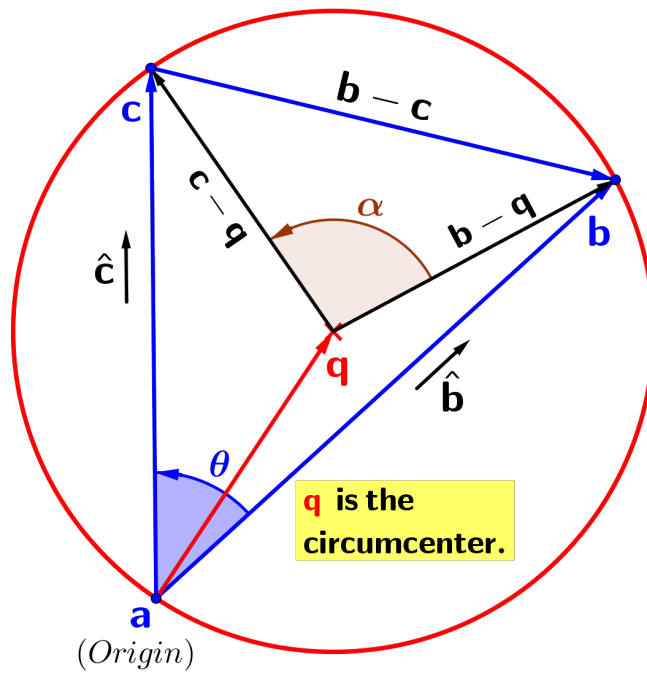


Rotations of Vectors via Geometric Algebra:

Explanation, and Usage in Solving
Classic Geometric "Construction" Problems



Geometric-Algebra Formulas for Plane (2D) Geometry

The Geometric Product, and Relations Derived from It

For any two vectors \mathbf{a} and \mathbf{b} ,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{ab} + \mathbf{ba} = 2\mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{ab} - \mathbf{ba} = 2\mathbf{a} \wedge \mathbf{b}$$

$$\mathbf{ab} = 2\mathbf{a} \cdot \mathbf{b} + \mathbf{ba}$$

$$\mathbf{ab} = 2\mathbf{a} \wedge \mathbf{b} - \mathbf{ba}$$

Definitions of Inner and Outer Products (Macdonald A. 2010 p. 101.)

The inner product

The inner product of a j -vector A and a k -vector B is

$A \cdot B = \langle AB \rangle_{k-j}$. Note that if $j > k$, then the inner product doesn't exist.

However, in such a case $B \cdot A = \langle BA \rangle_{j-k}$ does exist.

The outer product

The outer product of a j -vector A and a k -vector B is

$A \wedge B = \langle AB \rangle_{k+j}$.

Relations Involving the Outer Product and the Unit Bivector, \mathbf{i} .

For any two vectors \mathbf{a} and \mathbf{b} ,

$$\mathbf{ia} = -\mathbf{ai}$$

$$\mathbf{a} \wedge \mathbf{b} = [(\mathbf{ai}) \cdot \mathbf{b}] \mathbf{i} = -[\mathbf{a} \cdot (\mathbf{bi})] \mathbf{i} = -\mathbf{b} \wedge \mathbf{a}$$

Equality of Multivectors

For any two multivectors \mathcal{M} and \mathcal{N} ,

$\mathcal{M} = \mathcal{N}$ if and only if for all k , $\langle \mathcal{M} \rangle_k = \langle \mathcal{N} \rangle_k$.

Formulas Derived from Projections of Vectors and Equality of Multivectors

Any two vectors \mathbf{a} and \mathbf{b} can be written in the form of "Fourier expansions" with respect to a third vector, \mathbf{v} :

$$\mathbf{a} = (\mathbf{a} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + [\mathbf{a} \cdot (\hat{\mathbf{v}}\mathbf{i})] \hat{\mathbf{v}}\mathbf{i} \text{ and } \mathbf{b} = (\mathbf{b} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + [\mathbf{b} \cdot (\hat{\mathbf{v}}\mathbf{i})] \hat{\mathbf{v}}\mathbf{i}.$$

Using these expansions,

$$\mathbf{ab} = \{(\mathbf{a} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + [\mathbf{a} \cdot (\hat{\mathbf{v}}\mathbf{i})] \hat{\mathbf{v}}\mathbf{i}\} \{(\mathbf{b} \cdot \hat{\mathbf{v}}) \hat{\mathbf{v}} + [\mathbf{b} \cdot (\hat{\mathbf{v}}\mathbf{i})] \hat{\mathbf{v}}\mathbf{i}\}$$

Equating the scalar parts of both sides of that equation,

$$\mathbf{a} \cdot \mathbf{b} = [\mathbf{a} \cdot \hat{\mathbf{v}}] [\mathbf{b} \cdot \hat{\mathbf{v}}] + [\mathbf{a} \cdot (\hat{\mathbf{v}}i)] [\mathbf{b} \cdot (\hat{\mathbf{v}}i)], \text{ and}$$

$$\mathbf{a} \wedge \mathbf{b} = \{[\mathbf{a} \cdot \hat{\mathbf{v}}] [\mathbf{b} \cdot (\hat{\mathbf{v}}i)] - [\mathbf{a} \cdot (\hat{\mathbf{v}}i)] [\mathbf{b} \cdot (\hat{\mathbf{v}}i)]\} i.$$

Also, $a^2 = [\mathbf{a} \cdot \hat{\mathbf{v}}]^2 + [\mathbf{a} \cdot (\hat{\mathbf{v}}i)]^2$, and $b^2 = [\mathbf{b} \cdot \hat{\mathbf{v}}]^2 + [\mathbf{b} \cdot (\hat{\mathbf{v}}i)]^2$.

Reflections of Vectors, Geometric Products, and Rotation operators

For any vector \mathbf{a} , the product $\hat{\mathbf{v}}\mathbf{a}\hat{\mathbf{v}}$ is the reflection of \mathbf{a} with respect to the direction $\hat{\mathbf{v}}$.

For any two vectors \mathbf{a} and \mathbf{b} , $\hat{\mathbf{v}}\mathbf{a}\mathbf{b}\hat{\mathbf{v}} = \mathbf{b}\mathbf{a}$, and $\mathbf{v}\mathbf{a}\mathbf{b}\mathbf{v} = v^2\mathbf{b}\mathbf{a}$.

Therefore, $\hat{\mathbf{v}}e^{\theta i}\hat{\mathbf{v}} = e^{-\theta i}$, and $\mathbf{v}e^{\theta i}\mathbf{v} = v^2e^{-\theta i}$.

The trouble with the very notion of 'application' [of mathematics to science, or of one branch of mathematics to another] is that it is a one-way concept: we apply A to B. To counter this I have invented the word 'interapplicability'. Mathematicians who work in these domains more often speak of correspondence, which is a symmetric notion. Although I began this discussion [of Descartes's contributions] in terms of application, we might better have spoken of Descartes establishing a correspondence between arithmetic and geometry. I once in conversation spoke of Descartes arithmetizing geometry; the person with whom I was speaking, having in mind the way algebraic problems of the day could now be solved geometrically, observed that Descartes had geometrized algebra. Exactly so.¹

¹Hacking, Ian. 2014. Why is there Philosophy of Mathematics at All?. pp. 20-21. Cambridge University Press, New York. I am grateful to "Lorena", a student in philosophy of Mathematics at *La Universidad Nacional Autónoma de México*, for introducing me to this book.

Rotations of Vectors via Geometric Algebra: Explanation, and Usage in Solving Classic Geometric "Construction" Problems

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1 Introduction

{Author's note, 27 January 2016:

This document, in all of its versions, has been prepared for two very different audiences: for my fellow students of GA, and for experts who are preparing materials for us, and need to know which GA concepts we understand and apply readily, and which ones we do not. This new version should be useful to both audiences. It presents a much easier way to find the circumcenter of a triangle (pp. 22ff), but preserves the previous solution as Appendix A so that my earlier oversights will be clear to readers. As a plus, that solution makes useful, time-saving observations on inverses of multivectors, and on transformations of vector expressions. □ }

{Author's note, 27 January 2016:

I hope that the new material in this version will help students avoid forming one of my own bad habits when using GA: the tendency to believe that GA can't possibly be as convenient as is claimed! That tendency can cause unnecessary work by leading us to translate geometric products into quantities that are more familiar, but much less efficient. In 4.2, we'll learn time-saving maneuvers that are available to us if we accept that GA's theorems mean what they say.

A new section (2) provides background that will help us to formulate key aspects of geometry problems in ways that can be manipulated via GA to find solutions. The first three problems in this new version were added after I learned of the solutions posted on line by Professor Ramon González Calvet, for the incenter and circumcenter of a triangle. Professor González's solutions do not use rotations; the contrast between his insights and the ones used here is instructive.

A useful resource not mentioned in the previous version is "Find tangents to a circle from a point, using Geometric Algebra" (as GeoGebra worksheet, as YouTube video). The present version solves that problem using rotations. □ }

{Introduction to the version of 31 December 2015:}

This document is part of a series of resources that I am preparing in support of Professor David Hestenes's goal of using Geometric Algebra (GA) to integrate high-school algebra, geometry, trigonometry, and physics into a coherent curriculum.

I will be grateful for any comments, suggestions, and corrections.

One important piece of advice before we start:

Don't let yourself be intimidated by the equations that arise in problems like these!

We'll learn to recognize patterns (for example, products of four vectors that represent a simple rotation) that will help us simplify complicated equations readily. Also, I've presented more than one way to solve each problem. In addi-

tion to the way that appeared most reasonable and efficient, I've also included at least one way "sub-optimal" way, so that students can see that they needn't worry about having to find "the way" to get the job done.

An additional benefit of presenting those sub-optimal ways (some of which, like 4.5.2 border on the absurd) is that they help demonstrate the coherence and flexibility of GA's capacities for expressing and manipulating rotations.

Readers are encouraged to study the following GeoGebra worksheets and videos before beginning:

"Answering Two Common Objections to Geometric Algebra"

[As GeoGebra worksheet](#)

[As YouTube video.](#)

"Geometric Algebra: Find unknown vector from two dot products"

[As GeoGebra worksheet](#)

[As YouTube video](#)

For an more-complete treatment of rotations in plane geometry, be sure to read Hestenes D. 1999, pp. 78-92. His section on circles (pp. 87-89) is especially relevant to the present document. Macdonald A. 2010 is invaluable in many respects, and González Calvet R. 2001, [Treatise of Plane Geometry through Geometric Algebra](#) is a must-read.

What we'll see in this document ...

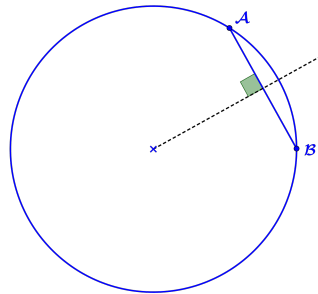
- Rotations of vectors as a natural development of the inner, outer, and geometric products
- How "construction" problems of classical geometry can be formulated in terms of rotations and dilations of vectors, then solved by
 - recognizing the geometric content of equations that arise; and
 - making use of postulates about equality of multivectors.
- How Professor Alan Macdonald's definitions of the inner and outer products simplify the solution process.
- How to simplify complicated products by
 - using basic identities; and
 - recognizing products that represent rotations and reflections
- **Most importantly:** That all this stuff about angles, exponents, and geometric products really is coherent, and terms like $e^{\theta i}$ really do follow the rules of exponents.

The author may be contacted at QueLaMateNoTeMate.webs.com.

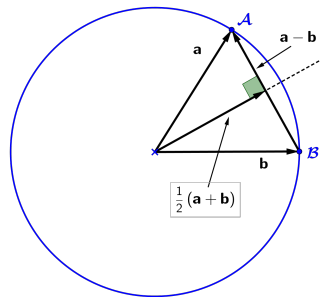
2 Important Facts about Tangents, Chords, and Angles

This information provided in this section will help us to formulate problems in symbolic terms that can be manipulated via GA to find solutions. Some of the relationships listed here are provided for completeness, and are not used in this document.

1. The mediatrix (perpendicular bisector) of a chord of a circle passes through the circle's center.



This important result from classical geometry can be proven simply via GA. Using the circle's center as the origin, the chord \overline{AB} becomes the vector $\mathbf{a} - \mathbf{b}$, and the vector from the center of the circle to the midpoint of \overline{AB} is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.



According to the postulates of GA, two vectors are perpendicular if and only if their dot product is zero, so let's find $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$. The vectors \mathbf{a} and \mathbf{b} are radii of the same circle, so $|\mathbf{a}| = |\mathbf{b}|$. Therefore,

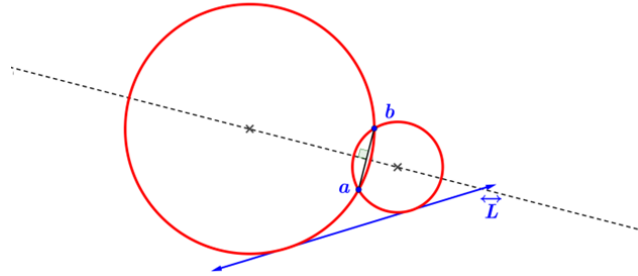
$$\begin{aligned} (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= a^2 - b^2 \\ &= 0. \quad \square \end{aligned}$$

The symbol " \square " is an alternative to "QED": both are used to show that the proof has been completed.

This result shows that the line that passes through the circle's center and the chord's midpoint is perpendicular to the chord. Therefore, that line is the chord's mediatrix.

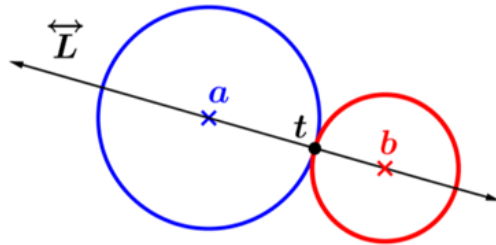
2. Two consequences of the perpendicularity between any chord and the radius drawn to its midpoint are

- (a) that the line connecting the centers of two intersecting circles is perpendicular to their common chord;

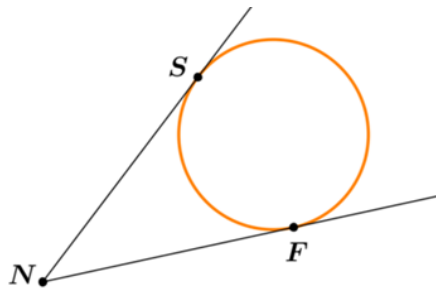


and

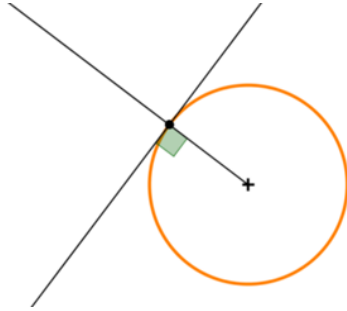
- (b) that the line connecting the centers of two tangent circles passes through the point of tangency.



3. The two tangents drawn to a circle from an external point are of equal length: $NS = NF$.

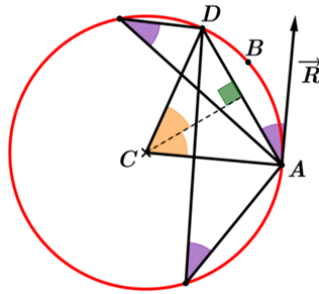


4. A tangent to a circle is perpendicular to the line that passes through the circle's center and the point of tangency.

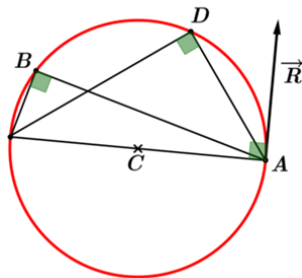


For a GA proof of this relationship, see "Find tangents to a circle from a point, using Geometric Algebra" (as GeoGebra worksheet, as YouTube video).

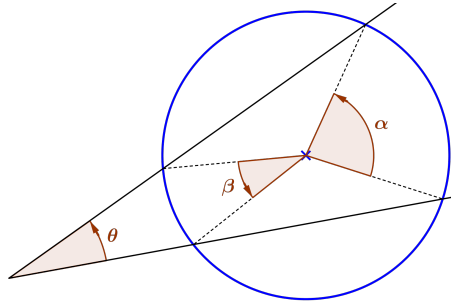
5. An angle inscribed in a circle is equal to half the central angle that subtends the same arc. For example, all of the purple angles—including the angle formed by the segment \overline{BA} and the ray \vec{R} subtend the same arc (ABD). For details, see Hestenes D. 1999, p. 89. Therefore, all are equal to each other, and measure $\frac{1}{2}\angle DCA$.



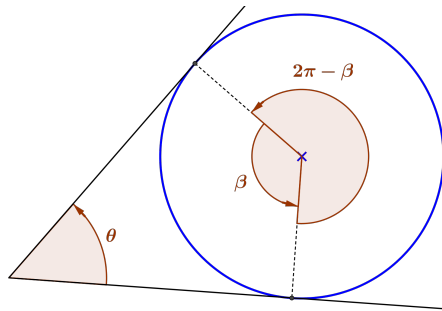
A corollary is that any angle inscribed in a semicircle is a right angle:



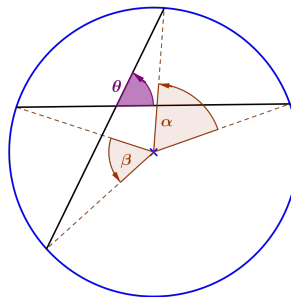
6. The measure of an angle whose vertex lies outside a circle is equal to half the difference between the measures of the central angles that subtend the arcs cut by the rays that form the angle's sides. For example, $\theta = \frac{1}{2}(\alpha - \beta)$.



(Note that GA's sign convention for angles is that counter-clockwise rotations are positive.) A special case of this relationship is the angle between the tangents drawn to a circle from an external point: $\theta = \frac{1}{2}[(2\pi - \beta) - \beta] = \pi - \beta$.



- The measure of an angle whose vertex lies inside a circle is equal to half the sum the measures of the central angles that subtend the arcs cut by the lines that form the angle's sides. For example, $\theta = \frac{1}{2}(\alpha + \beta)$.



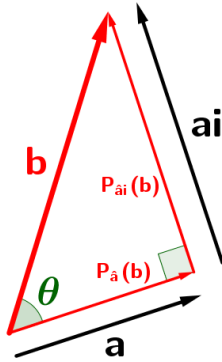
3 The Relationship between the Geometric Product and Rotations

Let's begin by reviewing a variation on a sequence of operations that we saw in the worksheet and video, "Answering Two Common Objections to Geometric Algebra":

<https://www.youtube.com/watch?v=oB0DZiF86Ns>
<http://tube.geogebra.org/material/simple/id/1565271>.

Given any two vectors \mathbf{a} and \mathbf{b} , we can express \mathbf{b} as the vector sum of its projections upon \mathbf{a} and \mathbf{ai} , where \mathbf{i} is the unit bivector of the plane that contains \mathbf{a} and \mathbf{b} . (Please recall that \mathbf{ai} is the 90° counter-clockwise rotation of \mathbf{a} .)

$$\begin{aligned}\mathbf{b} &= P_{\mathbf{a}}(\mathbf{b}) + P_{\mathbf{ai}}(\mathbf{b}) \\ &= (\mathbf{b} \cdot \hat{\mathbf{a}}) \hat{\mathbf{a}} + [\mathbf{b} \cdot (\hat{\mathbf{ai}})] \hat{\mathbf{ai}} \\ &= \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} \{ \hat{\mathbf{b}} \cdot \hat{\mathbf{a}} + [\hat{\mathbf{b}} \cdot (\hat{\mathbf{ai}})] \hat{\mathbf{i}} \}.\end{aligned}$$



Looking now at our diagram, we can see that $\hat{\mathbf{b}} \cdot \hat{\mathbf{a}} = \cos \theta$ and $\hat{\mathbf{b}} \cdot (\hat{\mathbf{ai}}) = \sin \theta$.

Therefore, we can write that last result as

$$\mathbf{b} = \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} (\cos \theta + \sin \theta \mathbf{i}),$$

one geometrical interpretation of which is that \mathbf{b} is obtained by rotating \mathbf{a} counterclockwise through the angle θ , and dilating it by the factor $|\mathbf{b}|/|\mathbf{a}|$. (Or by dilating, then rotating; the operations commute.)

Now, we write $\cos \theta + \sin \theta \mathbf{i}$ in exponential form, as $e^{\theta \mathbf{i}}$, to obtain

$$\mathbf{b} = \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} e^{\theta \mathbf{i}}.$$

Before we proceed, we should do a few trial calculations to assure ourselves that the preceding formulas work, and that we understand them. For example, we know that if we rotate the vector \mathbf{a} clockwise by 90° ($\pi/2$ radians), then dilate it by a factor of 2, we obtain the vector $2\mathbf{ai}$. Let's see whether that's the

result that we obtain from our formulas:

$$\frac{2}{1} \mathbf{a} \left(\cos \frac{\pi}{2} + \sin \frac{\pi}{2} \mathbf{i} \right) = 2\mathbf{a} (0 + 1\mathbf{i}) = 2\mathbf{a}\mathbf{i}. \checkmark$$

We also know that if we rotate \mathbf{a} through the angle θ , then rotate the resulting vector through the angle ϕ , the result should be the vector \mathbf{a} rotated through the angle $\theta + \phi$. Is that what the formulas tell us? Using the symbol \mathbf{a}' to represent the vector that's obtained by the rotation through θ , we have

$$\mathbf{a}' = \mathbf{a} (\cos \theta + \sin \theta \mathbf{i}).$$

The rotation of that vector (\mathbf{a}') through the angle ϕ is then

$$\begin{aligned} \mathbf{a}'' &= \mathbf{a}' (\cos \phi + \sin \phi \mathbf{i}) \\ &= \underbrace{[\mathbf{a} (\cos \theta + \sin \theta \mathbf{i})]}_{\mathbf{a}'} (\cos \phi + \sin \phi \mathbf{i}) \\ &= \mathbf{a} [\cos \theta \cos \phi + \sin \theta \sin \phi \mathbf{i}^2 + (\cos \theta \sin \phi + \sin \theta \cos \phi) \mathbf{i}] \\ &= \mathbf{a} [(\cos \theta \cos \phi - \sin \theta \sin \phi) + (\cos \theta \sin \phi + \sin \theta \cos \phi) \mathbf{i}] \\ &= \mathbf{a} [\cos (\theta + \phi) + \sin (\theta + \phi) \mathbf{i}], \end{aligned}$$

which is \mathbf{a} rotated through the angle $\theta + \phi$. We could have obtained that same result using the exponential form:

$$\begin{aligned} \mathbf{a}'' &= \underbrace{[\mathbf{a} e^{\theta \mathbf{i}}]}_{\mathbf{a}'} e^{\phi \mathbf{i}} \\ &= \mathbf{a} e^{(\theta + \phi) \mathbf{i}} \\ &= \mathbf{a} [\cos (\theta + \phi) + \sin (\theta + \phi) \mathbf{i}]. \end{aligned}$$

These encouraging results should help us see that the formulas that we've developed thus far really do provide convenient ways of expressing and manipulating rotations of vectors symbolically. But now, we'll do something that at first sight seems pointless, even if correct. Starting from our result $\mathbf{b} = \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} e^{\theta \mathbf{i}}$, where θ is the angle from \mathbf{a} to \mathbf{b} , we can see that

$$\begin{aligned} \mathbf{a}\mathbf{b} &= \mathbf{a} \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} e^{\theta \mathbf{i}} \\ &= \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a}^2 e^{\theta \mathbf{i}} \\ &= \frac{|\mathbf{b}|}{|\mathbf{a}|} |\mathbf{a}|^2 e^{\theta \mathbf{i}} \\ &= |\mathbf{a}| |\mathbf{b}| e^{\theta \mathbf{i}}. \end{aligned}$$

So, the geometric product $\mathbf{a}\mathbf{b}$ is equal to $|\mathbf{a}| |\mathbf{b}| e^{\theta \mathbf{i}}$. This result, too, deserves some discussion before we continue. Let's start by considering what happens to

some third vector \mathbf{v} , coplanar with \mathbf{a} and \mathbf{b} , when we right-multiply that vector by \mathbf{ab} :

$$\mathbf{v}(\mathbf{ab}) = \mathbf{v} [|\mathbf{a}||\mathbf{b}|e^{\theta i}] = |\mathbf{a}||\mathbf{b}| [\mathbf{v}e^{\theta i}],$$

meaning that when right-multiplied by \mathbf{ab} , the vector \mathbf{v} is dilated by the scalar factor $|\mathbf{a}||\mathbf{b}|$, then rotated through the angle θ . Hildebrand and Oldenburg 2015 were referring to this property of the geometric product when they said that geometric products are used for handling transformations. We should also note that both $\frac{|\mathbf{b}|}{|\mathbf{a}|}e^{\theta i}$ and $|\mathbf{a}||\mathbf{b}|e^{\theta i}$ have the form of the product of (1) a scalar dilation factor; and (2) an operator that rotates by an angle equal to that between \mathbf{a} and \mathbf{b} . Incidentally, this result also shows that the product of any three coplanar vectors is a vector. We'll make use of that important fact later.

Of course we should also note that because $\mathbf{ab} = |\mathbf{a}||\mathbf{b}|e^{\theta i}$,

$$\frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|} = e^{\theta i}.$$

When we need to rotate vectors to solve "construction" problems later in this video, we'll use that equivalence as a convenient way to obtain an expression for $e^{\theta i}$.

Those of you who've studied rotations may be objecting that the way to rotate a vector \mathbf{v} through the angle θ is to multiply \mathbf{v} on the left by $e^{-\frac{\theta}{2}i}$, and on the right by $e^{\frac{\theta}{2}i}$:

$$e^{-\frac{\theta}{2}i} \mathbf{v} e^{\frac{\theta}{2}i} = \mathbf{v}, \text{ rotated ccw by } \theta.$$

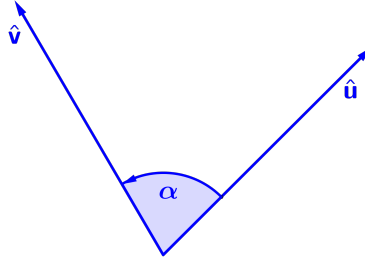
That's true: it's the form used in 3-D (and higher), with i being the unit bivector for the plane of rotation. But in 2-D, $e^{-\frac{\theta}{2}i} \mathbf{v} e^{\frac{\theta}{2}i}$ reduces to $e^{\theta i}$, as we'll now see:

$$\begin{aligned} e^{-\frac{\theta}{2}i} \mathbf{v} e^{\frac{\theta}{2}i} &= \left[\cos\left(-\frac{\theta}{2}\right) + \sin\left(-\frac{\theta}{2}\right) i \right] \mathbf{v} \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \\ &= \left[\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right) i \right] \mathbf{v} \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \\ &= \left[\cos\left(\frac{\theta}{2}\right) \mathbf{v} - \sin\left(\frac{\theta}{2}\right) i \mathbf{v} \right] \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \\ &= \left[\cos\left(\frac{\theta}{2}\right) \mathbf{v} + \sin\left(\frac{\theta}{2}\right) \underbrace{\mathbf{v}i}_{=-i\mathbf{v}} \right] \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \\ &= \mathbf{v} \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \left[\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) i \right] \\ &= \mathbf{v} e^{\frac{\theta}{2}i} e^{\frac{\theta}{2}i} \\ &= \mathbf{v} e^{\left[\frac{\theta}{2}i + \frac{\theta}{2}i\right]} \\ &= \mathbf{v} e^{\theta i}. \end{aligned}$$

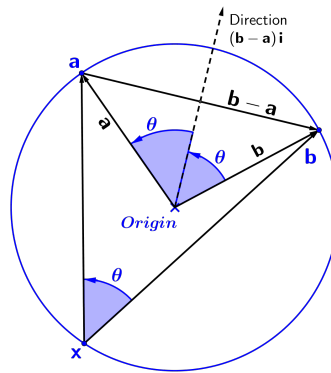
More generally, the product of any two vectors evaluates to the sum of a scalar and a bivector, so in plane (2-D) geometry, the product of any even number of vectors must also evaluate to the sum of a scalar and a bivector, while the product of any odd number of vectors evaluates to a vector. This fact will be important to us later. For example, see the margin note "A word about what we're working toward here" (p. 37).

This is a very brief —and not entirely satisfactory — explanation of what i represents in 3-D rotations. For details, see Macdonald 2010, pp. 89-91 and 125-127.

Before we leave the subject of rotations, we should treat the square root of a rotation operator, such as the square root of the operator $e^{\alpha i}$ that rotates \hat{u} into \hat{v} in the following figure.

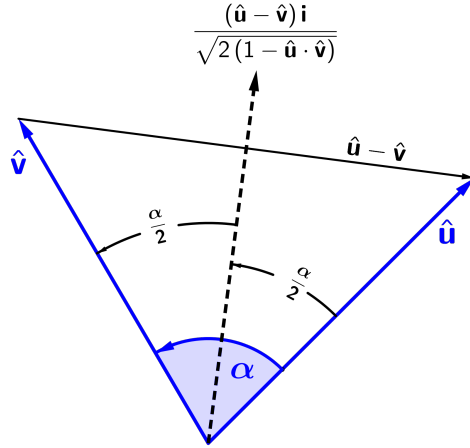


There are times when we might wish to express the square root of that operator in terms of \hat{u} and \hat{v} . For example, knowing that the central angle formed by vectors \mathbf{a} and \mathbf{b} in the following figure is 2θ (page 11),



how could we express the operator that rotates $\mathbf{b} - \mathbf{x}$ into $\mathbf{a} - \mathbf{x}$? That operator involves a dilation as well as a rotation (except when $|\mathbf{b} - \mathbf{x}| = |\mathbf{a} - \mathbf{x}|$), so let's see first how we'd express the pure rotation $e^{\theta i}$ in terms of $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

Let's address that question by adding a few elements to our diagram in which the operator $e^{\alpha i}$ rotated \hat{u} into \hat{v} .



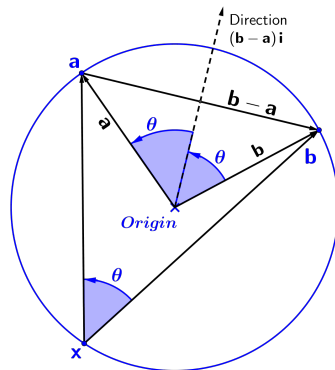
We see that the unit vector that bisects α is $\frac{(\hat{u} - \hat{v}) \mathbf{i}}{\sqrt{2(1 - \hat{u} \cdot \hat{v})}}$. Two rotations through the angle $\alpha/2$ produce a rotation through α , so

$$\begin{aligned} \sqrt{e^{\alpha \mathbf{i}}} &= e^{\frac{\alpha}{2} \mathbf{i}} \\ &= \text{either of } \hat{u} \left[\frac{(\hat{u} - \hat{v}) \mathbf{i}}{\sqrt{2(1 - \hat{u} \cdot \hat{v})}} \right] \text{ and } \left[\frac{(\hat{u} - \hat{v}) \mathbf{i}}{\sqrt{2(1 - \hat{u} \cdot \hat{v})}} \right] \hat{v}, \end{aligned}$$

both of which reduce to

$$e^{\frac{\alpha}{2} \mathbf{i}} = \frac{\mathbf{i} - \hat{u} \hat{v} \mathbf{i}}{\sqrt{2(1 - \hat{u} \cdot \hat{v})}}.$$

Therefore, in the case of our diagram with the circle that passes through the points \mathbf{a} , \mathbf{b} , and \mathbf{x} ,



we multiply the rotation operator $e^{\frac{\alpha}{2} \mathbf{i}}$ by the dilation factor $\frac{|\mathbf{a} - \mathbf{x}|}{|\mathbf{b} - \mathbf{x}|}$, and find

Note how the familiar scalar identity $(p^m)^n \equiv p^{mn}$ applies to rotation operators as well:

$$\begin{aligned} \sqrt{e^{\alpha \mathbf{i}}} &= \left[e^{\alpha \mathbf{i}} \right]^{\frac{1}{2}} \\ &= e^{\frac{1}{2}(\alpha \mathbf{i})} \\ &= e^{\frac{\alpha}{2} \mathbf{i}}. \end{aligned}$$

that

$$\begin{aligned} \mathbf{a} - \mathbf{x} &= \left[\frac{|\mathbf{a} - \mathbf{x}|}{|\mathbf{b} - \mathbf{x}|} \right] e^{\frac{\alpha}{2}i} \\ &= \left[\frac{|\mathbf{a} - \mathbf{x}|}{|\mathbf{b} - \mathbf{x}|} \right] \left[\frac{i - \hat{\mathbf{b}}\hat{\mathbf{a}}i}{\sqrt{2(1 - \hat{\mathbf{b}} \cdot \hat{\mathbf{a}})}} \right]. \end{aligned}$$

Now that we know how to find the square root of an operator that produces a pure rotation, how can we find the square root of an operator of the form $\lambda e^{\theta i}$ (with scalar λ), which combines a dilation and a rotation? The exponents in rotation operators obey all of the usual rules about exponents, so

$$\begin{aligned} \sqrt{\lambda e^{\theta i}} &= \sqrt{\lambda} \sqrt{e^{\theta i}} \\ &= \sqrt{\lambda} e^{\frac{\theta}{2}i}. \end{aligned}$$

4 Using Exponential Expressions of Rotations to Solve Geometry Problems

In this section, we'll see that GA's ability to manipulate rotations algebraically enables us to solve a given problem in many different ways. We'll make extensive use of our results $\mathbf{b} = \frac{|\mathbf{b}|}{|\mathbf{a}|} \mathbf{a} e^{\theta i}$, $\mathbf{a}\mathbf{b} = |\mathbf{a}||\mathbf{b}| e^{\theta i}$, and $\mathbf{v}(\mathbf{a}\mathbf{b}) = \mathbf{v}[|\mathbf{a}||\mathbf{b}| e^{\theta i}]$.

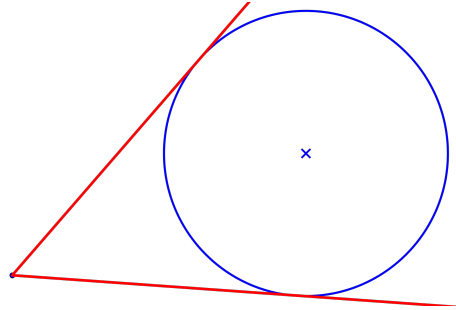
One important piece of advice before we start:

Don't let yourself be intimidated by the equations that arise in problems like these!

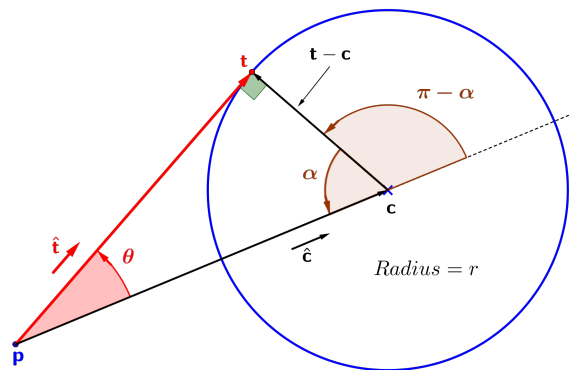
We'll learn to recognize patterns (for example, products of four vectors that represent a simple rotation) that will help us simplify complicated equations readily.

4.1 Problem 1

Derive equations for the tangents to a circle from a point outside it.



Identifying Potentially Useful Elements of the Problem We'll begin by identifying elements of the problem that might be sufficient to provide sufficient information for constructing one of the two tangents. The point t is a point of tangency. In deriving our solution, we'll use the same symbol—that is, t —to denote both the point of tangency and the vector to that point from the external point p . We'll rely upon context to tell the reader whether t is being used to refer to the point, or to the vector.



We can solve the problem if we can identify either of the angles θ and α , or either of the vectors t and \hat{t} . What elements of the problem might help us identify θ , α , t , and \hat{t} , and how are all of those quantities related?

Firstly, because ptc is a right triangle, $t^2 = c^2 - r^2$. From Section 2, we find that $\theta = \frac{1}{2}[(\pi - \alpha) - \alpha] = \frac{\pi}{2} - \alpha$. Therefore, $\theta + \alpha = \frac{\pi}{2}$. Finally, $e^{\theta i} = \hat{c}\hat{t}$, and $e^{\alpha i} = \frac{t-c}{|t-c|}(-\hat{c}) = \frac{(c-t)\hat{c}}{r}$.

$$|t - c| = r.$$

Formulating a Strategy Now, let's combine that information in a way that might lead us to a solution. A reasonable way to begin is by noting that

because $\theta + \alpha = \frac{\pi}{2}$, we can write

$$\begin{aligned} e^{(\theta+\alpha)\mathbf{i}} &= e^{\frac{\pi}{2}\mathbf{i}}, \\ [e^{\theta\mathbf{i}}] [e^{\alpha\mathbf{i}}] &= \mathbf{i}, \\ [\hat{\mathbf{c}}\hat{\mathbf{t}}] \left[\frac{(\mathbf{c}-\mathbf{t})\hat{\mathbf{c}}}{r} \right] &= \mathbf{i}. \end{aligned}$$

To put that result in a more-convenient form, we'll multiply both sides by $c^2 r |\mathbf{t}|$ (which is also $|c|^2 r |\mathbf{t}|$) to obtain

$$\mathbf{c}\mathbf{t}(\mathbf{c}-\mathbf{t})\mathbf{c} = c^2 r |\mathbf{t}| \mathbf{i}.$$

The symbol " \equiv " is used to communicate that the quantities $\mathbf{u}\mathbf{v}\mathbf{w}\mathbf{u}$ and $u^2\mathbf{v}\mathbf{w}$ are "equivalent", or "identical": they're equal for any three coplanar vectors, not just for some three particular vectors of interest.

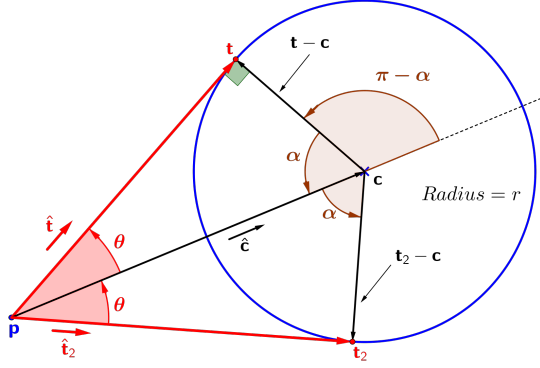
As shown in the Appendix (7.3), for any three coplanar vectors \mathbf{u}, \mathbf{v} and \mathbf{w} , $\mathbf{u}\mathbf{v}\mathbf{w}\mathbf{u} \equiv u^2\mathbf{v}\mathbf{w}$. (Note the reversal of the order of \mathbf{v} and \mathbf{w} .) That identity enables us to make the following simplifications:

$$\begin{aligned} \mathbf{c}\mathbf{t}(\mathbf{c}-\mathbf{t})\mathbf{c} &= c^2 r |\mathbf{t}| \mathbf{i}, \\ \mathbf{c}[\mathbf{t}(\mathbf{c}-\mathbf{t})]\mathbf{c} &= c^2 r |\mathbf{t}| \mathbf{i}, \\ c^2(\mathbf{c}\mathbf{t}-t^2) &= c^2 r |\mathbf{t}| \mathbf{i}, \\ \mathbf{c}\mathbf{t}-t^2 &= r |\mathbf{t}| \mathbf{i}. \end{aligned}$$

Solving the Equation That last result is quite concise, but is it useful to us? Yes: to obtain \mathbf{t} from $\mathbf{c}\mathbf{t}$, we just left-multiply by \mathbf{c}^{-1} . The other quantities in the result that we just obtained are already known to us: $t^2 = c^2 - r^2$, so $|\mathbf{t}| = \sqrt{c^2 - r^2}$. The rest is straightforward:

$$\begin{aligned} \mathbf{c}\mathbf{t}-t^2 &= r |\mathbf{t}| \mathbf{i}, \\ \mathbf{c}\mathbf{t} &= c^2 - r^2 + r \left[\sqrt{c^2 - r^2} \right] \mathbf{i}, \\ \mathbf{c}^{-1}\mathbf{c}\mathbf{t} &= \underbrace{\left(\frac{\mathbf{c}}{c^2} \right)}_{=\mathbf{c}^{-1}} \left\{ c^2 - r^2 + r \left[\sqrt{c^2 - r^2} \right] \mathbf{i} \right\}, \\ \mathbf{t} &= (c^2 - r^2) \left(\frac{\mathbf{c}}{c^2} \right) + r \left[\sqrt{c^2 - r^2} \right] \left(\frac{\mathbf{c}}{c^2} \right) \mathbf{i}, \\ &= \left[1 - \left(\frac{r}{|c|} \right)^2 \right] \mathbf{c} + \left[\frac{r}{|c|} \sqrt{1 - \left(\frac{r}{|c|} \right)^2} \right] \mathbf{c}\mathbf{i}. \end{aligned}$$

We've now identified one of the points of tangency. Finding the other (\mathbf{t}_2) turns out to be easier.



In this case, $e^{\theta i} = \hat{t}_2 \hat{c}$, and $e^{\alpha i} = (-\hat{c}) \left[\frac{t_2 - c}{r} \right] = \frac{\hat{c}(c - t_2)}{r}$. Using those expressions in our equation $[e^{\theta i}] [e^{\alpha i}] = i$,

$$\begin{aligned} [\hat{t}_2 \hat{c}] \left[\frac{\hat{c}(c - t_2)}{r} \right] &= i, \\ \frac{\hat{t}_2 \hat{c} \hat{c} (c - t_2)}{r} &= i \\ \hat{t}_2 c - \hat{t}_2 t_2 &= r i, \\ t_2 c - t_2 t_2 &= r |t_2| i, \\ t_2 c - t_2^2 &= r |t_2| i, \\ t_2 c &= t_2^2 + r |t_2| i, \\ \therefore t_2 c &= c^2 - r^2 + r \left[\sqrt{c^2 - r^2} \right] i. \end{aligned}$$

Remember: $\hat{c}\hat{c} = 1$, and for any vector v , $v\hat{v} = \hat{v}v = |v|$.

Comparing that equation to the analogous one that we obtained when finding t ,

$$ct = c^2 - r^2 + r \left[\sqrt{c^2 - r^2} \right] i,$$

we see the right-hand sides are identical, but the order of the multiplication on the left-hand side is inverted. To solve for t , we left-multiplied by c^{-1} , but we'll right-multiply by c^{-1} to solve for t_2 . What difference will that make? Let's see...

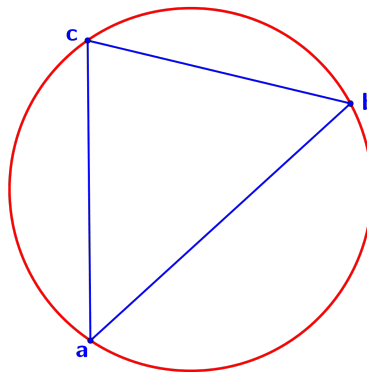
$$\begin{aligned} t_2 &= \left\{ c^2 - r^2 + r \left[\sqrt{c^2 - r^2} \right] i \right\} \underbrace{\left(\frac{c}{c^2} \right)}_{=c^{-1}} \\ &= \left[1 - \left(\frac{r}{|c|} \right)^2 \right] c + \left[\frac{r}{|c|} \sqrt{1 - \left(\frac{r}{|c|} \right)^2} \right] ic, \\ &= \left[1 - \left(\frac{r}{|c|} \right)^2 \right] c + \left[\frac{r}{|c|} \sqrt{1 - \left(\frac{r}{|c|} \right)^2} \right] \underbrace{(-ci)}_{=ci}, \\ &= \left[1 - \left(\frac{r}{|c|} \right)^2 \right] c - \left[\frac{r}{|c|} \sqrt{1 - \left(\frac{r}{|c|} \right)^2} \right] ci. \end{aligned}$$

Right-multiplying by c^{-1} , rather than left-multiplying, has inverted the sign of the ci term.

As a contrast that might prove useful, this problem is solved without using rotations in "Find tangents to a circle from a point, using Geometric Algebra" (as GeoGebra worksheet, as YouTube video).

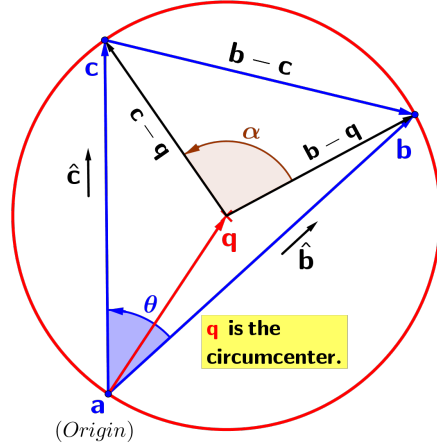
4.2 Problem 2

Derive an equation for the position of the circumcenter of a triangle with respect to one of its vertices.



Identifying Potentially Useful Elements of the Problem This problem can be solved in several different ways. For example, because the mediatrix of any chord in a circle passes through the circle's center (1), we can find the circumcenter of our triangle by finding the point at which the mediatrices of any two sides of the triangle intersect. We can also find the circumcenter via simple trigonometry.

However, we wish to solve this problem by using rotations, so let's begin by asking, "Is the circumcenter involved in the rotation of any identifiable vectors?" As soon as we add the circumcenter to our previous diagram, and draw vectors from the circumcenter to any two of the given triangle's vertices,



we can see that the required circle converts the given triangle into "three inscribed angles" (5). Therefore, we can choose any of the triangle's vertices the origin (we've chosen \mathbf{a}), and write $\alpha = 2\theta$, from which $e^{\theta i} e^{\theta i} = e^{2\theta i}$.

Formulating a Strategy One strategy is to express the equality of angles that we've just identified in terms of products of vectors, in order to obtain an equation involving \mathbf{q} :

$$e^{\alpha i} = e^{2\theta i}$$

$$e^{\alpha i} = [e^{\theta i}] [e^{\theta i}]$$

$$\left[\frac{\mathbf{b} - \mathbf{q}}{|\mathbf{b} - \mathbf{q}|} \right] \left[\frac{\mathbf{c} - \mathbf{q}}{|\mathbf{c} - \mathbf{q}|} \right] = [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] .$$

Then, we'd expand both sides of the equation that we've just obtained, after which we'd use other manipulations to identify \mathbf{q} .

We'll use that strategy in later problems, but before we dive into it here, we should also note that $\mathbf{c} - \mathbf{q}$ is a pure rotation of $\mathbf{b} - \mathbf{q}$ because both are radii of the same circle:

$$[\mathbf{b} - \mathbf{q}] e^{\alpha i} = \mathbf{c} - \mathbf{q}$$

$$[\mathbf{b} - \mathbf{q}] [e^{\theta i}] [e^{\theta i}] = \mathbf{c} - \mathbf{q}$$

$$[\mathbf{b} - \mathbf{q}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] = \mathbf{c} - \mathbf{q},$$

from which

$$\mathbf{b} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] - \mathbf{q} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] = \mathbf{c} - \mathbf{q},$$

$$\mathbf{q} - \mathbf{q} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] = \mathbf{c} - \mathbf{b} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}],$$

$$\mathbf{q} (1 - \hat{\mathbf{b}\hat{\mathbf{c}}}\hat{\mathbf{b}\hat{\mathbf{c}}}) = \mathbf{c} - \mathbf{b} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}]$$

$$\mathbf{q} (1 - \hat{\mathbf{b}\hat{\mathbf{c}}}\hat{\mathbf{b}\hat{\mathbf{c}}}) = \mathbf{c} - \hat{\mathbf{c}\hat{\mathbf{b}}}\hat{\mathbf{c}}.$$

Two methods of solving problems via rotations:

- Equate two expressions for the same angle (in this case, α); and
- Write one vector as a "rotation + dilation" of the other. In this case, $\mathbf{c} - \mathbf{q}$ is a pure rotation of $\mathbf{b} - \mathbf{q}$ because both are radii of the same circle.

Quite often, the equation obtained via one of these methods is much easier to solve than that obtained via the other.

$$\mathbf{b} [\hat{\mathbf{b}\hat{\mathbf{c}}}] [\hat{\mathbf{b}\hat{\mathbf{c}}}] = (\mathbf{b}\hat{\mathbf{b}}) \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}$$

$$= |\mathbf{b}| \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}$$

$$= \hat{\mathbf{c}} (|\mathbf{b}| \hat{\mathbf{b}}) \hat{\mathbf{c}}$$

$$= \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}.$$

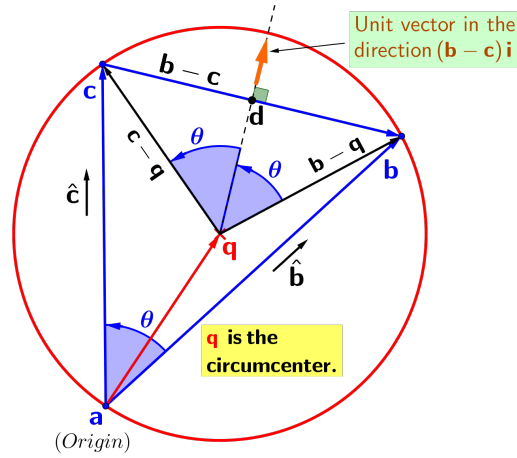
Appendix A (p. 60) solves for q in this way. Although it's more complicated than the ways that we'll see here shortly, it's recommended (along with the comments thereon) for its time-saving pointers regarding inverses of multivectors, and for its observations on geometric interpretation of results and transformations of vectors.

Is that result helpful? Yes, because $1 - \hat{b}\hat{c}\hat{b}$ has a multiplicative inverse in GA. Therefore, we can write the following in a purely formal way, then identify what that inverse is, precisely:

$$q \left(1 - \hat{b}\hat{c}\hat{b}\right) \left(1 - \hat{b}\hat{c}\hat{b}\right)^{-1} = (c - \hat{c}b\hat{c}) \left(1 - \hat{b}\hat{c}\hat{b}\right)^{-1}$$

$$\therefore q = (c - \hat{c}b\hat{c}) \left(1 - \hat{b}\hat{c}\hat{b}\right)^{-1}.$$

However, is there an even-easier way? Let's add a few more elements to our diagram, then examine it again:



We see now that a rotation through our angle θ , in combination with a dilation by the scalar factor $\frac{|d-q|}{|b-q|}$, will transform the vector $b-q$ into $d-q$:

$$d-q = \frac{|d-q|}{|b-q|} (b-q) e^{\theta i}$$

$$= \frac{|d-q|}{|b-q|} (b-q) \hat{b}\hat{c}.$$

Determining the value of $\frac{|d-q|}{|b-q|}$ is easy enough: $d-q$ is the projection of $b-q$ upon $(b-c)i$, so

$$|d-q| = |b-q| \cos \theta$$

$$= |b-q| \hat{b} \cdot \hat{c};$$

$$\therefore \frac{|d-q|}{|b-q|} = \hat{b} \cdot \hat{c}.$$

The next few paragraphs indicate how we'd use ordinary trigonometry to identify the position of the incenter with respect to the midpoint of segment \overline{bc} . Note also that

$$\frac{|b-c|}{2} = |b-d|$$

$$= |b-q| \left[(\hat{b}i) \cdot \hat{c} \right]$$

$$\therefore |b-q| = \frac{|b-c|}{2 (\hat{b}i) \cdot \hat{c}}.$$

Putting these observations and results together, our equation

$$d-q = \frac{|d-q|}{|b-q|} (b-q) \hat{b}\hat{c}$$

becomes

$$\mathbf{d} - \mathbf{q} = (\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) (\mathbf{b} - \mathbf{q}) \hat{\mathbf{b}}\hat{\mathbf{c}}.$$

To solve that equation for \mathbf{q} , we rearrange it as

$$\mathbf{q} \left[(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) \hat{\mathbf{b}}\hat{\mathbf{c}} - 1 \right] = (\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) \hat{\mathbf{b}}\hat{\mathbf{c}} - \mathbf{d},$$

then right-multiply both sides by $\left[(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) \hat{\mathbf{b}}\hat{\mathbf{c}} - 1 \right]^{-1}$.

That task promises to be somewhat easier than finding \mathbf{q} via the equation that we obtained previously, which was

$$\mathbf{q} = (\mathbf{c} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}) \left(1 - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} \right)^{-1}.$$

Still, we can do even better. Let's recall the first strategy that we identified for finding \mathbf{q} : that of writing the rotation operator $e^{2\theta\mathbf{i}}$ as the product of the unit vector in the direction $\mathbf{b} - \mathbf{q}$ and the unit vector in the direction $\mathbf{c} - \mathbf{q}$. Combining that idea with the experience we've gained subsequently from formulating two alternative strategies, we can see potential benefit in writing

The unit vector in the direction $(\mathbf{b} - \mathbf{c})\mathbf{i}$ is the rotation, through the angle θ , of the unit vector in the direction $\mathbf{b} - \mathbf{q}$.

That is,

$$\frac{(\mathbf{b} - \mathbf{c})\mathbf{i}}{|\mathbf{b} - \mathbf{c}|} = \left[\frac{\mathbf{b} - \mathbf{q}}{|\mathbf{b} - \mathbf{q}|} \right] e^{\theta\mathbf{i}}$$

from which

$$\begin{aligned} \frac{|\mathbf{b} - \mathbf{q}|}{|\mathbf{b} - \mathbf{c}|} (\mathbf{b} - \mathbf{c})\mathbf{i} &= (\mathbf{b} - \mathbf{q}) \hat{\mathbf{b}}\hat{\mathbf{c}}, \text{ and} \\ \mathbf{q}\hat{\mathbf{b}}\hat{\mathbf{c}} &= \mathbf{b}\hat{\mathbf{b}}\hat{\mathbf{c}} - \left[\frac{(\mathbf{b} - \mathbf{c})\mathbf{i}}{2(\hat{\mathbf{b}}\mathbf{i}) \cdot \hat{\mathbf{c}}} \right] \\ &= |\mathbf{b}|\hat{\mathbf{c}} - \left[\frac{(\mathbf{b} - \mathbf{c})\mathbf{i}}{2(\hat{\mathbf{b}}\mathbf{i}) \cdot \hat{\mathbf{c}}} \right]. \end{aligned}$$

From our previous work, we can derive that

$$\frac{|\mathbf{b} - \mathbf{q}|}{|\mathbf{b} - \mathbf{c}|} = \frac{1}{2(\hat{\mathbf{b}}\mathbf{i}) \cdot \hat{\mathbf{c}}}.$$

That equation is the one that we shall now solve.

Solving the Equation In the equation

$$\mathbf{q}\hat{\mathbf{b}}\hat{\mathbf{c}} = |\mathbf{b}|\hat{\mathbf{c}} - \left[\frac{(\mathbf{b} - \mathbf{c})\mathbf{i}}{2(\hat{\mathbf{b}}\mathbf{i}) \cdot \hat{\mathbf{c}}} \right],$$

\mathbf{q} is right-multiplied by $\hat{\mathbf{b}}\hat{\mathbf{c}}$. We can undo those operations by right-multiplying by $\hat{\mathbf{c}}\hat{\mathbf{b}}$, which of course happens to be $(\hat{\mathbf{b}}\hat{\mathbf{c}})^{-1}$:

Appendix A (p. 60) discusses in detail many aspects of finding inverses of multivectors. One quick way to see that $(\hat{\mathbf{b}}\hat{\mathbf{c}})^{-1}$ is $\hat{\mathbf{c}}\hat{\mathbf{b}}$ is by noting that $\hat{\mathbf{b}}\hat{\mathbf{c}} = e^{\theta\mathbf{i}}$, the multiplicative inverse of which is $e^{-\theta\mathbf{i}}$, which in turn, as we can deduce from our diagrams, is $\hat{\mathbf{c}}\hat{\mathbf{b}}$.

$$\begin{aligned} (\hat{\mathbf{q}}\hat{\mathbf{b}}\hat{\mathbf{c}})\hat{\mathbf{c}}\hat{\mathbf{b}} &= \left\{ |\mathbf{b}| \hat{\mathbf{c}} - \left[\frac{(\mathbf{b}-\mathbf{c})\mathbf{i}}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \right\} \hat{\mathbf{c}}\hat{\mathbf{b}} \\ \mathbf{q} &= |\mathbf{b}| \hat{\mathbf{c}}\hat{\mathbf{c}}\hat{\mathbf{b}} - \left[\frac{(\mathbf{b}-\mathbf{c})\mathbf{i}\hat{\mathbf{c}}\hat{\mathbf{b}}}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \\ &= \mathbf{b} + \left[\frac{\mathbf{c}\hat{\mathbf{c}}\hat{\mathbf{b}} - \mathbf{b}\hat{\mathbf{c}}\hat{\mathbf{b}}}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \mathbf{i} \\ &= \mathbf{b} + \left[\frac{|\mathbf{c}|\hat{\mathbf{b}} - \mathbf{b}(2\hat{\mathbf{c}}\cdot\hat{\mathbf{b}} - \hat{\mathbf{b}}\hat{\mathbf{c}})}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \mathbf{i} \\ &= \mathbf{b} + \left[\frac{|\mathbf{c}|\hat{\mathbf{b}} + |\mathbf{b}|\hat{\mathbf{c}} - 2(\hat{\mathbf{b}}\cdot\hat{\mathbf{c}})\mathbf{b}}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \mathbf{i} \end{aligned}$$

This answer is satisfactory for computing \mathbf{q} , but we can transform it into a version that's more useful and informative.

Interpreting the Solution, and Transforming It into a More-Useful Form Readers are encouraged to review the extensive treatment that this subject is given in Appendix A (p. 60), for the version of the solution obtained by solving the equation

$$\mathbf{q} = (\mathbf{c} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}) (1 - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}})^{-1}.$$

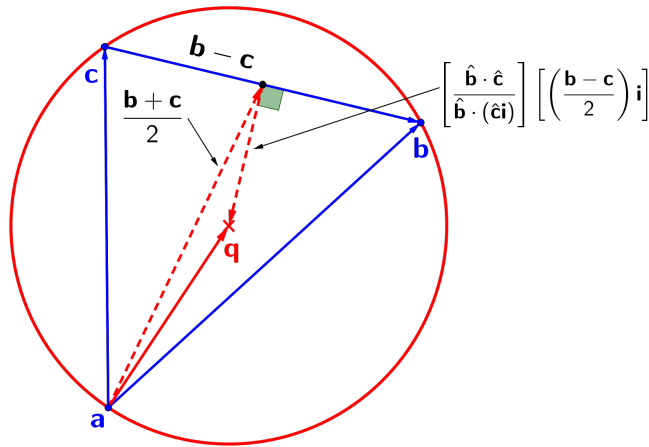
Here, we'll transform the solution that we just obtained into a form that shows that \mathbf{q} lies along the mediatrix of segment $\overline{\mathbf{bc}}$. We begin the transformation by going back a few steps to

$$\mathbf{q} = \mathbf{b} + \left[\frac{\mathbf{c}\hat{\mathbf{c}}\hat{\mathbf{b}} - \mathbf{b}\hat{\mathbf{c}}\hat{\mathbf{b}}}{2(\hat{\mathbf{b}}\mathbf{i})\cdot\hat{\mathbf{c}}} \right] \mathbf{i},$$

then continuing

$$\begin{aligned}
\mathbf{q} &= \mathbf{b} + \left[\frac{c\hat{c}\hat{b} - b(2\hat{b} \cdot \hat{c} - \hat{b}\hat{c})}{2(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \mathbf{i} \\
&= \mathbf{b} + \left[\frac{c\hat{c}\hat{b} + b\hat{b}\hat{c} - 2(\hat{b} \cdot \hat{c})b}{2(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \mathbf{i} \\
&= \mathbf{b} + \left[\frac{c(\hat{c} \cdot \hat{b} + \hat{c} \wedge \hat{b}) + b(\hat{b} \cdot \hat{c} + \hat{b} \wedge \hat{c}) - 2(\hat{b} \cdot \hat{c})b}{2(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \mathbf{i} \\
&= \mathbf{b} + \left[\frac{(c-b)(\hat{b} \cdot \hat{c}) + (b-c)(\hat{b} \wedge \hat{c})}{2(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \mathbf{i} \\
&= \mathbf{b} + \left[\frac{(c-b)(\hat{b} \cdot \hat{c}) + (b-c)[(\hat{b}\mathbf{i}) \cdot \hat{c}]\mathbf{i}}{2(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \mathbf{i} \\
&= \mathbf{b} + \left[\frac{\hat{b} \cdot \hat{c}}{(\hat{b}\mathbf{i}) \cdot \hat{c}} \right] \left(\frac{c-b}{2} \right) \mathbf{i} + \frac{c-b}{2} \\
&= \frac{\mathbf{b} + \mathbf{c}}{2} + \left[\frac{\hat{b} \cdot \hat{c}}{\hat{b} \cdot (\hat{c}\mathbf{i})} \right] \left[\left(\frac{b-c}{2} \right) \mathbf{i} \right].
\end{aligned}$$

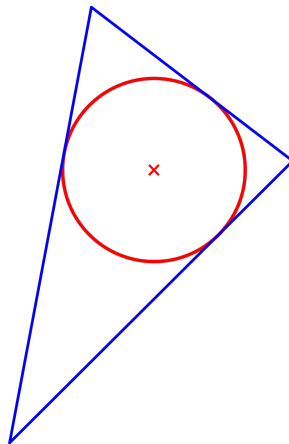
The geometric significance of that version is shown in the following figure:



As we knew from the classical solution and from trigonometry, \mathbf{q} lies along the mediatrix of segment \overline{bc} .

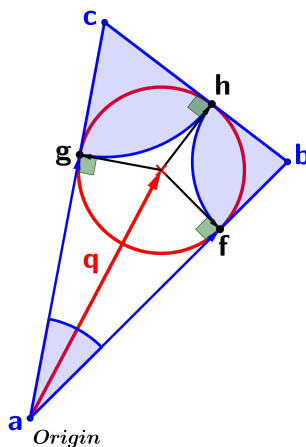
4.3 Problem 3

Derive an equation for the position of the incenter of a triangle with respect to one of its vertices.

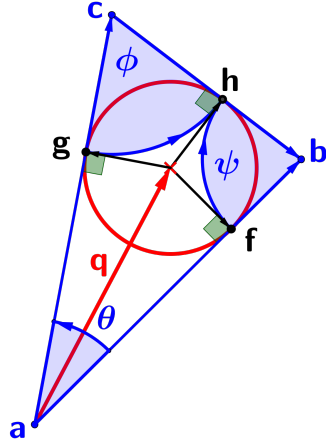


Identifying Potentially Useful Elements of the Problem Although the problem is posed as one of finding the incenter, we can see that each of the triangle's vertices is a point from which tangents are drawn to the required circle. Thus, we have three cases of Problem 1 (4.1). So, let's choose one of the vertices as the origin, then identify elements that might be useful.

One key fact is that from Problem 1, we know that the incenter must lie along the bisector of the angle formed by the tangents drawn to the circle. Thus the incenter lies along the direction $\frac{\hat{b} + \hat{c}}{2}$. A second is that the radii from the incenter to the points of tangency are perpendicular to the triangle's sides. A third is that the lengths of the tangents from each vertex are equal. From the latter, we can deduce that the length of the segment \overline{cb} is equal to the sum of the lengths of segments \overline{cg} and \overline{bf} .



We can also see several rotations that we might be able to formulate via GA and use to find the answer, so let's add a few more elements to our figure so that we can treat those rotations more precisely.



Although we drew this diagram in order to examine rotations, we can see that it also helps us refine our initial observations about lengths of segments. Why not do so now, before moving on to the rotations?

We noted that the sides of the triangle are perpendicular to the radii drawn from the incenter, so we know that vector \mathbf{f} is \mathbf{q} 's projection upon $\hat{\mathbf{b}}$:

$$\mathbf{f} = (\mathbf{q} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}.$$

Because the two tangents drawn to a circle from a given point are equal in length, we know that $|\mathbf{g}| = |\mathbf{f}|$. Therefore,

$$\mathbf{g} = (\mathbf{q} \cdot \hat{\mathbf{b}}) \hat{\mathbf{c}}.$$

Our initial observation that the length of segment $\overline{\mathbf{bc}}$ is the sum of the lengths of segments $\overline{\mathbf{cg}}$ and $\overline{\mathbf{bf}}$ can also be translated into a "GA-friendly" vector equation:

$$\begin{aligned} |\mathbf{c} - \mathbf{b}| &= |\mathbf{g} - \mathbf{c}| + |\mathbf{f} - \mathbf{b}| \\ &= |\mathbf{c}| - |\mathbf{g}| + |\mathbf{b}| - |\mathbf{f}| \\ &= |\mathbf{c}| - \mathbf{q} \cdot \hat{\mathbf{b}} + |\mathbf{b}| - \mathbf{q} \cdot \hat{\mathbf{b}}. \end{aligned}$$

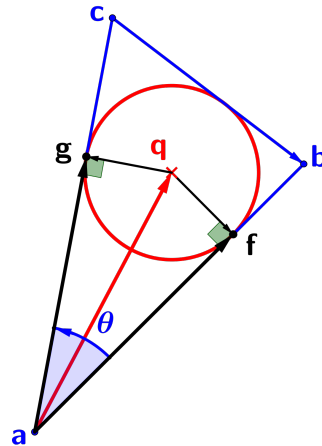
Therefore,

$$\mathbf{q} \cdot \hat{\mathbf{b}} = \frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{2}.$$

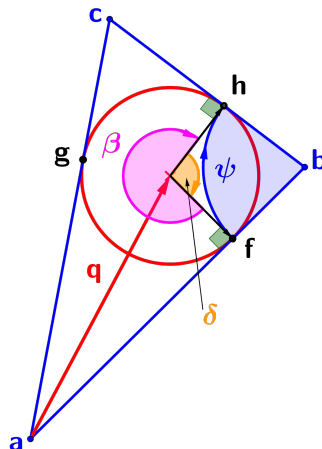
That result is sure to be important; we could solve for \mathbf{q} immediately if we knew the geometric product $\mathbf{q}\hat{\mathbf{b}}$, and all we need to do in order to form that product is to determine the outer product $\mathbf{q} \wedge \hat{\mathbf{b}}$. That outer product will probably arise somewhere in the expressions for rotations that we intend to examine, so let's turn to those now.

One of those rotations is that of vector \mathbf{f} through the angle θ to give vector \mathbf{g} :

$$\mathbf{g} = \mathbf{f}e^{\theta i}, \text{ with } e^{\theta i} = \hat{\mathbf{b}}\hat{\mathbf{c}}.$$



A more-exotic example is illustrated by the following diagram.



We express the vector \mathbf{h} in two ways: as \mathbf{q} plus the rotation of $\mathbf{f} - \mathbf{q}$ through the angle β ,

$$\mathbf{h} = \mathbf{q} + (\mathbf{f} - \mathbf{q})e^{\beta i}.$$

and as the vector \mathbf{b} plus the rotation of $\mathbf{f} - \mathbf{b}$ through the angle ψ :

$$\mathbf{h} = \mathbf{b} + (\mathbf{f} - \mathbf{b})e^{\psi i}.$$

The rotation operator $e^{\psi i}$ can be written as the geometric product $(-\hat{\mathbf{b}}) \left[\frac{\mathbf{c} - \mathbf{b}}{|\mathbf{c} - \mathbf{b}|} \right] = \frac{|\mathbf{b}| - \hat{\mathbf{b}}\mathbf{c}}{|\mathbf{c} - \mathbf{b}|}$. To derive an expression for β , we use the theorem about the measure of an angle drawn from a point exterior to a circle (p. 11). The angles β and δ are shown as negative (clockwise) angles in our diagram, so $\beta + \delta = -2\pi$. Therefore,

$$\begin{aligned} \psi &= \frac{1}{2} \left[\beta - \underbrace{(-2\pi - \beta)}_{=\delta} \right], \\ &= \beta + \pi, \end{aligned}$$

from which $\beta = \psi - \pi$ (which is clearly a negative angle).

These observations appear to have provided enough information —and in "GA-friendly form"—to solve the problem, so let's formulate a strategy.

Formulating a Strategy Our observations have suggested two strategies:

1. Identify \mathbf{q} from the known value of $\mathbf{q} \cdot \hat{\mathbf{b}}$ and from the value of $\mathbf{q} \cdot \hat{\mathbf{b}}$, which is still unknown, but which we should be able to determine by analyzing rotations; and
2. Equate the two expressions that we obtained for the vector h :

$$\mathbf{h} = \mathbf{b} + (\mathbf{f} - \mathbf{b}) e^{\psi i}.$$

and

$$\mathbf{h} = \mathbf{q} + (\mathbf{f} - \mathbf{q}) e^{\beta i}.$$

$$\text{with } \mathbf{f} = (\mathbf{q} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} = \left[\frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{2} \right] \hat{\mathbf{b}}, \text{ and}$$

$$e^{\beta i} = e^{(\psi - \pi)i} = e^{\psi i} \underbrace{e^{(-\pi)i}}_{=-1} = -e^{\psi i} = \frac{\hat{\mathbf{b}}\mathbf{c} - |\mathbf{b}|}{|\mathbf{c} - \mathbf{b}|}.$$

We'll use the first strategy because it appears to be simpler. However, this is a good moment to note that neither of the strategies makes use of an important observation that we made earlier: the point \mathbf{q} lies along the direction $\frac{\hat{\mathbf{b}} + \hat{\mathbf{c}}}{2}$. We can express that observation in terms such as

$$\mathbf{q} \wedge (\hat{\mathbf{b}} + \hat{\mathbf{c}}) = 0, \text{ and } \mathbf{q} \wedge \hat{\mathbf{b}} + \mathbf{q} \wedge \hat{\mathbf{c}} = 0.$$

Let's summarize the information that we've identified as relevant to the strategy we've chosen:

Although the solution via the second strategy is not presented in this document, that strategy does work. Equating the two expressions for h gives $\mathbf{q} + (\mathbf{f} - \mathbf{q}) e^{\beta i} = \mathbf{b} + (\mathbf{f} - \mathbf{b}) e^{\psi i}$, $\mathbf{q} + (\mathbf{q} - \mathbf{f}) e^{\psi i} = \mathbf{b} + (\mathbf{f} - \mathbf{b}) e^{\psi i}$, and $\mathbf{q}(1 + \mathbf{b}) e^{\psi i} = \mathbf{b}(1 - e^{\psi i}) + 2\mathbf{f}e^{\psi i}$. From there, we'd right-multiply both sides by $(1 + e^{\psi i})^{-1}$ to solve for \mathbf{q} . See Appendix A (p. 60) for the method.

An important piece of information that neither of our strategies uses.

$$\mathbf{q} \cdot \hat{\mathbf{b}} = \frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{2}$$

$$\mathbf{q} \wedge (\hat{\mathbf{b}} + \hat{\mathbf{c}}) = 0, \text{ or equivalently, } \mathbf{q} \wedge \hat{\mathbf{b}} + \mathbf{q} \wedge \hat{\mathbf{c}} = 0.$$

Having formed $\mathbf{q} \wedge (\hat{\mathbf{b}} + \hat{\mathbf{c}})$ so easily, and recognizing that we'd need to work a bit to find $\mathbf{q} \wedge \hat{\mathbf{c}}$, we might ask now whether we'd be better off finding $\mathbf{q} \cdot (\hat{\mathbf{b}} + \hat{\mathbf{c}})$, so that we can then find \mathbf{q} from the geometric product $\mathbf{q} (\hat{\mathbf{b}} + \hat{\mathbf{c}})$. This moment in our solution process is where our initial exploration of lengths of segments pays off: we found that $|\mathbf{g}| = |\mathbf{f}| = \mathbf{q} \cdot \hat{\mathbf{b}} = \frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{2}$. That additional information makes the route clear to us:

$$\begin{aligned} \mathbf{q} \cdot (\hat{\mathbf{b}} + \hat{\mathbf{c}}) &= \mathbf{q} \cdot \hat{\mathbf{b}} + \mathbf{q} \cdot \hat{\mathbf{c}} = |\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|; \\ \mathbf{q} \wedge (\hat{\mathbf{b}} + \hat{\mathbf{c}}) &= 0; \\ \therefore \mathbf{q} (\hat{\mathbf{b}} + \hat{\mathbf{c}}) &= \mathbf{q} \cdot \hat{\mathbf{b}} + \mathbf{q} \cdot \hat{\mathbf{c}} = |\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|. \end{aligned}$$

We'll find \mathbf{q} by solving that equation.

Solving the Equation Our equation is

$$\mathbf{q} (\hat{\mathbf{b}} + \hat{\mathbf{c}}) = \mathbf{q} \cdot \hat{\mathbf{b}} + \mathbf{q} \cdot \hat{\mathbf{c}} = |\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|,$$

which we solve via

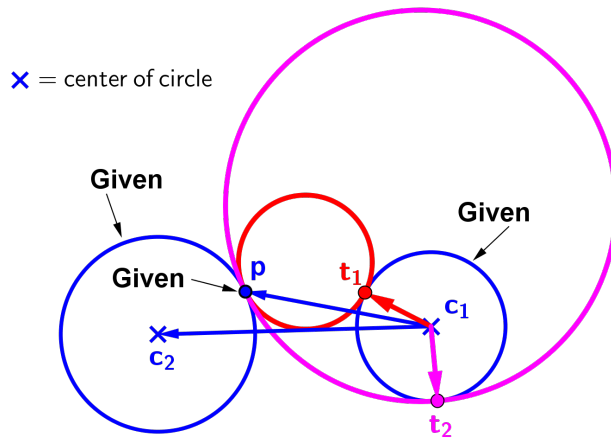
$$\begin{aligned} \mathbf{q} (\hat{\mathbf{b}} + \hat{\mathbf{c}}) (\hat{\mathbf{b}} + \hat{\mathbf{c}})^{-1} &= [|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|] (\hat{\mathbf{b}} + \hat{\mathbf{c}})^{-1} \\ \mathbf{q} &= [|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|] \left[\frac{\hat{\mathbf{b}} + \hat{\mathbf{c}}}{(\hat{\mathbf{b}} + \hat{\mathbf{c}})^2} \right] \\ &= [|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|] \left[\frac{\hat{\mathbf{b}} + \hat{\mathbf{c}}}{2 + 2\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}} \right] \\ &= \left[\frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}} \right] \left[\frac{\hat{\mathbf{b}} + \hat{\mathbf{c}}}{2} \right]. \end{aligned}$$

Interpreting the Solution The incenter lies along the bisector of the angle formed by sides \mathbf{b} and \mathbf{c} , at a distance from point \mathbf{a} equal to $\left[\frac{|\mathbf{b}| + |\mathbf{c}| - |\mathbf{c} - \mathbf{b}|}{1 + \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}} \right]$ times the length of $\frac{\hat{\mathbf{b}} + \hat{\mathbf{c}}}{2}$.

Because the assumptions that we made about the vertex \mathbf{a} apply to all three vertices of any triangle, our solution is valid for all vertices of every triangle. Therefore, the incenter is the point of intersection of the bisectors of the three angles formed by the sides of the triangle. This result is the same as that obtained via classical geometry.

4.4 Problem 4

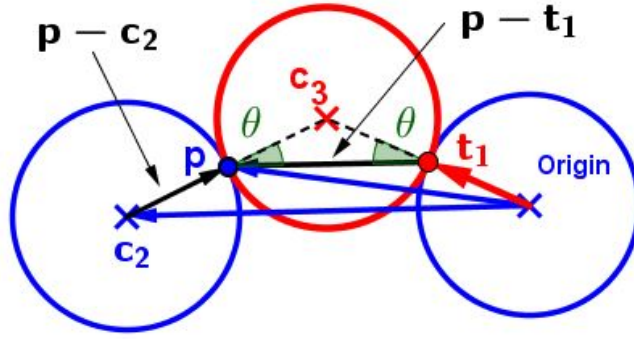
Given two circles, and a point p on one of them, construct the circles that are tangent to both of the given circles, with p being one of the points of tangency.



This problem has two solutions (i.e., the red and magenta circles). We'll find them in four ways, using two different concepts.

4.4.1 Solution Concept 1

In this first solution, we won't think about the problem in terms of rotating vectors; instead, we'll use the expressions that we've developed as a means of expressing angles between pairs of vectors in a convenient way. From there, we'll go on to solve for the vectors from the origin to the points of tangency between the given circles and the ones that we're asked to construct. Please note that the solution presented here, although it uses the same ideas as the one presented in "Answering Two Common Objections to Geometric Algebra" ([on YouTube](#), [on GeoGebraTube](#)), is considerably "cleaner" because we're using the starting point of the vectors t_1 and t_2 is used as the origin, rather than the center of the circle on which p is located.



The angle between the directions of vectors t_1 and $p-t_1$ is θ , in the positive (i.e., ccw) direction. Because the triangle pt_1c_2 is isosceles, the angle between the directions of t_1-p and $p-c_2$ is θ as well. Also, $|t_1| = r_1$, and $|p-c_2| = r_2$. Therefore,

$$\begin{aligned} t_1 (p - t_1) &= |t_1| |p - t_1| e^{\theta i} \\ &= r_1 |p - t_1| e^{\theta i}, \end{aligned}$$

Additional details on the maneuvers needed here can be found in Appendix C (page 74) and in "Answering Two Common Objections to Geometric Algebra":

[On YouTube](#)

[On GeoGebraTube](#).

In those sources, you'll also find other ways to solve such equations.

and

$$\begin{aligned} (t_1 - p) (p - c_2) &= |t_2| |p - c_2| e^{\theta i} \\ &= r_2 |p - t_1| e^{\theta i}, \end{aligned}$$

from which

$$(t_1 - p) (p - c_2) = \frac{r_2}{r_1} t_1 (p - t_1).$$

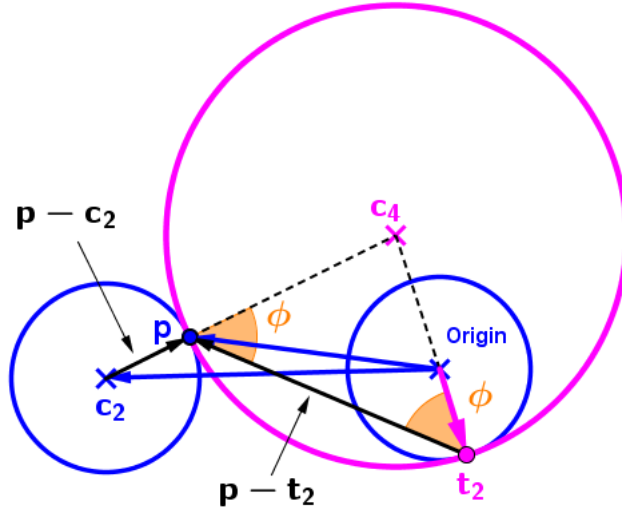
Expanding both sides, recognizing that $t_1^2 = r_1^2$, and rearranging, we obtain

$$t_1 \left[\left(\frac{r_2}{r_1} - 1 \right) p + c_2 \right] = r_1 r_2 + p c_2 - p^2.$$

Now, we right-multiply both sides by the inverse of $\left[\left(\frac{r_2}{r_1} - 1 \right) p + c_2 \right]$ to solve for t_1 . (Recall that the inverse of a vector v is $v/|v|^2$.) After rearranging the right-hand side, we arrive at

$$t_1 = \frac{\left[r_2^2 - r_1 r_2 + c_2^2 - \left(\frac{r_2}{r_1} - 1 \right) p^2 + 2 \left(\frac{r_2}{r_1} - 1 \right) p \cdot c_2 \right] p + \left[r_1 r_2 - \frac{r_2}{r_1} p^2 \right] c_2}{\left(\frac{r_2}{r_1} - 1 \right)^2 p^2 + c_2^2 + 2 \left(\frac{r_2}{r_1} - 1 \right) p \cdot c_2}.$$

To find t_2 , we recognize that the angles ϕ are equal.



Using the same ideas as in the solution for t_1 , we write

$$\begin{aligned} -t_2(\mathbf{p} - \mathbf{t}_2) &= |-t_2| |\mathbf{p} - \mathbf{t}_2| e^{\phi i} \\ &= r_1 |\mathbf{p} - \mathbf{t}_2| e^{\phi i}; \end{aligned}$$

and

$$\begin{aligned} (\mathbf{t}_2 - \mathbf{p})(\mathbf{p} - \mathbf{c}_2) &= |t_2| |\mathbf{p} - \mathbf{c}_2| e^{\phi i} \\ &= r_2 |\mathbf{p} - \mathbf{t}_2| e^{\phi i}, \end{aligned}$$

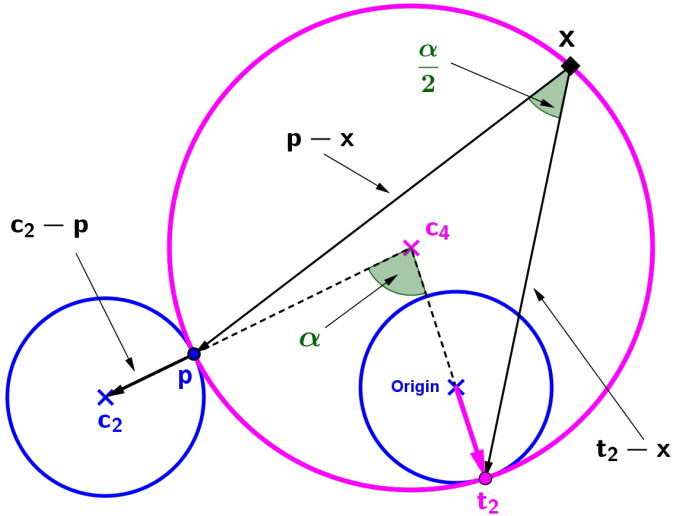
which leads to

$$t_2 \left[\left(\frac{r_2}{r_1} + 1 \right) \mathbf{p} - \mathbf{c}_2 \right] = r_1 r_2 - \mathbf{p} \mathbf{c}_2 + p^2,$$

and

$$t_2 = \frac{\left[r_2^2 + r_1 r_2 + c_2^2 + \left(\frac{r_2}{r_1} + 1 \right) p^2 - 2 \left(\frac{r_2}{r_1} + 1 \right) \mathbf{p} \cdot \mathbf{c}_2 \right] \mathbf{p} + \left[\frac{r_2}{r_1} p^2 - r_1 r_2 \right] \mathbf{c}_2}{\left(\frac{r_2}{r_1} + 1 \right)^2 p^2 + c_2^2 - 2 \left(\frac{r_2}{r_1} + 1 \right) \mathbf{p} \cdot \mathbf{c}_2}.$$

You've probably been thinking that the problem asks us to do more than find t_1 and t_2 : we're required to identify (that is, to give equations for) the tangent circles. So, now that we've found the points of tangency, how do we proceed?



One possibility is given in Hestenes D. 1999, pp. 88-89. We'll use the magenta circle as our example. Knowing t_2 , we can determine the angle α . Thus, we know that every point x on the magenta circle satisfies the condition expressed by the equation

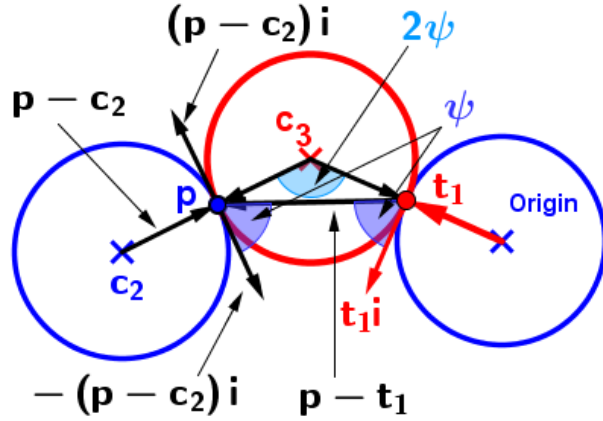
We can also express $e^{\frac{\alpha}{2}i}$ directly in terms of the vectors $c_2 - p$ and t_2 . See page 16 .

$$(p - x)^{-1} (t_2 - x) = \delta e^{\frac{\alpha}{2}i},$$

where δ is a scalar, $-\infty < \delta < \infty$. To each finite value of δ , there corresponds a unique point x ; the absolute value of δ increases without limit ("goes to infinity") as x approaches p .

4.4.2 Solution Concept 2

In our second Solution Concept, we makes life more difficult for ourselves—deliberately—in order to demonstrate ideas that will prove helpful in more-difficult problems later on. We begin by re-examining our figure, and noting from plane geometry, the angle between the vectors $p - c_3$ and $t_1 - c_3$ is 2ψ .



Thus

$$\mathbf{t}_1 - \mathbf{c}_3 = (\mathbf{p} - \mathbf{c}_3) e^{2\psi i}.$$

To obtain an expression for $e^{\psi i}$, we can use either

$$\frac{[-(\mathbf{p} - \mathbf{c}_2) i] (\mathbf{t}_1 - \mathbf{p})}{|(-\mathbf{p} - \mathbf{c}_2) i| |\mathbf{t}_1 - \mathbf{p}|} \quad \text{or} \quad \frac{(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i}{|\mathbf{p} - \mathbf{t}_1| |\mathbf{t}_1 i|}.$$

We'll opt for the latter, because it promises to be simpler to use. From the preceding, we can see that we can obtain the vector $\mathbf{t}_1 - \mathbf{c}_3$ by rotating the vector $\mathbf{p} - \mathbf{c}_3$ counterclockwise through the angle ψ twice:

$$(\mathbf{p} - \mathbf{c}_3) e^{\psi i} e^{\psi i} = \mathbf{t}_1 - \mathbf{c}_3.$$

That information doesn't appear useful until we recognize that $\mathbf{p} - \mathbf{c}_3 = -\frac{r_3}{r_2} (\mathbf{p} - \mathbf{c}_2)$, and $\mathbf{t}_1 - \mathbf{c}_3 = -\frac{r_3}{r_1} \mathbf{t}_1$. Making these substitutions, and using the expression that we chose for $e^{\psi i}$, the previous equation becomes

$$\underbrace{-\frac{r_3}{r_2} (\mathbf{p} - \mathbf{c}_2)}_{=\mathbf{p}-\mathbf{c}_3} \left[\frac{(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i}{|\mathbf{p} - \mathbf{t}_1| |\mathbf{t}_1 i|} \right] \left[\frac{(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i}{|\mathbf{p} - \mathbf{t}_1| |\mathbf{t}_1 i|} \right] = -\frac{r_3}{r_1} \mathbf{t}_1.$$

We can simplify that result, using $|\mathbf{t}_1 i| = r_1$, thereby finding that

$$(\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1.$$

Next, we eliminate the two factors i on the left-hand side by "bringing them together". To do so, we just make a series of "switches" of place between one of the i 's and an adjacent vector. We use the identity $i\mathbf{v} \equiv -\mathbf{v}i$ to keep track of sign changes:

$$\underbrace{(\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 i}_{\text{Repeating the previous equation}} = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1,$$

A word about what we're working toward here: We're going to try to form an equation in which one side is a product of vectors, and the other is either a pure scalar or a pure bivector. Then, we'll use postulates about the equality of multivectors to obtain an equation that we can solve simply for \mathbf{t}_1 .

"Switching places of i 's and vectors" is a common and important maneuver that we'll use many times in this document. You'll learn to simplify it quite soon: just examine the term on which you're working, and count the number of "switches" that will be needed to bring the i 's together within that term to make a "-1". That is, an " i^2 ".

If that number is even, then the sign of the term inverts; if odd, the sign remains unchanged. In the present example, we made two switches, so the sign inverted.

$$-(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1 \underbrace{(\mathbf{p} - \mathbf{t}_1)\mathbf{i}\mathbf{t}_1\mathbf{i}}_{1st\ switch} = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1,$$

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) \underbrace{\mathbf{t}_1\mathbf{i}}_{2nd} \mathbf{i} = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1,$$

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1\mathbf{i}^2 = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1,$$

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)(-1) = r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1,$$

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1.$$

Right-multiplying now by \mathbf{t}_1^{-1} , we obtain the key equation for Concept 2:

The key equation for Concept 2

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2.$$

The right-hand side is a scalar. That result deserves several comments. The first is that as we saw earlier, the geometric product of any three coplanar vectors is another vector in the same plane. Therefore, the geometric product of any four coplanar vectors is the sum of a scalar and a bivector, only. More to the point, because $(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)$ evaluates to a scalar, its bivector part is zero:

$$\langle (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) \rangle_2 = 0.$$

We'll see, shortly, how to make use of that fact, but first let's note another important aspect of our key equation: it contains the sequence of factors $(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)$, which is of the form $\mathbf{u}\mathbf{v}\mathbf{u}$. That's noteworthy because for any two vectors \mathbf{v} and $\hat{\mathbf{u}}$, the product $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ is the reflection of \mathbf{v} with respect to $\hat{\mathbf{u}}$. Hence, $(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)$ is the reflection of \mathbf{t}_1 with respect to $(\mathbf{p} - \mathbf{t}_1)$, and multiplied by the factor $|\mathbf{p} - \mathbf{t}_1|^2$. Based upon those observations, and upon $|\mathbf{p} - \mathbf{c}_2| = r_2$ and $\mathbf{t}_1 = r_1$, we can see that the equation

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2$$

tells us that $(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)$ is equal to $|\mathbf{p} - \mathbf{t}_1|^2 \mathbf{t}_1$ in magnitude, and is parallel to $\mathbf{p} - \mathbf{c}_2$, but opposite in direction.

Now that we've discussed some of the geometric significance of the equation

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2,$$

Now we see why we wanted (p. 37) an equation in which one side was either a pure scalar or a pure bivector: The left-hand side of our key equation is the product of an even number of vectors, so it must evaluate to a multivector that's the sum of a scalar and a bivector. Because the right-hand side is a pure scalar, the bivector part of the left-hand side must be zero.

we'll solve that equation in three ways.

Concept 2, Solution Method 1 As noted above, the right-hand side of our key equation (Page 38)

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2$$

is a scalar. Therefore,

$$\langle (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) \rangle_2 = 0.$$

To use that fact, we'll begin by expanding the left-hand side, then simplifying. (Again, $\mathbf{t}_1^2 = r_1^2$.)

$$\begin{aligned} (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) &= \mathbf{p}\mathbf{p}\mathbf{t}_1\mathbf{p} - \mathbf{p}\mathbf{p}\mathbf{t}_1\mathbf{t}_1 - \mathbf{p}\mathbf{t}_1\mathbf{t}_1\mathbf{p} + \mathbf{p}\mathbf{t}_1\mathbf{t}_1\mathbf{t}_1 - \mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{p} \\ &\quad + \mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{t}_1 + \mathbf{c}_2\mathbf{t}_1\mathbf{t}_1\mathbf{p} - \mathbf{c}_2\mathbf{t}_1\mathbf{t}_1\mathbf{t}_1 \\ &= p^2\mathbf{t}_1\mathbf{p} - 2r_1^2p^2 + r_1^2\mathbf{p}\mathbf{t}_1 - \mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{p} + 2r_1^2\mathbf{c}_2\mathbf{p} \\ &\quad - r_1^2\mathbf{c}_2\mathbf{t}_1. \end{aligned}$$

Now, we need to identify the bivector part of the simplified expansion. The bivector part of a sum of terms is the sum of the terms' respective bivector parts. (Note that $r_1^2p^2$ is a scalar, so its bivector part is zero.) The only term whose bivector part might cause us some trouble is $\mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{p}$. What is $\langle \mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{p} \rangle_2$?

Several different ways of identifying it are presented in the Appendix (7.2). The most straightforward way uses the identity that for any two vectors \mathbf{u} and \mathbf{v} , $\mathbf{u}\mathbf{v} \equiv 2\mathbf{u} \cdot \mathbf{v} - \mathbf{v}\mathbf{u}$. Therefore,

$$\begin{aligned} \langle \mathbf{c}_2\mathbf{p}\mathbf{t}_1\mathbf{p} \rangle_2 &= \langle \mathbf{c}_2(2\mathbf{p} \cdot \mathbf{t}_1 - \mathbf{t}_1\mathbf{p})\mathbf{p} \rangle_2 \\ &= \langle 2(\mathbf{p} \cdot \mathbf{t}_1)\mathbf{c}_2\mathbf{p} - p^2\mathbf{c}_2\mathbf{t}_1 \rangle_2 \\ &= 2(\mathbf{p} \cdot \mathbf{t}_1)\mathbf{c}_2 \wedge \mathbf{p} - p^2\mathbf{c}_2 \wedge \mathbf{t}_1. \end{aligned}$$

Using this expression, and our identity that for any two vectors \mathbf{u} and \mathbf{v} , $\mathbf{u} \wedge \mathbf{v} = [(\mathbf{u}\mathbf{i}) \cdot \mathbf{v}]\mathbf{i}$, we arrive at

$$\begin{aligned} \langle (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) \rangle_2 &= (p^2 - r_1^2)\mathbf{c}_2 \wedge \mathbf{t}_1 - (p^2 - r_1^2)\mathbf{p} \wedge \mathbf{t}_1 \\ &\quad - 2(\mathbf{p} \cdot \mathbf{t}_1)\mathbf{c}_2 \wedge \mathbf{p} + 2r_1^2\mathbf{c}_2 \wedge \mathbf{p} \\ &= (p^2 - r_1^2)[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{i} - (p^2 - r_1^2)[(\mathbf{p}\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{i} \\ &\quad - 2(\mathbf{p} \cdot \mathbf{t}_1)[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}]\mathbf{i} + 2r_1^2[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}]\mathbf{i}. \end{aligned}$$

Now, we can make use of the fact that $\langle (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) \rangle_2$ is zero:

$$(p^2 - r_1^2)[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{i} - (p^2 - r_1^2)[(\mathbf{p}\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{i} - 2(\mathbf{p} \cdot \mathbf{t}_1)[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}]\mathbf{i} + 2r_1^2[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}]\mathbf{i} = 0,$$

which (after eliminating the factor \mathbf{i}) we can rearrange as follows :

$$\underbrace{\{[2(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}]\mathbf{p} + (p^2 - r_1^2)(\mathbf{p}\mathbf{i} - \mathbf{c}_2\mathbf{i})\}}_{\text{We'll call this vector "z"}} \cdot \mathbf{t}_1 = 2r_1^2(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}.$$

Therefore,

$$\mathbf{t}_1 \cdot \hat{\mathbf{z}} = \frac{2r_1^2(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}}{|\mathbf{z}|}.$$

That result is useful to us because we also know that $\mathbf{t}_1 = (\mathbf{t}_1 \cdot \hat{\mathbf{z}})\hat{\mathbf{z}} + [\mathbf{t}_1 \cdot (\hat{\mathbf{z}}\mathbf{i})]\hat{\mathbf{z}}\mathbf{i}$, and $\mathbf{t}_1^2 = r_1^2$. From those facts, we can show that

$$\mathbf{t}_1 \cdot (\hat{\mathbf{z}}\mathbf{i}) = \pm \sqrt{r_1^2 - (\mathbf{t}_1 \cdot \hat{\mathbf{z}})^2}.$$

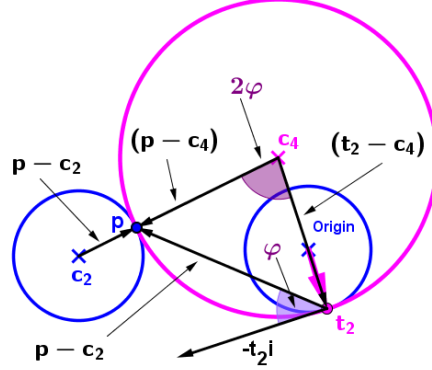
An important identity that's useful in solving equations that arise when working with rotations:

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} &= [(\mathbf{u}\mathbf{i}) \cdot \mathbf{v}]\mathbf{i} \\ &= -[\mathbf{u} \cdot (\mathbf{v}\mathbf{i})]\mathbf{i}. \end{aligned}$$

But which root is correct: the positive, or the negative? To answer that question, let's attempt to identify the vector \mathbf{t}_2 .

Looking again at our diagram, we find that

$$(\mathbf{p} - \mathbf{c}_4) e^{2\varphi i} = \mathbf{t}_2 - \mathbf{c}_4.$$



Proceeding now in the same way that we did to find \mathbf{t}_1 ,

$$\begin{aligned} \left[-\frac{r_4}{r_2} (\mathbf{p} - \mathbf{c}_2) \right] e^{\varphi i} e^{\varphi i} &= \frac{r_4}{r_1} \mathbf{t}_2 \\ \left[-\frac{r_4}{r_2} (\mathbf{p} - \mathbf{c}_2) \right] \underbrace{\left[\frac{(\mathbf{p} - \mathbf{t}_2) (-\mathbf{t}_2 i)}{|\mathbf{p} - \mathbf{t}_2| |-\mathbf{t}_2 i|} \right]}_{=e^{\varphi i}} \left[\frac{(\mathbf{p} - \mathbf{t}_2) (-\mathbf{t}_2 i)}{|(-\mathbf{t}_2 i)| |-\mathbf{t}_2 i|} \right] &= \frac{r_4}{r_1} \mathbf{t}_2 \\ (\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 (\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 &= \frac{r_2}{r_1} |\mathbf{p} - \mathbf{t}_2|^2 |\mathbf{t}_2|^2 \mathbf{t}_2 \end{aligned}$$

Therefore,

$$(\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 (\mathbf{p} - \mathbf{t}_2) = r_1 r_2 |\mathbf{p} - \mathbf{t}_2|^2.$$

Let's compare that equation to the corresponding one that we obtained for \mathbf{t}_1 (Page 38):

$$(\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2$$

The right-hand sides of the two equations have opposite algebraic signs. Those signs, plus the fact that the right-hand sides are scalars, tell us that the vector $(\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 (\mathbf{p} - \mathbf{t}_2)$ is parallel to $\mathbf{p} - \mathbf{c}_2$, while the vector $(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1)$ is anti-parallel to it. Therefore, \mathbf{t}_1 and \mathbf{t}_2 are not the same vector.

Nevertheless, both of those vectors have the same length ($= r_1$), and both have the same projection upon \mathbf{z} , as we can show by recognizing that $\langle (\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 (\mathbf{p} - \mathbf{t}_2) \rangle_2 = 0$, then proceeding as we did for \mathbf{t}_1 . In this way we arrive at

$$\underbrace{\{ [2(\mathbf{c}_2 i) \cdot \mathbf{p}] \mathbf{p} + (p^2 - r_1^2) (\mathbf{p} i - \mathbf{c}_2 i) \}}_{\text{This is the same vector "z" as for } \mathbf{t}_1} \cdot \mathbf{t}_2 = \underbrace{2r_1^2 (\mathbf{c}_2 i) \cdot \mathbf{p}}_{=\mathbf{t}_1 \cdot \mathbf{z}}$$

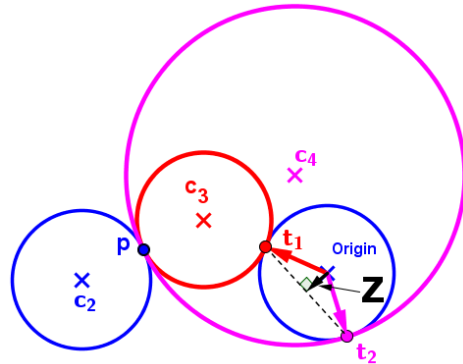
$$\therefore \mathbf{t}_2 \cdot \hat{\mathbf{z}} = \frac{2r_1^2 (\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}}{|\mathbf{z}|} = \mathbf{t}_1 \cdot \hat{\mathbf{z}}.$$

Their components perpendicular to \mathbf{z} are of equal length $\left(= \sqrt{r_1^2 - (\mathbf{t}_1 \cdot \hat{\mathbf{z}})^2} \right)$, but are oppositely directed. (That's why the vectors $(\mathbf{p} - \mathbf{t}_2) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1)$ and $(\mathbf{p} - \mathbf{t}_2) \mathbf{t}_2 (\mathbf{p} - \mathbf{t}_2)$ are oppositely directed.) Therefore, the solution obtained by using the present method is

$$\mathbf{t} = \left[\frac{2r_1^2 (\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}}{|\mathbf{z}|} \right] \hat{\mathbf{z}} \pm \left\{ \sqrt{r_1^2 - \left[\frac{2r_1^2 (\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}}{|\mathbf{z}|} \right]^2} \right\} \hat{\mathbf{z}} \mathbf{i},$$

where

$$\mathbf{z} = [2 (\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}] \mathbf{p} + (p^2 - r_1^2) (\mathbf{p} \mathbf{i} - \mathbf{c}_2 \mathbf{i}).$$



That form of the solution is satisfactory for computing \mathbf{t} ; if we wished, we could transform it into another form by using ideas presented in the worksheet and video, "Find unknown vector from two dot products" (as [GeoGebra worksheet](#), as [YouTube video](#)).

Concept 2, Solution Method 2 We'll start from our key equation for this concept (Page 38):

$$(\mathbf{p} - \mathbf{c}_2) (\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2.$$

In Solution Method 1 for this concept, we used the fact that the left-hand side evaluates to a scalar, and thereby arrived at an equation for the dot product $\mathbf{t}_1 \cdot \mathbf{z}$. However, we made no use of the fact the left-hand side evaluates to the specific scalar $-r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2$. In the present Method, we'll use that information to find \mathbf{t}_1 's dot product with a second vector, which we'll call $\hat{\mathbf{w}}_1$. Knowing \mathbf{t}_1 's dot products with those two vectors, we'll be able to find \mathbf{t}_1 using methods developed in the worksheet and video, "Find unknown vector from two dot products" ([As GeoGebra worksheet](#), [As YouTube video](#)).

From Method 1, we also know that $(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1)$ can be expanded as

$$\begin{aligned} (\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1)\mathbf{t}_1(\mathbf{p} - \mathbf{t}_1) &= p^2\mathbf{t}_1\mathbf{p} + r_1^2\mathbf{p}\mathbf{t}_1 - r_1^2\mathbf{c}_2\mathbf{t}_1 + 2r_1^2\mathbf{c}_2\mathbf{p} - 2r_1^2p^2 \\ &\quad - \underbrace{2(\mathbf{p} \cdot \mathbf{t}_1)\mathbf{c}_2\mathbf{p} - p^2\mathbf{c}_2\mathbf{t}_1}_{=c_2\mathbf{p}\mathbf{t}_1\mathbf{p}}. \end{aligned}$$

Because $|\mathbf{p} - \mathbf{t}_1|^2$ is $(\mathbf{p} - \mathbf{t}_1)^2$,

$$p^2\mathbf{t}_1\mathbf{p} + r_1^2\mathbf{p}\mathbf{t}_1 - r_1^2\mathbf{c}_2\mathbf{t}_1 + 2r_1^2\mathbf{c}_2\mathbf{p} - 2r_1^2p^2 - 2(\mathbf{p} \cdot \mathbf{t}_1)\mathbf{c}_2\mathbf{p} - p^2\mathbf{c}_2\mathbf{t}_1 = -r_1r_2 \underbrace{(p^2 - 2\mathbf{p} \cdot \mathbf{t}_1 + r_1^2)}_{=(\mathbf{p}-\mathbf{t}_1)^2}.$$

Equating the scalar parts of the two sides, and rearranging, we obtain

$$\underbrace{\{[p^2 + r_1^2 - 2(r_1r_2 + \mathbf{c}_2 \cdot \mathbf{p})]\mathbf{p} + (p^2 - r_1^2)\mathbf{c}_2\}}_{\text{We'll call this } \mathbf{w}_1} \cdot \mathbf{t}_1 = 2r_1^2(p^2 - \mathbf{c}_2 \cdot \mathbf{p}) - r_1r_2(p^2 + r_1^2).$$

Similarly, starting from

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_2)\mathbf{t}_2(\mathbf{p} - \mathbf{t}_2) = r_1r_2|\mathbf{p} - \mathbf{t}_2|^2,$$

we find that

$$\underbrace{\{[p^2 + r_1^2 + 2(r_1r_2 - \mathbf{c}_2 \cdot \mathbf{p})]\mathbf{p} + (p^2 - r_1^2)\mathbf{c}_2\}}_{\text{We'll call this } \mathbf{w}_2} \cdot \mathbf{t}_2 = 2r_1^2(p^2 - \mathbf{c}_2 \cdot \mathbf{p}) + r_1r_2(p^2 + r_1^2).$$

We now have a pair of "dot-product" equations for each of the vectors \mathbf{t}_1 and \mathbf{t}_2 :

$$\begin{aligned} \mathbf{z} \cdot \mathbf{t}_1 &= 2r_1^2(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}, \\ \mathbf{w}_1 \cdot \mathbf{t}_1 &= 2r_1^2(p^2 - \mathbf{c}_2 \cdot \mathbf{p}) - r_1r_2(p^2 + r_1^2). \end{aligned}$$

$$\begin{aligned} \mathbf{z} \cdot \mathbf{t}_2 &= 2r_1^2(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{p}, \\ \mathbf{w}_2 \cdot \mathbf{t}_2 &= 2r_1^2(p^2 - \mathbf{c}_2 \cdot \mathbf{p}) + r_1r_2(p^2 + r_1^2). \end{aligned}$$

In the worksheet and video "Find unknown vector from two dot products", we learned that given the dot products of an unknown vector \mathbf{x} with two known vectors \mathbf{a} and \mathbf{b} , we can find \mathbf{x} by writing it as a linear combination of \mathbf{a} and \mathbf{b} ,

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b},$$

from which we can then form the pair of linear equations

$$\begin{aligned} \mathbf{a} \cdot \mathbf{x} &= \alpha a^2 + \beta\mathbf{a} \cdot \mathbf{b}, \\ \mathbf{b} \cdot \mathbf{x} &= \alpha\mathbf{a} \cdot \mathbf{b} + \beta b^2. \end{aligned}$$

Solving those equations for α and β , we obtain

$$\alpha = \frac{b^2 \mathbf{a} \cdot \mathbf{x} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \cdot \mathbf{x})}{a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2},$$

$$\beta = \frac{a^2 \mathbf{b} \cdot \mathbf{x} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{x})}{a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2}.$$

Making use of that solution, we find that

$$\mathbf{t}_1 = \alpha_1 \mathbf{w}_1 + \beta_1 \mathbf{z},$$

where

$$\alpha_1 = \frac{z^2 \mathbf{w}_1 \cdot \mathbf{t}_1 - (\mathbf{w}_1 \cdot \mathbf{z})(\mathbf{z} \cdot \mathbf{t}_1)}{w_1^2 z^2 - (\mathbf{w}_1 \cdot \mathbf{z})^2},$$

$$\beta = \frac{w_1^2 \mathbf{z} \cdot \mathbf{t}_1 - (\mathbf{w}_1 \cdot \mathbf{z})(\mathbf{w}_1 \cdot \mathbf{t}_1)}{w_1^2 z^2 - (\mathbf{w}_1 \cdot \mathbf{z})^2},$$

$$\mathbf{w}_1 = [p^2 + r_1^2 - 2(r_1 r_2 + \mathbf{c}_2 \cdot \mathbf{p})] \mathbf{p} + (p^2 - r_1^2) \mathbf{c}_2,$$

$$\mathbf{w}_2 = [p^2 + r_1^2 + 2(r_1 r_2 - \mathbf{c}_2 \cdot \mathbf{p})] \mathbf{p} + (p^2 - r_1^2) \mathbf{c}_2,$$

$$\mathbf{z} = [2(\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}] \mathbf{p} + (p^2 - r_1^2) (\mathbf{p} \mathbf{i} - \mathbf{c}_2 \mathbf{i}),$$

$$\mathbf{w}_1 \cdot \mathbf{t}_1 = 2r_1^2 (p^2 - \mathbf{c}_2 \cdot \mathbf{p}) - r_1 r_2 (p^2 + r_1^2),$$

$$\mathbf{w}_1 \cdot \mathbf{t}_2 = 2r_1^2 (p^2 - \mathbf{c}_2 \cdot \mathbf{p}) + r_1 r_2 (p^2 + r_1^2),$$

$$\mathbf{z} \cdot \mathbf{t}_1 = \mathbf{z} \cdot \mathbf{t}_2 = 2r_1^2 (\mathbf{c}_2 \mathbf{i}) \cdot \mathbf{p}.$$

Concept 2, Solution Method 3 Anyone who's studied Geometric Algebra—even casually—knows that the method usually prescribed for solving for an unknown vector \mathbf{x} in a given problem is to find \mathbf{x} 's inner and outer products with a known vector \mathbf{b} , then proceed as follows:

$$\mathbf{x} \cdot \mathbf{b} + \mathbf{x} \wedge \mathbf{b} = \mathbf{x} \mathbf{b}$$

$$\mathbf{x} = (\mathbf{x} \mathbf{b}) \mathbf{b}^{-1}$$

Very antiseptic-looking! However, in this section, we'll learn that $\mathbf{x} \cdot \mathbf{b}$ and $\mathbf{x} \wedge \mathbf{b}$ can take quite-complicated forms.

As we did in previous sections, we'll start with our key equation:

$$(\mathbf{p} - \mathbf{c}_2)(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1) = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2.$$

Again, we'll expand the left-hand side. But this time, we'll maintain $\mathbf{p} - \mathbf{c}_2$ intact, as a single factor:

$$(\mathbf{p} - \mathbf{c}_2) \underbrace{(\mathbf{p} \mathbf{t}_1 \mathbf{p} - 2r_1^2 \mathbf{p} + r_1^2 \mathbf{t}_1)}_{=(\mathbf{p} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{p} - \mathbf{t}_1)} = -r_1 r_2 |\mathbf{p} - \mathbf{t}_1|^2.$$

Examining the left-hand side, we see an interesting possibility: if we left-multiply by $(\mathbf{p} - \mathbf{c}_2)^{-1}$, and right-multiply by \mathbf{p} , we'll obtain a sum of three terms: a scalar, and two terms that are scalar multiples of geometric products of \mathbf{p} and \mathbf{t}_1 . On the right-hand side, we'll obtain a scalar multiple of the product $(\mathbf{p} - \mathbf{c}_2)\mathbf{p}$:

$$\begin{aligned} (\mathbf{p} - \mathbf{c}_2)^{-1} [(\mathbf{p} - \mathbf{c}_2)(\mathbf{p}\mathbf{t}_1\mathbf{p} - 2r_1^2\mathbf{p} + r_1^2\mathbf{t}_1)] \mathbf{p} &= (\mathbf{p} - \mathbf{c}_2)^{-1} [-r_1r_2|\mathbf{p} - \mathbf{t}_1|^2] \mathbf{p} \\ p^2\mathbf{p}\mathbf{t}_1 + r_1^2\mathbf{t}_1\mathbf{p} - 2r_1^2p^2 &= -\frac{r_1r_2|\mathbf{p} - \mathbf{t}_1|^2}{(\mathbf{p} - \mathbf{c}_2)^2}(\mathbf{p} - \mathbf{c}_2)\mathbf{p} = \frac{r_1r_2|\mathbf{p} - \mathbf{t}_1|^2}{(\mathbf{p} - \mathbf{c}_2)^2}(\mathbf{c}_2\mathbf{p} - p^2). \end{aligned}$$

Now, we can develop expressions for $\mathbf{t}_1 \cdot \mathbf{p}$ and $\mathbf{t}_1 \wedge \mathbf{p}$. Equating the scalar parts of both sides, and solving for $\mathbf{t}_1 \cdot \mathbf{p}$, we find that

$$\mathbf{t}_1 \cdot \mathbf{p} = \frac{2r_1^2p^2(\mathbf{p} - \mathbf{c}_2)^2 - r_1r_2(p^2 + r_1^2)(p^2 - \mathbf{c}_2 \cdot \mathbf{p})}{(p^2 + r_1^2)(\mathbf{p} - \mathbf{c}_2)^2 - 2r_1r_2(p^2 - \mathbf{c}_2 \cdot \mathbf{p})}.$$

Equating the bivector parts of both sides, we obtain

$$\mathbf{t}_1 \wedge \mathbf{p} = \frac{r_1r_2(p^2 - 2\mathbf{t}_1 \cdot \mathbf{p} + r_1^2)}{(p^2 - r_1^2)(\mathbf{p} - \mathbf{c}_2)^2} \underbrace{[(\mathbf{p}\mathbf{i}) \cdot \mathbf{c}_2]}_{=\mathbf{p} \wedge \mathbf{c}_2} \mathbf{i}.$$

Note that our expression for $\mathbf{t}_1 \wedge \mathbf{p}$ contains the quantity $\mathbf{t}_1 \cdot \mathbf{p}$. That's fine, because the expression for $\mathbf{t}_1 \cdot \mathbf{p}$ allows us to calculate the value of $\mathbf{t}_1 \cdot \mathbf{p}$ directly from known quantities (r_1 , r_2 , \mathbf{p} , and \mathbf{c}_2), after which we'd use that value to calculate $\mathbf{t}_1 \wedge \mathbf{p}$. If we wished, we could also do the work symbolically. That is, we could substitute our expression for $\mathbf{t}_1 \cdot \mathbf{p}$ into our expression for $\mathbf{t}_1 \wedge \mathbf{p}$, then do the algebra to obtain an expression for $\mathbf{t}_1 \wedge \mathbf{p}$ in terms of the known quantities.

Having obtained these expressions for $\mathbf{t}_1 \cdot \mathbf{p}$ and $\mathbf{t}_1 \wedge \mathbf{p}$, how do we use them to determine \mathbf{t}_1 ? This is where we need to understand the meaning of the symbol "+" in the "antiseptic" version of the solution

$$\begin{aligned} \mathbf{t}_1 \cdot \mathbf{p} + \mathbf{t}_1 \wedge \mathbf{p} &= \mathbf{t}_1\mathbf{p} \\ \mathbf{t}_1 &= (\mathbf{t}_1\mathbf{p})\mathbf{p}^{-1}. \end{aligned}$$

We'll see a brief explanation here; more details are given in the worksheet and video "Answering Two Common Objections to Geometric Algebra" ([As GeoGebra worksheet](#), [As YouTube video](#)). To solve for \mathbf{t}_1 , starting from the preceding equations, we write

$$\begin{aligned} \mathbf{t}_1 &= \underbrace{(\mathbf{t}_1 \cdot \mathbf{p} + \mathbf{t}_1 \wedge \mathbf{p})}_{=\mathbf{t}_1\mathbf{p}} \mathbf{p}^{-1} \\ &= (\mathbf{t}_1 \cdot \mathbf{p})\mathbf{p}^{-1} + (\mathbf{t}_1 \wedge \mathbf{p})\mathbf{p}^{-1} \\ &= \left[\frac{\mathbf{t}_1 \cdot \mathbf{p}}{p^2} \right] \mathbf{p} + \left\{ \frac{\mathbf{t}_1 \wedge \mathbf{p}}{p^2} \right\} \mathbf{p}. \end{aligned}$$

Every exterior product (including $\mathbf{t}_1 \wedge \mathbf{p}$) is a scalar multiple of the unit bivector \mathbf{i} . Therefore, the second term on the right-hand side of the above equation is a scalar multiple of $\mathbf{i}\mathbf{p}$, which is a vector. For that reason, the second term is also a scalar multiple of $-\mathbf{p}\mathbf{i}$, where $\mathbf{p}\mathbf{i}$ is the 90° counter-clockwise rotation of \mathbf{p} .

$$\text{Therefore, the solution } \mathbf{t}_1 = \left[\frac{\mathbf{t}_1 \cdot \mathbf{p}}{p^2} \right] \mathbf{p} + \left\{ \frac{\mathbf{t}_1 \wedge \mathbf{p}}{p^2} \right\} \mathbf{p}$$

expresses \mathbf{t}_1 as a linear combination of the vectors \mathbf{p} and $\mathbf{p}\mathbf{i}$.

The analysis that we've just seen shows why we can view the operation represented by the symbol "+" in the definition $\mathbf{x} \cdot \mathbf{b} + \mathbf{x} \wedge \mathbf{p} = \mathbf{x}\mathbf{b}$ as a "latent vector addition" that becomes "activated" when right-multiplied by a vector. When multiplied by the vector \mathbf{b}^{-1} , the result is \mathbf{x} .

With that understanding, we can see how to put the product $(\mathbf{t}_1 \wedge \mathbf{p})\mathbf{p}^{-1}$ into a form that's useful to us:

$$\begin{aligned} (\mathbf{t}_1 \wedge \mathbf{p})\mathbf{p}^{-1} &= \left\{ \frac{r_1 r_2 (p^2 - 2\mathbf{t}_1 \cdot \mathbf{p} + r_1^2)}{(p^2 + r_1^2)(\mathbf{p} - \mathbf{c}_2)^2} [(\mathbf{p}\mathbf{i}) \cdot \mathbf{c}_2] \mathbf{i} \right\} \underbrace{\frac{\mathbf{p}}{p^2}}_{=\mathbf{p}^{-1}} \\ &= - \left\{ \frac{r_1 r_2 (p^2 - 2\mathbf{t}_1 \cdot \mathbf{p} + r_1^2)}{p^2 (p^2 + r_1^2)(\mathbf{p} - \mathbf{c}_2)^2} [(\mathbf{p}\mathbf{i}) \cdot \mathbf{c}_2] \right\} \mathbf{p}\mathbf{i}. \end{aligned}$$

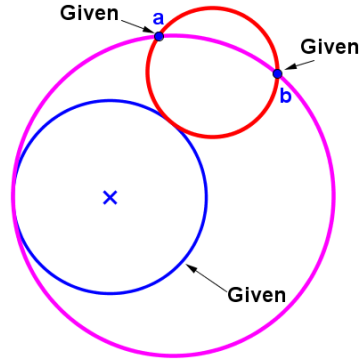
Therefore, the solution given by the present method is

$$\begin{aligned} \mathbf{t}_1 \cdot \mathbf{p} &= \left\{ \frac{2r_1^2 p^2 (\mathbf{p} - \mathbf{c}_2)^2 - r_1 r_2 (p^2 + r_1^2) (p^2 - \mathbf{c}_2 \cdot \mathbf{p})}{p^2 [(p^2 + r_1^2)(\mathbf{p} - \mathbf{c}_2)^2 - 2r_1 r_2 (p^2 - \mathbf{c}_2 \cdot \mathbf{p})]} \right\} \mathbf{p} \\ &\quad \underbrace{\hspace{10em}}_{=(\mathbf{t}_1 \cdot \mathbf{p})/p^2} \\ &\quad - \left\{ \frac{r_1 r_2 (p^2 - 2\mathbf{t}_1 \cdot \mathbf{p} + r_1^2)}{p^2 (p^2 + r_1^2)(\mathbf{p} - \mathbf{c}_2)^2} [(\mathbf{p}\mathbf{i}) \cdot \mathbf{c}_2] \right\} \mathbf{p}\mathbf{i}. \end{aligned}$$

Again, this solution could be simplified algebraically. (We'll omit the solution for \mathbf{t}_2 .)

4.5 Problem 5

Given a circle and two points outside of it, identify the circles that are tangent to the given one, and that pass through both of the given points.



We'll see three ways to arrive at equations that can be solved for the points of tangency by using Method 1 from Problem 4, Concept 2 (4.4.2).

4.5.1 Solution Concept 1

How would we proceed after identifying the points of tangency?

As is known from classical geometry, we can construct a circle if we know any three points that lie on it. One of the GA expressions of that truth is that if c , d , and e are three known points on a circle, then every other point x on that circle must satisfy the condition expressed by the "cross-ratio" equation

$$\frac{(c - x)^{-1} (d - x)}{(c - e)^{-1} (d - e)} = a \text{ scalar.}$$

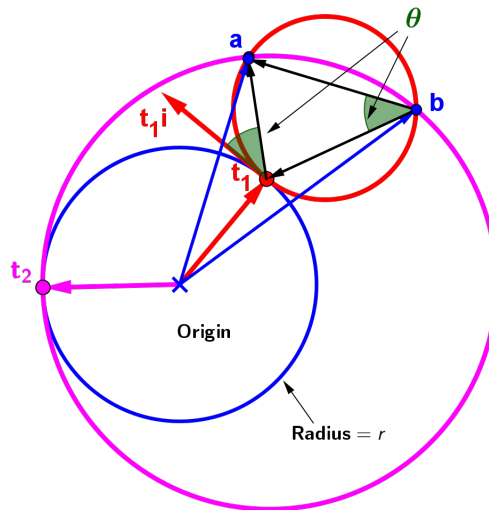
For details, see Hestenes D. 1999, p. 89.

In our case, for the circle that contains t_1 , the known points would be a , b , and t_1 , so one version of the cross-ratio equation for that circle would be

$$\frac{(a - x)^{-1} (b - x)}{(a - t_1)^{-1} (b - t_1)} = a \text{ scalar.}$$

Other versions can be obtained by interchanging a , b , and t_1 .

This way is the simplest: we'll begin by identifying the elements shown in the following figure:



Now, we obtain two expressions for $e^{\theta i}$, and equate them to each other. We'll use the fact that $|t_1 i| = |t_1| = r$:

$$\frac{(a - b)(t_1 - b)}{|a - b| |t_1 - b|} = \frac{(a - t_1) t_1 i}{|a - t_1| |t_1 i|}.$$

Both sides are expressions for $e^{\theta i}$

Next, we form an equation in which one side is a product of the vectors that are involved, and the other side is either a scalar or a bivector:

$$\begin{aligned}
(\mathbf{a} - \mathbf{t}_1) \frac{(\mathbf{a} - \mathbf{b})(\mathbf{t}_1 - \mathbf{b})}{|\mathbf{a} - \mathbf{b}| |\mathbf{t}_1 - \mathbf{b}|} \mathbf{t}_1 i &= (\mathbf{a} - \mathbf{t}_1) \frac{(\mathbf{a} - \mathbf{t}_1) \mathbf{t}_1 i}{|\mathbf{a} - \mathbf{t}_1| |\mathbf{t}_1 i|} \mathbf{t}_1 i \\
(\mathbf{a} - \mathbf{t}_1) (\mathbf{a} - \mathbf{b})(\mathbf{t}_1 - \mathbf{b}) \mathbf{t}_1 i (-i) &= |\mathbf{a} - \mathbf{t}_1| |\mathbf{t}_1 - \mathbf{b}| (\mathbf{a} - \mathbf{t}_1) r (-i) \\
(\mathbf{a} - \mathbf{t}_1) (\mathbf{a} - \mathbf{b})(\mathbf{t}_1 - \mathbf{b}) \mathbf{t}_1 &= -|\mathbf{a} - \mathbf{t}_1| |\mathbf{t}_1 - \mathbf{b}| (\mathbf{a} - \mathbf{t}_1) r i.
\end{aligned}$$

The right-hand side is a bivector, so $\langle (\mathbf{a} - \mathbf{t}_1) (\mathbf{a} - \mathbf{b})(\mathbf{t}_1 - \mathbf{b}) \mathbf{t}_1 \rangle_0 = 0$. A convenient way to expand $(\mathbf{a} - \mathbf{t}_1) (\mathbf{a} - \mathbf{b})(\mathbf{t}_1 - \mathbf{b}) \mathbf{t}_1$ is via

$$[a^2 - \mathbf{a}\mathbf{b} - \mathbf{t}_1\mathbf{a} + \mathbf{t}_1\mathbf{b}] [t_1^2 + \mathbf{t}_1\mathbf{b}],$$

which works out to

$$(b^2 - r^2) \mathbf{a}\mathbf{t}_1 - (a^2 - r^2) \mathbf{b}\mathbf{t}_1 + r^2 (a^2 - b^2) - r^2 \mathbf{a}\mathbf{b} + \mathbf{t}_1 \mathbf{a}\mathbf{b}\mathbf{t}_1.$$

Therefore,

$$\langle (b^2 - r^2) \mathbf{a}\mathbf{t}_1 - (a^2 - r^2) \mathbf{b}\mathbf{t}_1 + r^2 (a^2 - b^2) - r^2 \mathbf{a}\mathbf{b} + \mathbf{t}_1 \mathbf{a}\mathbf{b}\mathbf{t}_1 \rangle_0 = 0.$$

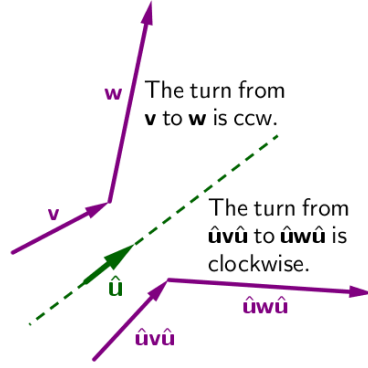
The term $\mathbf{t}_1 \mathbf{a}\mathbf{b}\mathbf{t}_1$ is interesting. At the beginning of 4.4.2, we saw that the product $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ evaluates to a vector: specifically, the reflection of \mathbf{v} with respect to $\hat{\mathbf{u}}$. Similarly, the product $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}}$ is the reflection of the geometric product $\mathbf{v}\mathbf{w}$. But let's see exactly why that is, and what it means. We'll discover that the scalar part of $\mathbf{v}\mathbf{w}$: is unaffected by the reflection, but the bivector part is reversed, so that $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} = \mathbf{w}\mathbf{v}$:

$$\begin{aligned}
\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} &= \hat{\mathbf{u}} (\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}) \hat{\mathbf{u}} \\
&= \hat{\mathbf{u}} (\mathbf{v} \cdot \mathbf{w}) \hat{\mathbf{u}} + \hat{\mathbf{u}} (\mathbf{v} \wedge \mathbf{w}) \hat{\mathbf{u}} \\
&= \hat{\mathbf{u}}^2 (\mathbf{v} \cdot \mathbf{w}) + \hat{\mathbf{u}} [(\mathbf{v}i) \cdot \mathbf{w}] i \hat{\mathbf{u}} \\
&= \mathbf{v} \cdot \mathbf{w} + \hat{\mathbf{u}} [-\mathbf{v} \cdot (\mathbf{w}i)] (-\hat{\mathbf{u}}i) \\
&= \mathbf{v} \cdot \mathbf{w} + \hat{\mathbf{u}}^2 [(\mathbf{w}i) \cdot \mathbf{v}] i \\
&= \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \wedge \mathbf{v} \\
&= \mathbf{w}\mathbf{v}.
\end{aligned}$$

Another interesting aspect of the product $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}}$ is that the reflection of the exterior product of \mathbf{v} and \mathbf{w} is equal to the exterior product of the two vectors' reflections:

$$\begin{aligned}
\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} &= \hat{\mathbf{u}}\mathbf{v} (\hat{\mathbf{u}}\hat{\mathbf{u}}) \mathbf{w}\hat{\mathbf{u}} \\
&= (\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}) (\hat{\mathbf{u}}\mathbf{w}\hat{\mathbf{u}}).
\end{aligned}$$

That observation provides a geometric interpretation of why reflecting a bivector changes its sign: the direction of the turn from \mathbf{v} to \mathbf{w} reverses.



Returning now to $\mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1$, we see that

$$\begin{aligned} \mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1 &= |\mathbf{t}_1| \hat{\mathbf{t}}_1 \mathbf{a} \mathbf{b} |\mathbf{t}_1| \hat{\mathbf{t}}_1 \\ &= |\mathbf{t}_1|^2 \hat{\mathbf{t}}_1 \mathbf{a} \mathbf{b} \hat{\mathbf{t}}_1 \\ &= r^2 \mathbf{b} \mathbf{a}. \end{aligned}$$

We derived that equivalence so we could deal with the term $\mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1$ in the equation

$$\langle (b^2 - r^2) \mathbf{a} \mathbf{t}_1 - (a^2 - r^2) \mathbf{b} \mathbf{t}_1 + r^2 (a^2 - b^2) - r^2 \mathbf{a} \cdot \mathbf{b} + \mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1 \rangle_0 = 0.$$

Making that substitution, and taking the scalar part of each term,

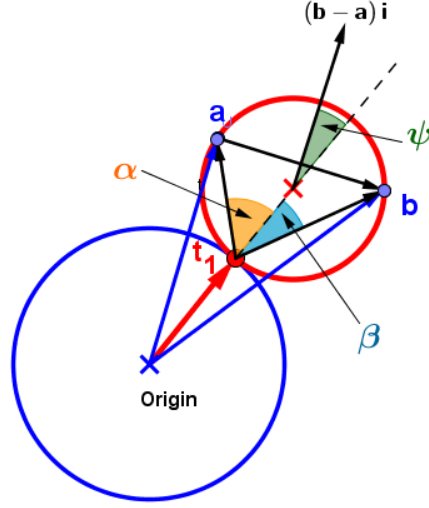
$$\begin{aligned} (b^2 - r^2) \mathbf{a} \cdot \mathbf{t}_1 - (a^2 - r^2) \mathbf{b} \cdot \mathbf{t}_1 + r^2 (a^2 - b^2) - r^2 \mathbf{a} \cdot \mathbf{b} + r^2 \mathbf{a} \cdot \mathbf{b} &= 0, \\ \{(b^2 - r^2) \mathbf{a} - (a^2 - r^2) \mathbf{b}\} \cdot \mathbf{t}_1 &= r^2 (b^2 - a^2). \end{aligned}$$

We can solve that equation as we did in Problem 4, Concept 2, Method 1. Rather than do so immediately, we'll first derive it starting from a different concept, which will help show that all this stuff about angles, exponents, and geometric products really is coherent, and that terms like $e^{\theta i}$ really do follow the rules of exponents.

4.5.2 Solution Concept 2

This is one of the "sub-optimal" (to the point of absurdity!) solution strategies, referred to in the Introduction, that help demonstrate the coherence and flexibility of GA's capacities for expressing and manipulating rotations.

By adding the mediatrix of the segment $\mathbf{a} \mathbf{b}$ to our previous diagram, and analyzing a bit,



we can show, using plane geometry or Hestenes 1999 pp. 87-90, that

$$\psi = \alpha + \beta,$$

where the algebraic signs of α and β follow GA's usual right-hand convention. Therefore, $e^{\psi i} = e^{(\alpha+\beta)i} = e^{\alpha i} e^{\beta i}$, which we'll express as

$$\underbrace{\left[\frac{t_1 [(b-a) i]}{|t_1| |(b-a) i|} \right]}_{=e^{\psi i}} = \underbrace{\left[\frac{t_1 (a-t_1)}{|t_1| |a-t_1|} \right]}_{=e^{\alpha i}} \underbrace{\left[\frac{t_1 (b-t_1)}{|t_1| |b-t_1|} \right]}_{=e^{\beta i}}.$$

Now, using manipulations that we've seen previously,

$$t_1^{-1} \left[\frac{t_1 [(b-a) i]}{|t_1| |(b-a) i|} \right] = t_1^{-1} \left[\frac{t_1 (a-t_1)}{|t_1| |a-t_1|} \right] \left[\frac{t_1 (b-t_1)}{|t_1| |b-t_1|} \right],$$

$$\frac{(b-a) i}{|b-a|} = \frac{(a-t_1) t_1 (b-t_1)}{r |a-t_1| |b-t_1|}$$

$$(b-a) \frac{(b-a) i}{|b-a|} = (b-a) \frac{(a-t_1) t_1 (b-t_1)}{r |a-t_1| |b-t_1|}$$

$$\frac{|b-a|^2 i}{|b-a|} = \frac{(b-a) (a-t_1) t_1 (b-t_1)}{r |a-t_1| |b-t_1|}$$

we arrive at

$$(b-a) (a-t_1) t_1 (b-t_1) = r |b-a| |a-t_1| |b-t_1| i.$$

For t_2 , the corresponding equation is

$$(b-a) (a-t_2) t_2 (b-t_2) = r |b-a| |a-t_2| |b-t_2| i.$$

The right-hand sides of each of those equations is a bivector. Therefore (in the case of \mathbf{t}_1),

$$\langle (\mathbf{b} - \mathbf{a})(\mathbf{a} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{b} - \mathbf{t}_1) \rangle_0 = 0.$$

Expanding the left-hand side, we obtain

$$\mathbf{b} \mathbf{a} \mathbf{t}_1 \mathbf{b} - r^2 \mathbf{b} \mathbf{a} - r^2 b^2 + r^2 \mathbf{b} \mathbf{t}_1 - a^2 \mathbf{t}_1 \mathbf{b} + r^2 a^2 + r^2 \mathbf{b} \mathbf{a} - r^2 \mathbf{a} \mathbf{t}_1.$$

From the work that we did earlier in this problem, on the product $\mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1$, we know that $\langle \mathbf{t}_1 \mathbf{a} \mathbf{b} \mathbf{t}_1 \rangle_0 = b^2 \mathbf{a} \cdot \mathbf{t}_1$. Using that result, we find that

$$\langle (\mathbf{b} - \mathbf{a})(\mathbf{a} - \mathbf{t}_1) \mathbf{t}_1 (\mathbf{b} - \mathbf{t}_1) \rangle_0 = b^2 \mathbf{a} \cdot \mathbf{t}_1 - r^2 b^2 + r^2 \mathbf{b} \cdot \mathbf{t}_1 - a^2 \mathbf{t}_1 \cdot \mathbf{b} + r^2 a^2 - r^2 \mathbf{a} \cdot \mathbf{t}_1,$$

which we set equal to zero, then rearrange as

$$\{(b^2 - r^2) \mathbf{a} - (a^2 - r^2) \mathbf{b}\} \cdot \mathbf{t}_1 = r^2 (b^2 - a^2).$$

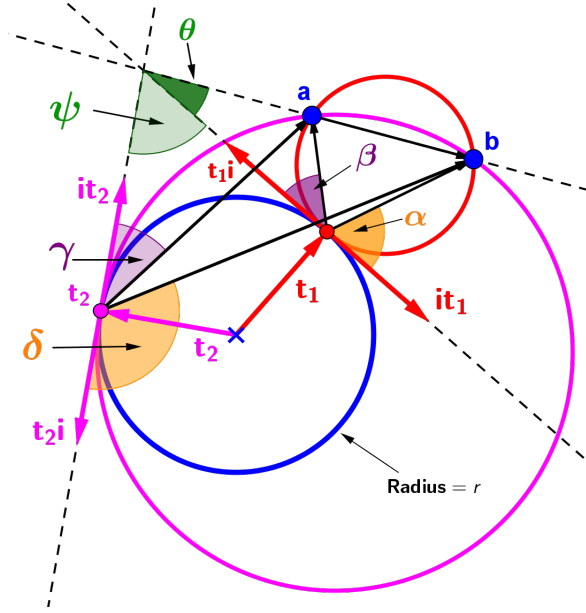
This is the same equation at which we arrived in 4.5.1. From 4.4.2, we know that there are two solutions, which turn out to be

$$\mathbf{t} = \frac{r^2 (b^2 - a^2)}{|z|} \hat{z} \pm \left\{ \sqrt{r_1^2 - \left[\frac{r^2 (b^2 - a^2)}{|z|} \right]^2} \right\} \hat{z} i,$$

where $\mathbf{z} = (b^2 - r^2) \mathbf{a} - (a^2 - r^2) \mathbf{b}$.

4.5.3 Solution Concept 3

This concept serves to demonstrate the coherence and flexibility of GA's capacities for expressing and manipulating rotations without being quite as "absurd" as the previous. Actually, it's quite practical. Here, we'll treat only the solution for \mathbf{t}_1 .



From classical plane geometry, we know that $\theta = \frac{1}{2}(\alpha - \beta)$, where the positive direction of each angle is given by GA's usual right-hand rule. Using ideas that we've seen several times now, we write

$$\begin{aligned}
 e^{\theta i} &= e^{\left(\frac{\alpha}{2} - \frac{\beta}{2}\right) i} \\
 e^{\theta i} &= \left[e^{\frac{\alpha}{2} i} \right] \left[e^{-\frac{\beta}{2} i} \right] \\
 \underbrace{\frac{(it_1)(b-a)}{|it_1||b-a|}}_{=e^{\theta i}} &= \underbrace{\left[\frac{(it_1)(b-t_1)}{|it_1||b-t_1|} \right]}_{=e^{\frac{\alpha}{2} i}} \underbrace{\left[\frac{(t_1 i)(a-t_1)}{|t_1 i||a-t_1|} \right]}_{=e^{-\frac{\beta}{2} i}} \\
 \frac{b-a}{|b-a|} &= \left[\frac{b-t_1}{|b-t_1|} \right] \left[\frac{(t_1 i)(a-t_1)}{r_1 |a-t_1|} \right] \\
 (b-a)(b-t_1)t_1 i(a-t_1) &= r_1 |b-a| |b-t_1| |a-t_1| \\
 (b-a)(b-t_1)t_1(a-t_1)i &= -r_1 |b-a| |b-t_1| |a-t_1| \\
 (b-a)(b-t_1)t_1(a-t_1) &= \underbrace{r_1 |b-a| |b-t_1| |a-t_1| i}_{\text{A bivector}} \\
 \langle (b-a)(b-t_1)t_1(a-t_1) \rangle_0 &= 0.
 \end{aligned}$$

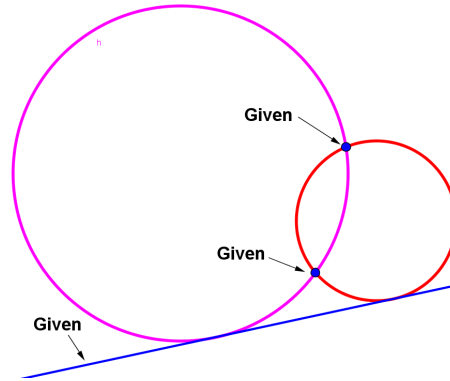
Now, we proceed as we did in Concept 1 and Concept 2 for this problem. Again, we'd arrive at equation

$$\{(b^2 - r^2)\mathbf{a} - (a^2 - r^2)\mathbf{b}\} \cdot \mathbf{t}_1 = r^2(b^2 - a^2),$$

from which we'd find \mathbf{t}_1 and \mathbf{t}_2 as before.

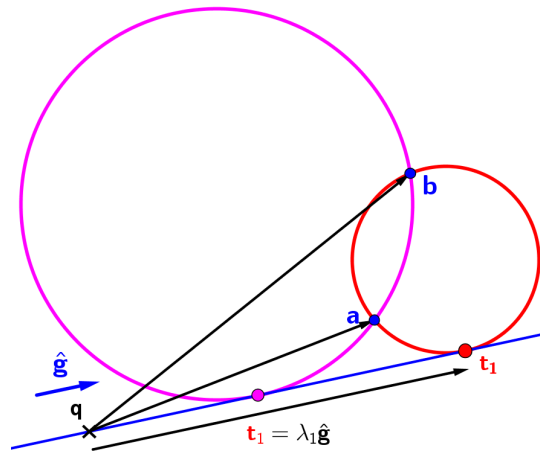
4.6 Problem 6

Given two points on the same side of a given line, find the circles tangent to the given line, and that pass through the given points.



The term "a given line" needs some explanation in the context of a problem like this one. In classical geometry, the line and points would be presented to us on a sheet of paper: we wouldn't need to do anything to characterize their positions before getting down to work with a ruler, a compass, and a good, sharp pencil. However, to solve the problem via GA, someone needs to specify the location and orientation of the line for us in terms of quantities that GA can manipulate.

A reasonable way to do so (we'll see another one in 4.6.4) is by using some convenient point \mathbf{q} on the line as our origin, and specifying direction via the vector $\hat{\mathbf{g}}$:

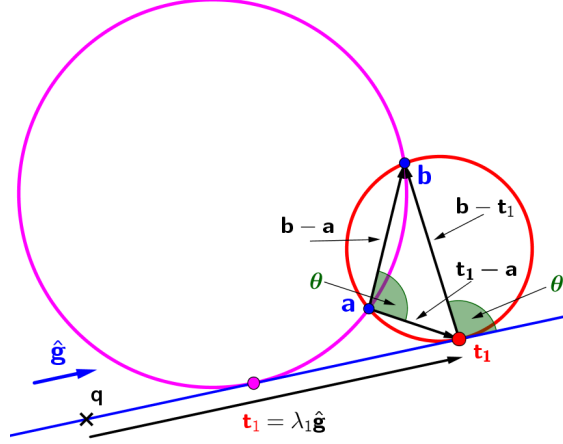


Having found the point of tangency for one of the required circles, we'd give the equation for that circle in the form of a cross ratio. (See p. 46 .)

Then, we can solve for the two points of tangency using either of the methods that we saw in Problem 5.

4.6.1 Solution Concept 1

In our first solution for the present problem, the vector \hat{g} plays the same role that $t_1 i$ did in Problem 5:



To solve this problem, we equate two expressions for $e^{\theta i}$, then proceed to obtain an equation in which one side is either a scalar (only) or a bivector(only):

$$\frac{\hat{g}(b - t_1)}{|b - t_1|} = \frac{(t_1 - a)(b - a)}{|t_1 - a||b - a|},$$

$$\frac{\hat{g}(b - \lambda_1 \hat{g})}{|b - \lambda_1 \hat{g}|} = \frac{(\lambda_1 \hat{g} - a)(b - a)}{|\lambda_1 \hat{g} - a||b - a|},$$

$$(b - \lambda_1 \hat{g}) \hat{g} (\lambda_1 \hat{g} - a)(b - a) = |b - a| |\lambda_1 \hat{g} - a| |b - \lambda_1 \hat{g}|.$$

The right-hand side is a scalar, so the bivector part of the left-hand side is equal to zero. The expansion of the left-hand side is

$$\lambda_1 b^2 - \lambda_1 b a - b \hat{g} a b + a^2 b \hat{g} - \lambda_1^2 \hat{g} b + \lambda_1^2 \hat{g} a + \lambda_1 a b - \lambda_1 a^2.$$

From our work in Problem 5, Solution Concept 2, we know that $b \hat{g} a b = -b^2 \hat{g} a$. Using that fact, the bivector part of the left-hand side is

$$2\lambda_1 a \wedge b + b^2 \hat{g} \wedge a - a^2 \hat{g} \wedge b - \lambda_1^2 \hat{g} \wedge b + \lambda_1^2 \hat{g} \wedge a.$$

Setting it equal to zero, we arrive (after some rearranging) at

$$\lambda_1^2 [\hat{g} \wedge a - \hat{g} \wedge b] + \lambda_1 [2a \wedge b] + [b^2 \hat{g} \wedge a - a^2 \hat{g} \wedge b] = 0,$$

and

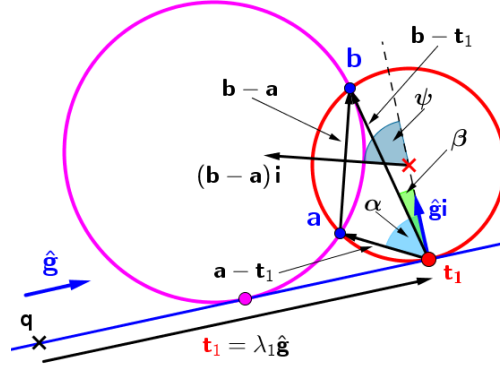
$$\lambda_1^2 [(\hat{g}i) \cdot a - (\hat{g}i) \cdot b] + \lambda_1 [2(\mathbf{a}i) \cdot b] + [b^2 (\hat{g}i) \cdot a - a^2 (\hat{g}i) \cdot b] = 0.$$

The solutions to that quadratic are

$$\lambda = \frac{-(\mathbf{a}i) \cdot b \pm \sqrt{[(\mathbf{a}i) \cdot b]^2 + (a^2 + b^2)[(\hat{g}i) \cdot a][(\hat{g}i) \cdot b] - a^2[(\hat{g}i) \cdot b]^2 - b^2[(\hat{g}i) \cdot a]^2}}{(\hat{g}i) \cdot a - (\hat{g}i) \cdot b}.$$

4.6.2 Solution Concept 2

We could have derived the same quadratic using the method that we saw in Problem 5, Solution Concept 2. The vector $\hat{g}i$ plays (in some respects) the same role that t_1 did in Problem 5, Solution Concept 2:



As in Problem 5, $\psi = \alpha + \beta$, where the algebraic signs of α and β follow GA's usual right-hand convention. Therefore,

$$e^{\psi i} = e^{(\alpha + \beta)i} = e^{\alpha i} e^{\beta i},$$

which we'll express as

$$\underbrace{\frac{[\hat{g}i][(b-a)i]}{[\hat{g}i][(b-a)i]}}_{=e^{\psi i}} = \underbrace{\frac{[\hat{g}i](a-t_1)}{[\hat{g}i](a-t_1)}}_{=e^{\alpha i}} \underbrace{\frac{[\hat{g}i](b-t_1)}{[\hat{g}i](b-t_1)}}_{=e^{\beta i}}.$$

From there,

$$\underbrace{\frac{[\hat{g}i][(b-a)i]}{[\hat{g}i][(b-a)i]}}_{=e^{\psi i}} = \underbrace{\frac{[\hat{g}i](a-t_1)}{[\hat{g}i](a-t_1)}}_{=e^{\alpha i}} \underbrace{\frac{[\hat{g}i](b-t_1)}{[\hat{g}i](b-t_1)}}_{=e^{\beta i}},$$

$$\frac{(b-a)i}{|b-a|} = \frac{(a-t_1)\hat{g}i(b-t_1)}{|a-t_1||b-t_1|},$$

$$\frac{(\mathbf{b} - \mathbf{a})i}{|\mathbf{b} - \mathbf{a}|} = -\frac{(\mathbf{a} - \mathbf{t}_1)\hat{\mathbf{g}}(\mathbf{b} - \mathbf{t}_1)i}{|\mathbf{a} - \mathbf{t}_1||\mathbf{b} - \mathbf{t}_1|},$$

$$\frac{(\mathbf{a} - \mathbf{t}_1)\hat{\mathbf{g}}(\mathbf{b} - \mathbf{t}_1)}{|\mathbf{a} - \mathbf{t}_1||\mathbf{b} - \mathbf{t}_1|} = -\frac{(\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|},$$

$$(\mathbf{b} - \mathbf{a})(\mathbf{a} - \mathbf{t}_1)\hat{\mathbf{g}}(\mathbf{b} - \mathbf{t}_1) = -|\mathbf{b} - \mathbf{a}||\mathbf{a} - \mathbf{t}_1||\mathbf{b} - \mathbf{t}_1|.$$

Here's the expansion of the left-hand side:

$$\mathbf{b}\mathbf{a}\hat{\mathbf{g}}\mathbf{b} - \mathbf{b}\mathbf{a}\hat{\mathbf{g}}\mathbf{t}_1 - \mathbf{b}\mathbf{t}_1\hat{\mathbf{g}}\mathbf{b} + \mathbf{b}\mathbf{t}_1\hat{\mathbf{g}}\mathbf{t}_1 - \mathbf{a}^2\hat{\mathbf{g}}\mathbf{b} + \mathbf{a}^2\hat{\mathbf{g}}\mathbf{t}_1 + \mathbf{a}\mathbf{t}_1\hat{\mathbf{g}}\mathbf{b} - \mathbf{a}\mathbf{t}_1\hat{\mathbf{g}}\mathbf{t}_1.$$

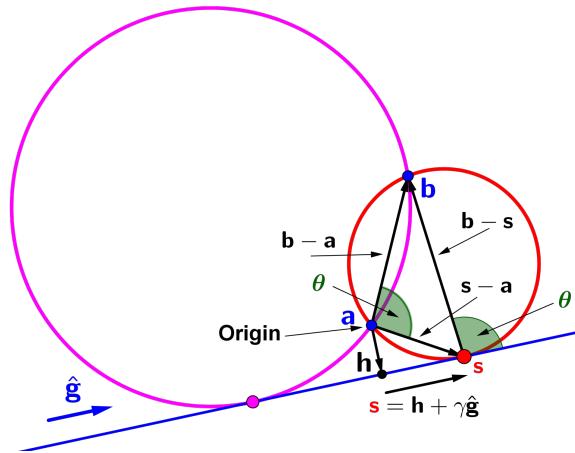
From our work in 4.5.2, we know that $\mathbf{b}\mathbf{a}\hat{\mathbf{g}}\mathbf{b} = b^2\hat{\mathbf{g}}\mathbf{a}$. Using that fact, and making the substitution $\mathbf{t}_1 = \lambda_1\hat{\mathbf{g}}$, the left-hand side becomes

$$b^2\hat{\mathbf{g}}\mathbf{a} - \lambda_1\mathbf{b}\mathbf{a} - \lambda_1b^2 + \lambda_1^2\mathbf{b}\hat{\mathbf{g}} - \mathbf{a}^2\hat{\mathbf{g}}\mathbf{b} + \lambda_1a^2 + \lambda_1\mathbf{a}\mathbf{b} - \lambda_1^2\mathbf{a}\hat{\mathbf{g}}.$$

Setting the bivector part of that expression equal to zero, we obtain the same quadratic that we solved at the end of Solution Concept 1.

4.6.3 Solution Concept 3

As mentioned at the beginning of this problem, the location and orientation of the given line need to be specified in terms of quantities that GA can manipulate. In this third Solution Concept, we specify the line's orientation via the unit vector of the line's direction, and the line's position via the vector \mathbf{h} . We also take the unusual step of using \mathbf{a} as the origin, thereby simplifying the solution process.



Equating two expressions for $e^{\theta i}$, we have

$$\frac{\hat{\mathbf{g}}(\mathbf{b} - \mathbf{s})}{|\mathbf{b} - \mathbf{s}|} = \frac{\mathbf{s}\mathbf{b}}{|\mathbf{s}||\mathbf{b}|},$$

from which we then obtain

$$\mathbf{s}\hat{\mathbf{g}}(\mathbf{b} - \mathbf{s})\mathbf{b} = \underbrace{|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|}_{\text{scalar}}.$$

Now, as in previous solutions of our three problems, we expand the left-hand side, then find its bivector part, and set it equal to zero:

$$\langle b^2 \mathbf{s}\hat{\mathbf{g}} - (2\mathbf{s} \cdot \hat{\mathbf{g}} - \hat{\mathbf{g}}\mathbf{s})\mathbf{s}\mathbf{b} \rangle_2 = 0,$$

$$\langle b^2 \mathbf{s}\hat{\mathbf{g}} - (2\mathbf{s} \cdot \hat{\mathbf{g}})\mathbf{s}\mathbf{b} - s^2 \hat{\mathbf{g}}\mathbf{b} \rangle_2 = 0,$$

$$b^2 \mathbf{s} \wedge \hat{\mathbf{g}} - (2\mathbf{s} \cdot \hat{\mathbf{g}})\mathbf{s} \wedge \mathbf{b} - s^2 \hat{\mathbf{g}} \wedge \mathbf{b} \rangle_2 = 0.$$

From our diagram, $\mathbf{s} = \mathbf{h} + \gamma\hat{\mathbf{g}}$. Making this substitution, the preceding equation becomes (after some manipulation)

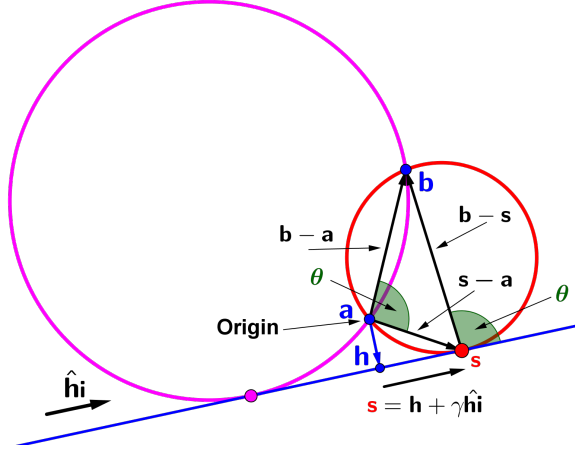
$$[(\hat{\mathbf{g}}\mathbf{i}) \cdot \mathbf{b}]\gamma^2 + 2[(\mathbf{h}\mathbf{i}) \cdot \mathbf{b}]\gamma + (\hat{\mathbf{g}}\mathbf{i}) \cdot (b^2\mathbf{h} - h^2\mathbf{b}).$$

The solutions to that quadratic are

$$\gamma = \frac{-(\mathbf{h}\mathbf{i}) \cdot \mathbf{b} \pm \sqrt{[(\mathbf{h}\mathbf{i}) \cdot \mathbf{b}]^2 - [(\hat{\mathbf{g}}\mathbf{i}) \cdot \mathbf{b}][(\hat{\mathbf{g}}\mathbf{i}) \cdot (b^2\mathbf{h} - h^2\mathbf{b})]}}{(\hat{\mathbf{g}}\mathbf{i}) \cdot \mathbf{b}}.$$

4.6.4 Solution Concept 4

This concept is closely related to the previous one, and very much in the spirit of GA: Since we're expressing the position of the line by means of a vector (\mathbf{h}) that's perpendicular to the line, why use a separate vector ($\hat{\mathbf{g}}$) to express the line's direction? Instead, we can use the vector $\hat{\mathbf{h}}\mathbf{i}$:



Equating two expressions for $e^{\theta i}$, we have

$$\frac{\hat{h}i(\mathbf{b} - \mathbf{s})}{|\mathbf{b} - \mathbf{s}|} = \frac{\mathbf{s}\mathbf{b}}{|\mathbf{s}||\mathbf{b}|},$$

from which we then obtain

$$\mathbf{b}\mathbf{s}\hat{h}i(\mathbf{b} - \mathbf{s}) = \underbrace{|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|}_{\text{scalar}},$$

$$\mathbf{b}\mathbf{s}\hat{h}(\mathbf{s} - \mathbf{b})i = |\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|,$$

$$\mathbf{b}\mathbf{s}\hat{h}(\mathbf{s} - \mathbf{b}) = -\underbrace{|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|i}_{\text{bivector}},$$

$$\mathbf{b}\mathbf{s}\hat{h}\mathbf{s} - \underbrace{\mathbf{b}\mathbf{s}\hat{h}\mathbf{b}}_{=b^2\hat{h}\mathbf{s}} = -|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|i,$$

$$\mathbf{b}\mathbf{s} \left[\underbrace{2\hat{h} \cdot \mathbf{s} - \mathbf{s}\hat{h}}_{=\hat{h}\mathbf{s}} \right] - b^2\hat{h}\mathbf{s} = -|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|i,$$

$$2(\hat{h} \cdot \mathbf{s})\mathbf{b}\mathbf{s} - s^2\mathbf{b}\hat{h} - b^2\hat{h}\mathbf{s} = -|\mathbf{s}||\mathbf{b} - \mathbf{s}||\mathbf{b}|i.$$

The right-hand side is a bivector, so the scalar part of the left-hand side is zero:

$$2(\hat{h} \cdot \mathbf{s})\mathbf{b} \cdot \mathbf{s} - s^2\mathbf{b} \cdot \hat{h} - b^2\hat{h} \cdot \mathbf{s} = 0.$$

From our diagram, $\mathbf{s} = \mathbf{h} + \gamma\hat{h}i$. Therefore, $\hat{h} \cdot \mathbf{s} = \hat{h} \cdot \mathbf{h} = |\mathbf{h}|$, and $s^2 = h^2 + \gamma^2$. Making those substitutions,

$$2|\mathbf{h}| \left[\mathbf{b} \cdot \mathbf{h} + \gamma \mathbf{b} \cdot (\hat{\mathbf{h}}i) \right] - (h^2 + \gamma^2) \mathbf{b} \cdot \hat{\mathbf{h}} - b^2 |\mathbf{h}| = 0,$$

$$\left[\hat{\mathbf{h}} \cdot \mathbf{b} \right] \gamma^2 - \left[2|\mathbf{h}| \mathbf{b} \cdot (\hat{\mathbf{h}}i) \right] \gamma + b^2 |\mathbf{h}| + \underbrace{h^2 \mathbf{b} \cdot \hat{\mathbf{h}}}_{=|\mathbf{h}|^2 \mathbf{b} \cdot \hat{\mathbf{h}}} - 2|\mathbf{h}| \mathbf{b} \cdot \mathbf{h} = 0,$$

$$\left[\hat{\mathbf{h}} \cdot \mathbf{b} \right] \gamma^2 - [2(\mathbf{h}i) \cdot \mathbf{b}] \gamma + b^2 |\mathbf{h}| - |\mathbf{h}| \mathbf{h} \cdot \mathbf{b} = 0$$

The solutions to that quadratic are

$$\gamma = \frac{(\mathbf{h}i) \cdot \mathbf{b} \pm \sqrt{[(\mathbf{h}i) \cdot \mathbf{b}]^2 + [\mathbf{h} \cdot \mathbf{b}]^2 - b^2 [\mathbf{h} \cdot \mathbf{b}]}}{\hat{\mathbf{h}} \cdot \mathbf{b}}.$$

We can simplify that equation by recalling that the vector \mathbf{b} is the vector sum of its projections upon the directions $\hat{\mathbf{h}}$ and $\hat{\mathbf{h}}i$:

$$\mathbf{b} = (\mathbf{b} \cdot \hat{\mathbf{h}}) \hat{\mathbf{h}} + [\mathbf{b} \cdot (\hat{\mathbf{h}}i)] \hat{\mathbf{h}}i,$$

from which $(\mathbf{b} \cdot \hat{\mathbf{h}}) \mathbf{h} + [\mathbf{b} \cdot (\hat{\mathbf{h}}i)] \mathbf{h}i = |\mathbf{h}| \mathbf{b}$. Therefore, $[(\mathbf{h}i) \cdot \mathbf{b}]^2 + [\mathbf{h} \cdot \mathbf{b}]^2 = b^2 h^2$. Using that substitution, the solutions to our quadratic become

$$\begin{aligned} \gamma &= \frac{(\mathbf{h}i) \cdot \mathbf{b} \pm \sqrt{b^2 h^2 - b^2 [\mathbf{h} \cdot \mathbf{b}]}}{\hat{\mathbf{h}} \cdot \mathbf{b}} \\ &= \frac{(\mathbf{h}i) \cdot \mathbf{b} \pm |\mathbf{b}| \sqrt{h^2 - [\mathbf{h} \cdot \mathbf{b}]}}{\hat{\mathbf{h}} \cdot \mathbf{b}}. \end{aligned}$$

Comparing that version of the solution to the following, which we obtained by using $\hat{\mathbf{g}}$ to give the direction of the line,

$$\gamma = \frac{-(\mathbf{h}i) \cdot \mathbf{b} \pm \sqrt{[(\mathbf{h}i) \cdot \mathbf{b}]^2 - [(\hat{\mathbf{g}}i) \cdot \mathbf{b}] [(\hat{\mathbf{g}}i) \cdot (b^2 \mathbf{h} - h^2 \mathbf{b})]}}{(\hat{\mathbf{g}}i) \cdot \mathbf{b}},$$

we can see that the " $\hat{\mathbf{h}}$ version" is much "cleaner", although I must point out that we could also have cleaned-up the $\hat{\mathbf{g}}$ version with a bit of effort.

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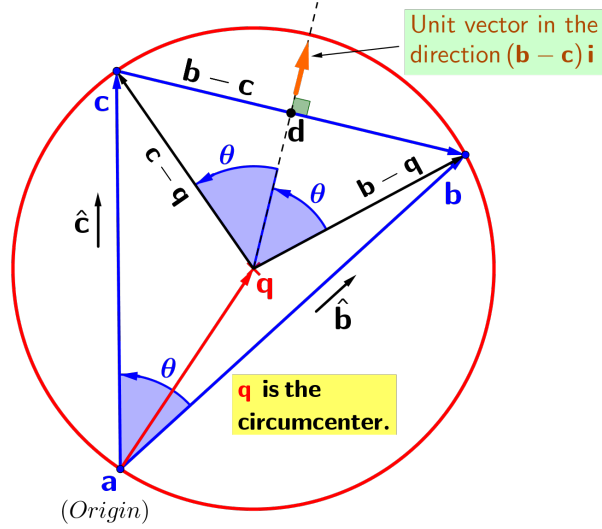
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6 Appendix A: Finding the Circumcenter Using the Inverse of a Multivector

This Appendix presents a solution method that takes considerably more work than that given in the main text (pp. 23 ff), but which has useful, time-saving pointers and observations about inverses of multivectors and transformations of expressions. We begin with the following diagram:



In the main text, we noted that the angle between the vectors $b-q$ and $c-q$ is 2θ , and therefore that

$$\begin{aligned} [b-q] e^{\alpha i} &= c-q \\ [b-q] [e^{\theta i}] [e^{\theta i}] &= c-q \\ [b-q] [\hat{b}\hat{c}] [\hat{b}\hat{c}] &= c-q. \end{aligned}$$

Because $c-q$ and $b-q$ are radii of the same circle, $c-q$ is a pure rotation (that is, without any dilation) of $b-q$. Therefore,

$$\begin{aligned} [b-q] e^{\alpha i} &= c-q \\ [b-q] [e^{\theta i}] [e^{\theta i}] &= c-q \\ [b-q] [\hat{b}\hat{c}] [\hat{b}\hat{c}] &= c-q, \end{aligned}$$

from which

$$\begin{aligned} b [\hat{b}\hat{c}] [\hat{b}\hat{c}] &= (b\hat{b}) \hat{c}\hat{b}\hat{c} \\ &= |b| \hat{c}\hat{b}\hat{c} \\ &= \hat{c} (|b| \hat{b}) \hat{c} \\ &= \hat{c} b \hat{c}. \end{aligned}$$

$$\begin{aligned} b [\hat{b}\hat{c}] [\hat{b}\hat{c}] - q [\hat{b}\hat{c}] [\hat{b}\hat{c}] &= c-q, \\ q - q [\hat{b}\hat{c}] [\hat{b}\hat{c}] &= c - b [\hat{b}\hat{c}] [\hat{b}\hat{c}], \\ q (1 - \hat{b}\hat{c}\hat{b}\hat{c}) &= c - b [\hat{b}\hat{c}] [\hat{b}\hat{c}] \\ q (1 - \hat{b}\hat{c}\hat{b}\hat{c}) &= c - \hat{c} b \hat{c}. \end{aligned}$$

Is that result helpful? Yes, because $1 - \hat{b}\hat{c}\hat{b}\hat{c}$ has a multiplicative inverse in GA. Therefore, we can write the following in a purely formal way, then identify what that inverse is, precisely:

$$\begin{aligned} \mathbf{q} \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right) \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^{-1} &= (c - \hat{c}\hat{b}\hat{c}) \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^{-1} \\ \therefore \mathbf{q} &= (c - \hat{c}\hat{b}\hat{c}) \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^{-1}. \end{aligned}$$

Now here is where we can cause ourselves much unnecessary work if we don't familiarize ourselves with GA's theorems about inverses of multivectors. Based upon those theorems (Hestenes D. pp. 37, 45-46), the multiplicative inverse \mathcal{M}^{-1} of a multivector \mathcal{M} is

$$\mathcal{M}^{-1} = \frac{\mathcal{M}^\dagger}{\langle \mathcal{M}^\dagger \mathcal{M} \rangle_0},$$

where \mathcal{M}^\dagger is the "reverse" of \mathcal{M} . GA's theorems also tell us that the reverse of the sum of two multivectors \mathcal{A} and \mathcal{B} is

$$(\mathcal{A} + \mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger.$$

Every scalar is a multivector, and so is the product $\hat{b}\hat{c}\hat{b}\hat{c}$. Therefore,

$$\left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger = 1^\dagger - \left(\hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger.$$

Next, we use the fact that the reverse of a scalar is that same scalar, and that the reverse of a geometric product of vectors is that same product written in reverse order:

$$\begin{aligned} \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger &= 1^\dagger - \left(\hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger \\ &= 1 - \hat{c}\hat{b}\hat{c}\hat{b}. \end{aligned}$$

Now we can write

$$\begin{aligned} \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^{-1} &= \frac{\left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger}{\langle \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)^\dagger \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right) \rangle_0} \\ &= \frac{1 - \hat{c}\hat{b}\hat{c}\hat{b}}{\langle \left(1 - \hat{c}\hat{b}\hat{c}\hat{b}\right) \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right) \rangle_0}. \end{aligned}$$

We'll leave the numerator as-is, but we'll expand and simplify the denominator. First, we'll see an efficient procedure for effecting the expansion and simplification, after which we'll see my original, less-inspired way, so that students can see that they needn't find "the" way to get the job done.

The efficient procedure begins by expanding $\left(1 - \hat{c}\hat{b}\hat{c}\hat{b}\right) \left(1 - \hat{b}\hat{c}\hat{b}\hat{c}\right)$ as the product of two binomials:

I wasted a great deal of time in my original solution because I thought —mistakenly—that I needed to identify the scalar and bivector parts of $\hat{b}\hat{c}\hat{b}\hat{c}$ explicitly in order to find the inverse of $1 - \hat{b}\hat{c}\hat{b}\hat{c}$.

$$(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) = 1 - \hat{c}\hat{b}\hat{c}\hat{b} - \hat{b}\hat{c}\hat{b}\hat{c} + \underbrace{(\hat{c}\hat{b}\hat{c}\hat{b})(\hat{b}\hat{c}\hat{b}\hat{c})}_{\text{Simplifies to 1}}.$$

$\hat{b}\hat{c}\hat{b}$ evaluates to a vector
(page 15) .

Next, we write $\hat{c}\hat{b}\hat{c}\hat{b}$ as $\hat{c}(\hat{b}\hat{c}\hat{b})$, and $\hat{b}\hat{c}\hat{b}\hat{c}$ as $(\hat{b}\hat{c}\hat{b})\hat{c}$.

$$(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) = 2 - \hat{c}(\hat{b}\hat{c}\hat{b}) - (\hat{b}\hat{c}\hat{b})\hat{c}.$$

Now, we recall that the denominator is $\langle(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})\rangle_0$ rather than $(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})$ itself, so the bivector terms —, that is, the terms $\hat{c} \wedge (\hat{b}\hat{c}\hat{b})$ and $(\hat{b}\hat{c}\hat{b}) \wedge \hat{c}$ — in the geometric products $\hat{c}(\hat{b}\hat{c}\hat{b})$, and $\hat{b}\hat{c}\hat{b}\hat{c}$ as $(\hat{b}\hat{c}\hat{b})\hat{c}$ don't concern us:

$$\langle(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})\rangle_0 = 2 - \hat{c} \cdot (\hat{b}\hat{c}\hat{b}) - (\hat{b}\hat{c}\hat{b}) \cdot \hat{c}.$$

The two dot products are equal, so we write

$$\langle(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})\rangle_0 = 2 - 2 \{ \hat{c} \cdot (\hat{b}\hat{c}\hat{b}) \}.$$

Two indispensable identities:

$$uv = 2\mathbf{u} \cdot \mathbf{v} - v\mathbf{u}$$

$$\text{and } uv = 2\mathbf{u} \wedge \mathbf{v} + v\mathbf{u}.$$

For any two vectors \mathbf{u} and \mathbf{v} ,

$$\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} + [\mathbf{u} \cdot (\hat{\mathbf{v}}i)]\hat{\mathbf{v}}i.$$

Therefore, for the unit vector $\hat{\mathbf{u}}$,

$$\hat{\mathbf{u}}^2 = 1 = [\hat{\mathbf{u}} \cdot \hat{\mathbf{v}}]^2 + [\mathbf{u} \cdot (\hat{\mathbf{v}}i)]^2.$$

Finally, we expand $\hat{b}\hat{c}\hat{b}$, (for example, as $2\hat{c} \cdot \hat{\mathbf{b}} - \hat{\mathbf{b}}\hat{c}$), then simplify:

$$\begin{aligned} \langle(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})\rangle_0 &= 2 - 2 \{ \hat{c} \cdot [(2\hat{\mathbf{b}} \cdot \hat{c} - \hat{\mathbf{b}}\hat{c})\hat{\mathbf{b}}] \} \\ &= 2 - 2 \left\{ \hat{c} \cdot \left[(2\hat{\mathbf{b}} \cdot \hat{c})\hat{\mathbf{b}} - \underbrace{\hat{\mathbf{b}}\hat{c}}_{=\hat{c}} \right] \right\} \\ &= 2 - 2 \left\{ 2(\hat{\mathbf{b}} \cdot \hat{c})^2 - 1 \right\} \\ &= 4 - 4(\hat{\mathbf{b}} \cdot \hat{c})^2 \\ &= 4 \left[1 - (\hat{\mathbf{b}} \cdot \hat{c})^2 \right] \\ &= (\text{also}) 4 \left[(\hat{\mathbf{b}}i) \cdot \hat{c} \right]^2 \text{ and } 4 \left[\hat{\mathbf{b}} \cdot (\hat{c}i) \right]^2. \end{aligned}$$

My less-efficient procedure, mentioned earlier, expresses $\hat{b}\hat{c}\hat{b}\hat{c}$ as $(\hat{b}\hat{c})^2$, which then becomes

$$\begin{aligned} \hat{b}\hat{c}\hat{b}\hat{c} &= (\hat{b}\hat{c})^2 \\ &= \{ \hat{\mathbf{b}} \cdot \hat{c} + \hat{\mathbf{b}} \wedge \hat{c} \}^2 \\ &= \{ \hat{\mathbf{b}} \cdot \hat{c} + [(\hat{\mathbf{b}}i) \cdot \hat{c}]i \}^2 \\ &= [\hat{\mathbf{b}} \cdot \hat{c}]^2 - [(\hat{\mathbf{b}}i) \cdot \hat{c}]^2 + 2[\hat{\mathbf{b}} \cdot \hat{c}][(\hat{\mathbf{b}}i) \cdot \hat{c}]i. \end{aligned}$$

Similarly,

$$\begin{aligned}
\hat{c}\hat{b}\hat{c}\hat{b} &= (\hat{c}\hat{b})^2 \\
&= \{\hat{c} \cdot \hat{b} + \hat{c} \wedge \hat{b}\}^2 \\
&= \{\hat{c} \cdot \hat{b} + [(\hat{c}i) \cdot \hat{b}] i\}^2 \\
&= [\hat{c} \cdot \hat{b}]^2 - [(\hat{c}i) \cdot \hat{b}]^2 + 2[\hat{c} \cdot \hat{b}] [(\hat{c}i) \cdot \hat{b}] i.
\end{aligned}$$

Because $[(\hat{c}i) \cdot \hat{b}] = -[(\hat{b}i) \cdot \hat{c}]$,

$$\hat{c}\hat{b}\hat{c}\hat{b} + \hat{b}\hat{c}\hat{b}\hat{c} = 2 \left\{ [\hat{c} \cdot \hat{b}]^2 - [(\hat{c}i) \cdot \hat{b}]^2 \right\}.$$

The bivector terms have canceled out, but they would not have entered into $\langle (1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) \rangle_0$ anyway.

We'll use that result in our expansion of $(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c})$, then simplify:

$$\begin{aligned}
(1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) &= 1 - \hat{c}\hat{b}\hat{c}\hat{b} - \hat{b}\hat{c}\hat{b}\hat{c} + (\hat{c}\hat{b}\hat{c}\hat{b})(\hat{b}\hat{c}\hat{b}\hat{c}) \\
&= 2 - 2(\hat{c}\hat{b}\hat{c}\hat{b} + \hat{b}\hat{c}\hat{b}\hat{c}) \\
&= 2 - 2 \left\{ [\hat{c} \cdot \hat{b}]^2 - [(\hat{c}i) \cdot \hat{b}]^2 \right\} \\
&= 2 - 2 \left\{ 1 - 2 [(\hat{c}i) \cdot \hat{b}]^2 \right\} \\
&= 4 [(\hat{c}i) \cdot \hat{b}]^2,
\end{aligned}$$

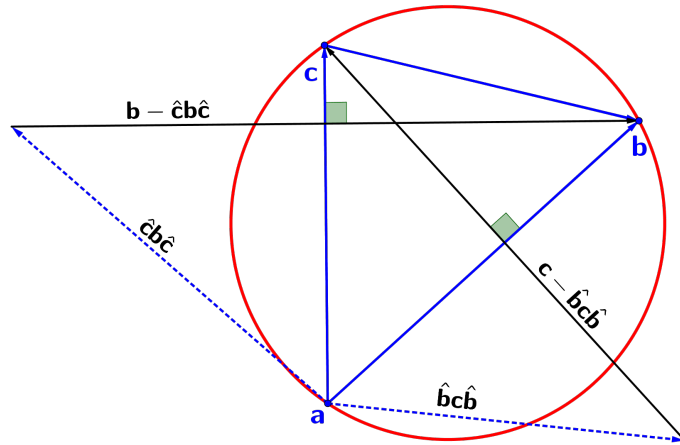
Having shown that $\langle (1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) \rangle_0 = 4 [(\hat{c}i) \cdot \hat{b}]^2 = 4 [(\hat{b}i) \cdot \hat{c}]^2$, we can identify $(1 - \hat{b}\hat{c}\hat{b}\hat{c})^{-1}$ explicitly:

$$\begin{aligned}
(1 - \hat{b}\hat{c}\hat{b}\hat{c})^{-1} &= \frac{1 - \hat{c}\hat{b}\hat{c}\hat{b}}{\langle (1 - \hat{c}\hat{b}\hat{c}\hat{b})(1 - \hat{b}\hat{c}\hat{b}\hat{c}) \rangle_0} \\
&= \frac{1 - \hat{c}\hat{b}\hat{c}\hat{b}}{4 [(\hat{b}i) \cdot \hat{c}]^2}.
\end{aligned}$$

Now we're ready to solve for the circumcenter, \mathbf{q} , via

$$\begin{aligned}
 \mathbf{q} &= (\mathbf{c} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}) \left(1 - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}\right)^{-1} \\
 &= (\mathbf{c} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}) \left\{ \frac{1 - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2} \right\} \\
 &= \frac{\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}} + \mathbf{b} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2} \\
 &= \frac{\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2} + \frac{\mathbf{b} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2}.
 \end{aligned}$$

We'll transform that result after examining it a bit. The vector $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}$ is the reflection of \mathbf{c} with respect to the direction of \mathbf{b} (7.1). Similarly, $\hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}$ is the reflection of \mathbf{b} with respect to the direction of \mathbf{c} (page 73). Thus, the vectors $\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}$ and $\mathbf{b} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}$ are as shown:



The perpendicularities shown here can be demonstrated in several ways. One of the easiest is by using the definition of the dot product given on p. 51 of Macdonald A. 2010. For example,

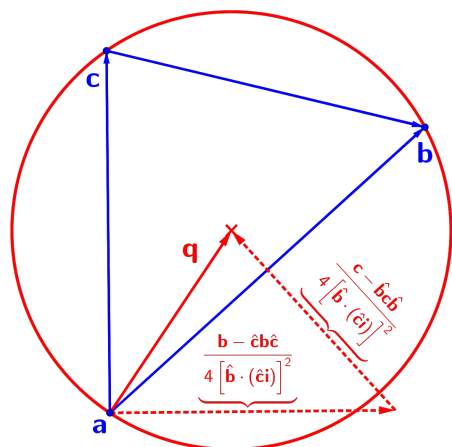
$$\begin{aligned}
 (\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}) \cdot \mathbf{b} &= \langle (\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}) \mathbf{b} \rangle_0 \\
 &= \langle \mathbf{c}\mathbf{b} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}\mathbf{b} \rangle_0 \\
 &= \langle \mathbf{c}\mathbf{b} - \mathbf{b}\mathbf{c} \rangle_0 \\
 &= 0.
 \end{aligned}$$

$$\mathbf{c}\mathbf{b} - \mathbf{b}\mathbf{c} = 2\mathbf{c} \wedge \mathbf{b}.$$

Therefore, our result

$$\mathbf{q} = \frac{\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2} + \frac{\mathbf{b} - \hat{\mathbf{c}}\mathbf{b}\hat{\mathbf{c}}}{4 \left[\left(\hat{\mathbf{b}}\mathbf{i} \right) \cdot \hat{\mathbf{c}} \right]^2}$$

is as illustrated below:

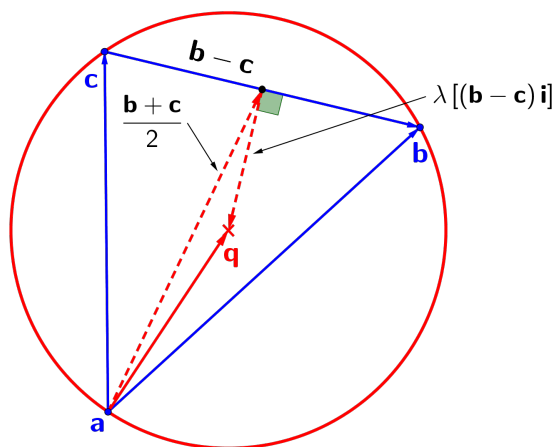


As our diagram suggests, our answer has the form of a linear combination of bi and ci .

Interpreting the Solution, and Transforming It into a More-Useful Form

Form We shouldn't be surprised to find that the vector q is the sum of vectors that are perpendicular to the triangle's sides, because we know that every vector v can be written as a linear combination of any two non-parallel vectors that are coplanar with it. The sides of a triangle cannot be parallel to each other; therefore, neither can the vectors that are perpendicular to them. Hence, the vector q is guaranteed to be some linear combination of the vectors that are perpendicular to the triangle's sides.

However, that rather dismissive analysis cannot have told us the whole story about the vector q , because the point q is the point of intersection of the mediatrices of the triangle's sides. Therefore, we must be able to write q as $q = \lambda[(b - c) i]$, where λ is some scalar:



So, what is the value of λ ? To determine it, let's go back to our equation

$$\mathbf{q} = \frac{\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}} + \mathbf{b} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}}{4 \left[(\hat{\mathbf{b}}\hat{\mathbf{i}}) \cdot \hat{\mathbf{c}} \right]^2}.$$

The transformations that we'll use to introduce $\mathbf{b}\hat{\mathbf{i}}$ and $\mathbf{c}\hat{\mathbf{i}}$ are useful in other contexts as well. For example, the projection of a vector \mathbf{u} upon a vector \mathbf{v} is $[(\mathbf{u} \cdot \hat{\mathbf{v}})] \hat{\mathbf{v}}$. If we need to express that projection in terms of $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}\hat{\mathbf{i}}$, we can do so via either of the transformations given here. However, those may not be the most-efficient methods.

That equation doesn't mention $\mathbf{b}\hat{\mathbf{i}}$ or $\mathbf{c}\hat{\mathbf{i}}$ explicitly, but we can introduce them in at least two ways. One is to recognize that $\hat{\mathbf{b}}$ can be written as

$$\hat{\mathbf{b}} = [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] \hat{\mathbf{c}} + [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \hat{\mathbf{c}}\hat{\mathbf{i}}.$$

Using that substitution, we can transform the vector $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}$ into a linear combination of \mathbf{c} and $\mathbf{c}\hat{\mathbf{i}}$:

$$\begin{aligned} \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}} &= \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] \hat{\mathbf{c}} + [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \hat{\mathbf{c}}\hat{\mathbf{i}} \right\} \mathbf{c} \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] \hat{\mathbf{c}} + [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \hat{\mathbf{c}}\hat{\mathbf{i}} \right\} \\ &= (\text{after simplifying}) \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}]^2 - [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 \right\} \mathbf{c} + 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} \mathbf{c}\hat{\mathbf{i}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}} &= \mathbf{c} - \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}]^2 - [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 \right\} \mathbf{c} - 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} \mathbf{c}\hat{\mathbf{i}} \\ &= \left\{ \underbrace{1 - [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}]^2}_{=[\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2} + [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 \right\} \mathbf{c} - 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} \mathbf{c}\hat{\mathbf{i}} \\ &= 2 [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 \mathbf{c} - 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} \mathbf{c}\hat{\mathbf{i}}. \end{aligned}$$

We can transform $\mathbf{b} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}$, similarly, by writing $\hat{\mathbf{c}}$ as

$$\hat{\mathbf{c}} = [\hat{\mathbf{c}} \cdot \hat{\mathbf{b}}] \hat{\mathbf{b}} + [\hat{\mathbf{c}} \cdot (\hat{\mathbf{b}}\hat{\mathbf{i}})] \hat{\mathbf{b}}\hat{\mathbf{i}}.$$

After expanding and simplifying, we obtain

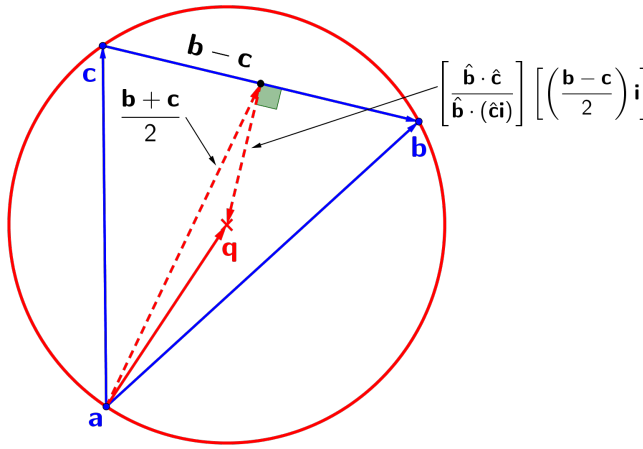
$$\begin{aligned} \mathbf{b} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} &= 2 [\hat{\mathbf{c}} \cdot (\hat{\mathbf{b}}\hat{\mathbf{i}})]^2 \mathbf{b} - 2 \left\{ [\hat{\mathbf{c}} \cdot \hat{\mathbf{b}}] [\hat{\mathbf{c}} \cdot (\hat{\mathbf{b}}\hat{\mathbf{i}})] \right\} \mathbf{b}\hat{\mathbf{i}} \\ &= 2 [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 \mathbf{b} + 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} \mathbf{b}\hat{\mathbf{i}}. \end{aligned}$$

Using these transformations of $\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}$ and $\mathbf{b} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}$,

$$\mathbf{c} - \hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}} + \mathbf{b} - \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} = 2 [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})]^2 (\mathbf{b} + \mathbf{c}) + 2 \left\{ [\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}] [\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\hat{\mathbf{i}})] \right\} [(\mathbf{b} - \mathbf{c}) \hat{\mathbf{i}}].$$

Now we can put our expression for q in the desired form:

$$\begin{aligned}
 q &= \frac{c - \hat{b}\hat{c}\hat{b} + b - \hat{c}\hat{b}\hat{c}}{4 \left[(\hat{b}\hat{i}) \cdot \hat{c} \right]^2} \\
 &= \frac{2 \left[\hat{b} \cdot (\hat{c}\hat{i}) \right]^2 (b+c) + 2 \left\{ \left[\hat{b} \cdot \hat{c} \right] \left[\hat{b} \cdot (\hat{c}\hat{i}) \right] \right\} [(b-c)\hat{i}]}{4 \left[(\hat{b}\hat{i}) \cdot \hat{c} \right]^2} \\
 &= \frac{b+c}{2} + \left[\frac{\hat{b} \cdot \hat{c}}{\hat{b} \cdot (\hat{c}\hat{i})} \right] \left[\frac{(b-c)\hat{i}}{2} \right].
 \end{aligned}$$



A second way of transforming $\hat{b}\hat{c}\hat{b}$ and $\hat{c}\hat{b}\hat{c}$ provides a surprising example of GA's flexibility. Treating $\hat{b}\hat{c}\hat{b}$ first,

$$\begin{aligned}
 \hat{b}\hat{c}\hat{b} &= \hat{b}(|c|\hat{c})\hat{b} \\
 &= (|c|)\hat{b}\hat{c}\hat{b} \\
 &= (c\hat{c})\hat{b}\hat{c}\hat{b} \\
 &= c(\hat{c}\hat{b}\hat{c}\hat{b}) \\
 &= c \underbrace{\left\{ \left[\hat{b} \cdot \hat{c} \right]^2 - \left[(\hat{b}\hat{i}) \cdot \hat{c} \right]^2 + 2 \left[\hat{b} \cdot \hat{c} \right] \left[(\hat{c}\hat{i}) \cdot \hat{b} \right] \hat{i} \right\}}_{\text{We saw this expansion earlier.}} \\
 &= \left\{ \left[\hat{b} \cdot \hat{c} \right]^2 - \left[\hat{b} \cdot (\hat{c}\hat{i}) \right]^2 \right\} c + 2 \left[\hat{b} \cdot \hat{c} \right] \left[\hat{b} \cdot (\hat{c}\hat{i}) \right] \hat{c}\hat{i},
 \end{aligned}$$

I confess that when I first saw this possibility, I had a hard time believing that it could be legitimate.

which is the result that we obtained by substituting $\left[\hat{b} \cdot \hat{c} \right] \hat{c} + \left[\hat{b} \cdot (\hat{c}\hat{i}) \right] \hat{c}\hat{i}$

for $\hat{\mathbf{b}}$ in $\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}$. Applying the same ideas to $\hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}}$,

$$\begin{aligned}
 \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} &= \hat{\mathbf{c}} \left(|\hat{\mathbf{b}}| \hat{\mathbf{b}} \right) \hat{\mathbf{c}} \\
 &= (|\hat{\mathbf{b}}|) \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\
 &= (\hat{\mathbf{b}}\hat{\mathbf{b}}) \hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} \\
 &= \hat{\mathbf{b}} \left(\hat{\mathbf{b}}\hat{\mathbf{c}}\hat{\mathbf{b}}\hat{\mathbf{c}} \right) \\
 &= \hat{\mathbf{b}} \underbrace{\left\{ \left[\hat{\mathbf{c}} \cdot \hat{\mathbf{b}} \right]^2 - \left[(\hat{\mathbf{c}}\mathbf{i}) \cdot \hat{\mathbf{b}} \right]^2 + 2 \left[\hat{\mathbf{c}} \cdot \hat{\mathbf{b}} \right] \left[(\hat{\mathbf{b}}\mathbf{i}) \cdot \hat{\mathbf{c}} \right] \mathbf{i} \right\}}_{\text{We saw this expansion earlier.}} \\
 &= \left\{ \left[\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} \right]^2 - \left[\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\mathbf{i}) \right]^2 \right\} \hat{\mathbf{b}} - 2 \left[\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} \right] \left[\hat{\mathbf{b}} \cdot (\hat{\mathbf{c}}\mathbf{i}) \right] \mathbf{c}\mathbf{i},
 \end{aligned}$$

Again, this is the result that we obtained earlier.

7 Appendix B: Useful Transformations of Products of Three and Four vectors

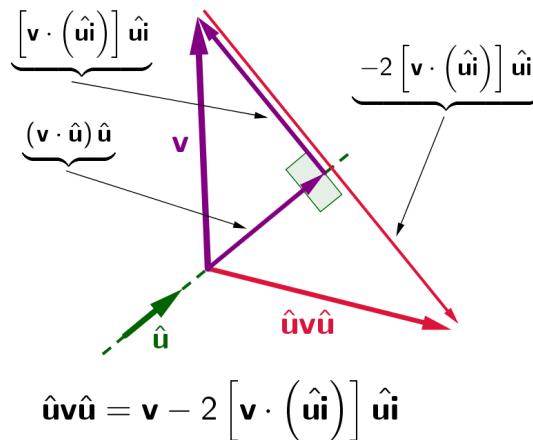
Readers may wish to consult Macdonald A. 2010's definitions of the dot and wedge products (p. 101) regarding some steps in the transformations. Those definitions can be invaluable as sources for ideas about transformations.

Note that some of this material is taken directly from the main text, without modification.

7.1 Transforming Products of the Form $\hat{u}v\hat{u}$

As noted in 4.4.2, for any two vectors \hat{u} and v , the product $\hat{u}v\hat{u}$ (which evaluates to a vector) is the reflection of v with respect to \hat{u} . Therefore,

$$\hat{u}v\hat{u} = v - 2[v \cdot (\hat{u}\hat{i})]\hat{u}\hat{i}.$$



We also note that because $u = |u|\hat{u}$,

$$uvu = u^2(\hat{u}v\hat{u}) = u^2v - 2[v \cdot (ui)]ui.$$

7.2 Useful Transformations of $\langle c_2 p t_1 p \rangle_2$

In 4.4.2 , we used the identity $uv \equiv 2u \cdot v - vu$ to derive an expression for $\langle c_2 p t_1 p \rangle_2$ as follows:

$$\begin{aligned}
 \langle c_2 p t_1 p \rangle_2 &= \langle c_2 (2p \cdot t_1 - t_1 p) p \rangle_2 \\
 &= \langle 2(p \cdot t_1) c_2 p - p^2 c_2 t_1 \rangle_2 \\
 &= 2(p \cdot t_1) c_2 \wedge p - p^2 c_2 \wedge t_1 \\
 &= 2(p i \cdot t_1) [(c_2 i) \cdot p] i - p^2 [(c_2 i) \cdot t_1] i.
 \end{aligned}$$

In this section, we'll derive other expressions that are equivalent to $\langle c_2 p t_1 p \rangle_2$. For convenience, we'll omit the subscripts from c_2 and t_1 .

Version 1

This version may be the simplest. In the main text, we saw that we wish to obtain equivalents that contain only two types of terms: products of the known vectors c and p , and products of t with those known vectors. We can do so as follows:

$$\begin{aligned}
 c p t p &= c (p t) p \\
 &= c (p \cdot t + p \wedge t) p \\
 &= (p \cdot t) c p + c (p \wedge t) p \\
 &= (p \cdot t) c p + c \underbrace{[(p i) \cdot t] i p}_{=p \wedge t} \\
 &= (p \cdot t) c p + \underbrace{[(p i) \cdot t] c i p}_{a \text{ scalar}} \\
 &= (p \cdot t) c p + [(p i) \cdot t] c (-p i) \\
 &= (p \cdot t) c p - [(p i) \cdot t] c p i.
 \end{aligned}$$

We need to find the bivector part of that result:

$$\begin{aligned}
 \langle c p t p \rangle_2 &= \langle (t \cdot p) c p - [t \cdot (p i)] c p i \rangle_2 \\
 &= t \cdot p \langle c p \rangle_2 - [t \cdot (p i)] \langle c p i \rangle_2 \\
 &= (t \cdot p) \underbrace{[c i \cdot (p)] i}_{=c \wedge p} - [t \cdot (p i)] \langle (c \cdot p + c \wedge p) i \rangle_2 \\
 &= (t \cdot p) \underbrace{[c i \cdot (p)] i}_{=c \wedge p} - [t \cdot (p i)] [\langle (c \cdot p) i \rangle_2 + \langle (c \wedge p) i \rangle_2].
 \end{aligned}$$

Now, we note that $(c \wedge p) i$ is a scalar, so $\langle (c \wedge p) i \rangle_2 = 0$:

$$\begin{aligned}
 (c \wedge p) i &= [c i \cdot p] i i \\
 &= - (c i) \cdot p.
 \end{aligned}$$

Therefore,

$$\langle \mathbf{cptp} \rangle_2 = (\mathbf{t} \cdot \mathbf{p}) [\mathbf{ci} \cdot (\mathbf{p})] \mathbf{i} - [\mathbf{t} \cdot (\mathbf{pi})] (\mathbf{c} \cdot \mathbf{p}) \mathbf{i}.$$

Version 2

$$\begin{aligned} \mathbf{cptp} &= \mathbf{cp} \{ (\mathbf{t} \cdot \hat{\mathbf{p}}) \hat{\mathbf{p}} + [\mathbf{t} \cdot (\hat{\mathbf{p}}\mathbf{i})] \hat{\mathbf{p}}\mathbf{i} \} \\ &= \mathbf{c} \left\{ (\mathbf{t} \cdot \hat{\mathbf{p}}) \underbrace{\mathbf{p}\hat{\mathbf{p}}\mathbf{p}}_{=|\mathbf{p}|^2\hat{\mathbf{p}}} + [\mathbf{t} \cdot (\hat{\mathbf{p}}\mathbf{i})] \mathbf{p}\hat{\mathbf{p}}\mathbf{i} \right\} \\ &= \mathbf{c} \{ (\mathbf{t} \cdot \mathbf{p}) \mathbf{p} - [\mathbf{t} \cdot (\hat{\mathbf{p}}\mathbf{i})] \mathbf{p}\hat{\mathbf{p}}\mathbf{i} \} \\ &= \mathbf{c} \{ (\mathbf{t} \cdot \mathbf{p}) \mathbf{p} - [\mathbf{t} \cdot (\mathbf{pi})] \mathbf{pi} \} \\ &= (\mathbf{t} \cdot \mathbf{p}) \mathbf{cp} - [\mathbf{t} \cdot (\mathbf{pi})] \mathbf{cpi}. \end{aligned}$$

We arrived at that same expression in Version 1, then proceeded from there to obtain

$$\langle \mathbf{cptp} \rangle_2 = (\mathbf{t} \cdot \mathbf{p}) [\mathbf{ci} \cdot (\mathbf{p})] \mathbf{i} - [\mathbf{t} \cdot (\mathbf{pi})] (\mathbf{c} \cdot \mathbf{p}) \mathbf{i},$$

Version 3

In the main text, we used the identity $\mathbf{ab} = 2\mathbf{a} \cdot \mathbf{b} - 2\mathbf{ba}$ to find an equivalent for $\langle \mathbf{cptp} \rangle_2$. Now, we'll use the identity $\mathbf{ab} = 2\mathbf{a} \wedge \mathbf{b} + 2\mathbf{ba}$.

$$\begin{aligned} \mathbf{cptp} &= \mathbf{c}(\mathbf{pt})\mathbf{p} \\ &= \mathbf{c}[2\mathbf{p} \wedge \mathbf{t} + \mathbf{tp}]\mathbf{p} \\ &= 2[(\mathbf{pi}) \cdot \mathbf{t}] \mathbf{cip} + p^2 \mathbf{cp}. \end{aligned}$$

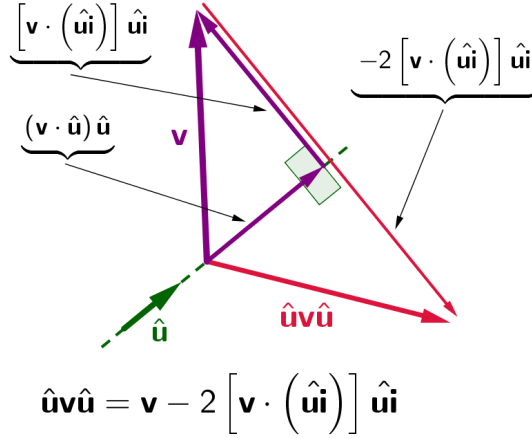
Now, using ideas that we saw in Versions 1 and 2, we find that

$$\langle \mathbf{cptp} \rangle_2 = p^2 [(\mathbf{ci}) \cdot \mathbf{t}] \mathbf{i} - 2\mathbf{c} \cdot \mathbf{p} [(\mathbf{pi}) \cdot \mathbf{t}] \mathbf{i}.$$

Version 4

As noted in 4.4.2, for any two vectors $\hat{\mathbf{u}}$ and \mathbf{v} , the product $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ (which evaluates to a vector) is the reflection of \mathbf{v} with respect to $\hat{\mathbf{u}}$. Therefore,

$$\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}} = \mathbf{v} - 2 \left[\mathbf{v} \cdot (\hat{\mathbf{u}}\mathbf{i}) \right] \hat{\mathbf{u}}\mathbf{i}.$$



We also note that because $\mathbf{u} = |\mathbf{u}|\hat{u}$,

$$\mathbf{u}v\mathbf{u} = u^2 (\hat{u}v\hat{u}) = u^2 \mathbf{v} - 2[\mathbf{v} \cdot (\mathbf{u}i)] \mathbf{u}i.$$

From that result, we can see that

$$\begin{aligned} \mathbf{c}p\mathbf{t}p &= p^2 \mathbf{c} \left\{ \mathbf{t} - 2 \left[\mathbf{t} \cdot (\hat{p}i) \right] \hat{p}i \right\} \\ &= p^2 \mathbf{c}t - 2[\mathbf{t} \cdot (\mathbf{p}i)] \mathbf{c}p\mathbf{i}. \end{aligned}$$

We want to find the bivector part:

$$\begin{aligned} \langle \mathbf{c}p\mathbf{t}p \rangle_2 &= p^2 \mathbf{c} \wedge \mathbf{t} - 2[\mathbf{t} \cdot (\mathbf{p}i)] \langle \mathbf{c}p\mathbf{i} \rangle_2 \\ &= p^2 [\mathbf{c}i \cdot \mathbf{t}] i - 2[\mathbf{t} \cdot (\mathbf{p}i)] (\mathbf{c} \cdot \mathbf{p}) i. \end{aligned}$$

See Version 1 for details on handling of $\langle \mathbf{c}p\mathbf{i} \rangle_2$.

Version 5

The method we'll use for our last version is nothing fancy: just "brute force and ignorance". But for that same reason, the method is useful when dealing with products of four distinct vectors, in which cases there are no symmetries, etc. of which we can take advantage:

$$\begin{aligned} \mathbf{c}p\mathbf{t}p &= \{\mathbf{c}p\} \{\mathbf{t}p\} \\ &= \left\{ \mathbf{c} \cdot \mathbf{p} + \underbrace{[(\mathbf{c}i) \cdot \mathbf{p}] i}_{=\mathbf{c} \wedge \mathbf{p}} \right\} \left\{ \mathbf{t} \cdot \mathbf{p} + \underbrace{[(\mathbf{t}i) \cdot \mathbf{p}] i}_{=\mathbf{t} \wedge \mathbf{p}} \right\}. \end{aligned}$$

The expansion of that product will produce three types of terms: scalars with scalars; bivectors with bivectors; and scalars with bivectors. Only the last of these evaluate to bivectors, so

$$\langle \mathbf{cptp} \rangle_2 = (\mathbf{c} \cdot \mathbf{p}) [(\mathbf{ti}) \cdot \mathbf{p}] \mathbf{i} + [(\mathbf{ci}) \cdot \mathbf{p}] \mathbf{i} (\mathbf{t} \cdot \mathbf{p}).$$

As noted earlier in this Appendix, we want an equivalent of $\langle \mathbf{cptp} \rangle_2$ that contains only two types of products: those of known vectors \mathbf{c} and \mathbf{p} , and those of \mathbf{t} with \mathbf{c} and \mathbf{p} . For that reason, we transform $(\mathbf{ti}) \cdot \mathbf{p}$ into $-(\mathbf{t}) \cdot \mathbf{pi}$, to give us

$$\langle \mathbf{cptp} \rangle_2 = [(\mathbf{ci}) \cdot \mathbf{p}] (\mathbf{t} \cdot \mathbf{p}) \mathbf{i} - (\mathbf{c} \cdot \mathbf{p}) [(\mathbf{pi}) \cdot \mathbf{t}] \mathbf{i}.$$

7.3 Transformations of Products of the Form $uvwu$

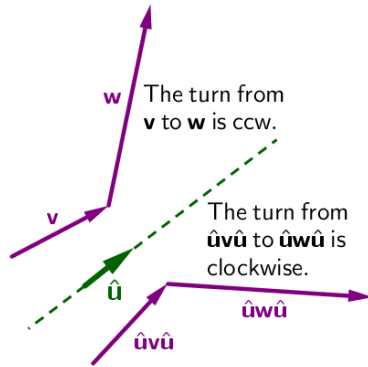
At the beginning of 4.4.2, we saw that the product $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ evaluates to a vector: specifically, the reflection of \mathbf{v} with respect to $\hat{\mathbf{u}}$. Similarly, the product $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}}$ is the reflection of the geometric product \mathbf{vw} . But let's see exactly why that is, and what it means. We'll discover that the scalar part of \mathbf{vw} : is unaffected by the reflection, but the bivector part is reversed, so that $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} = \mathbf{wv}$:

$$\begin{aligned} \hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} &= \hat{\mathbf{u}} (\mathbf{v} \cdot \mathbf{w} + \mathbf{v} \wedge \mathbf{w}) \hat{\mathbf{u}} \\ &= \hat{\mathbf{u}} (\mathbf{v} \cdot \mathbf{w}) \hat{\mathbf{u}} + \hat{\mathbf{u}} (\mathbf{v} \wedge \mathbf{w}) \hat{\mathbf{u}} \\ &= \hat{\mathbf{u}}^2 (\mathbf{v} \cdot \mathbf{w}) + \hat{\mathbf{u}} [(\mathbf{vi}) \cdot \mathbf{w}] \mathbf{i} \hat{\mathbf{u}} \\ &= \mathbf{v} \cdot \mathbf{w} + \hat{\mathbf{u}} [-\mathbf{v} \cdot (\mathbf{wi})] (-\hat{\mathbf{u}}\mathbf{i}) \\ &= \mathbf{v} \cdot \mathbf{w} + \hat{\mathbf{u}}^2 [(\mathbf{wi}) \cdot \mathbf{v}] \mathbf{i} \\ &= \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \wedge \mathbf{v} \\ &= \mathbf{wv}. \end{aligned}$$

Another interesting aspect of the product $\hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}}$ is that the reflection of the exterior product of \mathbf{v} and \mathbf{w} is equal to the exterior product of the two vectors' reflections:

$$\begin{aligned} \hat{\mathbf{u}}\mathbf{v}\mathbf{w}\hat{\mathbf{u}} &= \hat{\mathbf{u}}\mathbf{v} (\hat{\mathbf{u}}\hat{\mathbf{u}}) \mathbf{w}\hat{\mathbf{u}} \\ &= (\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}) (\hat{\mathbf{u}}\mathbf{w}\hat{\mathbf{u}}). \end{aligned}$$

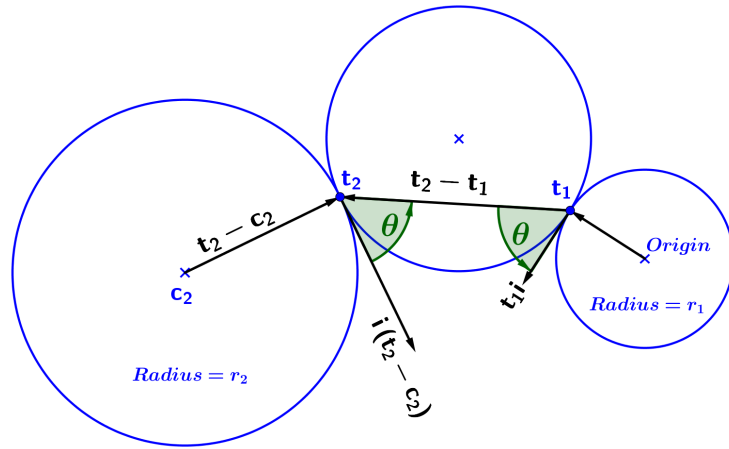
That observation provides a geometric interpretation of why reflecting a bivector changes its sign: the direction of the turn from \mathbf{v} to \mathbf{w} reverses.



8 Appendix C: Another Derivation of the Relationship between Points of Tangency among Circles

In this Appendix, we'll encounter a potential difficulty that can arise when solving problems involving tangency, after which we'll deal with that difficulty via a maneuver that is often available to us. The problem that we'll solve is

Using the information shown for the situation in the following diagram, derive an equation for t_2 in terms of t_1 , c_2 , r_1 , and r_2 .



We'll begin by equating two expressions for the rotation operator $e^{\theta i}$:

$$\frac{[i(t_2 - c_2)](t_1 - t_2)}{|t_2 - c_2||t_1 - t_2|} = \frac{(t_2 - t_1)t_1i}{|t_2 - t_1||t_1|}.$$

Making the substitutions $|t_1| = r_1$ and $|t_2 - c_2| = r_2$, and switching places between vectors and the bivector i while changing algebraic signs accordingly (page 37), we obtain

NOTE: $t_1^2 = r_1^2$, but $t_2^2 \neq r_2^2$.

$$\frac{(t_2 - c_2)(t_1 - t_2)i}{r_2|t_1 - t_2|} = \frac{(t_2 - t_1)t_1i}{|t_2 - t_1|r_1}, \text{ then}$$

$$(t_2 - c_2)(t_1 - t_2) = \left(\frac{r_2}{r_1}\right)(t_2 - t_1)t_1, \text{ and finally}$$

$$\left(1 - \frac{r_2}{r_1}\right)t_2t_1 + c_2t_2 - t_2^2 = c_2t_1 - r_1r_2.$$

This is the maneuver mentioned in the introduction to this Appendix.

The terms c_2t_2 and t_2^2 are troublesome. What might we do about them? There should be some relationship between those terms and r_2 because

$$(\mathbf{t}_2 - \mathbf{c}_2)^2 = r_2^2:$$

$$\begin{aligned} (\mathbf{t}_2 - \mathbf{c}_2)^2 &= r_2^2 \\ \mathbf{t}_2^2 - 2\mathbf{t}_2 \cdot \mathbf{c}_2 + \mathbf{c}_2^2 &= r_2^2 \\ \therefore 2\mathbf{t}_2 \cdot \mathbf{c}_2 - \mathbf{t}_2^2 &= \mathbf{c}_2^2 - r_2^2. \end{aligned}$$

Does that result help us to deal with the troublesome combination " $\mathbf{c}_2\mathbf{t}_2 - \mathbf{t}_2^2$ "? Since the relationship we've just found between \mathbf{c}_2 , \mathbf{t}_2 , and r_2 involves $\mathbf{t}_2 \cdot \mathbf{c}_2$ rather than $\mathbf{t}_2\mathbf{c}_2$, let's write $\mathbf{c}_2\mathbf{t}_2 - \mathbf{t}_2^2$ as

$$\mathbf{c}_2\mathbf{t}_2 - \mathbf{t}_2^2 = \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 - \mathbf{t}_2^2.$$

We see now that there might be a role here for the relationship $2\mathbf{t}_2 \cdot \mathbf{c}_2 - \mathbf{t}_2^2 = \mathbf{c}_2^2 - r_2^2$ that we found between \mathbf{c}_2 , \mathbf{t}_2 , and r_2 .

$$\begin{aligned} \mathbf{c}_2\mathbf{t}_2 - \mathbf{t}_2^2 &= \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 - \mathbf{t}_2^2 \\ &= 2\mathbf{c}_2 \cdot \mathbf{t}_2 - \mathbf{t}_2^2 - \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 \\ &= \mathbf{c}_2^2 - r_2^2 - \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2. \end{aligned}$$

With this substitution, our earlier equation

$$\left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 + \mathbf{c}_2\mathbf{t}_2 - \mathbf{t}_2^2 = \mathbf{c}_2\mathbf{t}_1 - r_1r_2$$

becomes

$$\left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 + \mathbf{c}_2^2 - r_2^2 - \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 = \mathbf{c}_2\mathbf{t}_1 - r_1r_2.$$

This is a good time to pause and recall that we want to derive an equation for \mathbf{t}_2 . From experience, we know that to do so, we'll want to transform the equation we just obtained into one that has the form

$$A \text{ product of } \mathbf{t}_2 \text{ and some known vector} = \text{Some known multivector},$$

where the "known vector" may be a linear combination of known vectors. For example, of \mathbf{t}_1 —which we're treating as known—and \mathbf{c}_2 . Similarly, the known multivector may be the sum of known scalars, known bivectors, and products of known vectors. As a first step in writing an equation of that form, let's move all terms that don't involve \mathbf{t}_2 to the right-hand side:

$$\left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 - \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 = r_2^2 - r_1r_2 - \mathbf{c}_2^2 + \mathbf{c}_2\mathbf{t}_1.$$

The $\mathbf{t}_2\mathbf{t}_1$ term is a right-multiplication of \mathbf{t}_2 by the known vector \mathbf{t}_1 . Now, if we can transform $-\mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2$ into a right-multiplication of \mathbf{t}_2 , we'll have a left-hand side of the form that we desire. You've probably seen already what we need to do:

$$\begin{aligned} \left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 - \mathbf{c}_2 \cdot \mathbf{t}_2 + \mathbf{c}_2 \wedge \mathbf{t}_2 &= r_2^2 - r_1r_2 - c_2^2 + \mathbf{c}_2\mathbf{t}_1 \\ \left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 - \mathbf{c}_2 \cdot \mathbf{t}_2 - \mathbf{t}_2 \wedge \mathbf{c}_2 &= r_2^2 - r_1r_2 - c_2^2 + \mathbf{c}_2\mathbf{t}_1 \\ \left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_2\mathbf{t}_1 - \mathbf{t}_2\mathbf{c}_2 &= r_2^2 - r_1r_2 - c_2^2 + \mathbf{c}_2\mathbf{t}_1 \\ \mathbf{t}_2 \left[\left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_1 - \mathbf{c}_2 \right] &= r_2^2 - r_1r_2 - c_2^2 + \mathbf{c}_2\mathbf{t}_1. \end{aligned}$$

Now, we solve for \mathbf{t}_2 via

$$\mathbf{t}_2 = [r_2^2 - r_1r_2 - c_2^2 + \mathbf{c}_2\mathbf{t}_1] \left[\left(1 - \frac{r_2}{r_1}\right) \mathbf{t}_1 - \mathbf{c}_2 \right]^{-1}.$$

To effect the operations on the right-hand side, we may choose to write $\mathbf{c}_2\mathbf{t}_1$ as $\mathbf{c}_2 \cdot \mathbf{t}_1 + [(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{i}$. If we do so, then the expression for \mathbf{t}_2 becomes

$$\mathbf{t}_2 = \frac{\left[2r_2^2 - r_1r_2 + \left(\frac{r_2}{r_1} - 2\right)c_2^2 - \frac{r_2^3}{r_1}\right]\mathbf{t}_1 + (r_1^2 + c_2^2 - r_2^2)\mathbf{c}_2 + 2[(\mathbf{c}_2\mathbf{i}) \cdot \mathbf{t}_1]\mathbf{c}_2\mathbf{i}}{c_2^2 + (r_1 - r_2)^2 - \left[2\left(1 - \frac{r_2}{r_1}\right)c_2\right] \cdot \mathbf{t}_1}.$$

How would the equation for \mathbf{t}_3 in the following situation differ from the equation for \mathbf{t}_2 that we just derived?

