Is "Dark Energy" Just an Effect of Gravitational Time Dilation?

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Abstract

When an expanding uniform-density dust ball's radius doesn't sufficiently exceed the Schwarzschild value, its expansion rate will actually be increasing because the dominant gravitational time dilation effect diminishes as the dust ball expands. But such acceleration of expansion is absent in "comoving coordinates" because the "comoving" fixing of the 00 component of the metric tensor to unity extinguishes gravitational time dilation, as is evidenced in the "comoving" FLRW dust-ball model by the Newtonian form of its Friedmann equation of motion. Therefore we extend to all dust-ball initial conditions the singular Oppenheimer-Snyder transformation from "comoving" to "standard" coordinates which they carried out for a particular initial condition. In "standard" coordinates relativistic time dilation is manifest in the equations of motion of the dynamical radii of all of the dust ball's interior shells; the acceleration of expansion of the surface shell peaks when its radius is only fractionally larger than the dust ball's Schwarzschild radius. Even so, for a range of initial conditions a dust ball's expansion continues accelerating at all "standard" times, although that acceleration asymptotically decreases toward zero. Attempts to account for the observed acceleration of the expansion of the universe by fitting a nonzero "dark energy" cosmological constant thus seem to be quite unnecessary.

Introduction

The Friedmann equation for the spherically-symmetric, uniform-density FLRW dust-ball model in "comoving coordinates" is mathematically indistinguishable from the *strictly Newtonian* equation of motion for a test mass moving purely radially under the gravitational influence of a point mass [1, 2]. That seeming anomaly occurs because in "comoving coordinates" the metric component g_{00} *is fixed to unity* [3]; therefore since $(g_{00})^{-\frac{1}{2}}$ *is the gravitational time dilation factor* [4], "comoving coordinates" necessarily *extinguish* gravitational time dilation, *whose presence normally distinguishes GR from Newtonian gravitation* (in which there is *no* gravitational time dilation).

Gravitation *without* gravitational time dilation seems unlikely to be compatible with *GR physical principles*, however. In fact, in order to *accomplish* the fixing of the metric component g_{00} to unity, "comoving time" is defined by the clock readings of *an infinite number of different observers* [5], a "coordinate" definition *that is completely incompatible with Einstein's observer-to-coordinate-system paradigm*.

The *GR-unphysical nature of "comoving time"* is further *underlined* by the fact that the metric tensor for the FLRW dust-ball model in "comoving coordinates" has a *singularity* at the particular "comoving time" when the Newtonian-analog radially-moving test mass *coincides in location* with the Newtonian-analog point mass.

That *metric singularities* are indeed *GR-unphysical* follows via Einstein's equivalence principle from the fact that *coordinate-transformation Jacobian-matrix singularities* are incompatible with the tensor contraction theorem—the tensor contraction theorem is of course *indispensable* to the general covariance of the Einstein equation because the Einstein tensor is constructed from *contractions* of the Riemann tensor.

The incompatibility of coordinate-transformation Jacobian-matrix singularities with the tensor contraction theorem stems from the fact that the proof of the tensor contraction theorem requires the Jacobian matrix of any candidate coordinate transformation $\bar{x}^\alpha(x^\mu)$ (and of its inverse transformation $x^\nu(\bar{x}^\alpha)$) to satisfy the Jacobian-matrix relation [6],

$$(\partial\bar{x}^\alpha/\partial x^\mu)(\partial x^\nu/\partial\bar{x}^\alpha) = \delta^\nu_\mu, \tag{1}$$

which, if each component of the Jacobian matrix $\partial\bar{x}^\alpha/\partial x^\mu$ is well-defined in terms of the finite real numbers at a given space-time point x^μ, and *also* each component of its inverse matrix is thus well-defined in terms of the finite real numbers, follows at that space-time point from the chain rule of the calculus. However, because the right-hand side δ^ν_μ of Eq. (1) is *always* well-defined in terms of the finite real numbers, Eq. (1) *becomes self-inconsistent* at any space-time point x^μ where any component of the Jacobian matrix $\partial\bar{x}^\alpha/\partial x^\mu$ or any component of its inverse matrix *fails to be a well-defined finite real number*. Thus at a singularity of a coordinate transformation's Jacobian matrix or at a singularity of the inverse of that matrix the underpinning of the proof of the GR-indispensable tensor contraction theorem is destroyed.

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Einstein's equivalence principle implies that a metric tensor is at each space-time point the congruence transformation of the Minkowski metric tensor with the Jacobian matrix of some coordinate transformation [7]. Therefore given the foregoing discussion of GR-physical coordinate transformations, a metric tensor is GR-physical *only* at those space-time points where both it and its inverse consist solely of components which are well-defined finite real numbers and its signature is equal to the $\{+,-,-,-\}$ signature of the Minkowski metric tensor [8]. Thus the metric tensor of the FLRW dust-ball model in "comoving coordinates" is clearly GR-unphysical at its singularity in "comoving time", namely at the particular "comoving time" when its Newtonian-analog radially-moving test mass coincides in location with its Newtonian-analog point mass.

In a *GR-physical* coordinate system where g_{00} *isn't* fixed to unity, the FLRW dust-ball model *will* of course be affected by gravitational time dilation. Oppenheimer and Snyder transformed an FLRW dust-ball model with a very particular initial condition (namely that the radial velocity of the Newtonian-analog test mass initially vanishes) from GR-unphysical "comoving coordinates"—where its metric tensor is governed by the GR-unphysical Newtonian-analogous Friedmann equation—to "standard" coordinates [9], and found that in this *GR-physical* coordinate system gravitational time dilation *completely blocks* the Newtonian-analog radially-moving test mass from ever coming as near to the Newtonian-analog point mass as the Schwarzschild radius of that point mass [10, 11, 2], thus *preventing* the "comoving coordinate" metric *singularity* from *ever occurring* in *GR-physical* "standard" coordinates.

The time dilation *introduced* by the Oppenheimer-Snyder transformation from "comoving" to "standard" coordinates is entirely naturally *infinite* at the Schwarzschild radius of the Newtonian-analog point mass. That fact, however, makes the Oppenheimer-Snyder coordinate transformation a GR-unphysical *singular* one. Of course it is mathematically *obvious* that a GR-unphysical *singular* metric can *only* be transformed into a GR-physical *nonsingular* metric by a GR-unphysical *singular* coordinate transformation [11], such as the one of Oppenheimer and Snyder. If the relevant Einstein equations were all analytically solvable, the GR practitioner would have *no call* to *ever* become thus embroiled in GR-unphysical singular metrics or coordinate transformations. For example, the GR sensible approach to the FLRW dust ball model would be to straightaway solve its Einstein equation in a GR-physical coordinate system such as the "standard", "isotropic" or "harmonic" one, *shunning* the patently *GR-unphysical* $g_{00} = 1$ "comoving system" like the proverbial plague. Unfortunately, of course, the brutal fact of the matter is that it isn't known how to analytically solve the Einstein equation for the dust ball in *other* than GR-unphysical "comoving coordinates". But the GR practitioner certainly mustn't proceed under the misapprehension that the use of GR-unphysical $g_{00} = 1$ "coordinates" produces an Einstein-equation solution which is GR-physical; indeed the *resulting* very precisely Newtonian Friedmann equation for a test mass moving radially in the gravitational field of a point mass, which is bereft of *any trace* of purely relativistic phenomena such as gravitational or speed time dilation, but which *does* deliver a wholly GR-unphysical *metric singularity*, absolutely confirms *the opposite*.

Therefore if we want *GR-physical* analytic results of the FLRW dust ball model, we apparently have no viable option *other* than to follow the Oppenheimer-Snyder lead of (singularly) *transforming* the patently GR-unphysical singular "comoving" dust-ball metric to *a GR-physical coordinate system*, such as the "standard", "isotropic" or "harmonic" one, in which we *would* have analytically solved the dust-ball Einstein equation had we been *able* to do so.

The *experience* of Oppenheimer and Snyder clearly shows that this coordinate-transformation course *indeed fills in fundamental phenomena of GR physics* such as gravitational time dilation *that are entirely absent from GR-unphysical "comoving metric" results*. Proper understanding of gravitational time dilation in the dust-ball model is of considerable importance, for example, to working out the behavior of an expanding dust ball in the distant past [2]. Gravitational time dilation can, just on its face, be expected to *reverse* the intuitive Newtonian "deceleration of expansion" of any dust ball which isn't sufficiently larger than its Schwarzschild radius. Such implications of gravitational time dilation for dust balls are of special interest in light of observations that the universe is undergoing "acceleration of expansion" [12, 13] (which has popularly been modeled by fitting a nonzero value of the "cosmological constant", thereby producing a space-time permeating "expansive pressure" [14], the ether-reminiscent "dark energy").

Therefore this article will *extend* Oppenheimer and Snyder's transformation from their specialized case of *initially stationary* dust-ball uniform energy density [15] in "comoving coordinates" to *any initial rate of change* of dust-ball uniform energy density in those "coordinates". The specialized Oppenheimer-Snyder initial condition is guaranteed to be immediately followed by an epoch of *increasing* uniform energy density of the dust ball in "comoving coordinates", which implies a *contracting* dust-ball radius in *GR-physical* coordinates such as "standard" coordinates. Oppenheimer and Snyder's focus on "gravitational collapse" guided their selection of a specialized initial condition, but here we want to *as well* treat all cases of *expanding* dust balls, so we will avoid restricting the initial rate of change of dust-ball uniform energy density.

The extension to arbitrary initial rates of change of dust-ball uniform energy density results in *an extended form* of the particular Friedmann equation which was relevant for Oppenheimer and Snyder. Therefore, *before* we present the derivation of the extended Oppenheimer-Snyder transformation itself, we present in the next section important properties of the *solutions* of that extended Friedmann equation, as those solutions are *key constituents* of the GR-unphysical singular "comoving metric tensors" for dust balls which are to be (singularly) transformed to GR-physical nonsingular "standard" form.

Friedmann-equation solutions for general "comoving" dust balls

In GR-unphysical "comoving coordinates" all individual dust particles always have zero three-velocity [16], so a uniform energy-density dust ball of radius a never changes that radius in "comoving coordinates". However, the value of the uniform energy density within the dust ball *can* change in GR-unphysical "comoving time"; the Friedmann equation is a *consequence* of the Einstein equation in "comoving coordinates" which governs the evolution of the dust ball's uniform energy density and the accompanying "comoving metric tensor" within the dust ball. The dimensionless function which the first-order Friedmann differential equation in GR-unphysical "comoving time" describes can be conceptualized in different ways, one convenient one being as the cube root of the reciprocal of the ratio of the dust ball's uniform energy density to its initial uniform energy density [17],

$$R(t) = (\rho(t_0)/\rho(t))^{\frac{1}{3}}, \tag{2a}$$

so that $R(t_0) = 1$. But *in addition* to the Eq. (2a) relationship that is satisfied by $R(t)$, the *square* of $R(t)$ *also occurs as the unique "comoving time-dependent" factor* of *both* nontrivial components of the spherically-symmetric "comoving metric tensor" [18].

The Friedmann equation for $R(t)$ which follows from the Einstein equation for the uniform energy-density dust ball in "comoving coordinates" can be conveniently written as [19],

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma), \tag{2b}$$

for which, from Eq. (2a),

$$R(t_0) = 1. \tag{2c}$$

The convenient abbreviation ω^2 is defined by,

$$\omega^2 \stackrel{\text{def}}{=} (8\pi/3)G\rho(t_0)/c^2, \tag{2d}$$

and the dimensionless constant γ can be evaluated in terms of $(\dot{R}(t_0)/\omega)^2$ by specializing Eq. (2b) to the initial time $t = t_0$,

$$\gamma = (\dot{R}(t_0)/\omega)^2 - 1 = (\dot{\rho}(t_0)/(3\omega\rho(t_0)))^2 - 1, \tag{2e}$$

where the second equality follows from Eq. (2a). Thus γ *reflects* $\dot{\rho}(t_0)$, the initial rate of change of dust-ball uniform energy density, moreover,

$$\gamma \geq -1. \tag{2f}$$

Oppenheimer and Snyder deliberately *restricted* their work to $\dot{\rho}(t_0) = 0$, i.e., to $\gamma = -1$.

The *wholly Newtonian analog* of the Friedmann equation emerges upon taking the radial coordinate $r(t)$ of the purely radially moving Newtonian-analog test mass to be,

$$r(t) \stackrel{\text{def}}{=} aR(t), \tag{3a}$$

and the mass M of the Newtonian-analog point mass to be the initial effective mass of the dust ball, i.e.,

$$M = (4\pi/3)\rho(t_0)a^3/c^2 = \omega^2 a^3/(2G), \tag{3b}$$

which implies that,

$$\omega^2 = 2GM/a^3. \tag{3c}$$

Inserting the Eq. (3a) and (3c) substitutions for $R(t)$ and ω^2 into the Friedmann Eq. (2b) and also into Eq. (2e) for γ, and furthermore taking account of the Eq. (2c) initial condition yields,

$$\tfrac{1}{2}(\dot{r}(t))^2 - GM/r(t) = \tfrac{1}{2}(\dot{r}(t_0))^2 - GM/r(t_0), \tag{4a}$$

which when multiplied through by the arbitrary value m of the test mass yields the very familiar *conservation* of the *strictly Newtonian* kinetic plus gravitational potential energy of the test mass in the gravitational field of the point mass M.

Of course when the location of the test mass is coincident with that of the point mass, namely when $r(t) = 0$ (which is when $R(t) = 0$ in the language of the Friedmann equation), then $(\dot{r}(t))^2$ is *infinite* (and the *same* is true of $(\dot{R}(t))^2$ in Friedmann equation language). Furthermore, since $(R(t))^2$ is the "comoving" time-dependent *factor* of both nontrivial *components* of the "comoving metric" which applies within the dust ball, that metric is *singular* when $R(t) = 0$. In *addition*, Eq. (2a) tells us that $\rho(t)$ is *infinite* when $R(t) = 0$.

A simple alternative way to write the test-mass Eq. (4a) is readily verified to be,

$$(dr/dt)^2 = 2GM\big((1/r) + (\gamma/a)\big), \tag{4b}$$

where, of course, $a = r(t_0)$. In "standard" coordinates it will turn out that *the dynamically changing radius of the dust ball* obeys Eq. (4b) in the nonrelativistic limit $c \to \infty$, but for *finite* values of c, the *form* of Eq. (4b) is *modified* by *a reciprocal squared relativistic time dilation factor on its right-hand side* that makes $|dr/dt|$ not only less than c but as well *linearly diminishing toward zero* as r approaches the Schwarzschild radius value $r_S = 2GM/c^2$ of the dust ball.

The solution of the Friedmann equation can be *directly* expressed in terms of elementary functions *only* for "parabolic" initial conditions wherein $\gamma = 0$. In that case the Friedmann equation simplifies to,

$$(\dot{R}(t))^2 = \omega^2/R(t) \ \text{ or } \ \dot{R}(t) = \pm\omega/(R(t))^{\frac{1}{2}}, \tag{5a}$$

which with the initial condition $R(t_0) = 1$ yields the solution,

$$R(t) = (1 \pm \tfrac{3}{2}\omega(t - t_0))^{\frac{2}{3}}, \tag{5b}$$

where $\pm$ is the *sign* of $\dot{R}(t_0)$.

Even in those cases where $\gamma \neq 0$, however, the Friedmann equation *and* its initial condition $R(t_0) = 1$ can be cast into the integral form,

$$\int_1^{R(t)} R^{\frac{1}{2}} dR/(1 + \gamma R)^{\frac{1}{2}} = \pm\omega(t - t_0). \tag{6}$$

where $\pm$ is again the *sign* of $\dot{R}(t_0)$ when $\dot{R}(t_0) \neq 0$, and equals -1 in the case that $\dot{R}(t_0)$ vanishes (which is precisely the $\gamma = -1$ case that was treated by Oppenheimer and Snyder).

In the "parabolic" $\gamma = 0$ initial-condition case, Eq. (6) quickly leads to the solution for $R(t)$ that is given in Eq. (5b).

In the "hyperbolic" $\gamma > 0$ initial-condition case, the change of variable $R = [\sinh(u)]^2/\gamma$, i.e., $u = \sinh^{-1}((\gamma R)^{\frac{1}{2}})$, permits evaluation of the integral on the left side of Eq. (6) in terms of elementary functions. But the *consequence* of that evaluation is *only* an *implicit* algebraic expression for $R(t)$, namely,

$$(R(t))^{\frac{1}{2}}(1 + \gamma R(t))^{\frac{1}{2}} - (1 + \gamma)^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} \sinh^{-1}(\gamma^{\frac{1}{2}}) - \gamma^{-\frac{1}{2}} \sinh^{-1}((\gamma R(t))^{\frac{1}{2}}) = \pm\gamma\omega(t - t_0). \tag{7a}$$

Since $\sinh^{-1}(x) = \ln\big((1 + x^2)^{\frac{1}{2}} + x\big)$, we can also express Eq. (7a) in the form,

$$(R(t))^{\frac{1}{2}}(1 + \gamma R(t))^{\frac{1}{2}} - (1 + \gamma)^{\frac{1}{2}} + \gamma^{-\frac{1}{2}} \ln\left(\frac{(1+\gamma)^{\frac{1}{2}} + \gamma^{\frac{1}{2}}}{(1+\gamma R(t))^{\frac{1}{2}} + (\gamma R(t))^{\frac{1}{2}}}\right) = \pm\gamma\omega(t - t_0). \tag{7b}$$

In the "elliptic" $-1 \leq \gamma < 0$ initial-condition case, the change of variable $R = -[\sin(u)]^2/\gamma$, i.e., $u = \arcsin((-\gamma R)^{\frac{1}{2}})$, likewise permits evaluation of the integral on the left side of Eq. (6) in terms of elementary functions. The consequence of that evaluation is the following *implicit* algebraic expression for $R(t)$,

$$(R(t))^{\frac{1}{2}}(1 + \gamma R(t))^{\frac{1}{2}} - (1 + \gamma)^{\frac{1}{2}} + (-\gamma)^{-\frac{1}{2}} \arcsin((-\gamma)^{\frac{1}{2}}) - (-\gamma)^{-\frac{1}{2}} \arcsin((-\gamma R(t))^{\frac{1}{2}}) = \pm\gamma\omega(t - t_0). \tag{8}$$

We have pointed out that at the "comoving time" t when $R(t) = 0$ it is the case that $(\dot{R}(t))^2$ is *infinite* and the "comoving" metric is *singular*. With Eqs. (5b), (7b) and (8) in hand, we can now *explicitly* write down the *value* of the "comoving time" t_s when $R(t_s) = 0$, namely the *value* of the "comoving time" *when this singularity occurs*.

In the "parabolic" initial-condition case that $\gamma = 0$ we see from Eq. (5b) that if $R(t_s) = 0$,

$$t_s = t_0 \mp \tfrac{2}{3}\omega^{-1}. \tag{9a}$$

In the "hyperbolic" initial-condition case that $\gamma > 0$ we see from Eq. (7b) that if $R(t_s) = 0$,

$$t_s = t_0 \mp (\gamma\omega)^{-1}\left[(1+\gamma)^{\frac{1}{2}} - \gamma^{-\frac{1}{2}}\ln\left((1+\gamma)^{\frac{1}{2}} + \gamma^{\frac{1}{2}}\right)\right], \tag{9b}$$

In the "elliptic" initial-condition case that $-1 \le \gamma < 0$ we see from Eq. (8) that if $R(t_s) = 0$,

$$t_s = t_0 \mp (\gamma\omega)^{-1}\left[(1+\gamma)^{\frac{1}{2}} - (-\gamma)^{-\frac{1}{2}}\arcsin\left((-\gamma)^{\frac{1}{2}}\right)\right]. \tag{9c}$$

Having completed this comprehensive discussion of the character of the Friedmann equation and its solutions in the cases of "parabolic", "hyperbolic" and "elliptic" types of dust-ball initial conditions, we now turn to the *extension* to *all* of these types of dust-ball initial conditions of the Oppenheimer-Snyder transformation—the work of Oppenheimer and Snyder themselves was deliberately restricted to $\dot{\rho}(t_0) = 0$, which corresponds to the "elliptic" value -1 for the Friedmann-equation parameter γ.

The Oppenheimer-Snyder transformation for general "comoving" dust balls

The spherically-symmetric "comoving metric" for which the Einstein equation is solved in conjunction with the dust ball of radius a and uniform initial energy density $\rho(t_0)$ has the form [20],

$$ds^2 = (cdt)^2 - U(r,t)dr^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \tag{10a}$$

The result *in the region* $0 \le r \le a$ of solving the Einstein equation for this metric and the dust ball's uniform energy density $\rho(t)$ is [21],

$$V(r,t) = r^2(R(t))^2, \quad U(r,t) = (R(t))^2/(1+\gamma(\omega r/c)^2), \text{ and } \rho(t) = \rho(t_0)/(R(t))^3, \tag{10b}$$

where,

$$\omega^2 \stackrel{\text{def}}{=} (8\pi/3)G\rho(t_0)/c^2, \tag{10c}$$

and the dimensionless entity $R(t)$ satisfies the Friedmann equation,

$$(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma), \tag{10d}$$

with the initial condition $R(t_0) = 1$, which implies that $\gamma = (\dot{R}(t_0)/\omega)^2 - 1 = (\dot{\rho}(t_0)/(3\omega\rho(t_0)))^2 - 1 \ge -1$.

To carry out the Oppenheimer-Snyder mapping of the spherically-symmetric "comoving" coordinates (r,t) to the spherically-symmetric "standard" coordinates $(\bar{r},\bar{t})$ we write the invariant differential line element ds^2 of Eq. (10a) in terms of the metric tensors of *both* coordinate systems [2, 11],

$$\begin{aligned} ds^2 &= B(\bar{r},\bar{t})(cd\bar{t})^2 - A(\bar{r},\bar{t})(d\bar{r})^2 - \bar{r}^2((d\theta)^2 + (\sin\theta d\phi)^2) \\ &= (cdt)^2 - U(r,t)(dr)^2 - V(r,t)((d\theta)^2 + (\sin\theta d\phi)^2). \end{aligned} \tag{11a}$$

Eq. (11a) *constrains* the mapping vector $(\bar{r}(r,t), \bar{t}(r,t))$; thus comparing the last terms on the left and right-hand sides respectively of Eq. (11a) immediately yields,

$$\bar{r}(r,t) = (V(r,t))^{\frac{1}{2}} = rR(t), \tag{11b}$$

where we have used the Eq. (10b) relation $V(r,t) = r^2(R(t))^2$. Next we would like to obtain $\bar{t}$ as a function of r and t, just as has been done in Eq. (11b) for $\bar{r}$. Inspection of Eq. (11a), however, reveals that that task is completely entwined with the determination of B and A as functions of r and t; moreover $\bar{t}$ *itself doesn't occur in relations that can be extracted from* Eq. (11a), *only its partial derivatives* $(\partial\bar{t}/\partial t)$ *and* $(c(\partial\bar{t}/\partial r))$ *do*. We are thus faced with solving *both* simultaneous algebraic *and* first-order partial differential equations merely to obtain $\bar{t}(r,t)$! These considerations give us our first small taste of the formidably long and arduous path, so masterly pioneered by Oppenheimer and Snyder, which lies ahead.

We now present in more explicit detail the part of Eq. (11a) which still must be solved to obtain B, A and $\bar{t}$,

$$B[(\partial\bar{t}/\partial t)(cdt) + c(\partial\bar{t}/\partial r)dr]^2 - A[(1/c)(\partial\bar{r}/\partial t)(cdt) + (\partial\bar{r}/\partial r)dr]^2 = (cdt)^2 - U(r,t)(dr)^2. \tag{11c}$$

Since the three bilinear differential forms $(cdt)^2$, $(2c\,dt\,dr)$ and $(dr)^2$ are linearly independent, Eq. (11c) produces *the three simultaneous equations*,

$$B(\partial\bar{t}/\partial t)^2 - A((1/c)(\partial\bar{r}/\partial t))^2 = 1, \tag{12a}$$

$$B(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) - A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r) = 0, \tag{12b}$$

$$B(c(\partial\bar{t}/\partial r))^2 - A(\partial\bar{r}/\partial r)^2 = -U. \tag{12c}$$

We begin by solving Eq. (12b) for B in terms of A, $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$B = \frac{A((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}{(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))}. \tag{13a}$$

We substitute Eq. (13a) into Eq. (12a), and solve the result for A in terms of $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$A = \frac{1}{(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)/(c(\partial\bar{t}/\partial r)) - ((1/c)(\partial\bar{r}/\partial t))^2}. \tag{13b}$$

We now divide Eq. (12c) through by A, thus expressing it in terms of (B/A) and $(1/A)$. Into that we insert (B/A) obtained from Eq. (13a) and $(1/A)$ obtained from Eq. (13b). The resulting equation in terms of $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ is,

$$\frac{(1/c)(\partial\bar{r}/\partial t)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))}{(\partial\bar{t}/\partial t)} - (\partial\bar{r}/\partial r)^2 + U\left[\frac{(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)}{(c(\partial\bar{t}/\partial r))} - ((1/c)(\partial\bar{r}/\partial t))^2\right] = 0. \tag{14a}$$

We now multiply Eq. (14a) through by the factor $(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))$ to obtain the following homogeneous bilinear equation in the two variables $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$,

$$\begin{gathered}(1/c)(\partial\bar{r}/\partial t)(\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r))^2 - (\partial\bar{r}/\partial r)^2(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \\ U[(\partial\bar{r}/\partial r)(1/c)(\partial\bar{r}/\partial t)(\partial\bar{t}/\partial t)^2 - ((1/c)(\partial\bar{r}/\partial t))^2(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r))] = 0.\end{gathered} \tag{14b}$$

The structure of this homogeneous bilinear equation in the two variables $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ becomes more transparent after *dividing it through by the factor* $U(\partial\bar{r}/\partial r)((1/c)(\partial\bar{r}/\partial t))$ and rearranging the order of the terms to obtain,

$$(\partial\bar{t}/\partial t)^2 - \left(\left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right] + \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right]\right)(\partial\bar{t}/\partial t)(c(\partial\bar{t}/\partial r)) + \left[\frac{1}{U}\right](c(\partial\bar{t}/\partial r))^2 = 0. \tag{14c}$$

The homogeneous bilinear form in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$ on the left-hand side of Eq. (14c) can now be seen to *factor* into the *product* of two homogeneous *linear* forms in $(\partial\bar{t}/\partial t)$ and $(c(\partial\bar{t}/\partial r))$, namely,

$$\left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left((\partial\bar{t}/\partial t) - \left[\frac{(\partial\bar{r}/\partial r)}{U((1/c)(\partial\bar{r}/\partial t))}\right](c(\partial\bar{t}/\partial r))\right) = 0. \tag{14d}$$

Now the *first* factor on the left-hand side of Eq. (14d) turns out to *also* be a factor of the *denominator* of the solution for A which is given by Eq. (13b). Indeed, it is readily verified from Eq. (13b) that $(1/A)$ can be written in the form,

$$(1/A) = \left((\partial\bar{t}/\partial t) - \left[\frac{((1/c)(\partial\bar{r}/\partial t))}{(\partial\bar{r}/\partial r)}\right](c(\partial\bar{t}/\partial r))\right)\left(\frac{((1/c)(\partial\bar{r}/\partial t))(\partial\bar{r}/\partial r)}{(c(\partial\bar{t}/\partial r))}\right),$$

which implies that if the *first* factor on the left-hand side of Eq. (14d) vanished, then A would be *infinite*. Thus to obtain a finite value for the "standard" metric component A, it must be that the *second* factor on the left-hand side of Eq. (14d) vanishes, which implies the following homogeneous linear first-order partial differential equation for $\bar{t}(r,t)$,

$$((1/c)(\partial\bar{r}/\partial t))U(\partial\bar{t}/\partial t) = (\partial\bar{r}/\partial r)(c(\partial\bar{t}/\partial r)). \tag{15a}$$

Since $\bar{r}(r,t) = rR(t)$ from Eq. (11b) and $U(r,t) = (R(t))^2/(1+\gamma(\omega r/c)^2)$ from Eq. (10b), the Eq. (15a) partial differential equation for $\bar{t}(r,t)$ can be written,

$$[(r/c)/(1+\gamma(\omega r/c)^2)](\partial\bar{t}/\partial t) = [R(t)\dot{R}(t)]^{-1}(c(\partial\bar{t}/\partial r)). \tag{15b}$$

Now given a separable homogeneous linear first-order partial differential equation of the form,

$$\tau(r)(\partial\bar{t}/\partial t) = T(t)(c(\partial\bar{t}/\partial r)), \tag{16a}$$

there exists a large class of solutions for $\bar{t}(r,t)$. Indeed, for *any* differentiable dimensionless function $\chi(y)$ of dimensionless argument, Eq. (16a) is solved by,

$$\bar{t}(r,t) = t_1\chi\left(f^{(2)}\int_{t_2}^{t}T(t')dt' + (1/c)f^{(2)}\int_{r_0}^{r}\tau(r')dr'\right), \tag{16b}$$

where $f^{(2)}$, r_0, t_1 and t_2 are arbitrary constants with the dimensions of frequency squared, length and time respectively. That the $\bar{t}(r,t)$ of Eq. (16b) actually solves the partial differential equation of Eq. (16a) can be straightforwardly verified.

In the *particular* case of Eq. (15b), which of course is the partial differential equation *of actual interest to us*, $\tau(r) = (r/c)/(1+\gamma(\omega r/c)^2)$ and $T(t) = [R(t)\dot{R}(t)]^{-1}$. If we conveniently select the arbitrary constant r_0 of Eq. (16b) to be zero, then,

$$(1/c)\int_0^r[(r'/c)/(1+\gamma(\omega r'/c)^2)]dr' = \tfrac{1}{2}(\gamma\omega^2)^{-1}\ln(1+\gamma(\omega r/c)^2) = (\gamma\omega^2)^{-1}\ln[(1+\gamma(\omega r/c)^2)^{\frac{1}{2}}]. \tag{16c}$$

Similarly, if we conveniently select the arbitrary constant t_2 to be the time t_s of the "comoving" metric's GR-unphysical *singularity*, namely t_s is such that $R(t_s) = 0$ (the precise *value* of t_s is laid out in complete detail in Eqs. (9)), then,

$$\begin{gathered}\int_{t_s}^{t}[R(t')\dot{R}(t')]^{-1}dt' = \int_{t_s}^{t}[R(t')(\dot{R}(t'))^2]^{-1}\dot{R}(t')dt' = \\ (\omega^2)^{-1}\int_{t_s}^{t}[1+\gamma R(t')]^{-1}\dot{R}(t')dt' = (\omega^2)^{-1}\int_0^{R(t)}[1+\gamma R']^{-1}dR' = (\gamma\omega^2)^{-1}\ln(1+\gamma R(t)),\end{gathered} \tag{16d}$$

where we applied, in succession, the Eq. (10d) Friedmann equation of motion $(\dot{R}(t))^2 = \omega^2((1/R(t))+\gamma)$, the change of integration variable from t' to $R' = R(t')$ (for which $\dot{R}(t')dt' = dR'$), and the fact that $R(t_s) = 0$.

We now insert the integrations performed in Eqs. (16c) and (16d) into the Eq. (16b) solution prescription, and furthermore select the following convenient values for the two remaining arbitrary constants: $f^{(2)} = \gamma\omega^2$ and $t_1 = (1/\omega)$. This yields the solution $\bar{t}(r,t)$ of the Eq. (15b) partial differential equation as,

$$\bar{t}(r,t) = (1/\omega)\chi\left(\ln[(1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t))]\right), \tag{16e}$$

where $\chi(y)$ is an *arbitrary* differentiable dimensionless function of dimensionless argument. A tidier, more compact expression of this result is,

$$\bar{t}(r,t) = (1/\omega)\phi(u(r,t)), \tag{16f}$$

where $u(r,t)$ is defined as,

$$u(r,t) \stackrel{\text{def}}{=} (1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t)), \tag{16g}$$

and $\phi(u)$ is an *arbitrary* differentiable dimensionless function of positive dimensionless argument. It is well worth noting that the $\bar{t}(r,t)$ given by Eqs. (16f) and (16g) can straightforwardly be *verified* to satisfy Eq. (15b), provided that the Friedmann equation of motion $(\dot{R}(t))^2 = \omega^2((1/R(t))+\gamma)$ for $R(t)$ is taken into account.

In the region $0 \le r \le a$ we now have obtained the general form of $\bar{t}(r,t)$ given by Eqs. (16f) and (16g), albeit in terms of a function of one variable $\phi(u)$ *which hasn't yet been determined*, to which we can add the Eq. (11b) fact that $\bar{r}(r,t) = rR(t)$. With *those two pieces of information* we can *also* obtain the "standard" metric component functions A and B by applying Eqs. (13b) and (13a). The results for A and B aren't definitive *either* because they involve $\phi'(u)$ *which hasn't yet been determined*. Oppenheimer and Snyder insightfully *realized*, however, that A and B are in fact *uniquely determined at the* $r = a$ *surface of the dust ball* by the Birkhoff theorem. As a matter of fact, it turns out that A *conforms to the requirement of the Birkhoff theorem at* $r = a$ *regardless of what the function* $\phi(u)$ *is*. But that definitely *isn't* the case for B; the Birkhoff-theorem requirement for B at $r = a$ *determines* $|\phi'(u)|$, and thus *completes our knowledge* of the "comoving" to "standard" time transformation $\bar{t}(r,t)$. Not surprisingly, however, $\bar{t}(a,t)$ turns out to be *infinite* if $\bar{r}(a,t)$ isn't *larger* than the dust ball's Schwarzschild radius $a(\omega a/c)^2 = (8\pi/3)G\rho(t_0)a^3/c^4 = 2GM/c^2$ (since $M = (4\pi/3)\rho(t_0)a^3/c^2$), a *singular consequence* of the *restoration* in GR-physical "standard" coordinates of the *gravitational time dilation* that *simply doesn't exist* in the GR-unphysical "comoving coordinates".

Gravitational time dilation *stymies* the dust ball's *access* in "standard" coordinates to a radius as small or smaller than its Schwarzschild radius, *even in the arbitrarily distant past*.

To calculate the "standard" metric components A and B we require the partial derivatives of $\bar{t}(r,t)$ and $\bar{r}(r,t)$ which enter into Eqs. (13b) and (13a) for A and B respectively. We use Eqs. (16f) and (16g) for $\bar{t}(r,t)$ to calculate its two partial derivatives,

$$(c(\partial\bar{t}/\partial r)) = \gamma(\omega r/c)(1+\gamma(\omega r/c)^2)^{-\frac{1}{2}}(1+\gamma R(t))\phi'(u(r,t)), \tag{17a}$$

and,

$$(\partial\bar{t}/\partial t) = (\gamma\dot{R}(t)/\omega)(1+\gamma(\omega r/c)^2)^{\frac{1}{2}}\phi'(u(r,t)). \tag{17b}$$

We calculate the two partial derivatives of $\bar{r}(r,t)$ from its formula $\bar{r}(r,t) = rR(t)$,

$$(\partial\bar{r}/\partial r) = R(t), \tag{17c}$$

and,

$$((1/c)(\partial\bar{r}/\partial t)) = (r\dot{R}(t)/c), \tag{17d}$$

When Eqs. (17a) through (17d) are inserted into Eq. (13b) for A, *no* factors of $\phi'(u(r,t))$ survive. Indeed, with the help of the Friedmann equation, $(\dot{R}(t))^2 = \omega^2((1/R(t))+\gamma)$, $A(r,t)$ is seen to have the relatively simple form,

$$A(r,t) = \tfrac{1}{1-[(\omega r/c)^2/R(t)]}. \tag{18a}$$

At the *surface* of the dust ball, namely at $r = a$, the $A(a,t)$ implied by Eq. (18a) *automatically* has the Birkhoff-theorem mandated *empty-space Schwarzschild-metric A-component form* as a function of $\bar{r}(a,t) = aR(t)$,

$$A(a,t) = \tfrac{1}{1-[(\omega a/c)^2/R(t)]} = \tfrac{1}{1-[(\omega^2a^3)/(c^2\bar{r}(a,t))]} = \tfrac{1}{1-[(2GM)/(c^2\bar{r})]}, \tag{18b}$$

where the last equality in Eq. (18b) is the consequence of $\omega^2 = (8\pi/3)G\rho(t_0)/c^2$, which is the definition of ω^2 given by Eq. (10c), together with the definition of the mass of the dust ball, namely $M = (4\pi/3)\rho(t_0)a^3/c^2$. The last expression on the right-hand side of Eq. (18b) is indeed the familiar classic A-component of *the empty-space Schwarzschild metric tensor*, which is exactly what is mandated for $A(r,t)$ at the $r = a$ surface of the dust ball by the Birkhoff theorem. Therefore the Eq. (18a) result for the A-component of the "standard" metric tensor *doesn't provide any information about the not yet determined function $\phi(u)$*.

However when Eqs. (17a) through (17d) are inserted into Eq. (13a) for B, the result is,

$$B(r,t) = \tfrac{1}{((1/R(t))+\gamma)(1-[(\omega r/c)^2/R(t)])(\gamma\phi'(u(r,t)))^2}, \tag{18c}$$

which, in stark contrast with the $A(r,t)$ of Eq. (18a), *is explicitly dependent* on the not yet determined function $\phi(u)$. The *presence* within the structure of the $B(r,t)$ of Eq. (18c) of a not yet determined function is in fact *fortunate* because the Birkhoff theorem at the dust-ball's surface mandates that,

$$B(a,t) = (1-[(2GM)/(c^2\bar{r}(a,t))]) = (1-[(\omega a/c)^2/R(t)]), \tag{18d}$$

which is obliged to be be *consistent* with the result for $B(a,t)$ which is implied by $B(r,t)$ given by Eq. (18c). We therefore equate the second expression for $B(a,t)$ on the right-hand side of Eq. (18d) to the just discussed result for $B(a,t)$ which follows from the Eq. (18c) form for $B(r,t)$, and then solve the resulting equation for $\phi'(u(a,t))$. It turns out to be very convenient notationally to express that result for $\phi'(u(a,t))$ in terms of an intermediary definition, namely,

$$\phi'(u(a,t)) = F(R(t)), \tag{19a}$$

where,

$$F(s) \stackrel{\text{def}}{=} \pm\frac{1}{\gamma((1/s)+\gamma)^{\frac{1}{2}}(1-[(\omega a/c)^2/s])}. \tag{19b}$$

We note from Eq. (16g) that $u(a,t)$ is a simple inhomogeneous linear function of $R(t)$ which is readily inverted, namely,

$$R(t) = \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u(a,t)-1]. \tag{19c}$$

Therefore, from Eq. (19a),

$$\phi'(u(a,t)) = F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u(a,t)-1]\right), \tag{19d}$$

which provides us *the functional form* of $\phi'(u)$ as,

$$\phi'(u) = F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u - 1]\right), \tag{19e}$$

where *the functional form* of $F(s)$ is, of course, given by Eq. (19b).

In order to obtain $\bar{t}(r,t)$, however, we see from Eq. (16f) that we need *the functional form* of $\phi(u)$, which of course is,

$$\phi(u) = \phi(u_0) + \int_{u_0}^{u} du'\, \phi'(u'). \tag{20a}$$

In Eq. (20a) we conveniently *fix* the arbitrary constant u_0 to have the value of $u(r,t)$ (which is defined by Eq. (16g)) at the initial time t_0 at the surface of the dust ball (namely at $r = a$),

$$u_0 = u(a,t_0) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma R(t_0)) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma), \tag{20b}$$

with the last equality following from the fact that the initial time t_0 satisfies $R(t_0) = 1$ (see immediately below Eq. (10d)).

Of course the functional form of $\phi'(u')$ is obtained from Eq. (19e). We insert that functional form into Eq. (20a), which produces,

$$\phi(u) = \phi(u(a,t_0)) + \int_{u(a,t_0)}^{u} du'\, F\left(\gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u' - 1]\right). \tag{20c}$$

We can formally greatly simplify the integrand in Eq. (20c) by making the simple inhomogenous linear change of the integration variable from u' to s, where,

$$s = \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u' - 1], \tag{20d}$$

and therefore,

$$u' = [(1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma s)] \text{ and } du' = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}\gamma ds. \tag{20e}$$

With this change of the integration variable, Eq. (20c) becomes,

$$\phi(u) = \phi(u(a,t_0)) + (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}\gamma \int_1^{\sigma(u)} ds\, F(s), \tag{20f}$$

where,

$$\sigma(u) \stackrel{\text{def}}{=} \gamma^{-1}[(1+\gamma(\omega a/c)^2)^{-\frac{1}{2}}u - 1],$$

and we have used the fact given in Eq. (20b) that $u(a,t_0) = (1+\gamma(\omega a/c)^2)^{\frac{1}{2}}(1+\gamma)$.

With the previously unknown function $\phi(u)$ now in hand via Eq. (20f), we recall from Eq. (16f) that the GR-unphysical Oppenheimer-Snyder transformation $\bar{t}(r,t)$ from GR-unphysical "comoving space-time" to GR-physical "standard" time is given by $(1/\omega)\phi(u(r,t))$, where $u(r,t) = (1+\gamma(\omega r/c)^2)^{\frac{1}{2}}(1+\gamma R(t))$ as per Eq. (16g). Furthermore, the explicit form of the integrand $F(s)$ in Eq. (20f) is, of course, given by Eq. (19b). Putting all this together, we obtain,

$$\bar{t}(r,t) = \bar{t}(a,t_0) \pm \left((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/\omega\right) \int_1^{S(r,t)} \frac{ds}{((1/s)+\gamma)^{\frac{1}{2}}(1-[(\omega a/c)^2/s])}, \tag{20g}$$

where,

$$S(r,t) \stackrel{\text{def}}{=} \sigma(u(r,t)) = \gamma^{-1}\left[(1+\gamma R(t))\left(\frac{1+\gamma(\omega r/c)^2}{1+\gamma(\omega a/c)^2}\right)^{\frac{1}{2}} - 1\right], \tag{20h}$$

and $R(t)$ satisfies the Friedmann equation $(\dot{R}(t))^2 = \omega^2((1/R(t)) + \gamma)$ with the initial condition $R(t_0) = 1$. Eq. (20g) is restricted to $0 \le r \le a$, where a is the dust ball's unchanging radius in "comoving coordinates". The parameter ω^2 equals by definition $(8\pi/3)G\rho(t_0)/c^2$, where $\rho(t_0)$ is the initial uniform energy density of the dust ball, while the parameter γ equals $[(\dot{\rho}(t_0)/(3\omega\rho(t_0)))^2 - 1]$; note that $\gamma \ge -1$. The $\pm$ sign in Eq. (20g) has the value of the *sign* of $\dot{R}(t)$ when $\dot{R}(t) \ne 0$, and has the value -1 when $\dot{R}(t) = 0$.

It is clear from Eq. (20g) that $\bar{t}(r,t)$ *diverges to infinity whenever* $S(r,t) \le (\omega a/c)^2$ (it is readily checked that $(\omega a/c)^2$ is the Schwarzschild radius of the dust ball divided by its "comoving coordinate" radius a). Thus all "comoving" space-time points (r,t) for which $S(r,t) \le (\omega a/c)^2$ are mapped to "standard" time *infinity*, which makes all such "comoving" space-time points *inaccessible* in GR-physical "standard" coordinates.

We have pointed out that the "comoving time" t_s such that $R(t_s) = 0$ produces a singularity in the "comoving metric" described by Eqs. (10a) through (10d). It is readily checked from Eq. (20h) that

$S(r,t_s) \le 0 < (\omega a/c)^2$ for all r within the "comoving" dust ball, namely $0 \le r \le a$. Therefore the points of singularity of the "comoving" metric are *inaccessible* in GR-physical "standard" coordinates. That is how the GR-unphysical singular Oppenheimer-Snyder transformation *removes* the singularity present in the GR-unphysical "comoving" metric in the course of mapping it to the GR-physical nonsingular "standard" metric.

In the next section we use the extended Oppenheimer-Snyder transformation's $\bar{r}(r,t) = rR(t)$ of Eq. (11b) and $\bar{t}(r,t)$ of Eqs. (20g) and (20h) to study the dynamical behavior in "standard" coordinates of the radii of the dust ball's interior shells. Despite having partially developed the wherewithal *needed* for such inquiry, Oppenheimer and Snyder themselves *made no systematic effort to pursue it*, preferring instead to treat the *interior* of the dust ball *exclusively in terms of "comoving coordinates" and the consequent purely Newtonian Friedmann equation*, which GR-unphysically extinguishes all relativistic time dilation, forcing the dust ball's *interior* energy density and metric to both become GR-unphysically *singular* at the *GR-inaccessible* "comoving time" t_s (for which $R(t_s) = 0$)—that is mapped to "standard" time *infinity*!

Dust-ball interior-shell radial dynamics in "standard" coordinates

Because dust three-velocity is always zero everywhere in GR-unphysical "comoving coordinates" [16], the dust ball's interior-shell radii ϵa, $0 < \epsilon \le 1$, are *unchanging* in GR-unphysical "comoving time" t, which implies that the "comoving-coordinate" world lines of the dust-ball interior shells are $(\epsilon a, t)$, $0 < \epsilon \le 1$. The corresponding "standard-coordinate" world lines $(\bar{r}_{\epsilon a}(t), \bar{t}_{\epsilon a}(t))$, $0 < \epsilon \le 1$, of the dust-ball interior shells are obtained by applying the extended Oppenheimer-Snyder transformation $(\bar{r}(r,t), \bar{t}(r,t))$ to $(r,t) = (\epsilon a, t)$, $0 < \epsilon \le 1$; the components $\bar{r}_{\epsilon a}(t) = \bar{r}(\epsilon a, t)$ and $\bar{t}_{\epsilon a}(t) = \bar{t}(\epsilon a, t)$ of those "standard-coordinate" world lines of the dust-ball interior shells follow immediately from Eqs. (11b), (20g) and (20h),

$$\bar{r}_{\epsilon a}(t) = \epsilon a R(t) \ \ \& \ \ \bar{t}_{\epsilon a}(t) = \bar{t}(a,t_0) \pm \left((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/\omega\right) \int_1^{S(\epsilon a,t)} \frac{ds}{((1/s)+\gamma)^{\frac{1}{2}}(1-[(\omega a/c)^2/s])}, \tag{21a}$$

where,

$$S(\epsilon a,t) = \gamma^{-1}\left[(1+\gamma R(t))\left(\frac{1+\epsilon^2\gamma(\omega a/c)^2}{1+\gamma(\omega a/c)^2}\right)^{\frac{1}{2}} - 1\right]. \tag{21b}$$

If we change the integration variable in the integral that appears in Eq. (21a) from s to $\rho = \epsilon a s$, it becomes apparent that the "standard-coordinate" local shell times $\bar{t}_{\epsilon a}(t)$, $0 < \epsilon \le 1$, *can be regarded as being functions of their corresponding "standard-coordinate" shell radii* $\bar{r}_{\epsilon a}(t) = \epsilon a R(t)$, $0 < \epsilon \le 1$,

$$\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t))) = \bar{t}(a,t_0) \pm \left((1+\gamma(\omega a/c)^2)^{\frac{1}{2}}/(\epsilon\omega a)\right) \int_{\epsilon a}^{\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t))} \frac{d\rho}{((\epsilon a/\rho)+\gamma)^{\frac{1}{2}}(1-[\epsilon a(\omega a/c)^2/\rho])}, \tag{21c}$$

where,

$$\rho_{\epsilon a}(\bar{r}_{\epsilon a}(t)) = \epsilon a S(\epsilon a,t) = -(\epsilon a/\gamma) + \left(\frac{1+\epsilon^2\gamma(\omega a/c)^2}{1+\gamma(\omega a/c)^2}\right)^{\frac{1}{2}} (\bar{r}_{\epsilon a}(t) + (\epsilon a/\gamma)). \tag{21d}$$

Note that the complicated inhomogeneous linear function $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ given by Eq. (21d) greatly simplifies in two special instances, the first being when $\epsilon = 1$, namely the case of the dust ball's *surface shell*,

$$\rho_a(\bar{r}_a) = \bar{r}_a, \tag{21e}$$

and the second being the $c \to \infty$ nonrelativistic limit,

$$\lim_{c\to\infty} \rho_{\epsilon a}(\bar{r}_{\epsilon a}) = \bar{r}_{\epsilon a}. \tag{21f}$$

Eqs. (21c) and (21d) become more compact when they are expressed in terms of the dust ball's "standard-coordinate" Schwarzschild radius $r_S = a(\omega a/c)^2 = 2GM/c^2$ (where $M = (4\pi/3)a^3\rho(t_0)/c^2 = \omega^2 a^3/(2G)$) and the dimensionless constant $\alpha \stackrel{\text{def}}{=} \gamma(\omega a/c)^2 = \gamma(r_S/a)$, instead of being expressed in terms of ω and γ,

$$\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a})) = \bar{t}(a,t_0) \pm \left((1+\alpha)^{\frac{1}{2}}/(c\epsilon)\right) \int_{\epsilon a}^{\rho_{\epsilon a}(\bar{r}_{\epsilon a})} \frac{d\rho}{((\epsilon r_S/\rho)+\alpha)^{\frac{1}{2}}(1-(\epsilon r_S/\rho))}, \tag{21g}$$

where,

$$\rho_{\epsilon a}(\bar{r}_{\epsilon a}) = -(\epsilon r_S/\alpha) + \left(\frac{1+\epsilon^2\alpha}{1+\alpha}\right)^{\frac{1}{2}} (\bar{r}_{\epsilon a} + (\epsilon r_S/\alpha))\,. \tag{21h}$$

The integral in Eq. (21g) *diverges to infinity* unless $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ (and $a > r_S$). Provided that care is exercised *to respect that caveat*, the integral in Eq. (21g) can be evaluated in closed form; the resulting *completely analytic*, but complicated, expression for $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ is useful *for crafting numerical plots of* $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a})$. We shall calculate that entirely analytic result for $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ at the end of this section, but unfortunately its form is too complicated to readily lend itself *to direct interpretation*.

However the integral form of $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ that is given by Eqs. (21g) and (21h) *also* yields *a first-order differential equation for* $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a})$, namely,

$$d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a} = (d\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))/d\bar{r}_{\epsilon a})^{-1} = \pm c\epsilon\left(\frac{(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a})) + \alpha}{1+\epsilon^2\alpha}\right)^{\frac{1}{2}} (1 - (\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))), \tag{21i}$$

where $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ is given in detail by Eq. (21h). The *caveat* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$*, which carries over from* Eq. (21g), *must of course be respected*; Eq. (21i) together with that *caveat* implies the following *upper bounds on the radial speeds* $|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}|$, $0 < \epsilon \le 1$, *of the dust ball's interior shells*,

$$|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}| < c\left(\frac{\epsilon^2+\epsilon^2\alpha}{1+\epsilon^2\alpha}\right)^{\frac{1}{2}} (1 - (\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))). \tag{21j}$$

Since the Eq. (21h) relation of $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ to $\bar{r}_{\epsilon a}$ *is a linear one*, the upper bounds of Eq. (21j) drive $|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}|$ *linearly toward zero* as $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) \to \epsilon r_S+$. Therefore it isn't possible for $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ to become equal to or smaller than ϵr_S *in any finite interval of "standard" local shell time* $\Delta\bar{t}_{\epsilon a}$. Thus the Eq. (21j) *upper bounds on the shell speeds* $|d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}|$ and the *caveat* that $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ *mutually reinforce each other*. Such *zeroing of the approach speed to a GR-unphysical configuration* exemplifies the GR-crucial role of gravitational time dilation. (In the special case that $\epsilon = 1$, i.e., the case of the dust ball's *surface shell*, we see from Eq. (21e) that the *caveat* simplifies to $\bar{r}_a > r_S$, namely the dust ball's dynamical radius $\bar{r}_a$ *always exceeds its Schwarzschild radius* r_S; thus a dust ball *never produces an event horizon*.)

From Eq. (21f) and the facts that $r_S = 2GM/c^2$ and $\alpha = \gamma(r_S/a)$, the $c \to \infty$ *nonrelativistic limit* of the Eq. (21i) dust-ball shell radial equations of motion are readily seen to be,

$$d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a} = \pm\epsilon^{\frac{3}{2}}(2GM)^{\frac{1}{2}}\left((1/\bar{r}_{\epsilon a}) + (\gamma/(\epsilon a))\right)^{\frac{1}{2}}. \tag{21k}$$

These nonrelativistic-limit shell-radius equations of motion are *completely devoid* of the time-dilation *speed* and *configuration* constraints which feature so prominently in Eq. (21j). Squaring both sides of Eq. (21k) reveals that *each one* of these ϵ-labeled nonrelativistic-limit shell-radius equations of motion corresponds to the Newtonian Friedmann equation $(\dot{R}(\bar{t}_{\epsilon a}))^2 = \omega^2((1/R(\bar{t}_{\epsilon a})) + \gamma)$, where $\omega^2 = (2GM/a^3)$, via the simple scaling relationship $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a}) = \epsilon a R(\bar{t}_{\epsilon a})$, $0 < \epsilon \le 1$.

The Eq. (21i) dust-ball shell-radius first-order equation of motion in "standard" coordinates has a second-order form which illuminates *the modification of shell-radius acceleration caused by gravitational time dilation*. To obtain that form, both sides of the Eq. (21i) first-order equation are differentiated with respect to $\bar{t}_{\epsilon a}$, the "standard" local time at the shell, which produces, inter alia, an overall factor of $d\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}$ on the right-hand side of the result. That factor is then replaced by the right-hand side of the original Eq. (21i) first-order equation, yielding the second-order equation,

$$d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a} = (\epsilon/2)(c^2/r_S)\left[(-1+2\alpha)(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))^2 + 3(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))^3\right]\left[\frac{1-(\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))}{(1+\epsilon^2\alpha)^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}}\right], \tag{21l}$$

which of course has *the same caveat* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ as the Eq. (21i) first-order equation.

Just as was done with Eq. (21i), the $c \to \infty$ *nonrelativistic limit* of Eq. (21l) is readily worked out to be,

$$d^2\bar{r}_{\epsilon a}/d\bar{t}^2_{\epsilon a} = -\epsilon^3 GM/\bar{r}^2_{\epsilon a}. \tag{21m}$$

Eq. (21m) is straightforwardly verified to be *consistent* with the nonrelativistic limit of Eq. (21i) that is given by the Newtonian Eq. (21k). The Eq. (21m) rendition of the radial acceleration of the ϵ-labeled shell reflects the *effective* Newtonian mass $\epsilon^3 M$ which actually exerts force on that shell.

However *despite* the fact that the shell-radius acceleration $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$ of the nonrelativistic-limit Eq. (21m) *is always negative* (i.e., *inward*), the *full* Eq. (21l) implies, because $3 > (1-2\alpha)$ (since $\alpha > -1$), *that there always exists a range of* $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$ *values which* both *satisfy the caveat* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ and *produce positive* (i.e., *outward*) *shell-radius acceleration* $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$. We *furthermore* see that *for all initial conditions which satisfy* $\alpha \geq \frac{1}{2}$ *every shell-radius acceleration* $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$, $0 < \epsilon \leq 1$, *of the full* Eq. (21l) *is positive* (i.e., *outward*) at *all finite "standard" local times* $\bar{t}_{\epsilon a}$—notice from Eq. (21g) that for *finite* "standard" local times $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ the caveat $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ *is necessarily satisfied*.

That $\alpha \geq \frac{1}{2}$ causes *every* shell-radius acceleration $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$ to be *positive* (i.e., *outward*) at *all* finite "standard" local times $\bar{t}_{\epsilon a}$ apparently *eliminates any need whatsoever to fit a nonzero "dark energy" cosmological constant*.

From Eq. (21l) we clearly see that the shell-radius *accelerations* $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$, which are always *positive* for *some range of* $\bar{r}_{\epsilon a}$ values that *also* satisfy the *caveat* that $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$, definitely *as well* tend toward *zero* both when $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) \to \epsilon r_S$ and when $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) \to +\infty$. Therefore $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$ must have *a positive maximum* at a value of $\bar{r}_{\epsilon a}$ which *also* satisfies the crucial *caveat* that $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$. With algebraic effort we obtain from Eq. (21l) that the value of $\bar{r}_{\epsilon a}$ where $d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2$ attains this maximum satisfies the equation,

$$\rho_{\epsilon a}\big(\bar{r}_{\epsilon a}^{(\max d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2)}\big) = (3\epsilon r_S/2)\left(1 + \left[(3\alpha+2) + \left((3\alpha+2)^2+8\right)^{\frac{1}{2}}\right]^{-1}\right)^{-1}, \tag{21n}$$

where, of course, $\alpha > -1$. We note that $\rho_{\epsilon a}\big(\bar{r}_{\epsilon a}^{(\max d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2)}\big)$ *strictly increases with* α, and as $\alpha \to -1$ it tends toward the value ϵr_S, while as $\alpha \to +\infty$ it tends toward the value $3\epsilon r_S/2$. We thus see that $\rho_{\epsilon a}\big(\bar{r}_{\epsilon a}^{(\max d^2\bar{r}_{\epsilon a}/d\bar{t}_{\epsilon a}^2)}\big)$ *is under all circumstances only fractionally larger than the GR-inaccessible value* ϵr_S *of* $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$.

When we *specialize* Eq. (21n) to the $\epsilon = 1$ *dust-ball surface-shell case* using Eq. (21e), we see that the outward (i.e., positive) acceleration of the dust ball's dynamical radius r_a *always peaks when that radius is only fractionally larger than the dust ball's Schwarzschild radius* r_S. Thus every *expanding* dust ball experiences an outward "accelerative inflation" peak when its dynamical radius is only fractionally larger than its Schwarzschild radius. Of course such a peak in outward *acceleration* doesn't at all necessarily entail *any corresponding peak in outward expansion velocity* (which is the commonplace conception of "inflation"); there is *scope* for the expanding dust ball's shell radii $\bar{r}_{\epsilon a}$ *to continue their outward acceleration forever*, namely the situation delineated by Eq. (21l) when $\alpha \geq \frac{1}{2}$—although that continuing outward acceleration *decreases asymptotically toward zero with increasing "standard" local time*.

We now turn to the calculation of the (complicated) *entirely analytic result for* $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$ by carrying out the integration in Eq. (21g). To do so, we change the integration variable from ρ to $v = ((\epsilon r_S/\rho) + \alpha)^{\frac{1}{2}}$, which produces an integrand that is a rational function of v^2. Partial-fraction expansion of that integrand then yields,

$$\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a})) = \bar{t}(a,t_0) \pm 2(r_S/c)(1+\alpha)^{\frac{1}{2}} \int_{((r_S/a)+\alpha)^{\frac{1}{2}}}^{((\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a}))+\alpha)^{\frac{1}{2}}} dv \left[\tfrac{1}{\alpha - v^2} - \tfrac{1}{(\alpha-v^2)^2} - \tfrac{1}{1+\alpha-v^2}\right]. \tag{21o}$$

It is now straightforward to write down *an appropriate particular indefinite integral* $\mathrm{I}(v;\alpha)$ of the elementary three-term integrand which is given inside the square brackets of Eq. (21o),

$$\mathrm{I}(v;\alpha) = \mathrm{Ip}(v;\alpha) - \frac{\tanh^{-1}(v/(1+\alpha)^{\frac{1}{2}})}{(1+\alpha)^{\frac{1}{2}}}, \tag{21p}$$

where, of course, $\alpha > -1$, and $\tanh^{-1}(x) = \frac{1}{2}\ln\big((1+x)/(1-x)\big)$ *is rejected as undefined unless* $|x| < 1$ *to enforce the caveat that* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$. The *remaining* indefinite integral part $\mathrm{Ip}(v;\alpha)$ that is referred to in Eq. (21p) *assumes three different functional forms, depending on whether* α *is positive, zero or negative*,

$$\mathrm{Ip}(v;\alpha) = \begin{cases} \left(1 - \frac{1}{2\alpha}\right)\left(\frac{\tanh^{-1}(\alpha^{\frac{1}{2}}/v)}{\alpha^{\frac{1}{2}}}\right) - \left(\frac{1}{2\alpha}\right)\left(\frac{v}{\alpha - v^2}\right) & \text{if } \alpha > 0, \\ \frac{1}{v} + \frac{1}{3v^3} & \text{if } \alpha = 0, \\ \left(1 - \frac{1}{2\alpha}\right)\left(\frac{\arctan((-\alpha)^{\frac{1}{2}}/v)}{(-\alpha)^{\frac{1}{2}}}\right) - \left(\frac{1}{2\alpha}\right)\left(\frac{v}{\alpha - v^2}\right) & \text{if } \alpha < 0. \end{cases} \tag{21q}$$

With our appropriate particular indefinite integral $\mathrm{I}(v;\alpha)$ thus fully in hand, we are able rewrite Eq. (21o) in completely integrated, entirely analytic form,

$$\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a})) = \bar{t}(a,t_0) \pm 2(r_S/c)(1+\alpha)^{\frac{1}{2}}\left[\mathrm{I}\big(v = ((\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a})) + \alpha)^{\frac{1}{2}};\,\alpha\big) - \mathrm{I}\big(v = ((r_S/a) + \alpha)^{\frac{1}{2}};\,\alpha\big)\right]. \tag{21r}$$

We note that that the term $\mp 2(r_S/c)(1+\alpha)^{\frac{1}{2}}\,\mathrm{I}\big(v = ((r_S/a)+\alpha)^{\frac{1}{2}};\,\alpha\big)$ of Eq. (21r) *is completely independent of both $\bar{r}_{\epsilon a}$ and ϵ*, and therefore can be combined with $\bar{t}(a,t_0)$ to produce a new arbitrary constant $\bar{t}_0$. Eq. (21r) is thereupon reexpressed as simply,

$$\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a})) = \bar{t}_0 \pm 2(r_S/c)(1+\alpha)^{\frac{1}{2}}\,\mathrm{I}\big(v = ((\epsilon r_S/\rho_{\epsilon a}(\bar{r}_{\epsilon a})) + \alpha)^{\frac{1}{2}};\,\alpha\big). \tag{21s}$$

Of course the *caveats* $\rho_{\epsilon a}(\bar{r}_{\epsilon a}) > \epsilon r_S$ and $\alpha > -1$ must be respected when dealing with the *arguments* of the entirely analytic Eq. (21s) representation of $\bar{t}_{\epsilon a}(\rho_{\epsilon a}(\bar{r}_{\epsilon a}))$, which, employed in tandem with the Eq. (21h) function $\rho_{\epsilon a}(\bar{r}_{\epsilon a})$, is useful *for crafting plots of the dust ball's dynamical shell radii $\bar{r}_{\epsilon a}(\bar{t}_{\epsilon a})$*, $0 < \epsilon \le 1$.

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