On the a-posteriori Fourier method for solving ill-posed problems

Zhenming Zheng *and Xiangtuan Xiong†

Department of Mathematics, Northwest Normal University, Gansu, China

Abstract

The Fourier method is a convenient regularization method for solving a class of ill-posed problems. This class of ill-posed problems can be also formulated as the problem of ill-posed multiplication operator equation in the frequency domain. A recent work on the Morozov's discrepancy principle for the Fourier method are discussed in [2]. In this paper, we investigate the Fourier method within the framework of regularization theory thoroughly for solving the severely ill-posed problems. Many ill-posed examples are provided.

Keywords : ill-posedness, regularization, error estimate, source condition.

1 Introduction

The main goal of this paper consists of demonstrating the advantages of using the notation of spectral method within the framework of regularization theory [1] instead of the so-called Fourier method [2]. Moreover, we answer the question whether the obtained convergence orders in [2] are optimal.

Many PDE-based ill-posed problems in mathematical physics defined on a strip domain can be formulated as the problem for solving an operator equation. Therefore, the theory analysis such as error estimate for the spectral method is possible for this kind of ill-posed problems. As in [2], we discuss this kind of ill-posed problems, which involves the numerical computation of pseudodifferential operator.

Throughout this paper, we extend any function $h(v)$ to the whole real line by setting the function to be zero for $v < 0$ if necessary. Let

$$\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(v) e^{-i\xi v} dv \tag{1}$$

*zhenmingzheng@163.com

†xiongxt@gmail.com

be the Fourier transform of the function $h(v) \in L^2(\mathbb{R})$. The Sobolev function space $H^p(\mathbb{R})$ is defined by

$$H^p(\mathbb{R}) = \{h(x)|h \in L^2(\mathbb{R}), \|h\|_p := (\int_{-\infty}^{\infty} (1+\xi^2)^p |\hat{h}(\xi)|^2 d\xi)^{\frac{1}{2}} < \infty\}, \tag{2}$$

where $\hat{h}(\xi)$ is the Fourier transform of function $h(x)$, and $\|\cdot\| := \|\cdot\|_0$ denotes the norm in $L^2(\mathbb{R})$.

Consider the numerical computation of the pseudodifferential operator with an unbounded symbol $a(\xi, s)$ given by

$$f(v, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(\xi, s)\hat{g}(\xi)e^{i\xi v} dv, \tag{3}$$

where $0 \le s \le L$ is a fixed constant, $\hat{g}(\xi)$ is the Fourier transform of the exact data $g(v) \in L^2(\mathbb{R})$, the data $g(v)$ is only given approximately by $g_\delta(v) \in L^2(\mathbb{R})$ satisfying

$$\|g(\cdot) - g_\delta(\cdot)\| \le \delta, \tag{4}$$

where δ is the noise level and assumed to be known.

If $a(\xi, s)$ satisfies $|a(\xi, s)| = O(\exp(c|\xi|^b))$ as $|\xi| \to \infty$ with $c, b > 0$, we called the problem of numerical computation of above pseudodifferential operators severely ill-posed problem. Likewise, if $a(\xi, s)$ satisfies $|a(\xi, s)| = O(|\xi|^b)$ as $|\xi| \to \infty$ with $b > 0$, we called the problem of numerical computation of above pseudodifferential operators mildly ill-posed problem. In this paper, we only consider the severely ill-posed problem since the mildly ill-posed problems can be solved more easily.

A-priori information about unknown solution has been proved to be essential in the analysis of ill-posed problems in mathematical physics. Otherwise, without the a-priori information the convergence rate of the constructed regularization method is arbitrarily slow [1]. For analysis we assume that there holds the a-priori bound for the unknown solution

$$\|f(\cdot, L)\|_p \le E, \, p \ge 0. \tag{5}$$

Let $f_\alpha^\delta(\cdot, s)$ denote a regularization solution. If $0 \le s < L$, we called the $f_\alpha^\delta(\cdot, s)$ interior inversion. If $s = L$, we called the $f_\alpha^\delta(\cdot, s)$ boundary inversion.

1.1 Previous work on Fourier method

In [2], it is assumed that the symbol $a(\xi, s)$ satisfies the following growth condition:

$$\begin{aligned} &C_1 \exp(s\phi(|\xi|)) \le |a(\xi, s)| \le C_2 \exp(s\phi(|\xi|)), |\xi| \ge c_1, s \in [0, L], \\ &|a(\xi, s)| \le z(s), \; |\xi| < c_1, \end{aligned} \tag{6}$$

where the function $\phi(x)$ is strictly increasing with $\lim_{x\to\infty} \phi(x) = \infty$, the function $z(s) \in C[0, L]$ with $z(s) > 0$ and $C_1, C_2, c_1 > 0$ are constants.

Obviously, the ill-posedness of problem (1.3) is caused by the components of high frequency. If $c_1 \ne 0$, we only need to regularize the solution in the case of $|\xi| \ge c_1$ because for the case of $|\xi| \le c_1$ the problem (1.3) is well-posed which can be observed by the following derivation:

Let the sets $W(\xi)=\{\xi||\xi|\le c_1\}$ and $I(\xi)=\{\xi||\xi|\ge c_1\}$, and we have $L^2(\mathbb{R})=L^2(W)\bigoplus L^2(I)$. Consider the difference

$$\|\hat{f}(\xi,s)-\hat{f}(\xi,s)\|^2{}_{L^2(W)}=\int_W|a(\xi,s)|^2|\hat{g}(\xi)-\hat{g}_\delta(\xi)|^2d\xi$$
$$\le z^2(s)\delta^2\le C\delta^2.$$

The similar technique has been used [3][4]. Throughout this paper, we mainly deal with the case of $|\xi|\ge c_1$.

By Morozov's discrepancy principle, in [2] the authors derived a Hölder-type error estimate like $\|f_\alpha^\delta(\cdot,s)-f(\cdot,s)\|=O(E^{s/L}\delta^{1-\frac{s}{L}})$, $\delta\to 0$ under the a-priori bound (1.5) with $p=0$ by using Fourier method $f_\nu^\delta(\cdot,s)$ (Please see Theorem 3.1 in [2]). However, when $s=L$ at which we should be more interested in, we cannot get any convergence. To overcome the difficulty, usually, a stronger a-priori bound on the unknown solution should be added, i.e., we should require $p>0$ for (1.5) to obtain the convergence rate at $s=L$. In all, the authors in [2] has constructed the interior inversion by Fourier method. In the following, we will construct the interior and boundary boundary inversions by a spectral regularization method which is more suitable for solving problem (1.3).

1.2 Preliminary on the spectral method

Ill-posed operator equations arise in several contexts and various aspects have been treated in the literature [1][5][6]. We cannot give here an exhaustive survey. In this paper we are interesting in solving the solution $x^\dagger\in H_1$ of linear ill-posed problems by spectral cut-off method

$$Ax=y, \tag{7}$$

where $A:H_1\to H_2$ is a linear injective, closed operator between infinite-dimensional Hilbert spaces H_1 and H_2 with non-closed range $R(A)$. We suppose that $y^\delta\in H_2$ are the noisy data with

$$\|y-y^\delta\|\le\delta \tag{8}$$

and known noise level δ.

If we consider the ill-posed problem $Kx=y$ where only noisy data y^δ are available, K is a compact operator with singular system $\{\sigma_n,v_n,u_n\}_{n=1}^\infty$, since the ill-posedness is associated with the small singular values, an obvious idea is to truncate or damp the smaller singular values. That is the well-known truncated singular value decomposition (TSVD) method. These methods are simple and effective. In the practical computation, we can implement the spectral cut-off method by mollification method discovered by Hào [7].

For problem (1.7), most regularization operators can be written in the form,

$$R_\alpha:=g_\alpha(A^*A)A^* \tag{9}$$

with some function g_α satisfying

$$\lim_{\alpha\to 0}g_\alpha(\lambda)=\frac{1}{\lambda},$$

where the operator function $g_\alpha(A^*A)$ is well defined via the spectral representation $g_\alpha(A^*A) = \int_0^a g(\lambda)dE_\lambda$. Here $A^*A = \int_0^a \lambda dE_\lambda$, $\{E_\lambda\}$ denotes the spectral family of the operator A^*A and a is a constant satisfying $\|A^*A\| \leq a$ with $a = \infty$ if A^*A is unbounded.

Then for the regularization solution with unperturbed data, we have $x_\alpha := R_\alpha y$ and $x^\dagger - x_\alpha = r_\alpha(A^*A)x^\dagger$ with $r_\alpha(\lambda) = 1 - \lambda g_\alpha(\lambda)$. For example, for spectral cut-off method,

$$g_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \geq \alpha, \\ 0, & \lambda < \alpha. \end{cases} \tag{10}$$

$$r_\alpha(\lambda) = \begin{cases} 0, & \lambda \geq \alpha, \\ 1, & \lambda < \alpha. \end{cases} \tag{11}$$

In general, the exact solution $x^\dagger \in X$ is required to satisfy a so-called source condition, otherwise the convergence of the regularization method approximating the problem can be arbitrarily slow. For ill-posed problems, the source condition is chosen as

$$x^\dagger = [\varphi(A^*A)]^{1/2}\omega, \quad \|\omega\| \leq E, \tag{12}$$

i.e., $x^\dagger$ belongs to the source set

$$M_{\varphi,E} = \{[\varphi(A^*A)]^{1/2}\omega, \quad \omega \in X \text{ and } \|\omega\| \leq E\}, \tag{13}$$

where $\varphi(\lambda)$ satisfies some properties:

Assumption 1. $\lim_{\lambda\to 0}\varphi(\lambda) = 0$ and $\varphi(\lambda)$ is strict monotonically increasing, $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$ is convex.

For the stable approximate solution of problem (1.7) some regularization technique has to be applied, which provides regularized approximations $x_\alpha^\delta = R_\alpha^\delta y^\delta$ with property $x_\alpha^\delta \to x^\dagger$ as $\delta \to 0$.

Any operator $R : H_2 \to H_1$ can be considered as a special method for solving problem (1.7). the approximate solution to (1.7) is then given by Ry^δ. Consider the worst case error $\Delta(\delta, R)$ for identifying the solution $x^\dagger$ of problem (1.7) from noisy data y^δ under the assumptions $\|y - y^\delta\| \leq \delta$ and $x^\dagger$ belongs to a source set $M_{\varphi,E}$ which is defined by

$$\Delta(\delta, R) = \sup\{\|Ry^\delta - x^\dagger\| \big| x^\dagger \in M_{\varphi,E}, y^\delta \in H_2, \|y - y^\delta\| \leq \delta\}. \tag{14}$$

This worst case error characterizes the maximal error of the method R if the solution $x^\dagger$ of problem (1.7) varies in the set $M_{\varphi,E}$. An optimal method R_{opt} is characterized by $\Delta(\delta, R_{opt}) = \inf_R \Delta(\delta, R)$. It can easily be realized that

$$\inf_R \Delta(\delta, R) \geq \omega(\delta, M_{\varphi,E}), \tag{15}$$

where $\omega(\delta, M_{\varphi,E}) = \sup\{\|x\| \big| x \in M_{\varphi,E}, \|Ax\| \leq \delta\}$.

The following theorem and definition can be found in [8].

Theorem 1.1. *Let $M_{\varphi,E}$ is given by (1.13), and Assumption 1 be satisfied. If $\frac{\delta}{E} \in \sigma(A^*A\varphi(A^*A))$, then*

$$\omega(\delta, M_{\varphi,E}) = E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}, \tag{16}$$

where ρ is given by $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$.

Definition 1.1. *Let Assumption 1 be satisfied. Any regularization method R_α^δ for problem (4.1) with noisy data is called*

(i) optimal on the set $M_{\varphi,E}$ if $\|x_\alpha^\delta - x^\dagger\| \le E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$;

(ii) order optimal on the set $M_{\varphi,E}$ if $\|x_\alpha^\delta - x^\dagger\| \le cE\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}$ with $c \ge 1$.

2 Morozov's discrepancy principle for the spectral method

Now we prove some estimates for the spectral regularization method under the a-posteriori parameter choice rule.

By Morozov's discrepancy principle, we have a choice rule R

$$d(\alpha) := \|Ax_\alpha^\delta - y^\delta\| = \tau\delta, \tag{17}$$

where $\tau > 1$. Via the definition of x_α^δ, we have

$$\|Ax_\alpha^\delta - y^\delta\| = \|Ag_\alpha(A^*A)A^*y^\delta - y^\delta\| \le \sup_{\lambda\in\sigma(A^*A)} |1 - \lambda g(\lambda)|\|y^\delta\| \le \|y^\delta\|.$$

here it requires $\|y^\delta\| \ge \tau\delta$, when $\alpha \to 0$, $d(\alpha) \to 0$; when $\alpha \to \infty$,$d(\alpha) \to \|y^\delta\|$. According to the continuity of $d(\alpha)$ with respect with α, the parameter α chosen by the R exists.

First we have

$$\|x_\alpha^\delta - x_\alpha\| = \|g_\alpha(A^*A)A^*y^\delta - g_\alpha(A^*A)A^*y\| \le \sup_{\lambda\in\sigma(A^*A)} |\sqrt{\lambda}g_\alpha(\lambda)|\|g^\delta - g\| \le \frac{\delta}{\sqrt{\alpha}}. \tag{18}$$

Proposition 2.1. Let $x^\dagger \in M_{\varphi,E}$, Assumption 1 is satisfied, if α is chosen by the rule R, then there holds

$$\|x_\alpha - x^\dagger\| \le (\tau + 1)\omega(\delta, A, M_{\varphi,E}). \tag{19}$$

Proof. According to the expression of x_α, it yields

$$x^\dagger - x_\alpha = x^\dagger - R_\alpha y = (I - R_\alpha A)x^\dagger,$$

since $x^\dagger = [\varphi(A^*A)]^{1/2}v$, so $(I-R_\alpha A)[\varphi(A^*A)]^{1/2}x^\dagger = [\varphi(A^*A)]^{1/2}(I-R_\alpha A)v$. Let $w = (I-R_\alpha A)v$, then $\|w\| \le \|I - R_\alpha A\|\|v\| \le \|v\| \le E$. Noting the definition of $M_{\varphi,E}$, we obtain

$$(I - R_\alpha A)x^\dagger \in M_{\varphi,E}.$$

At the same time, by the rule R,

$$\begin{aligned}&\|A(I - R_\alpha A)x^\dagger\| = \|y - AR_\alpha y\| \le \|(I - AR_\alpha)(y^\delta - y)\| + \|(I - AR_\alpha)y^\delta\| \\ &\le \delta + \tau\delta = (1 + \tau)\delta.\end{aligned}$$

By the definition of $\omega(\delta, A, M_{\varphi,E})$, there holds

$$\omega(\delta, A, M_{\varphi,E}) := \sup \|x\| \big| x \in M_{\varphi,E}, \|Ax\| \le \delta,$$

we have

$$\|(I - R_\alpha A)x^\dagger\| \le (\tau + 1)\omega(\delta, A, M_{\varphi,E}),$$

i.e.

$$\|x^\dagger - x_\alpha\| \le (\tau + 1)\omega(\delta, A, M_{\varphi,E}). \tag{20}$$

Proposition 2.2. Let $x^\dagger \in M_{\varphi,E}$, and let Assumption 1 is satisfied, if α is chosen by the rule R, then

$$\frac{\delta}{\sqrt{\alpha}} \le \frac{E}{(\tau - 1)} \sqrt{\varphi(\alpha)}. \tag{21}$$

Proof. Let $G_\alpha = R_\alpha A = AR_\alpha$, then $d(\alpha) = \|(I - G_\alpha)y^\delta\|$, we get

$$\begin{aligned}\tau\delta &= \|(I - G_\alpha)y^\delta\| \le \|(I - G_\alpha)y\| + \|(I - G_\alpha)(y^\delta - y)\|, \\ &\le \|(I - G_\alpha)y\| + \delta.\end{aligned}$$

At the same time, there holds

$$\begin{aligned}&\|(I - G_\alpha)y\| = \|(I - G_\alpha)Ax^\dagger\| = \|(I - G_\alpha)A[\varphi(A^*A)]^{1/2}v\| \\ &\le \sup_{\lambda \in \sigma(A^*A)} |(1 - \lambda g_\alpha(\lambda))\sqrt{\lambda\varphi(\lambda)}|E \le \sqrt{\alpha\varphi(\alpha)}E.\end{aligned}$$

Hence

$$\frac{\delta}{\sqrt{\alpha}} \le \frac{E}{(\tau - 1)} \sqrt{\varphi(\alpha)}.$$

holds.

Theorem 2.1. Let $x^\dagger \in M_{\varphi,E}$,and Assumption 1 is satisfied, if α is chosen by the rule R, we have

$$\|x_\alpha^\delta - x^\dagger\| \le cE\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}, \tag{22}$$

where when $1 < C \le 2$, the constant $c = \tau + 1 + \frac{1}{\tau - 1}$; when $\tau > 2$, the constant $c = \tau + 2$.

Proof. By (2.5), we have

$$\varphi^{-1}(\frac{(\tau - 1)^2\delta^2}{E^2\alpha}) \le \alpha.$$

As $\rho(\lambda) = \lambda\varphi^{-1}(\lambda)$, it yields $\rho(\frac{(\tau-1)^2\delta^2}{E^2\alpha}) \le \frac{(\tau-1)^2\delta^2}{E^2\alpha}$. Because ρ is monotonic, when $1 < \tau \le 2$, we have

$$\frac{\delta}{\sqrt{\alpha}} \le \frac{E}{\tau - 1}\sqrt{\rho^{-1}(\frac{(\tau - 1)^2\delta^2}{E^2})} \le \frac{E}{\tau - 1}\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}. \tag{23}$$

when $\tau > 2$,because φ^{-1} is monotonic, we obtain

$$\varphi^{-1}(\frac{\delta^2}{E^2\alpha}) \leq \varphi^{-1}(\frac{(\tau-1)^2\delta^2}{E^2\alpha}) \leq \alpha.$$

Similarly

$$\frac{\delta}{\sqrt{\alpha}} \leq E\sqrt{\rho^{-1}(\frac{\delta^2}{E^2})}. \tag{24}$$

By using the Proposition 2.1 and (2.2), we have

$$\begin{aligned} &\|x^\dagger - x_\alpha^\delta\| \leq \|x^\dagger - x_\alpha\| + \|x_\alpha - x_\alpha^\delta\| \\ &\leq (\tau+1)\omega(\delta, A, M_{\varphi,E}). \end{aligned}$$

By Theorem 1.1, we obtain the (2.6).

3 Spectral regularization for problem (1.3)

Now in order to apply the spectral regularization method for problem (1.3), we formulate problem (1.3) as an operator equation in the frequency domain:

$$A(s)\hat{f} := a^{-1}(\xi, s)\hat{f}(\xi, s) = \hat{g}(\xi),$$

where $A(s) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a multiplication operator. Obviously $A(s) = a^{-1}(\xi, s)$, and the adjoint operator of $A(s)$ is $A^*(s) = \overline{a^{-1}(\xi, s)}$, where the symbol $\bar{\Pi}$ denotes the complex conjugate of Π. Therefore $A^*A(s) = |a^{-1}(\xi, s)|^2 = |a(\xi, s)|^{-2}$. In this section, we deal with problem (1.3) for two cases separately:

Case I (the case of the interior inversion): $p = 0$ in (1.5).

Case II(the case of the boundary inversion): $p > 0$ in (1.5).

For Case I, the authors in [2] has used the Fourier method for constructing the solution $f(\omega, s)$ with $0 \leq s < L$ and they obtained the Hölder-type error estimates. Now we investigate the spectral method for two cases.

First we can write the spectral regularization according to (1.10):

$$\hat{f}_\alpha^\delta(\xi, s) = \begin{cases} a(\xi, s)\hat{g}(\xi), & |a(\xi, s)| \leq \frac{1}{\sqrt{\alpha}}, \\ 0, & else. \end{cases} \tag{25}$$

Generally, the expression of $a(\xi, s)$ may be too complicate. However, by (1.6), the spectral method may be interpreted as the Fourier method. According to $|a(\xi, s)| \leq \frac{1}{\sqrt{\alpha}}$ we have $C_1 \exp(s\phi(|\xi|)) \leq \frac{1}{\sqrt{\alpha}}$. This implies that $|\xi| \leq \phi^{-1}(\frac{1}{s}\ln(\frac{1}{C_1\sqrt{\alpha}}))$, where $\phi^{-1}(\cdot)$ is the inverse function of $\phi(\cdot)$. Thus (3.1) is the Fourier method with the following form:

$$\hat{f}_\alpha^\delta(\xi, s) = \begin{cases} a(\xi, s)\hat{g}(\xi), & |\xi| \leq \phi^{-1}(\frac{1}{s}\ln(\frac{1}{C_1\sqrt{\alpha}})), \\ 0, & else. \end{cases} \tag{26}$$

Therefore, the regularization parameter ν in Fourier method [2] is related to the regularization parameter α in spectral regularization via $\nu = \phi^{-1}(\frac{1}{s}\ln(\frac{1}{C_1\sqrt{\alpha}}))$. However, via Morozov's discrepancy principle for choosing regularization parameter, the process of proof for error estimate is easier within the framework of spectral regularization.

In order to obtain the explicit expression of the error bound, we need to know the expression $\varphi(\cdot)$ for obtaining the function $\rho^{-1}(\cdot)$ in (2.5).

3.1 Interior inversion

In this subsection, we note that $p = 0$ in (1.5) and we want to recover the solution $\hat{f}(\xi, s)$ with $0 \le s < L$. First we use the equality

$$\hat{f}(\xi, s) = \frac{a(\xi, s)}{a(\xi, L)}\hat{f}(\xi, L).$$

Now (1.5) reads

$$\|\hat{f}(\xi, L)\|^2 = \|\frac{a(\xi, L)}{a(\xi, s)}\hat{f}(\xi, s)\|^2 \le E^2.$$

Denote the sets

$$M_1 = \{\hat{f}(\xi, s) \in L^2 \big| \|\frac{a(\xi, L)}{a(\xi, s)}\hat{f}(\xi, s)\|^2 \le E^2\}. \tag{27}$$

$$M = \{\hat{f}(\xi, s) \in L^2 \big| \|\frac{C_1}{C_2^2}|a(\xi, s)|^{\frac{L-s}{s}}\hat{f}(\xi, s)\|^2 \le E^2\}. \tag{28}$$

In fact, by $A^*A(s) = |a^{-1}(\xi, s)|^2 = |a(\xi, s)|^{-2}$, M is equivalent to M_2 given by in the form of (1.13)

$$M_2 = \{\hat{f}(\xi, s) \in L^2 \big| \|[\varphi^{(A^*A(s))}]^{-\frac{1}{2}}\hat{f}(\xi, s)\|^2 \le E^2\}, \tag{29}$$

where

$$\varphi(\lambda) = \frac{C_2^4}{C_1^2}\lambda^{\frac{L-s}{s}}. \tag{30}$$

Now we only need to show that every element from M_1 belongs to the set M. For any element $\hat{f}(\xi, s) \in M_1$, we have

$$\int |\frac{a(\xi, L)}{a(\xi, s)}|^2 |\hat{f}(\xi, s)|^2 d\xi \le E^2. \tag{31}$$

By (1.6), this implies

$$\int |\frac{C_1}{C_2}\exp((L-s)\phi(|\xi|))|^2 |\hat{f}(\xi, s)|^2 d\xi \le E^2. \tag{32}$$

On the other hand, for any element $\hat{f}(\xi, s) \in M$, we have two-side estimates

$$\int \frac{C_1^2}{C_2^4}|C_1\exp((L-s)\phi(|\xi|))|^2 |\hat{f}(\xi, s)|^2 d\xi \le \int \frac{C_1^2}{C_2^4}|a(\xi, s)|^{\frac{2(L-s)}{s}}|\hat{f}(\xi, s)|^2 d\xi$$

$$\leq \int \frac{C_1^2}{C_2^4}|C_2 \exp((L-s)\phi(|\xi|))|^2|\hat{f}(\xi,s)|^2 d\xi \leq E^2.$$

Thus we have shown that every element from M_1 belongs to the set $M = M_2$. Likewise, we can also show that every element from M_0 belongs to the set M_1 by constructing another a set M_0 similar to M. We have another $\varphi(\cdot)$ which differs from (3.6) only by a constant factor. Now we summarize what we have:

Conclusion 1. *If the a-priori bound (1.5) with* $p = 0$ *holds, for problem (1.3), the function* $\varphi(\cdot)$ *in the source set (3.5) has the form of* $\varphi(\lambda) = \gamma\lambda^{\frac{L-s}{s}}$ *with* γ *is a constant which only depends on* C_1 *and* C_2*. Thus, the function* $\rho(\lambda) = \lambda\varphi^{-1}(\lambda) = c_2\lambda^{\frac{L}{L-s}}$ *and* $\rho^{-1}(\lambda) = c_3\lambda^{\frac{L-s}{L}}$*. Therefore for problem (1.3) for solving* $\hat{f}(\xi,s)$*, the optimal convergence order for error estimates is given by*

$$E(\delta/E)^{\frac{L-s}{L}} = E^{\frac{s}{L}}\delta^{\frac{L-s}{L}}. \tag{33}$$

3.2 Boundary inversion

In this subsection, we note that $p > 0$ in (1.5) and we want to recover the solution $\hat{f}(\xi, L)$. Now (1.5) with $p > 0$ reads

$$\|\hat{f}(\xi,L)\|_p^2 = \|(1+\xi^2)^p \hat{f}(\xi,L)\|^2 \leq E^2.$$

Denote the sets

$$M_3 = \{\hat{f}(\xi,L) \in L^2 \big| \|(1+\xi^2)^p \hat{f}(\xi,L)\|^2 \leq E^2\}. \tag{34}$$

$$M_4 = \{\hat{f}(\xi,L) \in L^2 \big| \|[\phi^{-1}(\frac{\gamma_0}{2L}\ln([A^*A(L)]^{-1}))]^p \hat{f}(\xi,L)\|^2 \leq E^2\}, \tag{35}$$

where the $\phi^{-1}(\cdot)$ denotes the inverse function of $\phi(\cdot)$ and γ_0 is positive constant depends on C_1, C_1. Similar to above subsection, we can prove that every element from M_3 belongs to the set M_4. Thus we can obtain the expression of $\varphi(\lambda)$:

$$\varphi(\lambda) = [\phi^{-1}(\frac{\gamma_0}{2L}\ln([\lambda]^{-1}))]^{-2p}. \tag{36}$$

By a elementary calculation, it yields

$$\rho^{-1}(\lambda) = [\phi^{-1}(\tilde{c}\ln([\lambda]^{-1}))]^{-2p}, \tag{37}$$

where $\tilde{c}$ is a constant depends on γ_0, L.

Conclusion 2. *If the a-priori bound (1.5) with* $p > 0$ *holds, the optimal convergence order for solving* $\hat{f}(\xi, L)$ *is given by*

$$[\phi^{-1}(\tilde{c}\ln(E^2/\delta^2))]^{-p}. \tag{38}$$

International Journal of Mathematical Engineering and Science (IJMES)
Volume 5 Issue 1 (2016) ISSN : 2277-6982
http://www.ijmes.com/

4 Applications

Now we give some applications of theorem 2.1. Throughout this section, the $\tilde{C}, \tilde{C_1}, \tilde{C_2}$ are the positive constants which are not dependent on δ and E.

Exmaple 4.1. Consider the sideways heat equation:

$$\begin{aligned} & u_t - u_{xx} = 0, && x > 0, t > 0, \\ & u(x,0) = 0, && x > 0, \\ & u(L,t) = g(t), && t > 0, \\ & u(x,t)|_{x\to\infty} \ bounded, && \end{aligned} \tag{39}$$

where $g(t) \in L^2$ is given approximately by $g_\delta(t) \in L^2$. Here we want to determine the solution $u(x,t)$ for $0 \le x < 1$. As usual, we assume that the a-priori bound $\|u(0,\cdot)\|_p \le E$ holds.

The solution of problem (4.1) can be expressed as

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi t} e^{(L-x)\sqrt{i\xi}} \hat{g}(\xi) d\xi. \tag{40}$$

From (4.2), we have

$$\hat{u}(\xi,t) = e^{(L-x)\sqrt{i\xi}} \hat{g}(\xi). \tag{41}$$

Thus the symbol of pseudodifferential operator is given by

$$a(\xi,x) = e^{(L-x)\sqrt{i\xi}}. \tag{42}$$

It yields $|a(\xi,x)| = \exp((L-x)\sqrt{|\xi|/2})$.

Now we take $v = t, s = L - x, \phi(|\xi|) = \sqrt{|\xi|/2}, C_1 = C_2 = 1$ and let $f(v,s) = u(x,t), f_\alpha^\delta(v,s) = u_\alpha^\delta(x,t)$ where $u_\alpha^\delta(x,t)$ is the spectral regularization solution for problem (4.1). In this example, we have $\phi^{-1}(y) = 2y^2$. Therefore, according to (2.5) (3.9), we have the error estimate for the spectral method at $0 < x < L$

$$\|u_\alpha^\delta(x,\cdot) - u(x,\cdot)\| \le \tilde{C} E^{1-x/L} \delta^{x/L}. \tag{43}$$

Similarly according to (2.5)(3.14), we have the error estimate for the spectral method at $x = 0$

$$\|u_\alpha^\delta(L,\cdot) - u(L,\cdot)\| \le \tilde{C} [\ln(E/\delta)]^{-2p}. \tag{44}$$

Exmaple 4.2. Consider the Cauchy problem of the Laplace's equation:

$$\begin{aligned} & u_{xx} + u_{yy} = 0, && x \in (0,L), y \in \mathbb{R}, \\ & u(0,y) = g(y), && y \in \mathbb{R}, \\ & u_x(0,y) = 0, && y \in \mathbb{R}, \end{aligned} \tag{45}$$

where $g(y) \in L^2$ is given approximately by $g_\delta(y) \in L^2$. Here we want to determine the solution $u(x,y)$ for $0 \le x < L$. As usual, we assume that the a-priori bound $\|u(L,\cdot)\|_p \le E$ holds.

The solution of problem (4.7) can be expressed as

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi y}\cosh(x\xi)\hat{g}(\xi)d\xi. \tag{46}$$

From (4.8), we have

$$\hat{u}(\xi,x)=\cosh(x\xi)\hat{g}(\xi). \tag{47}$$

Thus the symbol of pseudodifferential operator is given by

$$a(\xi,x)=\cosh(x\xi). \tag{48}$$

It yields $|a(\xi,x)|=\cosh(x\xi)$ which satisfies

$$\frac{1}{2}e^{x|\xi|}\leq|a(\xi,x)|\leq e^{x|\xi|}. \tag{49}$$

Thus, we have $\phi(y)=y$. Now we take $v=y, s=x, C_1=1/2, C_2=1$ and let $f(v,s)=u(x,y), f_\alpha^\delta(v,s)=u_\alpha^\delta(x,y)$ where u_α^δ is the spectral regularization solution for problem (4.7). In this example, we have $\phi^{-1}(y)=y$. Therefore, according to (2.5) (3.9), we have the error estimate for the spectral method at $0<x<L$

$$\|u_\alpha^\delta(x,\cdot)-u(x,\cdot)\|\leq\tilde{C}E^{x/L}\delta^{1-x/L}. \tag{50}$$

Similarly according to (2.5)(3.14), we have the error estimate for the spectral method at $x=L$

$$\|u_\alpha^\delta(L,\cdot)-u(L,\cdot)\|\leq\tilde{C}[\ln(E/\delta)]^{-p}. \tag{51}$$

Exmaple 4.3. Consider the backward heat conduction problem:

$$\begin{aligned} &u_t-u_{xx}=0, && t\in(0,T), x\in\mathbb{R},\\ &u(x,T)=g(x), && y\in\mathbb{R} \end{aligned} \tag{52}$$

where $g(x)\in L^2$ is given approximately by $g_\delta(x)\in L^2$. Here we want to determine the solution $u(x,t)$ for $0\leq t<T$. As usual, we assume that the a-priori bound $\|u(\cdot,0)\|_p\leq E$ holds.

The solution of problem (4.14) can be expressed as

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\xi x}e^{\xi^2(T-t)}\hat{g}(\xi)d\xi. \tag{53}$$

From (4.15), we have

$$\hat{u}(\xi,t)=e^{\xi^2(T-t)}\hat{g}(\xi). \tag{54}$$

Thus the symbol of pseudodifferential operator is given by

$$a(\xi,x)=e^{\xi^2(T-t)}. \tag{55}$$

Thus, we have $\phi(y) = y^2, y > 0$. Now we take $L = T, v = x, s = T - t, C_1 = C_2 = 1$ and let $f(v, s) = u(x, t), f_\alpha^\delta(v, s) = u_\alpha^\delta(x, t)$ where u_α^δ is the spectral regularization solution for problem (4.14). In this example, we have $\phi^{-1}(y) = \sqrt{y}$. Therefore, according to (2.5) (3.9), we have the error estimate for the spectral method at $0 < t < T$

$$\|u_\alpha^\delta(x, \cdot) - u(x, \cdot)\| \le \tilde{C} E^{1-t/T} \delta^{t/T}. \tag{56}$$

Similarly according to (2.5)(3.14), we have the error estimate for the spectral method at $t = 0$

$$\|u_\alpha^\delta(\cdot, 0) - u(\cdot, 0)\| \le \tilde{C}[\ln(E/\delta)]^{-p/2}. \tag{57}$$

Example 4.4. Consider a two-layer body that consists of the first layer in $0 \le x \le l_1$ and the second layer in $l_1 \le x \le l_2$. The two layers are in perfect thermal contact at $x = l_1$.

Let $k_1, k_2 > 0$ be the thermal conductivities and $\alpha_1, \alpha_2 > 0$ be the thermal diffusivities of the first and the second layer, respectively. The temperature distributions in the first and the second layers, denoted by $u_1(x, t)$ and $u_2(x, t)$ respectively, satisfy the following partial differential equations in the two domains $D_1 := \{x \big| 0 \le x \le l_1\}$ and $D_2 := \{x \big| l_1 \le x \le l_2\}$:

$$\frac{\partial u_1}{\partial t} - \alpha_1 \frac{\partial^2 u_1}{\partial x^2} = 0, \qquad 0 < x < l_1, t > 0, \tag{58}$$

$$\frac{\partial u_2}{\partial t} - \alpha_2 \frac{\partial^2 u_2}{\partial x^2} = 0, \qquad l_1 < x < l_2, t > 0, \tag{59}$$

subject to the initial and boundary conditions

$$u_1(x, 0) = u_2(x, 0) = 0, \qquad 0 < x < l_2, \tag{60}$$

$$u_2(l_2, t) = g(t), \qquad t > 0, \tag{61}$$

$$\frac{\partial u_2}{\partial x}(l_2, t) = 0, \qquad t > 0, \tag{62}$$

$$u_1(l_1, t) = u_2(l_1, t), \qquad t > 0, \tag{63}$$

$$k_1 \frac{\partial u_1}{\partial x}(l_1, t) = k_2 \frac{\partial u_2}{\partial x}(l_1, t), \qquad t > 0. \tag{64}$$

We suppose that the exact data $g \in L^2(0, \infty)$ and thus it is natural to assume also that, for any fixed $x \in [0, l_2]$, the solutions $u_1(x, \cdot), u_2(x, \cdot)$ belong to $L^2(0, \infty)$. The inverse Cauchy problem is then to determine the solutions $u_1(x, t)$ for $0 \le x \le l_1$ in the space $L^2(0, \infty)$ from the given data $g(t) \in L^2(0, \infty)$ and the insulated condition at the accessible boundary $x = l_2$. In practice, the measurement of g contains error that gives $g^\delta(\cdot) \in L^2(0, \infty)$ satisfying

$$\|g^\delta(\cdot) - g(\cdot)\| \le \delta, \tag{65}$$

where the constant $\delta > 0$ represents a bound on the measurement error and $\| \cdot \|$ denotes the L^2-norm. Assume that there exists a constant $E > 0$ so that the following a-priori bound exists for the solution u_1 of the problem:

$$\|u_1(0, \cdot)\|_p \le E. \tag{66}$$

For $0 \le x \le l_1$, we have [9]

$$\hat{u}_1(x,\xi) = \cosh(\sqrt{\frac{i\xi}{\alpha_1}}(l_1 - x))\cosh(\sqrt{\frac{i\xi}{\alpha_2}}(l_2 - l_1))\hat{g}(\xi) + \frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}\sinh(\sqrt{\frac{i\xi}{\alpha_1}}(l_1 - x))\sinh(\sqrt{\frac{i\xi}{\alpha_2}}(l_2 - l_1))\hat{g}(\xi). \quad (67)$$

The solutions $u_1(x,t)$ can be recovered from taking the inverse Fourier transform. Denote

$$\kappa := \frac{k_2\sqrt{\alpha_1}}{k_1\sqrt{\alpha_2}}, \quad (68)$$

$$a(x,\xi) := \cosh(\sqrt{\frac{i\xi}{\alpha_1}}(l_1 - x))\cosh(\sqrt{\frac{i\xi}{\alpha_2}}(l_2 - l_1)) + \kappa\sinh(\sqrt{\frac{i\xi}{\alpha_1}}(l_1 - x))\sinh(\sqrt{\frac{i\xi}{\alpha_2}}(l_2 - l_1)), \quad (69)$$

Equations (2.27) can then be formulated in operator equations as follow:

$$\hat{u}_1(x,\xi) = a(x,\xi)\hat{g}(\xi). \quad (70)$$

For $a(x,\xi)$, we have [9] the following inequality: for $0 < x < l_1$, then

$$\tilde{C}_1 e^{(l_1-x)\sqrt{\frac{|\xi|}{2\alpha_1}}} e^{(l_2-l_1)\sqrt{\frac{|\xi|}{2\alpha_2}}} \le |a(x,\xi)| \le \tilde{C}_2 e^{(l_1-x)\sqrt{\frac{|\xi|}{2\alpha_1}}} e^{(l_2-l_1)\sqrt{\frac{|\xi|}{2\alpha_2}}}. \quad (71)$$

Now we take $v = t, L = \frac{l_1}{\sqrt{2\alpha_1}} + \frac{l_2-l_1}{\sqrt{2\alpha_2}}, s = L - \frac{l_1-x}{\sqrt{2\alpha_1}} + \frac{l_2-l_1}{\sqrt{2\alpha_2}} = \frac{x}{\sqrt{2\alpha_1}}, \phi(|\xi|) = \sqrt{|\xi|}$ and let $f(v,s) = u_1(x,t), f_\alpha^\delta(v,s) = u_{1,\alpha}^\delta(x,t)$ where $u_{1,\alpha}^\delta(x,t)$ is the spectral regularization solution for problem (4.20)-(4.26). In this example, we have $\phi^{-1}(y) = y^2$. Therefore, according to (2.5) (3.9), we have the error estimate for the spectral method at $0 < x < L$

$$\|u_{1,\alpha}^\delta(x,\cdot) - u(x,\cdot)\| \le \tilde{C}\delta^{\frac{\frac{x}{\sqrt{2\alpha_1}}}{\frac{l_1}{\sqrt{2\alpha_1}}+\frac{l_2-l_1}{\sqrt{2\alpha_2}}}} E^{1-\frac{\frac{x}{\sqrt{2\alpha_1}}}{\frac{l_1}{\sqrt{2\alpha_1}}+\frac{l_2-l_1}{\sqrt{2\alpha_2}}}}. \quad (72)$$

Similarly according to (2.5)(3.14), we have the error estimate for the spectral method at $x = 0$

$$\|u_{1,\alpha}^\delta(0,\cdot) - u(0,\cdot)\| \le \tilde{C}[\ln(E/\delta)]^{-2p}. \quad (73)$$

It is worthy of noting that the radial inverse heat conduction problem [10], the deblurring problem [11-12], the Cauchy problem for Helmholtz equation [3-4], the IHCP with variable coefficients [13] can be solved by the spectral regularization methods. There are many other ill-posed problems which we cannot give here an exhaustive survy.

5 Concluding Remark

Numerical solutions for many ill-posed problem can be recast as the numerical computation of a class of pseudodifferential operators. In this paper, for the numerical computation of a class of pseudodifferential operators, we pointed that the Fourier method provided in [2] can be fitted into the framework of regularization theory and the Fourier method can be considered as the spectral regularization method. Under the principle of Morozov's discrepancy, we derived the error bound for the numerical computation of a class of pseudodifferential operators. Relative to [2], our proposed methods and results are more general. For example, for our methods the adaptive balance principle [14] for choosing regularization parameter in the form of the operator equation can also be applied, however for the Fourier method the balance principle is not suitable.

References

[1] H. W. Engl, M. Hanke and A. Neubauer Regularization of Inverse Problems, Kluwer Academic Publisher, Dordrecht Boston London, 1996.

[2] C. L. Fu, Y. X. Zhang, H. Cheng and Y. J. Ma, The a posteriori Fourier method for solving ill-posed problems, Inverse Probl. 28(2012)095002(26pp).

[3] X.T. Xiong and C.L. Fu. Two approximate methods of a Cauchy problem for the Helmholtz equation, Comput. Appl. Math. 26(2007)285-307.

[4] T. Reginska and U. Tautenhahn, Conditional Stability Estimates and Regularization with Applications to Cauchy Problems for the Helmholtz Equation, Numer. Funct. Anal. Optim. 30(9-10)(2009)1065-1097.

[5] M. T. Nair and U. Tautenhahn, Lavrentiev Regularization for Linear Ill-Posed Problems under General Source Conditions, Z. Anal. Anw. 23(2004)167-185.

[6] M. T. Nair, Linear operator equations: Approximation and Regularization, World Scientific, 2009.

[7] D. N. Hào, Methods for Inverse Heat Conduction Problems. Peter Lang, Frankfurt am Main, 1998.

[8] U. Tautenhahn, Optimality for linear ill-posed problems under general source conditions, Numer. Funct. Anal. Optim. 19(1998)377-398.

[9] X. T. Xiong and Y. C. Hon, Regularization error analysis on a onedimensional inverse heat conduction problem in multilayer domain, Inverse Problem in Science and Engineering, 2013, http://dx.doi.org/10.1080/17415977.2013.788168.

[10] W. Cheng and C. L. Fu, Two regularization methods for an axissymmetric inverse heat conduction problem, J. Inverse and Ill-posed Problems. 17(2009)159-172.

[11] A. S. Carasso, Overcoming Hölder continuity in ill-posed continuation problems, SIAM J. Numer. Anal. 31(1994)1535-1557.

[12] X. T. Xiong, J. X. Wang and M. Li, An optimal method for fractional heat conduction problem backward in time, Appl. Anal. 91(2012)823-840.

[13] D. N. Hào and H.J. Reinhardt, On a sideways parabolic equation, Inverse Problems 13(1997) 297-309.

[14] S. Pereverzev and E. Shock, On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM J. Numer. Anal. 75(2006) 2060-2076.