-1<b_1<\frac{5\sqrt{3}-1}{5\sqrt{3}+1}\ ,a_2=\frac{2+\left(5\sqrt{3}-1\right)b_1}{5\sqrt{3}+1-2b_1}$$

(217) $$R=\left(\frac{1+c_2}{1+c_1}\right)\left(\frac{c_1^2-c_1+1}{c_2^2-c_2+1}\right),-1<c_1<\frac{8\sqrt{3}-1}{8\sqrt{3}+1}\ ,c_2=\frac{2+\left(8\sqrt{3}-1\right)c_1}{8\sqrt{3}+1-2c_1}$$

(218) $$\pi\ \tan^{-1}\left(\frac{a^2-1}{2a}\right)=\sum_{n=0}^{\infty}\frac{2^{2n+1}(n!)^2(a^{2n+1}-1)}{(2n+1)!\,(n+1)(a^2+1)^{n+1}}\ \ ,a>1$$

(219) $$\pi\ \tan^{-1}\left(\frac{a^2+2a-1}{1+2a-a^2}\right)=\sum_{n=0}^{\infty}\frac{2^{2n+2}(n!)^2}{(2n+1)!\,(n+1)}\left[\left(\frac{a^2}{1+a^2}\right)^{n+1}-\left(\frac{(1-a)^2}{2(1+a^2)}\right)^{n+1}\right]\ ,0<a<1$$

(220) $$\pi=4\tan^{-1}b+4\tan^{-1}(e^{-2ab})$$

donde

(221) $$a>0\,,0<b<1\,,b=\tanh(ab)$$

(222) $$\pi = 4\tan^{-1}\left(\frac{1}{\cosh x}\right) + 4\tan^{-1}\left(\left(\tanh\frac{x}{2}\right)^2\right) \quad ,x > 0$$

(223) $$\pi = 4\tan^{-1}(\tanh x) + 4\tan^{-1}(\cosh(2x) - \sinh(2x)) \quad ,x > 0$$

(224) $$\pi = 4\tan^{-1} a + 4\tan^{-1}\left(\frac{\cosh a}{2 + \cosh a}\right)$$

donde

(225) $$a = \frac{1}{1 + \cosh a} \quad ,a = \cfrac{1}{1 + \cosh\cfrac{1}{1 + \cosh\cfrac{1}{1 + \cdots}}} = 0.47304015 \ldots$$

(226) $$\pi = 4\tan^{-1} b + 4\tan^{-1}\left(\frac{\sinh b}{2 + \sinh b}\right)$$

donde

(227) $$b = \frac{1}{1 + \sinh b} \quad ,b = \cfrac{1}{1 + \sinh\cfrac{1}{1 + \sinh\cfrac{1}{1 + \cdots}}} = 0.60763540 \ldots$$

(228) $$\pi = 4\tan^{-1} c + 4\tan^{-1}\left(\frac{\tanh c}{2 + \tanh c}\right)$$

donde

(229) $$c = \frac{1}{1 + \tanh c} \quad ,c = \cfrac{1}{1 + \tanh\cfrac{1}{1 + \tanh\cfrac{1}{1 + \cdots}}} = 0.63923227 \ldots$$

(230) $$\pi = \frac{1}{a}\cosh^{-1}\left(\sqrt[3]{y(a) + \sqrt[3]{y(a) + \sqrt[3]{y(a) + \cdots}}}\right)$$

donde

(231) $$y(a) = \frac{1}{4}\left[\prod_{n=0}^{\infty}\left(1 + \frac{36a^2}{(2n+1)^2}\right) - \prod_{n=0}^{\infty}\left(1 + \frac{4a^2}{(2n+1)^2}\right)\right] \quad ,a > 0$$

(232) $$\pi = 4\tan^{-1} x + 4\tan^{-1}(\tanh x)$$

donde

(233) $$x = e^{-2e^{-2e^{-2\cdots}}} = 0.4263027 \ldots$$

(234) $$\pi = 6\sum_{n=1}^{\infty}\left(\frac{x^n}{n}\sin\left(\frac{n\pi}{3}\right) - \frac{(-1)^{n-1}}{2n-1}\left(\frac{2x-1}{\sqrt{3}}\right)^{2n-1}\right) \quad ,0 < x < 1$$

(235) $$\pi = 3\sum_{n=1}^{\infty}\left(\frac{x^n}{n}\sin\left(\frac{n\pi}{6}\right) - \frac{(-1)^{n-1}}{2n-1}\left(2x - \sqrt{3}\right)^{2n-1}\right) \quad ,\frac{\sqrt{3}-1}{2} < x < 1$$

(236) $$\ln\pi = 1 + \sum_{n=1}^{\infty}\left((1 - 2^{-n-1})\zeta(n+2) - (1 - 2^{-n})\zeta(n+1)\right)\sum_{k=1}^{n}\frac{1}{k+1}$$

(237) $$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} a^{2n+1}(1 + a^{2n+1})$$

donde

(238) $$a = \frac{1}{3}\left(-1 + \sqrt[3]{3\sqrt{33}+17} - \sqrt[3]{3\sqrt{33}-17}\right)$$

(239) $$\frac{1}{2\sqrt{3}} + \frac{4}{3}\left(\frac{1}{2\sqrt{3}} + \frac{4}{3}\left(\frac{1}{2\sqrt{3}} + \cdots\right)^3\right)^3 = \frac{27}{2\pi} \sum_{n=0}^{\infty} \left(\frac{3n+1}{9(3n+1)^2 - 1} - \frac{3n+2}{9(3n+2)^2 - 1}\right)$$

(240) $$\pi = 4 \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{x(\tanh(n+1) - \tanh(n))}{1 + x^2 \tanh(n+1)\tanh(n)}\right) + 4 \sum_{n=0}^{\infty} \tan^{-1}\left(\frac{(1-x^2)(\tanh(n+1) - \tanh(n))}{(1+x)^2 + (1-x)^2 \tanh(n+1)\tanh(n)}\right)$$

$$0 \le x \le 1$$

(241) $$\pi = \lim_{n\to\infty}\left(\frac{2^{n+3}}{\underbrace{\sqrt{2 - \sqrt{2 + \cdots + \sqrt{2}}}}_{n-radicales}} \ln\left(\frac{2}{\underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{n-radicales}}\right)\right)$$

(242) $$exp\left(\frac{\pi^2}{8}\right) = \lim_{n\to\infty}\left(\frac{2}{\underbrace{\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}_{n-radicales}}\right)^{2^{2n}}$$

(243) $$\frac{\sin(x\pi)}{x\pi} = \prod_{n=1}^{\infty}\left(1 - 4\left(\sin\left(\frac{x\pi}{5^n}\right)\right)^2 + \frac{16}{5}\left(\sin\left(\frac{x\pi}{5^n}\right)\right)^4\right) \quad , x \in \mathbb{R}$$

(244) $$\pi = \frac{1}{3} \sum_{n=0}^{\infty} \frac{n!\ 6^{-2n}}{(2n+1)!}\ e(n)$$

(245) $$e(n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} 6^{2n-2m}(2^{2m+1} + 3^{2m+1}) v(n,m)$$

(246) $$\pi = \frac{2}{553} \sum_{n=0}^{\infty} \frac{n!\ 553^{-2n}}{(2n+1)!}\ e(n)$$

(247) $$e(n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} 553^{2n-2m} (5 \cdot 79^{2m+1} + 2 \cdot 21^{2m+1}) v(n,m)$$

(248) $$\pi = \frac{1}{20} \sum_{n=0}^{\infty} \frac{n!\ 40^{-2n}}{(2n+1)!}\ e(n)$$

(249) $$e(n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} 40^{2n-2m} (20^{2m+1} + 8^{2m+1} + 5^{2m+1}) v(n,m)$$

(250) $$\pi = \frac{1}{3970} \sum_{n=0}^{\infty} \frac{n!\ 4^{-2n} 5^{-2n} 397^{-2n}}{(2n+1)!}\ e(n)$$

(251) $$e(n) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (4 \cdot 5 \cdot 397)^{2n-2m} (3 \cdot 5^{2m+1} \cdot 397^{2m+1} + 397^{2m+1} + 4^{2m+1}) v(n,m)$$

En las fórmulas (245),(247),(249),(251), se tiene:

(252) $$v(n,m) = \prod_{\substack{k=0 \\ k \neq m}}^{n} (2k+1) = \frac{(2n+1)!}{2^n n!\,(2m+1)} \quad , m = 0,1,2,\dots,n$$

(253) $$v(n,m+1) = \frac{2m+1}{2m+3}\ v(n,m) \quad , \qquad v(n,m) \in \mathbb{N}$$

(254) $$\pi = 2^{4n+1} \binom{2n}{n}^{-1} \sum_{k=0}^{\infty} \binom{n-1/2}{k} \frac{(-1)^k}{2k+2n+1} \quad , n \in \mathbb{N}_0$$

(255) $$\frac{2}{\sqrt{3}}\pi = 3^{3n+1} \binom{3n}{n}^{-1} \sum_{k=0}^{\infty} \binom{n-1/3}{k} \frac{(-1)^k}{3k+3n+1}$$

$$= 3^{3n+1} \binom{3n}{n}^{-1} \sum_{k=0}^{\infty} \binom{n-2/3}{k} \frac{(-1)^k}{3k+3n+2} \quad , n \in \mathbb{N}_0$$

(256) $$\frac{\pi}{\sqrt{2}} = 2^{6n+1} \binom{4n}{2n}^{-1} \sum_{k=0}^{\infty} \binom{n-1/4}{k} \frac{(-1)^k}{4k+4n+1}$$

$$= 2^{6n+1} \binom{4n}{2n}^{-1} \sum_{k=0}^{\infty} \binom{n-3/4}{k} \frac{(-1)^k}{4k+4n+3} \quad , n \in \mathbb{N}_0$$

(257) $$\pi = 3 \cdot 432^n \binom{2n}{n} \binom{3n}{2n}^{-1} \binom{6n}{3n}^{-1} \sum_{k=0}^{\infty} \binom{n-1/6}{k} \frac{(-1)^k}{6k+6n+1}$$

$$= 3 \cdot 432^n \binom{2n}{n} \binom{3n}{2n}^{-1} \binom{6n}{3n}^{-1} \sum_{k=0}^{\infty} \binom{n-5/6}{k} \frac{(-1)^k}{6k+6n+5} \quad , n \in \mathbb{N}_0$$

(258) $$\pi = 2^{4n}\binom{2n}{n}^{-1}\sum_{k=0}^{\infty}2^{-n}\sum_{m=0}^{k}\binom{k}{m}\binom{n-1/2}{k}\frac{(-1)^m}{2m+2n+1} \quad ,n\in\mathbb{N}_0$$

(259) $$\pi = 2^{4n+2}\left(\frac{n!}{(2n+1)!}\right)^2\prod_{k=1}^{\infty}\frac{4(k+1)^{2n+1}}{k^{2n-1}(2k+2n+1)^2} \quad ,n\in\mathbb{N}_0$$

(260) $$\frac{\pi}{\sqrt{3}} = \frac{3^{3n+2}\,n!}{2(3n+1)!\,(3n+2)}\prod_{k=1}^{\infty}\frac{9(k+1)^{2n+1}}{k^{2n-1}(3k+3n+1)(3k+3n+2)} \quad ,n\in\mathbb{N}_0$$

(261) $$\frac{\pi}{\sqrt{2}} = \frac{2^{6n+3}\,(2n)!}{(4n+1)!\,(4n+3)}\prod_{k=1}^{\infty}\frac{16(k+1)^{2n+1}}{k^{2n-1}(4k+4n+1)(4k+4n+3)} \quad ,n\in\mathbb{N}_0$$

(262) $$\pi = \frac{18\,(2n)!\,(3n)!\,432^n}{n!\,(6n+1)!\,(6n+5)}\prod_{k=1}^{\infty}\frac{36(k+1)^{2n+1}}{k^{2n-1}(6k+6n+1)(6k+6n+5)} \quad ,n\in\mathbb{N}_0$$

(263) $$I(k) = k!\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)_{k+2}} = k!\sum_{n=0}^{\infty}2^{-n-1}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{(2m+1)_{k+2}}$$
$$= k!\,(k+2)\sum_{n=0}^{\infty}\frac{8n+k+5}{(4n+1)_{k+4}}$$
$$= \frac{1}{(k+1)(k+2)}F\left(\frac{1}{2},1,1;\frac{k+3}{2},2+\frac{k}{2};-1\right) \quad ,k\in\mathbb{N}_0$$

(264) $$I(k+2) = \frac{1}{(k+2)(k+3)}+\frac{2(k+2)}{(k+3)}\,I(k+1)-\frac{2(k+1)}{(k+3)}\,I(k), k\in\mathbb{N}_0$$

(265) $$I(0) = \frac{\pi}{4}-\frac{\ln 2}{2}$$

(266) $$I(1) = \frac{1}{2}-\frac{\ln 2}{2}$$

(267) $$I(2) = \frac{5}{6}-\frac{\pi}{6}-\frac{\ln 2}{3}$$

(268) $$I(3) = \frac{5}{6}-\frac{\pi}{4}$$

(269) $$I(4) = \frac{23}{60}-\frac{\pi}{5}+\frac{2\ln 2}{5}$$

(270) $$I(5) = -\frac{79}{180}+\frac{2\ln 2}{3}$$

(271) $$I(6) = -\frac{134}{105}+\frac{2\pi}{7}+\frac{4\ln 2}{7}$$

(272) $$I(7) = -\frac{109}{70}+\frac{\pi}{2}$$

(273) $$\frac{\pi}{2e} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{(2k+1)(n-k)!}$$

(274) $$\frac{\pi}{2e}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{(2n+3)(2n-2k+1)!}+\sum_{n=0}^{\infty}(-1)^n\int_1^{\infty}\frac{\sin x}{x^{2n+1}}dx$$

(275) $$\frac{\pi}{2e}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{(2n+1)(2n-2k)!}+\sum_{n=0}^{\infty}\int_1^{\infty}\frac{\cos x}{x^{2n+2}}dx$$

(276) $$\pi=4\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^m}{2m+1}\left(\frac{\sqrt{\frac{n-m+1}{n-m+2}}-\sqrt{\frac{n-m}{n-m+1}}}{1+\sqrt{\frac{n-m}{n-m+2}}}\right)^{2m+1}$$

(277) $$\frac{4}{\pi}=\lim_{n\to\infty}\frac{1+\sum_{k=1}^{n}(2k-1)!!\,(2k)}{(2n+1)!!+\sum_{k=1}^{n}\left((-1)^k(2n+1)!!\right)/2k+1}$$

(278) $$\frac{\pi}{2}=\lim_{n\to\infty}\frac{4+\sum_{m=1}^{n}(2m+1)(2m+3)\prod_{k=1}^{m}(2k)^2}{3+\sum_{m=1}^{n}2(2(m+1)^2-1)\prod_{k=1}^{m}(4k^2-1)}$$

(279) $$\pi=\lim_{n\to\infty}\frac{2^{2(n+1)(3n+2)}}{\sum_{k=0}^{n}\binom{2k}{k}^3(42k+5)2^{2(n+1)(3n+2)-12k}}$$

(280) $$\pi=\ln\left(\frac{\cosh\left((m+1)\pi\right)}{\cosh(m\pi)}\right)-\ln\left(\frac{1+e^{-2(m+1)\pi}}{1+e^{-2m\pi}}\right)\quad ,m>0$$

(281) $$\pi=\sum_{n=1}^{\infty}\ln\left(1+\frac{8m+4}{(2n-1)^2+4m^2}\right)-\ln\left(\frac{1+e^{-2(m+1)\pi}}{1+e^{-2m\pi}}\right)\quad ,m>0$$

(282) $$\pi=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{n-k+1}\left(\frac{8m+4}{(2k+1)^2+4m^2}\right)^{n-k+1}-\ln\left(\frac{1+e^{-2(m+1)\pi}}{1+e^{-2m\pi}}\right)\quad ,m>3$$

(283) $$\pi=\ln\left(\frac{\sinh\left((m+1)\pi\right)}{\sinh(m\pi)}\right)-\ln\left(\frac{1-e^{-2(m+1)\pi}}{1-e^{-2m\pi}}\right)\quad ,m>0$$

(284) $$\pi=\ln\left(1+\frac{1}{m}\right)+\sum_{n=1}^{\infty}\ln\left(1+\frac{2m+1}{n^2+m^2}\right)-\ln\left(\frac{1-e^{-2(m+1)\pi}}{1-e^{-2m\pi}}\right)\quad ,m>0$$

(285) $$\pi=\ln\left(1+\frac{1}{m}\right)+\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{n-k+1}\left(\frac{2m+1}{(k+1)^2+m^2}\right)^{n-k+1}$$
$$-\ln\left(\frac{1-e^{-2(m+1)\pi}}{1-e^{-2m\pi}}\right)\quad ,m>2$$

(286) $$\frac{2\sqrt{3}}{27}\pi+\frac{4}{3}=\sum_{k=0}^{n}\binom{2k}{k}^{-1}+\binom{2n+2}{n+1}^{-1}F\left(1,n+2;n+\frac{3}{2};\frac{1}{4}\right)\quad ,n\in\mathbb{N}_0$$

(287) $$\frac{2\sqrt{3}}{27}\pi+\frac{2}{3}$$
$$=\sum_{k=1}^{n} k\binom{2k}{k}^{-1}$$
$$+(n+1)\binom{2n+2}{n+1}^{-1} F\left(1,n+2,n+2;n+1,n+\frac{3}{2};\frac{1}{4}\right)\ \ ,n\in\mathbb{N}_0$$

(288) $$\frac{\sqrt{3}}{9}\pi=\sum_{k=1}^{n}\frac{1}{k}\binom{2k}{k}^{-1}+(n+1)^{-1}\binom{2n+2}{n+1}^{-1} F\left(1,n+1;n+\frac{3}{2};\frac{1}{4}\right)\ \ ,n\in\mathbb{N}_0$$

(289) $$\frac{10\sqrt{3}}{81}\pi+\frac{4}{3}$$
$$=\sum_{k=1}^{n} k^2\binom{2k}{k}^{-1}$$
$$+(n+1)^2\binom{2n+2}{n+1}^{-1} F\left(1,n+2,n+2,n+2;n+1,n+1,n+\frac{3}{2};\frac{1}{4}\right)$$
$$n\in\mathbb{N}_0$$

(290) $$\frac{\pi^2}{18}=\sum_{k=1}^{n} k^{-2}\binom{2k}{k}^{-1}$$
$$+(n+1)^{-2}\binom{2n+2}{n+1}^{-1} F\left(1,n+1,n+1;n+2,n+\frac{3}{2};\frac{1}{4}\right)\ \ ,n\in\mathbb{N}_0$$

(291) $$\pi=4\tan^{-1}\left(\frac{\tanh 1}{\tan 1}\right)+4\sum_{n=1}^{\infty}\tan^{-1}(\tan(2^{-n})\tanh(2^{-n}))$$

(292) $$\pi=6\tan^{-1}\left(\frac{\tanh(1/\sqrt{3})}{\tan 1}\right)+6\sum_{n=1}^{\infty}\tan^{-1}\left(\tan(2^{-n})\tanh\left(\frac{2^{-n}}{\sqrt{3}}\right)\right)$$

(293) $$\pi=8\tan^{-1}\left(\frac{\tanh(\sqrt{2}-1)}{\tan 1}\right)+8\sum_{n=1}^{\infty}\tan^{-1}\left(\tan(2^{-n})\tanh\left(\frac{\sqrt{2}-1}{2^n}\right)\right)$$

(294) $$\pi=3\tan^{-1}\left(\frac{\tanh\sqrt{3}}{\tan 1}\right)+3\sum_{n=1}^{\infty}\tan^{-1}\left(\tan(2^{-n})\tanh\left(\frac{\sqrt{3}}{2^n}\right)\right)$$

(295) $$\pi=6\tan^{-1}\left(\ln\sqrt{\frac{\sqrt{3}+\tan 1}{\sqrt{3}-\tan 1}}\right)-6\sum_{n=1}^{\infty}\tan^{-1}(r(n)\tan(2^{-n}))$$

donde

(296) $$r(n)=\frac{\left(\sqrt{3}+\tan 1\right)^{2^{-n}}-\left(\sqrt{3}-\tan 1\right)^{2^{-n}}}{\left(\sqrt{3}+\tan 1\right)^{2^{-n}}+\left(\sqrt{3}-\tan 1\right)^{2^{-n}}}$$

(297) $$\pi=2+2\ \tanh^{-1}\left(\frac{a(e^2+1)-(e^2-1)}{(e^2+1)-a(e^2-1)}\right)$$

donde

(298) $$a=\sqrt{1-\prod_{n=0}^{\infty}\frac{(2n+1)^4}{((2n+1)^2+1)^2}}$$

(299) $$\pi=\frac{4}{m(m+3)}\sum_{k=1}^{m}(k+1)\tan^{-1}(k+1)+\frac{4}{m(m+3)}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(k+1)^{-2n}$$
$$m\in\mathbb{N}$$

(300) $$\pi=\frac{2}{m(m+2)}\sum_{k=1}^{m}(2k+1)\tan^{-1}(2k+1)$$
$$+\frac{2}{m(m+2)}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(2k+1)^{-2n}\qquad,m\in\mathbb{N}$$

(301) $$\pi=\frac{4}{m(m+3)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\sum_{k=1}^{m}(k+1)\left(\frac{(k+1)^2}{(k+1)^2+1}\right)^{n+\frac{1}{2}}$$
$$+\frac{4}{m(m+3)}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(k+1)^{-2n}\qquad,m\in\mathbb{N}$$

(302) $$\pi=\frac{2}{m(m+2)}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\sum_{k=1}^{m}(2k+1)\left(\frac{(2k+1)^2}{(2k+1)^2+1}\right)^{n+\frac{1}{2}}$$
$$+\frac{2}{m(m+2)}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(2k+1)^{-2n}\qquad,m\in\mathbb{N}$$

(303) $$\pi=\frac{4}{m(m+3)}\sum_{k=1}^{m}\frac{(k+1)^2}{\sqrt{(k+1)^2+1}}\ F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{(k+1)^2}{(k+1)^2+1}\right)$$
$$+\ \frac{4}{m(m+3)}\ \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(k+1)^{-2n}\quad,m\in\mathbb{N}$$

(304) $$\pi=\frac{2}{m(m+2)}\sum_{k=1}^{m}\frac{(2k+1)^2}{\sqrt{(2k+1)^2+1}}\ F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};\frac{(2k+1)^2}{(2k+1)^2+1}\right)$$
$$+\ \frac{2}{m(m+2)}\ \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{m}(2k+1)^{-2n}\quad,m\in\mathbb{N}$$

(305) $\pi = 3 + \sin(3) + \sin(3 + \sin(3)) + \sin(3 + \sin(3) + \sin(3 + \sin(3))) + \cdots$

(306) $\pi = 3 - \tan(3) - \tan(3 - \tan(3)) - \tan(3 - \tan(3) - \tan(3 - \tan(3))) - \cdots$

(307) $\pi$

$$= 4\left(\sqrt{2}\cos(1)\right)\left(\sqrt{2}\cos\left(\sqrt{2}\cos(1)\right)\right)\left(\sqrt{2}\cos\left(\left(\sqrt{2}\cos(1)\right)\left(\sqrt{2}\cos\left(\sqrt{2}\cos(1)\right)\right)\right)\right)\ldots$$

(308) $\frac{\pi}{2} = \frac{3}{2} + \cot\left(\frac{3}{2}\right) + \cot\left(\frac{3}{2} + \cot\left(\frac{3}{2}\right)\right) + \cot\left(\frac{3}{2} + \cot\left(\frac{3}{2}\right) + \cot\left(\frac{3}{2} + \cot\left(\frac{3}{2}\right)\right)\right) + \cdots$

(309) $\frac{\pi}{2} = \frac{3}{2} + \cos\left(\frac{3}{2}\right) + \cos\left(\frac{3}{2} + \cos\left(\frac{3}{2}\right)\right) + \cos\left(\frac{3}{2} + \cos\left(\frac{3}{2}\right) + \cos\left(\frac{3}{2} + \cos\left(\frac{3}{2}\right)\right)\right) + \cdots$

(310)
$$\frac{\pi^{2k} B_k}{(2k)!}\left(2^{2k-2} - \frac{1}{2}\right)\left(1 - 3^{1-2k}\right) = \sum_{n=0}^{\infty}\left(\frac{1/2}{(6n+1)^{2k}} - \frac{1/2}{(6n+2)^{2k}} - \frac{1}{(6n+3)^{2k}} - \frac{1/2}{(6n+4)^{2k}} + \frac{1/2}{(6n+5)^{2k}} + \frac{1}{(6n+6)^{2k}}\right) \quad , k \in \mathbb{N}$$

donde $B_k$ son los números de Bernoulli.

(311)
$$\pi = 2\prod_{n=1}^{\infty}\left(\left(\frac{n}{n+1}\right)^2\left(\frac{2n+3}{2n-1}\right)\right)^n$$

(312)
$$\pi = 2 + 2\sum_{n=1}^{\infty} 2^{-2n}\sum_{k=1}^{2^{n-1}}\left(-\sqrt{2^{2n} - (2k-2)^2} + 2\sqrt{2^{2n} - (2k-1)^2} - \sqrt{2^{2n} - (2k)^2}\right)$$

(313)
$$\frac{\pi}{6}\left[\left(\frac{1}{\sqrt{3}}\right)^{m+1} - \frac{\left(2-\sqrt{3}\right)^{m+1}}{2}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1}{\sqrt{3}}\right)^{2n+m+2} - \left(2-\sqrt{3}\right)^{2n+m+2}\right) \quad , m > -1$$

(314)
$$\pi = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(4e^{-(2n+1)x} + \frac{2E_n\, x^{2n+1}}{(2n)!}\right) \quad , 0 < x < \frac{\pi}{2}$$

donde $E_n$ son los números de Euler.

(315)
$$\pi = 4\sqrt{\frac{3}{5}} + 2\sum_{n=1}^{\infty}\left(\sqrt{\frac{(2n+3)(2n+4)E_{n+1}}{E_{n+2}}} - \sqrt{\frac{(2n+1)(2n+2)E_n}{E_{n+1}}}\right)$$

$E_n$ , números de Euler

(316)
$$\pi^2 = 15\prod_{n=1}^{\infty}\left(\frac{(n+2)(2n+3)B_{n+1}^2}{(n+1)(2n+1)B_n B_{n+2}}\right)$$

$B_n$ , números de Bernoulli

(317) $$\frac{2^{2n}B_n\pi^{2n}}{(2n)!} = 1 + \sum_{k=1}^{\infty}\big(k(k+1)\big)^{-2n} - \sum_{m=1}^{2n-1}(-1)^m\binom{2n}{m}\sum_{k=1}^{\infty}\frac{1}{k^{2n-m}(k+1)^m}$$

$$n \in \mathbb{N}$$

(318) $$\pi^2 = 9 + 3\sum_{k=1}^{\infty}\big(k(k+1)\big)^{-2}$$

(319) $$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\left(\frac{(a-2)n+b-1}{(2n+1)(an+b)}\right)^2 + \sum_{n=0}^{\infty}\frac{2}{(2n+1)(an+b)} - \sum_{n=0}^{\infty}\frac{1}{(an+b)^2}$$

$$a > 0\,, b > 0$$

(320) $$\frac{\pi}{4} + \sum_{n=0}^{\infty}\frac{(-1)^n}{an+b} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{(2n+1)(an+b)}} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{(a-2)n+b-1}{(an+b)\sqrt{2n+1}+(2n+1)\sqrt{an+b}}\right)^2$$

$$a > 0\,, b > 0$$

(321) $$\pi = \frac{1}{(1/2)_{n+1}(1/2)_{m+1}}\prod_{k=1}^{\infty}\frac{4(k+1)^{n+m+1}}{(2k+2n+1)(2k+2m+1)k^{n+m-1}}$$

$$n\,, m \in \mathbb{N}_0$$

(322) $$\pi\, e^{(n+m+1)\gamma} = \frac{1}{(1/2)_{n+1}(1/2)_{m+1}}\prod_{k=1}^{\infty}\frac{4k^2e^{(n+m+1)/k}}{(2k+2n+1)(2k+2m+1)}$$

donde $n, m \in \mathbb{N}_0$

(323) $$\pi = 8\sum_{k=1}^{\infty}\frac{1}{\sqrt{4k^2-1}\left(\sqrt{2k-1}+\sqrt{2k+1}\right)}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\sqrt{2m-1}}$$

(324) $$\frac{(2m)!}{2^{2m-1}B_m\,\pi^{2m}} = 1 - \sum_{n=2}^{\infty}\frac{1}{n^{2m}S_nS_{n-1}} \quad , m \in \mathbb{N}$$

donde

(325) $$S_n = \sum_{k=1}^{n}k^{-2m} \quad , B_m : números\ de\ Bernoulli$$

(326) $$\frac{2(2m)!}{(2^{2m}-1)B_m\,\pi^{2m}} = 1 - \sum_{n=2}^{\infty}\frac{1}{(2n-1)^{2m}S_nS_{n-1}} \quad , m \in \mathbb{N}$$

donde

(327) $$S_n = \sum_{k=1}^{n} (2k-1)^{-2m} \quad , B_m : \textit{números de Bernoulli}$$

(328) $$\frac{8}{\pi} = 3 - \sum_{n=2}^{\infty} \frac{1}{(4n-3)(4n-1)S_n S_{n-1}} \quad , S_n = \sum_{k=1}^{n} \frac{1}{(4k-3)(4k-1)}$$

(329) $$\pi = \frac{9}{4} + \frac{3}{2} \sum_{n=3}^{\infty} \left( \frac{1}{n} - \sin^{-1}\left(\frac{1}{n}\right) \right) + \frac{3}{2} \sum_{n=1}^{\infty} \binom{2n}{n} \frac{\zeta(2n+1)}{2^{2n}(2n+1)}$$

donde $\zeta(x)$ es la función zeta de Riemann.

(330) $$\pi = \frac{2}{\sqrt{1+x^2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left( \frac{1+x^{2n+1}}{(1+x^2)^n} \right) \quad , x > 0$$

(331) $$\pi = \frac{2}{\sqrt{10}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(3^{2n+1}+1)}{(2n+1)40^n}$$

(332) $$\pi^2 = \frac{(n!)^2 (m!)^2}{(1/2)_{n+m+1}^2} \sum_{k=0}^{\infty} \frac{(2n+1)_k (2m+1)_k (n+m+1)_k}{\left(n+m+\frac{3}{2}\right)_k (2n+2m+2)_k \, k!} \quad , n, m \in \mathbb{N}_0$$

(333) $$a^2 \pi^2 = \prod_{n=1}^{\infty} \frac{n^4((2n-1)^2+4a^2)^2}{(n^2+a^2)^2(2n-1)^4} - \prod_{n=1}^{\infty} \frac{n^4}{(n^2+a^2)^2} \quad , 0 < a \le 1$$

(334) $$\frac{\pi}{4} \ln\left( \frac{\tan(b/2)}{\tan(a/2)} \right) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \left( b - a + 2 \sum_{k=1}^{n} \frac{\cos\big(k(a+b)\big) \sin\big(k(b-a)\big)}{k} \right)$$

$$0 < a < b < \pi$$

(335) $$\left( \frac{\tan 1}{\tan(1/2)} \right)^{\pi} = e \prod_{n=1}^{\infty} (e^{1+2S_n})^{1/(2n+1)} \quad , S_n = \sum_{k=1}^{n} \frac{\sin(k)\cos(3k)}{k}$$

(336) $$\frac{\pi}{8} \ln\left( \frac{1+\sin b}{1+\sin a} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^{n} \frac{(-1)^{k-1} \sin\big(k(a+b)\big) \sin\big(k(a-b)\big)}{k}$$

$$0 < a < b < \pi/2$$

(337) $$\frac{4}{\pi^2} = 1 - 2^{-1} - \frac{2^{-2}}{2+\sqrt{2}} - \frac{2^{-4}}{2+\sqrt{2+\sqrt{2}}} - \frac{2^{-6}}{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} - \cdots$$

(338) $$\pi = \frac{\sin(a\pi)}{a} \prod_{n=1}^{\infty} exp\left( \sum_{k=1}^{n} \frac{1}{k} \left( \frac{a}{n-k+1} \right)^{2k} \right) \quad , 0 < a < 1$$

(339) $$\pi = 2 \prod_{n=1}^{\infty} exp\left( \sum_{k=1}^{n} \frac{1}{k(2n-2k+2)^{2k}} \right)$$

(340) $$\frac{1}{\pi} = \frac{a}{2\tan(a\pi/2)} \prod_{n=1}^{\infty} \left( exp\left( \sum_{k=1}^{n} \frac{(-1)^k}{k} \left( \frac{a}{n-k+1} \right)^{2k} \right) \right)^{(-1)^n} \quad , 0 < a < 1$$

(341) $$\pi = 4a + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1-\tan a}{1+\tan a}\right)^{2n+1} \quad ,0 \le a \le \pi/2$$

(342) $$\pi = \frac{8}{a}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{a}{16n^2-16n+3+a}\right)^m = \frac{8}{a}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{a}{16n^2-16n+3+a}\right)^m$$

$$a > 0$$

(343) $$\pi = \frac{512}{64-a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{64-a^2}{(32n-16-a)(32n-16+a)}\right)^m$$

$$= \frac{512}{64-a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{64-a^2}{(32n-16-a)(32n-16+a)}\right)^m \quad ,0 \le a < 8$$

(344) $$\pi = 8\sum_{n=1}^{\infty}\left(\frac{2^{2n}}{2^{4n}+1} + \sum_{k=1}^{2^{n-1}-1}\left(\frac{2^{2n}(2k+1)}{2^{4n}+(2k+1)^4} - \frac{2^{2n}(2k)}{2^{4n}+(2k)^4}\right)\right)$$

(345) $$\pi = 4 - 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\ln\left(1+\frac{1}{2n}\right)\right)^k$$

(346) $$\pi = 4 - 2\ln 2 + 2\ln\prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^{(-1)^{n-1}/n}$$

$$-2\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\ln\left(1+\frac{1}{2n}\right)\right)^k$$

(347) $$\pi^2 = 6 + 6\sum_{n=2}^{\infty}(\ln n!)\left(\frac{1}{n^2\ln n} - \frac{1}{(n+1)^2\ln(n+1)}\right)$$

(348) $$\frac{\pi}{6} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\csc\left(\frac{1}{2}\right)\right)\left(\frac{1}{2}\csc\left(\frac{1}{4}\csc\left(\frac{1}{2}\right)\right)\right)\left(\frac{1}{2}\csc\left(\frac{1}{8}\csc\left(\frac{1}{2}\right)\csc\left(\frac{1}{4}\csc\left(\frac{1}{2}\right)\right)\right)\right)\ldots$$

(349) $$\pi = 1 + 2\sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^2$$

(350) $$\pi = \frac{4}{3} + \frac{8}{3}\sum_{n=1}^{\infty}\left(\frac{\sin n}{n}\right)^3$$

(351) $$\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{\left((2n)!\right)^2(2n+1)} + \pi\sum_{n=1}^{\infty}\frac{1}{2^{2n}n\left((n-1)!\right)^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}e^{\sqrt{1-(k/n)^2}}$$

(352) $$\pi = m\sqrt{1-\sqrt{1-\sum_{n=1}^{\infty}\frac{12(mn)^2-90}{(mn)^4}}} \quad , \qquad m > \pi$$

(353) $$\frac{\pi}{512}\frac{\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2}}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)}{(32n-21)(32n-11)}$$

$$= \sum_{n=0}^{\infty} 2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(2k+1)}{(32k+11)(32k+21)}$$

(354) $$\frac{1}{\pi^{2n}} = \frac{2^{2n-1}B_n}{(2n)!}\left(1-\frac{1}{4^n}-\left(\frac{1}{9^n}-\frac{1}{16^n}\right)-\left(\frac{1}{16^n}-\frac{2}{36^n}+\frac{1}{64^n}\right)-\cdots\right)$$

$$= \frac{2^{2n-1}B_n}{(2n)!}\sum_{k=0}^{\infty} c_k(n) \quad , \qquad n \in \mathbb{N}$$

donde

(355) $$c_0(n) = 1 \ , \qquad c_m(n) = -\sum_{k=1}^{m}\frac{c_{m-k}(n)}{(k+1)^{2n}} \quad , m \in \mathbb{N}$$

En (354) $B_n$ son los números de Bernoulli.

(356) $$\frac{3\pi}{104}+\frac{3\ln 2}{26}-\frac{\ln 5}{13} = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{3n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{3^k 10^{n-k}}{3n-k+2}$$

(357) $$\pi = 8\sum_{n=0}^{\infty}\frac{s(n)}{(2n+1)5^{2n+1}}$$

donde

(358) $$s(n+2) = -6s(n+1) - 25s(n), s(0) = 2, s(1) = -2$$

(359) $$\frac{1}{\pi^m} = \frac{2^m m!}{(m+1)^{m+1}}\prod_{k=1}^{\infty}\left(\prod_{n=1}^{m}\left(1-\left(\frac{n}{k(m+1)}\right)^2\right)\right) \quad , m \in \mathbb{N}$$

(360) $$\pi = 2(-1)^m\left(\ln 2+\sum_{k=1}^{m}\frac{(-1)^k}{k}\right)+2(2m+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)(n+m+1)}$$

$m \in \mathbb{N}_0$

(361) $$\frac{1}{\pi} = \frac{1}{12}\prod_{n=1}^{\infty}\left(1+\frac{(3^{n+1}-1)\zeta(n+2)}{\sum_{k=1}^{n}4^{n-k+1}(3^k-1)\zeta(k+1)}\right)$$

$\zeta(x)$ función zeta de Riemann

(362) $$\pi^2 = 9\prod_{n=1}^{\infty}\left(1+\frac{(n!)^2}{(2n+1)(2n+2)A(n)}\right)$$

donde

(363) $$A(n) = \left((n-1)!\right)^2+\sum_{k=1}^{n-1}\left((k-1)!\right)^2(2k+1)_{2n-2k}$$

(364) $$A(n+1) = (2n+1)(2n+2)A(n)+(n!)^2 \quad , A(1) = 1$$

(365) $$\pi^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(6/7)^{k+1}}{(n-k+1)^2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(5/6)^{k}}{(n-k+1)^2}$$

(366) $$\frac{1}{\pi} = \frac{(1/2)_n^2}{(n-1)!}\sum_{k=0}^{\infty}\frac{(1/2)_{3k}^2\, T(n,k)}{(n+3k+2)!\,(3k+2)!} \quad , n \in \mathbb{N}$$

donde

(367) $$T(n,k) = (n+3k+1)(n+3k+2)(3k+1)(3k+2) + (n+3k+2)(3k+2)\left(3k+\frac{1}{2}\right)^2 + \left(3k+\frac{3}{2}\right)^2\left(3k+\frac{1}{2}\right)^2$$

(368) $$\pi = a(n)\sum_{k=2}^{\infty}\zeta(k)\left(\left(\frac{2^n+1}{2^{n+1}}\right)^{k-1} - \left(\frac{2^n-1}{2^{n+1}}\right)^{k-1}\right) \quad , n \in \mathbb{N}$$

donde

(369) $$a(1) = 1\,, a(n+1) = a(n) + \sqrt{1+(a(n))^2}$$

$\zeta(x)$ : function zeta de Riemann

(370) $$\pi = \frac{2^{4m+2} m (m!)^4}{\left((2m+1)!\right)^2}\prod_{n=1}^{\infty}\frac{2^{4m}(n+1)(mn+1)_m^4}{n(2mn+2)_{2m}^2} \quad , m \in \mathbb{N}$$

(371) $$\pi e = 4\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{1}{k!}$$

(372) $$\pi\zeta(3) = 4\sum_{n=1}^{\infty}\frac{1}{(n+1)^3}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1} + 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{1}{(k+1)^3}$$

(373) $$\pi^2 = 6\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{(r(n))^k}{n}$$

donde

(374) $$r(1) = 1\,, 0 < r(n) < 1\,, n \geq 2$$

(375) $$n(r(n))^{n+1} - (n+1)r(n) + 1 = 0 \quad , n \in \mathbb{N}$$

(376) $$\frac{1}{n+1} < r(n) < \frac{1}{n+1} + \frac{n}{n+1}(r(n-1))^{n+1} \quad , n \geq 2$$

(377) $$r(n) = \frac{1}{n+1} + \frac{n}{n+1}\left(\frac{1}{n+1} + \frac{n}{n+1}\left(\frac{1}{n+1} + \cdots\right)^{n+1}\right)^{n+1} \quad , n \in \mathbb{N}$$

(378) $$\pi = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{6n-2}(2^{4n-2}-1)B_{2n-1}u^{4n-2}}{(2n-1)(4n-2)!}$$

donde

(379) $$0 < u < 1\,, (\tanh u)(\tan u) = 1\,, u = 0.9375520343\ldots$$

En (378) $B_n$ son los números de Bernoulli.

(380) $$\pi=\frac{4}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{a_n\, v^{-n}}{2n+1}\quad ,\qquad v=\left(1+\left(1+(1+\cdots)^{2/7}\right)^{2/7}\right)^{2/7}=1.262\ldots$$

(381) $$\pi=\frac{6}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{a_n\, v^{-n}}{2n+1}\quad ,$$

$$v=\left(\sqrt{3}+\sqrt{3}\left(\sqrt{3}+\sqrt{3}(\sqrt{3}+\cdots)^{2/7}\right)^{2/7}\right)^{2/7}=1.524\ldots$$

(382) $$\pi=\frac{8}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{a_n\, v^{-n}}{2n+1}\quad ,$$

$$v=\left((1+\sqrt{2})+(1+\sqrt{2})\left((1+\sqrt{2})+\cdots\right)^{2/7}\right)^{2/7}=1.710\ldots$$

(383) $$\pi=\frac{12}{\sqrt{v}}\sum_{n=0}^{\infty}\frac{a_n\, v^{-n}}{2n+1}\quad ,$$

$$v=\left((2+\sqrt{3})+(2+\sqrt{3})\left((2+\sqrt{3})+\cdots\right)^{2/7}\right)^{2/7}=1.992\ldots$$

(384) $$\pi=4\sum_{n=0}^{\infty}\frac{a_n\left(x^{-n-1/2}+y^{-n-1/2}\right)}{2n+1}$$

donde

(385) $$x=\left(2+2(2+\cdots)^{2/7}\right)^{2/7}\ ,y=\left(3+3(3+\cdots)^{2/7}\right)^{2/7}$$

En las formulas (380)-(383), $a_n$ ,se define por:

(386) $$a_{n+7}=-(a_{n+2}+2a_{n+1}+a_n)\ ,a_0=a_1=0,a_2=5,a_3=7,a_4=a_5=a_6=0$$

(387) $$\frac{4}{\pi}=\frac{3}{2}\cdot\frac{5}{5+\frac{3}{2}}\cdot\frac{7}{7-\frac{3}{2}\frac{5}{5+\frac{3}{2}}}\cdot\frac{9}{9+\frac{3}{2}\frac{5}{5+\frac{3}{2}}\frac{7}{7-\frac{3}{2}\frac{5}{5+\frac{3}{2}}}}\ldots$$

(388) $$\frac{4}{\pi}=\frac{3}{3-u_1}\cdot\frac{5}{5+u_2}\cdot\frac{7}{7-u_3}\cdot\frac{9}{9+u_4}\ldots$$

donde

(389) $$u_{n+1}=\frac{3}{2}\cdot\frac{5}{5+u_2}\cdot\frac{7}{7-u_3}\cdot\ldots\frac{(2n+1)}{(2n+1)+(-1)^n u_n}\quad ,n\geq 1$$

(390) $$\frac{\pi(e-e^{-1})}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k+1)!}$$

(391) $$\frac{\pi(e+e^{-1})}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n)!}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{2k-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n}\frac{1}{(2k)!}$$

(392) $$\pi \sin^{-1} x = 4 \sum_{n=0}^{\infty} \frac{T_{2n+1}(x)}{(2n+1)^2}$$

(393) $$\pi^2 = 24 \sum_{n=0}^{\infty} \frac{T_{2n+1}(1/2)}{(2n+1)^2} = 16 \sum_{n=0}^{\infty} \frac{T_{2n+1}\left(1/\sqrt{2}\right)}{(2n+1)^2} = 12 \sum_{n=0}^{\infty} \frac{T_{2n+1}\left(\sqrt{3}/2\right)}{(2n+1)^2}$$

(394) $$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad , T_0(x) = 1, T_1(x) = x$$

(395) $$T_n(x) = F\left(n, -n; \frac{1}{2}; \frac{1-x}{2}\right)$$

(396) $$T_n(x) = x^n - \binom{n}{2} x^{n-2}(1-x^2) + \binom{n}{4} x^{n-4}(1-x^2)^2 - \cdots$$

$T_n(x)$ polinomios de Chebyshev , $F(a,b;c;x)$ función hipergeometrica de Gauss.

(397) $$\sum_{k=1}^{n} \frac{2^{2k-1} B_k \pi^{2k}}{(2k)!} = n + \frac{3}{4} - \sum_{k=2}^{\infty} \frac{1}{(k^2-1)k^{2n}} \quad , n \in \mathbb{N}$$

$B_k$ números de Bernoulli

(398) $$\pi = \frac{2^{4n+1}}{\binom{2n}{n}(2n+1)} \sum_{k=0}^{\infty} \frac{(2n-2k+1)\left(-\left(n+\frac{1}{2}\right)\right)_k}{k!\,(2n+2k+1)} \quad , n \in \mathbb{N}_0$$

(399) $$\pi - 2 \ln \prod_{k=1}^{\infty} \left(1 + \frac{2}{k(k+1)}\right) = -8 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(k^2+k+1)^{-(4n-1)}}{4n-1}$$

$$= -8 \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{\left((n-k)^2 + 3(n-k+1)\right)^{-(4k-1)}}{4k-1}$$

$$= -8 \sum_{n=1}^{\infty} \sum_{k=1}^{n} \frac{(k^2+k+1)^{-(4n-4k+3)}}{4n-4k+3}$$

(400) $$\ln\left(\frac{\pi}{2}\right) = 2 \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{8k^2-1}\right) + 4 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(8k^2-1)^{-(4n-1)}}{4n-1}$$

donde

(401) $$\pi^2 = 8 \sum_{n=0}^{\infty} \frac{e^{-(2n+1)}}{(2n+1)^2} + 4 \ln\left(\lim_{m\to\infty} \prod_{k=1}^{m} \left(\frac{1+e^{-k/m}}{1-e^{-k/m}}\right)^{1/m}\right)$$

(402) $$\frac{2^8}{\pi^2} = \sum_{n=0}^{\infty} 2^{-12n} \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k}^3 (42k+5)(42n-42k+5)$$

(403) $$\frac{1}{\pi} = \frac{5}{16} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \binom{2^k(2n-1)}{2^{k-1}(2n-1)}^3 \left(\frac{21 \cdot 2^k(2n-1)+5}{2^{2^{k+1}(6n-3)+4}}\right)$$

(404) $$\frac{4\pi^2}{27} = \sum_{n=0}^{\infty} \frac{(n!)^4}{\left((2n+1)!\right)^2} + 2 \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n+1)!} \sum_{k=1}^{n} \frac{\left((k-1)!\right)^2}{(2k-1)!}$$

(405) $$\frac{2}{5\pi^2}=\frac{1}{20}\sum_{n=0}^{\infty}2^{-12n}(-1)^n\binom{2n}{n}^5(8n+1)+\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}2^{-12k}\,(-1)^k\binom{2k}{k}^5(2n-1)$$

(406) $$\pi=\sum_{n=0}^{\infty}\binom{2n}{n}\frac{2^{-2n}}{2n+1}\left(\frac{1}{c}\right)^{2n+1}\left((as)^{2n+1}+(bs)^{2n+1}+(cs)^{2n+1}\right)$$

donde

(407) $$0<s<1\,,a>0,b>0,c>0,$$

(408) $$a\sqrt{1-s^2}<b<\frac{a}{\sqrt{1-s^2}}\ \ ,c=\sqrt{a^2+b^2-2ab\sqrt{1-s^2}}$$

(409) $$\pi=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{3^{n-k+1}-1}{4^{n-k+1}\,k^{n-k+2}}$$

(410) $$\pi=\frac{8}{3}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{(1/4)^k(3/4)^m}{n^{k+m}}$$

(411) $$\pi=3+\sum_{n=0}^{\infty}2^{-n-1}\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{(k+1)(k+2)(2k+3)}$$

(412) $$\frac{9801\sqrt{2}}{40\pi}=\sum_{n=0}^{\infty}\frac{A+Bn}{99^{4n+4}}\sum_{k=0}^{n}\frac{(1/4)_k(1/2)_k(3/4)_k}{(k!)^3}$$

donde

(413) $$A=10595371241\ \ ,B=253501284400$$

(414) $$\frac{5^4\sqrt{15}}{66\pi}=\sum_{n=0}^{\infty}\left(\frac{4}{125}\right)^n(121n+7)\sum_{k=0}^{n}\frac{(1/6)_k(1/2)_k(5/6)_k}{(k!)^3}$$

(415) $$\frac{3^6}{20\pi}=\sum_{n=0}^{\infty}\left(\frac{2}{27}\right)^n(75n+4)\sum_{k=0}^{n}\frac{(1/3)_k(1/2)_k(2/3)_k}{(k!)^3}$$

(416) $$\frac{1}{\pi}=\frac{(1/2)_n^2}{(n-1)!}\sum_{k=0}^{\infty}\frac{(1/2)_k^2}{(n+k)!\,k!}\quad ,n\in\mathbb{N}$$

(417) $$\pi=\sum_{m=0}^{\infty}\binom{2m}{m}2^{-2m}\left(\frac{2}{2m+1}+\sum_{k=0}^{n}\binom{2n}{2k}\frac{(-1)^k k!\,(1/2)_{m+n-k}}{(1/2)_{m+n+1}}\right)\quad ,n\in\mathbb{N}$$

(418) $$\pi\ =2\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{\frac{2n+5}{2n+1}}-1\right)+\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{\frac{2n+5}{2n+1}}-1\right)^2$$

(419) $$\pi=4\sum_{n=0}^{\infty}\left(\frac{1}{(2n+1)^2}+\sum_{k=2n(n+1)+1}^{2(n+1)(n+2)-1}\frac{(-1)^k}{2k+1}\right)$$

(420) $$\pi = 1 + \sqrt{1 - 8\sum_{n=0}^{\infty}\sum_{k=2n(n+1)+1}^{2(n+1)(n+2)-1}\frac{(-1)^k}{2k+1}}$$

(421) $$e^{\pi/4\sqrt{2}} = \sum_{k=0}^{\infty}\frac{a(k)}{2^k}$$

donde

(422) $$a(0) = 1 \ , a(k) = \frac{1}{k}\sum_{n=1}^{k}\frac{(2n-2)!\, n\, a(k-n)}{2^{2n-2}\big((n-1)!\big)^2(2n-1)} \ , k \geq 1$$

(423) $$\pi = \ln\big(3 - 2\sqrt{2}\big) + \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{n!}{2^n(1/4)_{n+1}}$$

(424) $$\pi\sqrt{3} = \frac{3^{3x+5}}{2(x+1)(3x+2)(3x+4)(3x+5)}\prod_{k=1}^{\infty}\frac{9(k+1)^3(k+3x+1)}{k(k+x+1)(3k+3x+4)(3k+3x+5)}$$

$$x > 0$$

(425) $$\frac{2^{13}}{\pi} = \sum_{n=1}^{\infty}\binom{2n}{n}^3\frac{(6n+7)(21n+5)(84n^2+156n+73)}{2^{12n}(n+1)^2}$$

(426) $$\frac{2}{\pi} = \sum_{n=0}^{\infty}\frac{(1/2)_n(-1/2)_n}{(n!)^2}$$

(427) $$\pi = 4\sum_{k=1}^{\infty}\frac{\sin(k/n)}{k(\cos(1/n)+\sin(1/n))^k} \ , \qquad n \in \mathbb{N}$$

(428) $$\pi = 3 + \frac{24}{5^2\cdot 7} + \frac{18}{5}\sum_{n=1}^{\infty}\frac{(-1)^n(36n^2-24n-13)}{(6n-1)(6n+1)(6n+5)(6n+7)}$$

(429) $$\pi = 2 + 4\sum_{n=1}^{\infty}\left(\frac{1}{2n+2} - \sum_{k=1}^{n}\frac{n(n+1)-k^2}{(n^2+k^2)((n+1)^2+k^2)}\right)$$

(430) $$\pi = 4\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{\sqrt{(n+1)^2-k^2}}{(n+1)^2} - \frac{\sqrt{n^2-k^2}}{n^2}\right)$$

(431) $$e^{\pi^2/12} = \prod_{n=1}^{\infty}\prod_{k=1}^{n}\left((n+1)^{1/(n+k+1)}n^{-1/(n+k)}k^{1/(n+k)(n+k+1)}\right)$$

$$= \prod_{n=1}^{\infty}\left((n+1)^{H_{2n+1}-H_{n+1}}n^{H_n-H_{2n}}\prod_{k=1}^{n}k^{1/(n+k)(n+k+1)}\right)$$

donde $H_n = \sum_{k=1}^{n} k^{-n}$

(432) $$\frac{9}{2\pi} = \sqrt{\frac{a}{a-1}} \cdot \sqrt{\frac{1}{2}+\frac{a}{4}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{a}{4}}} \cdot \ldots \quad , a = \sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\cdots}}}$$

(433) $$\frac{5\sqrt{3-\varphi}}{2\pi} = \sqrt{\frac{1}{2}+\frac{\varphi}{4}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\varphi}{4}}} \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{\varphi}{4}}}} \cdot \ldots$$

donde $\varphi = \frac{1+\sqrt{5}}{2}$

(434) $$\frac{\pi}{4} = \tan^{-1}(e^x) - \frac{x}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n E_n x^{2n+1}}{(2n+1)!} \quad , |x| < \pi/2$$

donde $E_n$ son los números de Euler.

(435) $$\pi = 3\ln 3 + 3\sum_{n=1}^{\infty} \frac{(-1)^n E_n (\ln 3)^{2n+1}}{(2n+1)!\, 2^{2n}}$$

$E_n$ , números de Euler

(436) $$\frac{\pi}{2 - 2\ln 2 + \ln 3} = 3\tan\left(\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty} 3^{-2n}\left(\frac{c_{4n}}{4n+1} - \frac{c_{4n+2}}{12n+9}\right)\right)$$

donde

(437) $$c_0 = 1\,, c_n = 1 - \sum_{k=1}^{n} \frac{c_{n-k}}{k} \quad , n \in \mathbb{N}$$

(438) $$\frac{1}{\pi} = \frac{1}{16}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 - \frac{9}{40} F\left(\begin{matrix} 1/2 & 1/2 & 1/4 \\ & 2 & 9/4 \end{matrix}\middle| 1\right)$$

(439) $$\frac{1}{\pi} = \frac{1}{4}\left(\frac{\Gamma(1/4)}{\Gamma(3/4)}\right)^2 - \frac{9}{5} F\left(\begin{matrix} 1/2 & 1/2 & 1/4 \\ & 1 & 9/4 \end{matrix}\middle| 1\right)$$

donde $F$ es la función hipergeométrica generalizada.

(440) $$\frac{\pi}{4}\tan^{-1}\left(\frac{1}{2m+1}\right) + \frac{1}{4}\ln\left(\frac{2}{(m+1)^2}\right)\ln\left(\frac{2(2m^2+2m+1)}{(m+1)^2}\right)$$
$$= \sum_{n=1}^{\infty} \frac{H_n - H_{2n} - 1/(2n)}{n(m+1)^{2n}} \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} m^{2n-2k} \quad , m \in \mathbb{N}$$

(441) $$\frac{\pi^2}{48} + \frac{1}{4}\ln\left(\frac{2\left(\sqrt{3}-1\right)^2}{3}\right)\ln\left(\frac{8}{3}\right)$$
$$= \sum_{n=1}^{\infty} \frac{H_n - H_{2n} - 1/(2n)}{n\, 3^n} \sum_{k=0}^{n} (-1)^k \binom{2n}{2k} \left(\sqrt{3}-1\right)^{2k}$$

(442) $$\frac{\pi^2}{24}+\frac{1}{4}\ln\left(8-4\sqrt{3}\right)\ln\left(16-8\sqrt{3}\right)$$
$$=\sum_{n=1}^{\infty}\frac{H_n-H_{2n}-1/(2n)}{n\left(2+\sqrt{3}\right)^{2n}}\sum_{k=0}^{n}(-1)^k\binom{2n}{2k}\left(\sqrt{3}+1\right)^{2k}$$

(443) $$\frac{\left(\Gamma(1/3)\right)^3}{2\pi\sqrt{3}\sqrt[3]{2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(3n+1)}$$

(444) $$\frac{\sqrt{3}\left(\Gamma(2/3)\right)^3}{\pi\sqrt[3]{4}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(3n+2)}$$

(445) $$\frac{\left(\Gamma(1/3)\right)^3}{2\pi\sqrt{3}\sqrt[3]{2}}-\frac{8\pi^2}{3\sqrt[3]{4}\left(\Gamma(1/3)\right)^3}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2(3n+1)(3n+2)}$$

(446) $$\frac{2\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}2^{-2n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\frac{1}{(3k+1)(3n-3k+2)}$$

(447) $$\pi=24\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}+\sqrt{6}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}+\sqrt{6}}\right)^n$$

(448) $$\pi=24\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}+\sqrt{6}}}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{a_n-b_n\sqrt{2}+c_n\sqrt{3}-d_n\sqrt{6}}{a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}}\right)\right)$$

donde

(449) $$\begin{cases} a_{n+1}=4a_n+2b_n+6d_n \\ b_{n+1}=a_n+4b_n+3c_n \\ c_{n+1}=2b_n+4c_n+2d_n \\ d_{n+1}=a_n+c_n+4d_n \\ a_1=4\,,b_1=1\,,c_1=0\,,d_1=1 \end{cases}$$

(450) $$\pi=6\sqrt{2-\sqrt{3}}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}\left(\frac{4-\sqrt{2}-\sqrt{6}}{2}\right)^n$$
$$=6\sqrt{2-\sqrt{3}}\left(1+\sum_{n=1}^{\infty}\frac{(n!)^2\left(a_n-b_n\sqrt{2}+c_n\sqrt{3}-d_n\sqrt{6}\right)}{2^n(2n+1)!}\right)$$

(451) $$\pi=\frac{6}{11}\sqrt{2-\sqrt{3}}\sum_{n=0}^{\infty}\frac{(n!)^2}{(2n+1)!}\left(\frac{4+\sqrt{2}+\sqrt{6}}{2}\right)^n$$
$$=\frac{6}{11}\sqrt{2-\sqrt{3}}\left(1+\sum_{n=1}^{\infty}\frac{(n!)^2\left(a_n+b_n\sqrt{2}+c_n\sqrt{3}+d_n\sqrt{6}\right)}{2^n(2n+1)!}\right)$$

En (450),(451), $a_n, b_n, c_n, d_n$ se definen por (449).

(452) $$\pi\sqrt{\frac{4-\sqrt{2}-\sqrt{6}}{4+\sqrt{2}+\sqrt{6}}}=\sum_{n=1}^{\infty}\frac{(2^{2n}-1)\zeta(2n)}{2^{6n-4}3^{2n-1}}$$

(453) $$\pi\sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{4-\sqrt{2}-\sqrt{6}}}=24-\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{6n-4}3^{2n-1}}$$

$\zeta(x)$ función zeta de Riemann

(454) $$\pi=2a\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n+1)a^{2n+2}}+\frac{2^{2n}(n!)^2a^{2n}}{(2n+1)!\,(1+a^2)^{n+1}}\right)\ ,a\geq 1$$

(455) $$\pi=A(p)+B(p)\ ,\qquad p\in\{2,4,6,8,\dots\}\ ,p\ número\ par$$

donde

(456) $$A(p)=\cfrac{4}{p+\cfrac{p}{3p^2+\cfrac{4}{5+\cfrac{9}{7p^2+\cfrac{16}{9+\cdots}}}}}$$

(457) $$B(p)=\cfrac{4(p-1)}{p+1+\cfrac{(p+1)(p-1)^2}{3(p+1)^2+\cfrac{4(p-1)^2}{5+\cfrac{9(p-1)^2}{7(p+1)^2+\cfrac{16(p-1)^2}{9+\cdots}}}}}$$

(458) $$\pi=\frac{9}{4}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{\left(\sqrt[3]{\sqrt{3}+i}+\sqrt[3]{\sqrt{3}-i}\right)^{2n+1}}{(2n+1)\left(\sqrt[3]{2}\right)^{14n+4}}=\frac{9\sqrt[6]{3}}{4\sqrt[3]{2}}\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{3}{256}\right)^{n/3}\frac{\rho^{2n+1}}{2n+1}$$

donde

(459) $$\sqrt[3]{\sqrt{3}+i}+\sqrt[3]{\sqrt{3}-i}=2\sqrt[6]{3}\sum_{n=0}^{\infty}(-1)^n\binom{1/3}{2n}3^{-n}=2\sqrt[6]{3}\,\rho$$

(460) $$\pi\sqrt{5}=2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(2^{2n+1}+1)}{20^n(2n+1)}$$

(461) $$\sqrt{\pi}=\sqrt{2\sqrt{3}}\left(1+\sum_{n=1}^{\infty}a_n\,3^{-n}\right)$$

(462) $$\sqrt{\pi}=2\sqrt{3}\sqrt{2-\sqrt{3}}\left(1+\sum_{n=1}^{\infty}a_n\left(7-4\sqrt{3}\right)^n\right)$$

(463) $$\sqrt{\pi}=2\sqrt{2\left(\sqrt{2}-1\right)}\left(1+\sum_{n=1}^{\infty}a_n\left(3-2\sqrt{2}\right)^n\right)$$

donde

(464) $$a_n=\frac{(-1)^n}{4n+2}-\frac{1}{2n}\sum_{m=1}^{n-1}\frac{(-1)^{n-m}(3m-n)a_m}{2n-2m+1}\quad ,n\in\mathbb{N}$$

(465) $$\pi = \frac{2}{5}\sum_{n=0}^{\infty} 10^{-n} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k 3^{2n-2k}}{2k+1}$$

(466) $$\pi = 8 \sum_{n=1}^{\infty} \frac{(16n^2+16n-3)}{(16n^2-9)(16n^2-1)} \sum_{k=1}^{n} \frac{1}{k}$$

(467) $$\frac{1}{\pi} = \frac{3}{10}\sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (1/4)_n (2n+1)}{(9/4)_n\ n!\ (n+1)!}$$

(468) $$\frac{1}{\pi} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (16n^2+36n+11)}{n!\ (n+1)!\,(4n+1)(4n+5)}$$

(469) $$\frac{1}{\pi} = \frac{3}{2}\sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (2n+1)}{n!\ (n+1)!\,(4n+1)(4n+5)}$$

(470) $$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (8n+1)}{(n!)^2(4n+1)(4n+5)}$$

(471) $$\frac{1}{\pi} = \frac{9}{4}\prod_{n=1}^{\infty} \frac{n(4n+3)^2}{(n+1)(4n+1)^2} - \frac{9}{5}\sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (1/4)_n}{(9/4)_n\ (n!)^2}$$

(472) $$\frac{1}{\pi} = \frac{9}{16}\prod_{n=1}^{\infty} \frac{n(4n+3)^2}{(n+1)(4n+1)^2} - \frac{9}{40}\sum_{n=0}^{\infty} \frac{(1/2)_n^2\ (1/4)_n}{(9/4)_n\ n!\,(n+1)!}$$

(473) $$\pi^2 = 6\sum_{n=1}^{\infty}\left(\ln\left(1+\frac{1}{n}\right)\right)^2 + 12\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{k+m}}{n^k\ (n+1)^m\ m\ (k+m)}$$

(474) $$\frac{\pi}{\left(\Gamma(3/4)\right)^4} = \prod_{n=1}^{\infty}\left(\frac{(4n-1)^2}{8n(2n-1)}\right)^2$$

(475) $$\frac{\pi}{\left(\Gamma(1/4)\right)^4} = \frac{1}{16}\prod_{n=1}^{\infty}\left(\frac{(4n-3)(4n+1)}{8n(2n-1)}\right)^2$$

$\Gamma(x)$ función gamma

(476) $$\pi = 4\tan^{-1}\left(\frac{2-\sqrt{\sqrt{5}-2}}{2+\sqrt{\sqrt{5}-2}}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{6n-5}(2n-1)}\binom{4n-2}{2n-1}$$

(477) $$\pi = 4\tan^{-1}\left(\frac{\sqrt{6}-\sqrt{\sqrt{10}-3}}{\sqrt{6}+\sqrt{\sqrt{10}-3}}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{4n-4}3^{2n-1}(2n-1)}\binom{4n-2}{2n-1}$$

(478) $$\pi = 4\tan^{-1}\left(\frac{2\sqrt{2\sqrt{17}-8}-\sqrt{17}+4}{2\sqrt{2\sqrt{17}-8}+\sqrt{17}-4}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{8n-6}(2n-1)}\binom{4n-2}{2n-1}$$

(479) $$\pi+4\ln\left(2-\sqrt{3}+2\sqrt{2-\sqrt{3}}\right)$$
$$=4\tan^{-1}\left(\frac{2-\sqrt{\sqrt{5}-2}}{2+\sqrt{\sqrt{5}-2}}\right)+\sum_{n=1}^{\infty}\frac{2^{-12n+12}}{(4n-3)}\binom{8n-6}{4n-3}$$

(480) $$\pi-4\ln\left(2-\sqrt{3}+2\sqrt{2-\sqrt{3}}\right)$$
$$=4\tan^{-1}\left(\frac{2-\sqrt{\sqrt{5}-2}}{2+\sqrt{\sqrt{5}-2}}\right)-\sum_{n=1}^{\infty}\frac{2^{-12n+6}}{(4n-1)}\binom{8n-2}{4n-1}$$

(481) $$\frac{96}{\pi^4}=\prod_{n=1}^{\infty}\frac{a_n}{\left((2n-1)!!\right)^4+a_n}$$

donde

(482) $$a_{n+1}=(2n+3)^4\left((2n-1)!!\right)^4+(2n+3)^4a_n\ \ ,a_1=3^4$$

(483) $$(2n-1)!!=1\cdot3\cdot5\cdot7\cdot\ldots(2n-1)\quad ,n\in\mathbb{N}$$

(484) $$\frac{96}{\pi^4}=\left(\frac{3^4}{(1)^4+3^4}\right)\left(\frac{2\cdot5^4\cdot41}{(1\cdot3)^4+2\cdot5^4\cdot41}\right)\left(\frac{7^5\cdot7333}{(1\cdot3\cdot5)^4+7^5\cdot7333}\right)\ldots$$

(485) $$\frac{\pi\ln\left(4-2\sqrt{2}\right)}{\left(\sqrt{2}-1\right)^3}=8\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{2}-1\right)^{2n}\sum_{k=0}^{n}\frac{1}{(2k+1)(n-k+1)}$$

(486) $$\pi=\frac{1}{2\ln(5/4)}\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\sum_{k=0}^{n}\frac{1}{(2k+1)(n-k+1)}$$
$$+\frac{4}{27\ln(10/9)}\sum_{n=0}^{\infty}\left(-\frac{1}{9}\right)^n\sum_{k=0}^{n}\frac{1}{(2k+1)(n-k+1)}$$

(487) $$\frac{\pi^2}{16}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\left(\frac{1/3}{9^n4^m}+\frac{1/14}{49^n4^m}+\frac{2/9}{9^{n+m}}+\frac{1/21}{49^n9^m}\right)$$

(488) $$\frac{\pi^2}{16}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\left(\frac{1}{4^{n+m+1}}+\frac{1}{9^{n+m+1}}+\frac{1/3}{9^n4^m}\right)$$

(489) $$\frac{\pi^2}{16}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+m}}{(2n+1)(2m+1)}\left(\frac{25}{49^{n+m+1}}+4\left(\frac{9}{6241}\right)^{n+m+1}\right.$$
$$\left.+\frac{60}{553}\left(\frac{9}{6241}\right)^n\left(\frac{1}{49}\right)^m\right)$$

(490) $$\pi=4\tan^{-1}(a(r,m))+4\sum_{n=0}^{\infty}\frac{(-1)^nr^{2n+1}\left(1-r^{(2n+1)m}\right)}{(2n+1)(1-r^{2n+1})}$$

donde

(491) $$a(r,m)\in\mathbb{Q},r\in\mathbb{Q}\,,0<r<1\,,m\in\mathbb{N}$$

(492) $$a\left(\frac{1}{2},1\right)=\frac{1}{3}\ ,a\left(\frac{1}{2},2\right)=\frac{1}{13}\ ,a\left(\frac{1}{2},3\right)=-\frac{1}{21}\ ,a\left(\frac{1}{2},4\right)=-\frac{37}{335}\ ,a\left(\frac{1}{2},5\right)=-\frac{1519}{10683}$$

(493) $$a\left(\frac{1}{3},1\right)=\frac{1}{2}\ ,a\left(\frac{1}{3},2\right)=\frac{7}{19}\ ,a\left(\frac{1}{3},3\right)=\frac{17}{52}\ ,a\left(\frac{1}{3},4\right)=\frac{1325}{4229}$$

(494) $$a\left(\frac{1}{4},1\right)=\frac{3}{5}\ ,a\left(\frac{1}{4},2\right)=\frac{43}{83}\ ,a\left(\frac{1}{4},3\right)=\frac{157}{315}\ ,a\left(\frac{1}{4},4\right)=\frac{39877}{80797}$$

(495) $$a\left(\frac{1}{5},1\right)=\frac{2}{3}\ ,a\left(\frac{1}{5},2\right)=\frac{47}{77}\ ,a\left(\frac{1}{5},3\right)=\frac{223}{372}$$

(496) $$\frac{1}{\pi}=\sum_{m\ par}\left(\frac{(2m)!}{m!\,(m+1)!}\right)^3\frac{P(m)}{2^{12m+13}}$$

(497) $$P(m)=21840m^4+67952m^3+73008m^2+29508m+2607\quad ,m\in\{0,2,4,\dots\}$$

(498) $$\frac{1}{\pi}=\sum_{m\ impar}\left(\frac{(2m-2)!}{(m-1)!\,m!}\right)^3\frac{Q(m)}{2^{12m+1}}$$

(499) $$Q(m)=21840m^4-19408m^3+192m^2-12m-5\quad ,m\in\{1,3,5,7,\dots\}$$

(500) $$P(m+8)=4P(m+6)-6P(m+4)+4P(m+2)-P(m)+8386560\quad ,m=2k\ ,k\in\mathbb{N}$$

(501) $$Q(m+8)=4Q(m+6)-6Q(m+4)+4Q(m+2)-Q(m)+8386560\quad ,m=2k+1\ ,k\in\mathbb{N}$$

(502) $$P(2k)=Q(2k+1)\quad ,k=0,1,2,3,\dots$$

(503) $$\frac{1}{\pi}=\frac{5}{16}\prod_{n=0}^{\infty}\left(\frac{a_{n+1}}{2^{12n+12}a_n}\right)=\frac{5}{16}\left(\frac{3\cdot 11\cdot 79}{2^9\cdot 5}\right)\left(\frac{5^2\cdot 17\cdot 8377}{2^{12}\cdot 11\cdot 79}\right)\dots$$

donde

(504) $$a_{n+1}=2^{12n+12}a_n+2^{6n(n+1)}\binom{2n+2}{n+1}^3(42n+47)\quad ,n=0,1,2,3,\dots$$

(505) $$\pi=\frac{5}{2}\ \ln\Big(a\,F\big(a\,F(a\,F(a\dots))\big)\Big)\quad ,a=\sqrt{\frac{5+\sqrt{5}}{2}}+\frac{1+\sqrt{5}}{2}$$

donde

(506) $$F(u)=\cfrac{1}{1+\cfrac{u^{-5}}{1+\cfrac{u^{-10}}{1+\cfrac{u^{-15}}{1+\cdots}}}}$$

(507) $$\pi=5\ \ln\Big(b\,G\big(b\,G(b\,G(b\dots))\big)\Big)\quad ,b=\sqrt{\frac{5-\sqrt{5}}{2}}+\frac{\sqrt{5}-1}{2}$$

donde

(508) $$G(u)=\cfrac{1}{1-\cfrac{u^{-5}}{1+\cfrac{u^{-10}}{1-\cfrac{u^{-15}}{1+\cdots}}}}$$

(509) $$\pi=8\sum_{n=1}^{\infty}\sum_{k=1}^{m}\frac{1}{(4(n-1)m+4k-3)(4(n-1)m+4k-1)}\quad ,m\in\mathbb{N}$$

(510) $$\pi=-3\ \ln\left(\frac{\sqrt{3}-1}{2}\ H\left(\frac{\sqrt{3}-1}{2}\ H\left(\frac{\sqrt{3}-1}{2}\ H\left(\frac{\sqrt{3}-1}{2}\ldots\right)\right)\right)\right)$$

donde

(511) $$H(u)=1-\cfrac{u^3-u^6}{1+\cfrac{u^6+u^{12}}{1-\cfrac{u^9-u^{18}}{1+\cdots}}}$$

(512) $$\frac{\pi}{6}+\tan^{-1}\left(\frac{y}{x}\right)=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2\sqrt{3}\cdot 3^{2n-1}}{3^{4n}+1}\right)-3\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{26\sqrt{3}\cdot 3^{6n-5}}{3^{12n-6}+1}\right)$$

donde

(513) $$x+iy=\cfrac{1}{1+\cfrac{a_1}{b_1+\cfrac{a_2}{1+\cfrac{a_3}{b_2+\cfrac{a_4}{1+\cdots}}}}}$$

(514) $$a_{2n-1}=(-1)^{n-1}3^n\sqrt{3}\,i-3\,,a_{2n}=(-1)^n3^n+1\,,b_n=3^{2n}\quad ,n\in\mathbb{N}$$

(515) $$x+i\,y=\cfrac{1}{1+\cfrac{3\sqrt{3}i-3}{3^2-\cfrac{2}{1-\cfrac{3^2\sqrt{3}i+3}{3^4+\cfrac{10}{1+\cdots}}}}}$$

(516) $$\pi=4\tan^{-1}\left(\prod_{n=1}^{\infty}\frac{((2mn)^2+1)\left(\left(m(2n-1)\right)^2-1\right)}{((2mn)^2-1)\left(\left(m(2n-1)\right)^2+1\right)}\right)$$

$$+4\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2m^2n^2}\right)\quad ,m\in\mathbb{N}$$

(517) $$\pi=3\ln\left(\frac{\sqrt{3}+1+\left(\sqrt{3}-1\right)\tan A}{\sqrt{3}-1+\left(\sqrt{3}+1\right)\tan A}\right)\quad ,A=\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{18n^2}\right)$$

(518) $$\pi = 4\ln\left(\frac{\sqrt{2}+\left(2-\sqrt{2}\right)\tan B}{2-\sqrt{2}+\sqrt{2}\ \tan B}\right) \quad ,B = \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{32n^2}\right)$$

(519) $$\pi = 6\ln\left(\frac{3-\sqrt{3}+\left(\sqrt{3}-1\right)\tan C}{\sqrt{3}-1+\left(3-\sqrt{3}\right)\tan C}\right) \quad ,C = \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{72n^2}\right)$$

(520) $$\frac{\pi}{8} = \sum_{n=0}^{\infty}\frac{1}{(4n+1)(4n+3)}$$

(521) $$\frac{\sqrt{3}}{6}\pi + \frac{\sqrt{3}}{4}\ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = \sum_{n=0}^{\infty}\frac{3^{-2n}}{(4n+1)(4n+3)}$$

(522) $$\frac{\sqrt{2}\left(\sqrt{2}+1\right)^2}{16}\pi + \left(\frac{\sqrt{2}+1}{2}\right)^2\ln\left(\sqrt{2}-1\right) = \sum_{n=0}^{\infty}\frac{\left(\sqrt{2}-1\right)^{4n}}{(4n+1)(4n+3)}$$

(523) $$\frac{5}{8}\pi - \ln 2 - \frac{3}{4}\ln 3 = \sum_{n=0}^{\infty}\frac{2^{-4n}+3^{-4n-1}}{(4n+1)(4n+3)}$$

(524) $$\frac{\pi}{6} + \tan^{-1}\left(\frac{y}{x}\right) = \sum_{n=0}^{\infty}T(n)$$

(525) $$\begin{aligned} &T(n) \\ &= \tan^{-1}\left(3^{-10n-11/2}\right) + \tan^{-1}\left(3^{-10n-13/2}\right) + \tan^{-1}\left(3^{-10n-17/2}\right) \\ &+ \tan^{-1}\left(3^{-10n-19/2}\right) - \tan^{-1}\left(3^{-10n-3/2}\right) - \tan^{-1}\left(3^{-10n-7/2}\right) \\ &- \tan^{-1}\left(3^{-10n-9/2}\right) - \tan^{-1}\left(3^{-10n-21/2}\right) \end{aligned}$$

(526) $$x + iy = \cfrac{1}{1 + \cfrac{i\sqrt{3}}{3 - \cfrac{1}{1 - \cfrac{i\sqrt{3}}{3^2 + \cfrac{1}{1 + \cfrac{i\sqrt{3}}{3^3 - \cfrac{1}{1 - \cdots}}}}}}}$$

(527) $$\begin{aligned} &\frac{\ln 2}{4} - \frac{\ln 3}{4} + \frac{3\sqrt{7}}{112}\pi - \frac{3\sqrt{7}}{28}\tan^{-1}\left(\frac{3-\sqrt{7}}{3+\sqrt{7}}\right) \\ &= \sum_{n=0}^{\infty}(-1)^n\, 2^{-2n-2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{3n+2-k} \end{aligned}$$

(528) $$-\frac{\ln 2}{8} + \frac{3\sqrt{7}}{112}\pi - \frac{3\sqrt{7}}{28}\tan^{-1}\left(\frac{5-\sqrt{7}}{5+\sqrt{7}}\right) = \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{3n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{2^{n-k}3^k}{3n+2-k}$$

(529) $$\frac{\ln 2}{8}-\frac{\ln 5}{4}+\frac{\ln 7}{8}+\frac{3\sqrt{7}}{112}\pi-\frac{3\sqrt{7}}{28}\tan^{-1}\left(\frac{\sqrt{7}-1}{\sqrt{7}+1}\right) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{5n+4}\sum_{k=0}^{n}\binom{n}{k}\frac{2^k}{3n+2-k}$$

(530) $$-\frac{\ln 2}{13}-\frac{\ln 3}{13}+\frac{\ln 13}{26}+\frac{3}{104}\pi-\frac{3}{26}\tan^{-1}\left(\frac{1}{5}\right)=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{5n+4}\sum_{k=0}^{n}\binom{n}{k}\frac{2^k5^{n-k}}{3n+2-k}$$

(531) $$\frac{\ln 2}{13}+\frac{\ln 5}{26}-\frac{\ln 7}{13}+\frac{3}{104}\pi-\frac{3}{26}\tan^{-1}\left(\frac{1}{3}\right)=\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5}\right)^{2n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{2^{n-k}}{3n+2-k}$$

(532) $$\frac{\ln 2}{56}-\frac{\ln 7}{56}+\frac{9\sqrt{31}}{1736}\pi-\frac{9\sqrt{31}}{868}\tan^{-1}\left(\frac{5}{\sqrt{31}}\right)=\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{5n+4}\sum_{k=0}^{n}\binom{n}{k}\frac{2^k15^{n-k}}{3n+2-k}$$

(533) $$\frac{\ln 5}{56}-\frac{\ln 2}{14}+\frac{9\sqrt{31}}{3472}\pi-\frac{9\sqrt{31}}{868}\tan^{-1}\left(\frac{7-\sqrt{31}}{7+\sqrt{31}}\right) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{5}\right)^{2n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{6^{n-k}}{3n+2-k}$$

(534) $$\frac{\ln 2}{28}-\frac{\ln 3}{14}+\frac{\ln 7}{56}+\frac{9\sqrt{31}}{3472}\pi-\frac{9\sqrt{31}}{868}\tan^{-1}\left(\frac{9-\sqrt{31}}{9+\sqrt{31}}\right) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6}\right)^{2n+2}\sum_{k=0}^{n}\binom{n}{k}\frac{5^{n-k}}{3n+2-k}$$

(535) $$\frac{13\ln 3}{42}-\frac{\ln 2}{7}+\frac{\sqrt{2}\ln\left(\sqrt{2}-1\right)}{14}-\frac{\sqrt{3}}{42}\pi = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{4n+4}\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}\frac{(-1)^m2^{3k-2m}}{5n+4-3k+m}=\sum_{n=0}^{\infty}\frac{c_n}{(n+4)2^{n+4}}$$

donde

(536) $$c_{n+5}=2c_{n+3}-c_{n+2}+2c_n\ \ ,n\in\mathbb{N}_0$$

(537) $$c_0=1,c_1=0,c_2=2,c_3=-1,c_4=4,c_5=-2,c_6=9$$

(538) $$\sqrt[3]{4}\left(\frac{\Gamma(4/3)}{\Gamma(5/6)}\right)^3\sqrt{\pi}=\frac{3}{16}\sum_{n=0}^{\infty}\frac{(5/6)_n^3(2n+1)}{2^{2n}(4/3)_n^3}+\frac{1}{3}\sum_{n=0}^{\infty}\frac{(1/2)_n(5/6)_n}{(4/3)_n(7/6)_n}$$

(539) $$\frac{2^{3x+1}\sqrt{2}(1+2x)^2}{\pi(1-2x)}\prod_{n=1}^{\infty}\frac{n^3(2n+2x+1)^2}{(2n+1)(2n-2x+1)(n+x)^3} = \sum_{n=0}^{\infty}\frac{(-1)^n\left(x+\frac{1}{2}\right)_n^3(6n+6x+1)}{2^{3n}(x+1)_n^3}+\frac{16x^2}{1-2x}\sum_{n=0}^{\infty}\frac{\left(x+\frac{1}{2}\right)_n^2}{2^n(x+1)_n\left(\frac{3}{2}-x\right)_n}$$

$$0<x<1/2$$

(540) $$\pi\sqrt{3}\ln\frac{4}{3} = 2\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n\sum_{k=0}^{n}\frac{1}{(2k+1)(n-k+1)}$$

(541) $$\frac{1}{\pi} = \frac{e^{\pi}+1}{2(e^{\pi}-1)} - \frac{1}{4(e^{\pi}-1)}\sum_{n=0}^{\infty}2^{-n}\sum_{k=1}^{2^n}\left(e^{\left(\frac{2k-2}{2^{n+1}}\right)\pi} - 2e^{\left(\frac{2k-1}{2^{n+1}}\right)\pi} + e^{\left(\frac{2k}{2^{n+1}}\right)\pi}\right)$$

(542) $$\pi = 2\sqrt{3} - \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty}2^{-n}\sum_{k=1}^{2^n}A(n,k)$$

donde

(543) $$A(n,k) = \frac{1}{1+\left(\frac{2k-2}{2^{n+1}}\right)+\left(\frac{2k-2}{2^{n+1}}\right)^2} - \frac{2}{1+\left(\frac{2k-1}{2^{n+1}}\right)+\left(\frac{2k-1}{2^{n+1}}\right)^2} + \frac{1}{1+\left(\frac{2k}{2^{n+1}}\right)+\left(\frac{2k}{2^{n+1}}\right)^2}$$

(544) $$\tan^{-1}\left(\frac{\sin(x\pi)}{\sinh(x\pi)}\right) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{x^2}{2n^2}\right) - \tan^{-1}\left(\frac{2x^2}{(2n-1)^2}\right)\right) \quad ,x \in \mathbb{R}$$

(545) $$\tan^{-1}\left(\frac{\overbrace{\sqrt{2-\sqrt{2+\cdots+\sqrt{2}}}}^{m-radicales}}{2\sinh(\pi 2^{-m-1})}\right) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{1}{2^{2n+3}n^2}\right) - \tan^{-1}\left(\frac{1}{2^{2m+1}(2n-1)^2}\right)\right) \quad ,m \in \mathbb{N}$$

(546) $$\pi = 8\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{a_n}{1+a_n^2}\right)$$

donde

(547) $$a_{n+1} = \frac{2a_1 + (1-a_1^2)a_n}{1-a_1^2-2a_1a_n} \quad ,n \in \mathbb{N} \quad ,a_1 \in (0,\infty)$$

(548) $$\pi^2\left(\frac{1}{6} - a + a^2\right) = \sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1-a_n^2}{1+a_n^2}\right)$$

donde

(549) $$0 < a < 1\,, a_1 = \tan(a\pi) \quad , a_{n+1} = \frac{a_1 + a_n}{1 - a_1a_n} \quad ,n \in \mathbb{N}$$

Casos particulares:

(550) $$\{(a,a_1)\} = \left\{\left(\frac{1}{12}, 2-\sqrt{3}\right), \left(\frac{1}{6}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{4}, 1\right), \left(\frac{1}{3}, \sqrt{3}\right), \left(\frac{5}{12}, 2+\sqrt{3}\right)\right\}$$

(551) $$\frac{\pi^2}{6} - \pi \tan^{-1}\left(\frac{1}{2}\right) + \left(\tan^{-1}\frac{1}{2}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{1-a_n^2}{1+a_n^2}\right)$$

donde

(552) $$a_{n+1} = \frac{1+2a_n}{2-a_n} \ , a_1 = \frac{1}{2} \ , n \in \mathbb{N}$$

(553) $$\pi = \frac{2^{2k+2}}{2^{2k}-1} a_k \prod_{n=1}^{\infty} \frac{2^{2k+2} n(n+1)}{2^{2k}(2n+1)^2 - 1} \ , k \in \mathbb{N}$$

donde

(554) $$a_1 = \frac{\sqrt{2}}{2} \ , a_{k+1} = \sqrt{\frac{1}{2} + \frac{1}{2} a_k} \ , k \in \mathbb{N}$$

(555) $$\pi = \frac{4096}{1023} \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}}}} \prod_{n=1}^{\infty} \frac{4096 n(n+1)}{1024(2n-1)^2 - 1}$$

(556) $$\pi = \frac{2^{2k}}{2^k - 1} a_k \prod_{n=1}^{\infty} \frac{2^{2k} n(n+1)}{2^{2k} n(n+1) + 2^k - 1} \ , k \in \mathbb{N}$$

donde

(557) $$a_1 = 1 \, , a_{k+1} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - a_k^2}} \ , k \in \mathbb{N}$$

(558) $$\pi = \frac{1024}{31} \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}}} \prod_{n=1}^{\infty} \frac{1024 n(n+1)}{1024 n(n+1) + 31}$$

(559) $$\pi = 2^{n+1} a_n \left(1 - \sum_{k=1}^{\infty} \frac{2}{2^{2n+2} k^2 - 1}\right) \ , n \in \mathbb{N}$$

donde

(560) $$a_1 = 1 \ , a_{n+1} = \frac{1}{a_n}\left(\sqrt{1 + a_n^2} - 1\right) \ , n \in \mathbb{N}$$

(561) $$\frac{1}{\pi} = \frac{1}{2^{16}} \sum_{k=0}^{\infty} \frac{171990k + 20433}{2^{12k}} \sum_{n=0}^{k} \binom{2n}{n}^3$$

(562) $$\frac{5}{2}\ln 2-\frac{3}{2}\ln 3+\left(\frac{\sqrt{3}}{6}-\frac{1}{4}\right)\pi=\sum_{n=1}^{\infty}\frac{n}{(2n-1)(3n-1)(4n-1)}$$

$$=\frac{1}{24}\sum_{n=0}^{\infty}\left(\frac{6}{2^n}-\frac{8}{3^n}+\frac{3}{4^n}\right)\zeta(n+2)$$

$\zeta(x)$ función zeta de Riemann

(563) $$\sum_{n=1}^{\infty}\frac{1}{n}\tan^{-1}\left(\frac{x}{n}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}\zeta(2n+2)}{2n+1} \quad ,|x|\leq 1$$

Casos particulares:

(564) $$\frac{\pi}{4}+\sum_{n=2}^{\infty}\frac{1}{n}\tan^{-1}\left(\frac{1}{n}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(2n+2)}{2n+1}$$

(565) $$\frac{\pi}{6}+\sum_{n=2}^{\infty}\frac{1}{n}\tan^{-1}\left(\frac{1}{n\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(2n+2)}{(2n+1)3^n}$$

(566) $$\frac{\pi}{8}+\sum_{n=2}^{\infty}\frac{1}{n}\tan^{-1}\left(\frac{\sqrt{2}-1}{n}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\sqrt{2}-1\right)^{2n+1}\zeta(2n+2)}{2n+1}$$

(567) $$\frac{\pi}{12}+\sum_{n=2}^{\infty}\frac{1}{n}\tan^{-1}\left(\frac{2-\sqrt{3}}{n}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(2-\sqrt{3}\right)^{2n+1}\zeta(2n+2)}{2n+1}$$

(568) $$x\pi=2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}a_n}{n} \quad ,|x|<1$$

donde

(569) $$a_1=\sin(x\pi)\,,b_1=\cos(x\pi)$$

(570) $$a_{n+1}=b_1a_n+a_1b_n \;, \qquad b_{n+1}=-a_1a_n+b_1b_n \;,n\in\mathbb{N}$$

(571) $$\pi=4\sum_{n=1}^{\infty}\frac{a_n}{2n-1} \quad ,0<x\leq 1$$

donde

(572) $$a_1=x\;,b_1=\sqrt{1-x^2}$$

(573) $$a_{n+1}=(1-2a_1^2)a_n+2a_1b_1b_n \;,b_{n+1}=-2a_1b_1a_n+(1-2a_1^2)b_n \;,n\in\mathbb{N}$$

(574) $$x\pi=2-4\sum_{n=1}^{\infty}\frac{b_n}{(2n-1)(2n+1)} \quad ,0<x\leq 1$$

(575) $$\pi\sqrt{1-x^2}=8\sum_{n=1}^{\infty}\frac{na_n}{(2n-1)(2n+1)} \quad ,0<x\leq 1$$

En fórmulas (574),(575) se tiene:

(576) $$a_1=2x\sqrt{1-x^2} \quad , \qquad b_1=1-2x^2$$

(577) $$a_{n+1} = b_1 a_n + a_1 b_n \quad , \qquad b_{n+1} = -a_1 a_n + b_1 b_n \ , n \in \mathbb{N}$$

(578) $$\frac{\pi^2}{16} = 2\left(\tan^{-1}\frac{1}{2}\right)^2 + 2\left(\tan^{-1}\frac{1}{3}\right)^2 - \left(\tan^{-1}\frac{1}{7}\right)^2$$

(579) $$\frac{\pi^2}{4} = 6\left(\tan^{-1}\frac{1}{\sqrt{2}}\right)^2 + 3\left(\tan^{-1}\frac{1}{2\sqrt{2}}\right)^2 - 2\left(\tan^{-1}\frac{2}{5\sqrt{2}}\right)^2$$

(580) $$\frac{\pi^2}{16} = 3\left(\tan^{-1}\frac{1}{2}\right)^2 + 3\left(\tan^{-1}\frac{1}{5}\right)^2 + 3\left(\tan^{-1}\frac{1}{8}\right)^2 - \left(\tan^{-1}\frac{3}{11}\right)^2 - \left(\tan^{-1}\frac{6}{17}\right)^2 - \left(\tan^{-1}\frac{3}{41}\right)^2$$

(581) $$\frac{\pi^2}{16} = \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{2n+1}{(n^2+n)^2+1}\right)\right)\left(\tan^{-1}\left(\frac{2n(n+1)+1}{(n^2+n)^2-1}\right)\right)$$

(582) $$\frac{\pi^2}{16} = \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{1}{n^2+n+1}\right)\right)\left(\tan^{-1}\left(\frac{2n+1}{n^2+n-1}\right)\right)$$

(583) $$\pi = 2\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) + \frac{2\sqrt{1-x^2}}{\sqrt{\frac{1}{2}+\frac{x}{2}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{x}{2}}}\ \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{x}{2}}}}\cdots}$$

$$0 < x \le 1$$

(584) $$\pi(x-y) + \sum_{n=1}^{\infty}\frac{1}{n}\left(e^{-2\pi yn} - e^{-2\pi xn}\right) = \ln\left(\frac{x}{y}\right) + \sum_{n=1}^{\infty}\ln\left(\frac{n^2+x^2}{n^2+y^2}\right)$$

$$x > 0 \, , y > 0$$

(585) $$\sqrt{a}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{e^{-an}\sin(an)}{an}\right) = \sqrt{\frac{b}{2\pi}}\left(\frac{\pi}{4} + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{2}{b^2n^2}\right)\right)$$

$$a > 0, b > 0, ab = 2\pi$$

(586) $$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{e^{-n}\sin n}{n} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{2\pi^2 n^2}\right)$$

(587) $$\frac{3(\sqrt{3}-1)}{\pi^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\Gamma\left(\frac{n+k}{6n}\right)\Gamma\left(\frac{5n-k}{6n}\right)\right)^{-1}$$

(588) $$\frac{\sqrt{\pi}}{\sqrt[4]{e}} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^n}{(2m)!\,(n-m)!\,(2n+1)} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Gamma\left(n+\frac{1}{2},1\right)$$

(589) $$\pi = 4(1-a^2)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}a^{2k-2}}{2k-1}b(k) \ , \sqrt{2}-1 < a < 1$$

donde

$$(590)\quad b(k)=1-\sum_{n=1}^{k-1}\frac{(-1)^{n-1}}{(2n)!}\left(1-\left(\frac{1-a^2}{2a}\right)^2\right)^n\prod_{m=1}^{n}((2k-1)^2-(2m-1)^2)\ ,k\geq 2$$

$$(591)\qquad \frac{4}{\pi}\tan^{-1}\left(\frac{1+b-z(1-b)}{1-b+z(1+b)}\right)=\sum_{n=0}^{\infty}\left(\frac{\tan^{-1}\left(\frac{b-z}{1+bz}\right)}{\tan^{-1}\left(\frac{1+b-z(1-b)}{1-b+z(1+b)}\right)}\right)^n$$

$$0<z<b<1$$

$$(592)\qquad \frac{6}{\pi}\tan^{-1}\left(\frac{1+b\sqrt{3}-z(\sqrt{3}-b)}{\sqrt{3}-b+z(1+b\sqrt{3})}\right)=\sum_{n=0}^{\infty}\left(\frac{\tan^{-1}\left(\frac{b-z}{1+bz}\right)}{\tan^{-1}\left(\frac{1+b\sqrt{3}-z(\sqrt{3}-b)}{\sqrt{3}-b+z(1+b\sqrt{3})}\right)}\right)^n$$

$$0<z<b<\sqrt{3}$$

$$(593)\qquad \sqrt{\pi}=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}e^{-k^2}}{(n-k)!}\left(\frac{1}{2n-2k+1}+\sum_{m=1}^{n-k}\binom{n-k}{m}\frac{(2k)^m}{2n-2k-m+1}\right)$$

$$(594)\quad \pi=1+2\sum_{n=1}^{\infty}2^{-n/2}\sum_{m=1}^{2^{n-1}}\left(\frac{1}{(1+(2m-1)2^{-n})\sqrt{2m-1}}-\frac{1}{(1+m2^{-n+1})\sqrt{2m}}\right)$$

$$(595)\qquad \frac{\pi}{\sin\left(\frac{2m+1}{2n}\pi\right)}=2n\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{2nk+2m+1}+\frac{1}{2nk+2n-2m-1}\right)$$

$$0<m<n$$

$$(596)\qquad \pi=\frac{\Gamma\left(\frac{1+x}{2}\right)\Gamma\left(\frac{1-x}{2}\right)}{\Gamma\left(1-\frac{x}{2}\right)\Gamma\left(1+\frac{x}{2}\right)}\prod_{n=0}^{\infty}\left(1-\left(\frac{x}{n+1}\right)^2\right)^{(-1)^n}\quad ,0<x<1$$

$$(597)\qquad \pi=\left(\frac{\Gamma\left(\frac{1+x}{2}\right)}{\Gamma\left(1+\frac{x}{2}\right)}\right)^2\prod_{n=0}^{\infty}\left(1+\frac{x}{n+1}\right)^{2(-1)^n}\quad x>-1$$

$$(598)\qquad \pi\ \frac{(1/2)_k(1/2)_{k+1}}{(k!)^2}=\prod_{n=0}^{\infty}\left(\frac{n+2k+2}{n+2k+1}\right)^{(-1)^n}\quad ,k\in\mathbb{N}_0$$

$$(599)\qquad \pi=\sum_{n=0}^{\infty}2^{-2n}\left(\sum_{k=0}^{2n}\binom{2n}{k}\frac{(-1)^k(6n-2k+3)}{(2k+1)(2n-k+1)}-\frac{1}{4n+3}\right)$$

$$(600)\qquad \pi=\sum_{n=0}^{\infty}2^{-2n}\left(\sum_{k=0}^{2n}\binom{2n+1}{k}\frac{(-1)^k(6n-2k+3)}{(2k+1)(2n+1)}-\frac{1}{4n+3}\right)$$

$$(601)\qquad \pi=\sum_{n=0}^{\infty}2^{-2n}\left(2\binom{2n+1/2}{2n}^{-1}+\binom{2n+3/2}{2n+1}^{-1}\right)$$

$$(602)\quad \frac{1}{\pi}=\frac{1}{2^k\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{k-radicale\ s}}\prod_{n=1}^{\infty}\frac{\overbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}^{(n+k)-radicales}}{2}\quad ,k\in\mathbb{N}$$

$$(603)\quad \frac{8\sqrt{\left(2-\sqrt{2+\sqrt{2}}\right)^3}}{3\pi}=\prod_{n=1}^{\infty}\left(1-\left(\frac{1}{16n}\right)^2\right)-\prod_{n=1}^{\infty}\left(1-\left(\frac{3}{16n}\right)^2\right)$$

$$(604)\quad \frac{8\sqrt{\left(2+\sqrt{2+\sqrt{2}}\right)^3}}{\pi}=21\prod_{n=1}^{\infty}\left(1-\left(\frac{7}{16n}\right)^2\right)+5\prod_{n=1}^{\infty}\left(1-\left(\frac{5}{16n}\right)^2\right)$$

$$(605)\quad \pi=\int_a^b\frac{1}{\sqrt{(x-a)(b-x)}}dx\quad ,0<a<b$$

$$(606)\quad \pi=2a\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{a^{2k}(1-a^2)^{n-k}}{2k+1}\quad ,0<a<1$$

$$(607)\quad \pi=4\tan^{-1}\left(\sqrt{\frac{c}{b-a-c}}\right)+\int_{a+c}^{b-c}\frac{1}{\sqrt{(x-a)(b-x)}}dx$$

$$0<a<b,0<c<(b-a)/2$$

$$(608)\quad \pi=4\tan^{-1}\left(\sqrt{\frac{a}{1+a}}\right)+\int_0^1\frac{1}{\sqrt{(x+a)(1+a-x)}}dx\quad ,a>0$$

$$(609)\quad \pi=4\tan^{-1}\left(\frac{a}{b}\right)+2\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\sum_{k=0}^{n}\binom{n}{k}\frac{(1-a^2-b^2)^{n-k}(b^{2k+1}-a^{2k+1})}{2k+1}$$

$$0<a<b\,,0<a^2+b^2<1$$

$$(610)\quad \pi=4\tan^{-1}\left(\sqrt{\frac{c}{a+b-c}}\right)+\int_{-a+c}^{b-c}\frac{1}{\sqrt{(x+a)(b-x)}}dx$$

$$0<b<a\ \ ,a^2+b^2<6ab\,,0<c<(a+b)/2$$

$$(611)\quad \pi=4\tan^{-1}\left(\sqrt{\frac{c}{a+b-c}}\right)+\frac{1}{\sqrt{ab}}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\int_{-a+c}^{b-c}\left(\frac{(a-b)x+x^2}{ab}\right)^n dx$$

donde

(612) $$\int_{-a+c}^{b-c} \left(\frac{(a-b)x+x^2}{ab}\right)^n dx$$

$$= \frac{1}{(ab)^n}\sum_{k=0}^{n}\binom{n}{k}\frac{(a-b)^{n-k}\left((b-c)^{n+k+1}-(-a+c)^{n+k+1}\right)}{n+k+1}$$

$$0<b<a, a^2+b^2<6ab, 0<c<(a+b)/2$$

(613) $$\pi = 4\tan^{-1} y + \int_{1+\frac{2y^2}{1+y^2}}^{3-\frac{2y^2}{1+y^2}} \frac{1}{\sqrt{(x-1)(3-x)}}dx \ ,0<y<1$$

(614) $$\pi = 4\tan^{-1}\left(\sqrt{\frac{c}{b-a-c}}\right)+\frac{1}{P}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}Q_n$$

donde

(615) $$Q_n = \int_{a+c}^{b-c}\left(\frac{P^2-(x-a)(b-x)}{P^2}\right)^n dx$$

$$= \sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^{k+m}\binom{n}{k}\binom{k}{m}\left(1+\frac{ab}{P^2}\right)^{n-k}\left(\frac{a+b}{P^2}\right)^{k-m}\left(\frac{1}{P^2}\right)^m D(k,m)$$

(616) $$D(k,m) = \frac{(b-c)^{k+m+1}-(a+c)^{k+m+1}}{k+m+1}$$

$$0<a<b, 0<c<\frac{(b-a)}{2}, P>(b-a)/2\sqrt{2}$$

(617) $$\pi = 3\int_{n+\frac{1}{2}}^{n+\frac{3}{2}}\frac{dx}{\sqrt{(x-n)(n+2-x)}} = 2\int_{n}^{n+1}\frac{dx}{\sqrt{(x-n)(n+2-x)}}$$

$$= 6\int_{n+1}^{n+2}\frac{dx}{\sqrt{(x-n)(n+4-x)}} = 6\int_{n+2}^{n+4}\frac{dx}{\sqrt{(x-n)(n+8-x)}}$$

$$n\in\mathbb{R}$$

(618) $$\pi = 12\sum_{k=1}^{n}\tan^{-1}\left(\frac{\sqrt{(n+k)(3n-k+1)}-\sqrt{(3n-k)(n+k-1)}}{\sqrt{(3n-k)(3n-k+1)}+\sqrt{(n+k)(n+k-1)}}\right)$$

$$n\in\mathbb{N}$$

# **to be continued**

# III. Referencias


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