Intuitionistic Fuzzy Hypermodules

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Abstract

The relationship between the intuitionistic fuzzy sets and the algebraic hyperstructures is described in this paper. The concept of the quasi-coincidence of an intuitionistic fuzzy interval valued with an interval-valued intuitionistic fuzzy set is introduced and this is a natural generalization of the quasi-coincidence of an intuitionistic fuzzy point in intuitionistic fuzzy sets. By using this new idea, the concept of interval-valued (α, β) - intuitionistic fuzzy sub - hypermodule of a hypermodule is defined. This newly defined interval-valued (α, β) - intuitionistic fuzzy sub - hypermodule is a generalization of the usual intuitionistic fuzzy sub - hypermodule.

Keywords : Intuitionistic fuzzy point, Interval-valued (α, β)- intuitionistic fuzzy sub - hypermodule, Interval-valued $(\in, \in \vee q)$- intuitionistic fuzzy sub - hypermodule.

1 Introduction:

The concept of hyperstructure was first introduced by F. Marty [10] at the eighth congress of Scandinavian Mathematics in 1934, where he defined hypergroups and started to analyze their properties. Now, the theory of algebraic hyperstructures has become a well-established branch in algebraic theory and it has an extensive applications in many branches of mathematics and applied science. Later on, people have developed the semi-hypergroups, which are the simplest algebraic hyperstructures having closure and associative properties. A comprehensive review of the theory of hyperstructures can be found in [8].

The canonical hypergroup is the special type of hypergroup, which was found by Mittas [11] who is the first one to study them extensively. The additive structure of hypermodules is just a canonical hypergroup. Several authors have studied this, for example Massouros, Corsini [8]; Davvaz, Ameri [1] et al.

After introducing, the celebrated concept of fuzzy sets, people have tried to implement the fuzzy concept of Zadeh [12] to classical mathematics. In this aspect, the concept of fuzzy subgroups was defined by Rosenfeld [9] and its structure was investigated. The fuzzy groups have been widely studied in [9]. In 1975, Zadeh further introduced the concept of interval valued fuzzy subsets where the values of the membership functions are intervals of numbers instead of the numbers alone. By using this new concept, Biswas defined the interval-valued fuzzy subgroups with the same nature of the fuzzy subgroups defined by Rosenfeld. A new type of fuzzy subgroup was therefore introduced by Bhakat and Das [6] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets.

On the other hand, Atanassov [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets by many authors. Some basic results on intuitionistic fuzzy sets were published in [3, 4] and the book [5] provides a comprehensive coverage of virtually all results in the area of the theory as well as the applications of intuitionistic fuzzy sets.

Actually, intuitionistic fuzzy sets are the objects of research by many scientists. In particular, intuitionistic fuzzy logic and the area of applications are been studied by Atanassov and co-workers (see [5]). Coker and Demirci [7] defined and studied the basic concept of intuitionistic fuzzy point.

In this paper, our aim is to introduce the concept of quasi-coincidence of an intuitionistic fuzzy interval valued with an interval-valued intuitionistic fuzzy set which generalizes the concept of quasi-coincidence of an intuitionistic fuzzy point in an intuitionistic fuzzy set. By using this new idea, we define an interval-valued (α, β)-intuitionistic fuzzy sub-hypermodules. Thus, this is a natural generalization of the intuitionistic fuzzy sub-hypermodules. We shall explore some of the interesting properties of interval-valued (α, β)-intuitionistic fuzzy sub-hypermodules. Moreover, some of the characterization theorems of such hypermodules will be given.

First, we present the basic definitions.

2 PRELIMINARIES

In this section, some of the basic definitions are summarized which are needed in the following sequel.

Definition 2.1 [13] A *hyperstructure* is a non-empty set H together with a mapping $\circ$:H$\times H \to P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H. If $x \in H$ and $A, B \in P^*(H)$, then $A \circ B$,$A \circ x$ and $x \circ B$ are defined as follows, $A \circ B = \bigcup_{a \in A, b \in B} a \circ b, A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$ respectively.

Definition 2.2 [13] A hyperstructure $(H, \circ)$ is said to be a *canonical hypergroup* if the following axioms are satisfied,

(i). For every $x, y, z \in H, x \circ (y \circ z) = (x \circ y) \circ z$,

(ii). For every $x, y \in H, x \circ y = y \circ x$,

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(iii). There exists an element $0 \in H$ such that $0 \circ x = x$, for all $x \in H$,
(iv). For every $x \in H$, there exists a unique element $x^{'} \in H$ such that $0 \in x \circ x^{'}$

Definition 2.3 [13] A *hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms,
(i). $(R, +)$ is a canonical hypergroup (here $-x$ for $x^{'}$),
(ii). $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element,
(iii). The multiplication is distributive w.r.t the hyperoperation "+".

Definition 2.4 [13] A non empty set M is called a *left hypermodule* over a hyperring R (R*-hypermodule*) if $(M, +)$ is a canonical hypergroup and there exists the map $\cdot : R \times M \to P^*(M)$ by $(r, m) \to r.m$ such that for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$, we have
(i). $r_1 \cdot (m_1 + m_2) = r_1 \cdot m_1 + r_1 \cdot m_2$,
(ii). $(r_1 + r_2) \cdot m_1 = r_1 \cdot m_1 + r_2 \cdot m_1$,
(iii). $(r_1 \cdot r_2) \cdot m_1 = r_1 \cdot (r_2 \cdot m_1)$.

Definition 2.5 [13] A fuzzy subset F of a hypermodule M over a hyperring R is a *fuzzy sub-hypermodule* of M if,
(i). $min\{F(x), F(y)\} \leq inf_{z \in x+y} F(z)$ for all $x, y \in M$,
(ii). $F(x) \leq F(-x)$ for all $x \in M$,
(iii). $F(x) \leq F(rx)$ for all $r \in R$ and $x \in M$.

Definition 2.6 [13] A fuzzy set F of a hypermodule M of the form,

$$F(y) = \begin{cases} t(\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases} \tag{1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by $U(x; t)$. A fuzzy point $U(x; t)$ is said to "belong to" a fuzzy set F, written as $U(x; t) \in F$ if $F(x) \geq t$. A fuzzy point $U(x; t)$ is said to "be quasi-coincident with" a fuzzy set F, written as $U(x; t)qF$ if $F(x) + t > 1$.

Remark 2.7 [13]
1. If $U(x; t) \in F$ or (and)$U(x; t)qF$, then we write it as $U(x; t) \in \vee qF, (U(x; t) \in \wedge qF)$.
2. The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.
3. Using the notion of "belongingness($\in$)" and "quasi-coincidence(q)" of fuzzy points with fuzzy subsets, we obtain the concept of an (α, β)- fuzzy subsemigroup, where α and β are any two of $\{\in, q, \in \vee q, \in \wedge q)$ with $\alpha \neq \in \wedge q$.
4. An interval number $\tilde{a}$ is an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers are denoted by $D[0, 1]$. The interval $[a, a]$ is identified with the number $a \in [0, 1]$.

Definition 2.8 [13] An *interval-valued fuzzy set* F on X is the set of the form,
$F = \{(x, [\mu_F^-(x), \mu_F^+(x)]) | x \in X\}$, where $\mu_F^-(x)$ and $\mu_F^+(x)$ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$.
Put $\tilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, then the set $F = \{(x, \tilde{\mu_F}(x)) | x \in X\}$, where $\tilde{\mu_F} : X \to D[0, 1]$.

Definition 2.9 [13] An interval-valued fuzzy set F of a hypermodule M of the form,

$$F(y) = \begin{cases} \tilde{t}(\neq [0,0]) & \text{if } y = x, \\ [0,0] & \text{if } y \neq x. \end{cases} \quad (2)$$

is said to be a *fuzzy interval value* with support x and interval value $\tilde{t}$ and is denoted by $U(x;\tilde{t})$.

A fuzzy interval value $U(x;\tilde{t})$ is said to be "belong to" an interval-valued fuzzy set F, written as $U(x;\tilde{t}) \in F$ if $\tilde{\mu_F}(x) \geq \tilde{t}$.

A fuzzy interval value $U(x;\tilde{t})$ is said to "be quasi-coincident with" an interval-valued fuzzy set F, written as $U(x;\tilde{t})qF$ if $\tilde{\mu_F}(x) + \tilde{t} > [1,1]$.

Definition 2.10 [13] An interval-valued fuzzy set F of M is called an *interval-valued* (α,β)*-fuzzy sub-hypermodule* of M, if for all $t, r \in (0,1], a \in R$ and $x, y \in M$,

(i). $U(x;\tilde{t})\alpha F$ and $U(y;\tilde{r})\alpha F$ imply $U(z;\text{r min}\{\tilde{t},\tilde{r}\})\beta F$, for all $z \in x + y$,

(ii). $U(x;\tilde{t})\alpha F$ implies $U(-x;\tilde{t})\beta F$,

(iii). $U(x;\tilde{t})\alpha F$ implies $U(ax;\tilde{t})\beta F$.

Definition 2.11 [13] An interval-valued fuzzy set F of M is said to be an *interval-valued fuzzy sub-hypermodule* of M, if for all $r \in (0,1], a \in R$ and $x, y \in M$, the following inequalities hold,

(i). r $min\{\tilde{\mu_F}(x), \tilde{\mu_F}(y)\} \leq \text{rinf}\{\tilde{\mu_F}(z) | z \in x + y\}$,

(ii). $\tilde{\mu_F}(x) \leq \tilde{\mu_F}(-x)$,

(iii). $\tilde{\mu_F}(x) \leq \tilde{\mu_F}(ax)$.

Definition 2.12 [13] An interval-valued fuzzy set F of M is called an *interval-valued* $(\in, \in \vee q)$*-fuzzy sub-hypermodule* of M, if for all $t, r \in (0,1], a \in R$ and $x, y \in M$,

(i). $U(x;\tilde{t}) \in F$ and $U(y;\tilde{r}) \in F$ imply $U(z;\text{r min}\{\tilde{t},\tilde{r}\}) \in \vee qF$, for all $z \in x + y$,

(ii). $U(x;\tilde{t}) \in F$ implies $U(-x;\tilde{t}) \in \vee qF$,

(iii). $U(x;\tilde{t}) \in F$ implies $U(ax;\tilde{t}) \in \vee qF$.

Definition 2.13 [2] An *Intuitionistic Fuzzy Set*(IFS)A in X is an object of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely,$\mu_A(x)$) and the degree of non-membership (namely,$\gamma_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.14 [2] Let A and B be two Intuitionistic Fuzzy Sets of the forms $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle | x \in X\}$. Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$,

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,

(c) The complement of A is denoted by $\bar{A}$ and is defined by $A = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in X\}$,

(d)$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle | x \in X\}$,

(e)$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle | x \in X\}$.

The Intuitionistic Fuzzy Sets $0_\sim = \{\langle x, 0, 1\rangle | x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle | x \in X\}$ represent the empty set and the whole set.

Definition 2.15 [7] Let $a, b \in [0,1]$ and $a+b \leq 1$. An *intuitionistic fuzzy point* is denoted by $x_{(a,b)}$ and is defined to be an intuitionistic fuzzy set of X given by,

$$x_{(a,b)}(y) = \begin{cases} (a,b) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x. \end{cases} \tag{3}$$

In this case, x is called the support of $x_{(a,b)}$, a is said to be the value of $x_{(a,b)}$ and b is the non-value of $x_{(a,b)}$.

An intuitionistic fuzzy point $x_{(a,b)}$ is said to be "belong to" an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, written as $x_{(a,b)} \in A$ if $\mu_A(x) \geq a$ and $\gamma_A(x) \leq b$.

3 Interval-Valued (α, β)- Intuitionistic Fuzzy Sub-hypermodules

In this section, we extend the concept of quasi-coincidence of a fuzzy interval value within an interval-valued fuzzy set to the concept of quasi-coincidence of an intuitionistic fuzzy point in an intuitionistic fuzzy set.

Definition 3.1 An intuitionistic fuzzy point $x_{(a,b)}$ is said to be "quasi-coincident" with an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, written as $x_{(a,b)}qA$ if $\mu_A(x) + a > 1$ and $\gamma_A(x) + b < 1$.

Example 3.2 Let $X = \{a, b, c\}$ be a non-empty set. Consider an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where $\mu_A(x) = \frac{0.6}{a} + \frac{0.54}{b} + \frac{0.7}{c}$ and $\gamma_A(x) = \frac{0.2}{a} + \frac{0.38}{b} + \frac{0.1}{c}$ for all $x \in X$ and an intuitionistic fuzzy point

$$x_{(a,b)}(y) = \begin{cases} (a,b) = (0.5, 0.4) & \text{if } y = x, \\ (0,1) & \text{if } y \neq x. \end{cases} \tag{4}$$

Here,the intuitionistic fuzzy point "quasi-coincidence" with A and "belongs" to A.

Definition 3.3 An intuitionistic fuzzy set A of a hypermodule M over a hyperring R is an *intuitionistic fuzzy sub-hypermodule* of M if for all $x, y \in M, r \in R$ and the following axioms are satisfied,

(i). $min\{\mu_A(x), \mu_A(y)\} \leq inf_{z \in x+y}\mu_A(z)$ and
$max\{\gamma_A(x), \gamma_A(y)\} \geq sup_{z \in x+y}\gamma_A(z)$,
(ii). $\mu_A(x) \leq \mu_A(-x)$ and $\gamma_A(x) \geq \gamma_A(-x)$,
(iii). $\mu_A(x) \leq \mu_A(rx)$ and $\gamma_A(x) \geq \gamma_A(rx)$.

If A is an intuitionistic fuzzy sub-hypermodule of M, then clearly we have,
$\mu_A(-x) = \mu_A(x), min\{\mu_A(x), \mu_A(y)\} \leq inf_{z \in x-y}\mu_A(z)$ and
$\gamma_A(-x) = \gamma_A(x), max\{\gamma_A(x), \gamma_A(y)\} \geq sup_{z \in x-y}\gamma_A(z)$ for all $x, y \in M$.

Definition 3.4 Let M be an R-hypermodule. Then, for an intuitionistic fuzzy set A of M, $t \in [0,1]$, the level subset $U(A;t)$ and the strong level subset $U(A;t^*)$ are defined by,
$U(A;t) = \{x \in M | \mu_A(x) \geq t \text{ and } \gamma_A(x) \leq t\}$ and $U(A;t^*) = \{x \in M | \mu_A(x) > t \text{ and } \gamma_A(x) < t\}$.

Definition 3.5 An *interval-valued intuitionistic fuzzy set* A on X is the set of the form,

$A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in X\}$, here $\tilde{\mu_A}(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\tilde{\gamma_A}(x) = [\gamma_A^-(x), \gamma_A^+(x)]$, where $\mu_A^-(x), \mu_A^+(x), \gamma_A^-(x)$ and $\gamma_A^+(x)$ are the member and non-member of an intuitionistic fuzzy set A on X such that $\mu_A^-(x) \leq \mu_A^+(x)$ and $\gamma_A^-(x) \geq \gamma_A^+(x)$ for all $x \in X$ and $\tilde{\mu_A} : X \to D[0,1]$, $\tilde{\gamma_A} : X \to D[0,1]$.

Definition 3.6 An interval-valued intuitionistic fuzzy set A of a hypermodule M of the form,

$$\tilde{\mu_A}(y) = \begin{cases} \tilde{t}(\neq [0,0]) & \text{if } y = x, \\ [0,0] & \text{if } y \neq x. \end{cases} \tag{5}$$

and

$$\tilde{\gamma_A}(y) = \begin{cases} \tilde{s}(\neq [1,1]) & \text{if } y = x, \\ [1,1] & \text{if } y \neq x. \end{cases} \tag{6}$$

is said to be an *intuitionistic fuzzy interval value* and is denoted by $x_{(\tilde{t},\tilde{s})}$ with support x, $\tilde{t}$ is called the value and $\tilde{s}$ is called the non-value of $x_{(\tilde{t},\tilde{s})}$.

An intuitionistic fuzzy interval value $x_{(\tilde{t},\tilde{s})}$ is said to "belong to" an interval-valued intuitionistic fuzzy set A, written as $x_{(\tilde{t},\tilde{s})} \in A$ if $\tilde{\mu_A}(x) \geq \tilde{t}$ and $\tilde{\gamma_A}(x) \leq \tilde{s}$.

An intuitionistic fuzzy interval value $x_{(\tilde{t},\tilde{s})}$ is said to "quasi-coincident" with an interval-valued intuitionistic fuzzy set A, written as $x_{(\tilde{t},\tilde{s})} qA$ if $\tilde{\mu_A}(x) + \tilde{t} > [1,1]$ and $\tilde{\gamma_A}(x) + \tilde{s} < [1,1]$.

Example 3.7 Let $X = \{1,2,3\}$ be a non-empty set, consider an interval-value intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in X\}$, where

$\tilde{\mu_A}(x) = \frac{0.65}{1} + \frac{0.7}{2} + \frac{0.8}{3}$ and $\tilde{\gamma_A}(x) = \frac{0.11}{1} + \frac{0.24}{2} + \frac{0.08}{3}$ for all $x \in X$ and an interval-value intuitionistic fuzzy set A is,

$$\tilde{\mu_A}(y) = \begin{cases} [0.5,0.6](\neq [0,0]) & \text{if } y = x, \\ [0,0] & \text{if } y \neq x. \end{cases} \tag{7}$$

and

$$\tilde{\gamma_A}(y) = \begin{cases} [0.32,0.45](\neq [1,1]) & \text{if } y = x, \\ [1,1] & \text{if } y \neq x. \end{cases} \tag{8}$$

Here, the intuitionistic fuzzy interval value "quasi-coincidence" with A and "belongs" to A.

Definition 3.8 An interval-valued intuitionistic fuzzy set A of M is called an *interval-valued (α, β)-intuitionistic fuzzy sub-hypermodule* of M, if for all $r, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in (0,1], a \in R$ and $x, y \in M$,

(i). $x_{(\tilde{a},\tilde{b})} \alpha A$ and $y_{(\tilde{c},\tilde{d})} \alpha A$ imply $z_{(rmin\{\tilde{a},\tilde{c}\}, rmax\{\tilde{b},\tilde{d}\})} \beta A$, for all $z \in x + y$,

(ii). $x_{(\tilde{a},\tilde{b})} \alpha A$ implies $-x_{(\tilde{a},\tilde{b})} \beta A$,

(iii). $x_{(\tilde{a},\tilde{b})} \alpha A$ implies $ax_{(\tilde{a},\tilde{b})} \beta A$.

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Remark 3.9 Let A be an interval-valued intuitionistic fuzzy set of M such that $\mu_A(x) \leq [0.4, 0.5]$ and $\gamma_A(x) > [0.5, 0.6]$ for all $x \in M$.

Suppose $x \in M$ and $t \in (0, 1]$ such that $x_{(\tilde{a},\tilde{b})} \in \wedge qA$, then
$\tilde{\mu_A}(x) \geq \tilde{a}$, $\tilde{\gamma_A}(x) \leq \tilde{b}$ and $\tilde{\mu_A}(x) + \tilde{a} > [1, 1]$ and $\tilde{\gamma_A}(x) + \tilde{b} < [1, 1]$.
Now, $[1, 1] < \tilde{\mu_A}(x) + \tilde{a} \leq \tilde{\mu_A}(x) + \tilde{\mu_A}(x) = 2\tilde{\mu}_A(x)$
$\Rightarrow$[1,1] $\leq 2\tilde{\mu}_A(x) \Rightarrow$[0.5,0.5] $\leq \tilde{\mu}_A(x)$ (ie). $\tilde{\mu}_A(x) > [0.5, 0.5]$
and then, $[1, 1] > \tilde{\gamma_A}(x) + \tilde{b} \geq \tilde{\gamma_A}(x) + \tilde{\gamma_A}(x) = 2\tilde{\gamma}_A(x)$
$\Rightarrow$[1,1] $\geq 2\tilde{\gamma}_A(x) \Rightarrow$[0.5,0.5] $\geq \tilde{\gamma}_A(x)$ (ie). $\tilde{\gamma}_A(x) \leq [0.5, 0.5]$.
From this, we get that $\{x_{(\tilde{a},\tilde{b})} | x_{(\tilde{a},\tilde{b})} \in \wedge qA\} = \phi$.
For this case, $\alpha \not\in \wedge q$.

Proposition 3.10 Every interval-valued $(\in \vee q, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Proof: Let A be an interval-valued $(\in \vee q, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

We have to prove that, A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

For that, let $x, y \in M$ and $r, a, b, c, d \in (0, 1]$ be such that $x_{(\tilde{a},\tilde{b})} \in A$ and $y_{(\tilde{c},\tilde{d})} \in A$.

Since, A is an interval-valued $(\in \vee q, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M, then $x_{(\tilde{a},\tilde{b})} \in \vee qA$ and $y_{(\tilde{c},\tilde{d})} \in \vee qA$ imply
$z_{(rmin\{\tilde{a},\tilde{c}\}, rmax\{\tilde{b},\tilde{d}\})} \in \vee qA$, for all $z \in x + y$——-(i)
then, $x_{(\tilde{a},\tilde{b})} \in A$ implies $-x_{(\tilde{a},\tilde{b})} \in \vee qA$ ———(ii)
and $x_{(\tilde{a},\tilde{b})} \in A$ implies $ax_{(\tilde{a},\tilde{b})} \in \vee qA$ ——-(iii)

From (i), (ii) and (iii), we get that A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Lemma 3.11 Every interval-valued $(\in, \in)$ intuitionistic fuzzy sub-hypermodule of M is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Lemma 3.12 If M_1 is a sub-hypermodule of M, then the characteristic function χ_{M_1} of M_1 is an interval-valued $(\in, \in)$ intuitionistic fuzzy sub-hypermodule of M.

Proposition 3.13 For any subset M_1 of M, χ_{M_1} is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M if and only if M_1 is a sub-hypermodule of M.

Proof: Let us assume that χ_{M_1} is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

We have to prove that, M_1 is a sub-hypermodule of M.
Let $x, y \in M_1$, then $x_{([1,1],[0,0])} \in \chi_{M_1}$ and $y_{([1,1],[0,0])} \in \chi_{M_1}$
which imply $z_{(rmin\{[1,1],[1,1]\}, rmax\{[0,0],[0,0]\})} \in \vee q\chi_{M_1}$ for all $z \in x + y$.
$\Rightarrow z_{([1,1],[0,0])} \in \vee q\chi_{M_1}$
$\Rightarrow \mu_{\chi_{M_1}}(z) > [0, 0]$ and $\gamma_{\chi_{M_1}}(z) < [1, 1]$ for all $z \in x + y$.
Thus, $x + y \in M_1$
Let $x \in M_1$, then $x_{([1,1],[0,0])} \in \chi_{M_1}$ and then $-x_{([1,1],[0,0])} \in \vee q\chi_{M_1}$

$\Rightarrow \mu_{\chi_{M_1}}(-x) > [0,0]$ and $\gamma_{\chi_{M_1}}(-x) < [1,1]$
Thus, $-x \in M_1$ and M_1 is a sub-hypergroup of $(M,+)$.
Hence, M_1 is a sub-hypermodule of M.
Conversely, let us assume that M_1 is a sub-hypermodule of M, then by using lemma 3.12, we get that the characteristic function χ_{M_1} of M_1 is an interval-valued $(\in,\in)$ intuitionistic fuzzy sub-hypermodule of M.
Then by using lemma 3.11, we get that χ_{M_1} is an interval-valued $(\in,\in\vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Proposition 3.14 Let A be a non-zero interval-valued (α,β) intuitionistic fuzzy sub-hypermodule of M. Then, the set
$A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in M\}$ where $\tilde{\mu_A}(x) > [0,0]$ and $\tilde{\gamma_A}(x) < [1,1]$ is a sub-hypermodule of M.
Proof: Let A be a non-zero interval-valued (α,β) intuitionistic fuzzy sub-hypermodule of M.
Claim: A is a sub-hypermodule of M.
Let $x, y \in A$, then $\tilde{\mu_A}(x) = \tilde{t} > [0,0]$ and $\tilde{\gamma_A}(x) = \tilde{s} < [1,1]$
also, $\tilde{\mu_A}(y) = \tilde{t_1} > [0,0]$ and $\tilde{\gamma_A}(y) = \tilde{s_1} < [1,1]$.
Assume that, $\tilde{\mu_A}(z) = [0,0]$ and $\tilde{\gamma_A}(z) = [1,1]$ for all $z \in x - y$.
If $\alpha \in \{\in, \in\vee q\}$, then $x_{(\tilde{t},\tilde{s})}\alpha A$ and $y_{(\tilde{t_1},\tilde{s_1})}\alpha A$
but $z_{(rmin\{\tilde{t},\tilde{t_1}\}, rmax\{\tilde{s},\tilde{s_1}\})}\overline{\beta}A$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$
which is a contradiction
Hence, for all $z \in x - y, \tilde{\mu_A}(z) > [0,0]$ and $\tilde{\gamma_A}(z) < [1,1]$
$\Rightarrow z \in A \Rightarrow x - y \subseteq A$. Thus, $(A,+)$ is a sub-hypergroup of M.
Hence, $(A,+)$ is a sub-hypermodule of M.

Definition 3.15 Let A be an interval-value intuitionistic fuzzy set. For every $t, s \in [0,1]$, the set $x_{(\tilde{t},\tilde{s})} = \{x \in M | \tilde{\mu_A}(x) \geq \tilde{t}, \tilde{\gamma_A}(x) \leq \tilde{s}\}$ is called the *interval-value level subset* of A.

Definition 3.16 An interval-valued intuitionistic fuzzy set A of a hypermodule M is said to be *proper* if ImA has at least two elements.

Definition 3.17 Two interval-valued intuitionistic fuzzy sets are said to be *equivalent* if they have same family of interval-value level subsets. Otherwise, they are said to be *non-equivalent*.

Proposition 3.18 Let M have proper sub-hypermodules. A proper interval-valued $(\in,\in)$ intuitionistic fuzzy sub-hypermodule A of M such that $cardIm$A ≥ 3 can be expressed as the union of 2 proper non-equivalent interval-valued $(\in,\in)$ intuitionistic fuzzy sub-hypermodules of M.
Proof: Let A be a proper interval-valued $(\in,\in)$ intuitionistic fuzzy sub-hypermodule of M with $Im\ \mu_A(x) = \{t_0, t_1, t_2, ...t_n\}$ and
$Im\ \gamma_A(x) = \{s_0, s_1, s_2, ...s_n\}$, where $n \geq 2$ with $\tilde{t_0} > \tilde{t_1} > \tilde{t_2} > ... > \tilde{t_n}$ and
$\tilde{s_0} < \tilde{s_1} < \tilde{s_2} < ... < \tilde{s_n}$. Then, $A_{(\tilde{t_0},\tilde{s_0})} \subseteq A_{(\tilde{t_1},\tilde{s_1})} \subseteq ... \subseteq A_{(\tilde{t_n},\tilde{s_n})} = M$ be the chain of interval-valued $(\in,\in)$ intuitionistic fuzzy sub-hypermodules of M.
Choose r_1, r_2, p_1, p_2 in $[0,1]$ such that $t_0 > t_1 > r_1 > t_2 > r_2 > ... > t_n$ and

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$s_0 < s_1 < p_1 < s_2 < p_2 < ... < s_n$.

Now, we define two interval-value intuitionistic fuzzy sets B, C in M by,

$$\tilde{\mu_B}(x) = \begin{cases} \tilde{r_1} & \text{if } x \in B_{(\tilde{t_1},\tilde{s_1})}, \\ \tilde{t_2} & \text{if } x \in B_{(\tilde{t_2},\tilde{s_2})} \setminus \ B_{(\tilde{t_1},\tilde{s_1})}, \\ \tilde{t_3} & \text{if } x \in B_{(\tilde{t_3},\tilde{s_3})} \setminus B_{(\tilde{t_2},\tilde{s_2})}, \\ . \\ . \\ . \\ \tilde{t_n} & \text{if } x \in B_{(\tilde{t_n},\tilde{s_n})} \setminus B_{(\tilde{t_{n-1}},\tilde{s_{n-1}})}. \end{cases} \tag{9}$$

and

$$\tilde{\gamma_B}(x) = \begin{cases} \tilde{p_1} & \text{if } x \in B_{(\tilde{t_1},\tilde{s_1})}, \\ \tilde{s_2} & \text{if } x \in B_{(\tilde{t_2},\tilde{s_2})} \setminus \ B_{(\tilde{t_1},\tilde{s_1})}, \\ . \\ . \\ . \\ \tilde{s_n} & \text{if } x \in B_{(\tilde{t_n},\tilde{s_n})} \setminus B_{(\tilde{t_{n-1}},\tilde{s_{n-1}})}. \end{cases} \tag{10}$$

Then,

$$\tilde{\mu_C}(x) = \begin{cases} \tilde{t_0} & \text{if } x \in C_{(\tilde{t_0},\tilde{s_0})}, \\ \tilde{t_1} & \text{if } x \in C_{(\tilde{t_1},\tilde{s_1})} \setminus \ C_{(\tilde{t_0},\tilde{s_0})}, \\ \tilde{r_2} & \text{if } x \in C_{(\tilde{t_3},\tilde{s_3})} \setminus C_{(\tilde{t_1},\tilde{s_1})}, \\ \tilde{t_4} & \text{if } x \in C_{(\tilde{t_4},\tilde{s_4})} \setminus C_{(\tilde{t_3},\tilde{s_3})}, \\ . \\ . \\ . \\ \tilde{t_n} & \text{if } x \in C_{(\tilde{t_n},\tilde{s_n})} \setminus C_{(\tilde{t_{n-1}},\tilde{s_{n-1}})}. \end{cases} \tag{11}$$

and

$$\tilde{\gamma_C}(x) = \begin{cases} \tilde{p_2} & \text{if } x \in C_{(\tilde{t_0},\tilde{s_0})}, \\ \tilde{s_1} & \text{if } x \in C_{(\tilde{t_1},\tilde{s_1})} \setminus \ C_{(\tilde{t_0},\tilde{s_0})}, \\ . \\ . \\ . \\ \tilde{s_n} & \text{if } x \in C_{(\tilde{t_n},\tilde{s_n})} \setminus C_{(\tilde{t_{n-1}},\tilde{s_{n-1}})}. \end{cases} \tag{12}$$

Now, let $x \in B_{(\tilde{t_1},\tilde{s_1})}$, then $\mu_B(x) \geq \tilde{t_1}$ and $\gamma_B(x) \leq \tilde{s_1}$
also, $x \in B_{(\tilde{t_2},\tilde{s_2})} \setminus \ B_{(\tilde{t_1},\tilde{s_1})}$
$\Rightarrow x \in B_{(\tilde{t_2},\tilde{s_2})}$, then $\mu_B(x) \geq \tilde{t_2}$ and $\gamma_B(x) \leq \tilde{s_2}$, proceeding like this we get , $B_{(\tilde{t_1},\tilde{s_1})} \subseteq B_{(\tilde{t_2},\tilde{s_2})} \subseteq \ ... \subseteq \ B_{(\tilde{t_n},\tilde{s_n})} = M$

Similarly, $C_{(\tilde{t_0},\tilde{s_0})} \subseteq \ C_{(\tilde{t_1},\tilde{s_1})} \subseteq \ C_{(\tilde{t_2},\tilde{s_2})} \subseteq \ ... \subseteq \ C_{(\tilde{t_n},\tilde{s_n})} = M$. Also, $B, C \leq A$.

Thus, B, C are non-equivalent. Hence, $B \cup C = A$.

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4 Interval-Valued $(\in, \in \vee q)$- Intuitionistic Fuzzy Sub-hypermodules

In this section, we extend the intuitionistic fuzzy sub-hypermodule to the interval-valued intuitionistic fuzzy sub-hypermodules of a hypermodule and also we concentrate on the interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodules of a hypermodule.

Definition 4.1 An interval-valued intuitionistic fuzzy set A of a hypermodule M is said to be an *interval-valued intuitionistic fuzzy sub-hypermodule* of M if for all $x, y \in M, a \in R, r \in (0, 1]$ and the following inequalities are satisfied,
(i). $rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\} \leq rinf_{z \in x+y}\tilde{\mu_A}(z)$ and $rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y)\} \geq rsup_{z \in x+y}\tilde{\gamma_A}(z)$,
(ii). $\tilde{\mu_A}(x) \leq \tilde{\mu_A}(-x)$ and $\tilde{\gamma_A}(x) \geq \tilde{\gamma_A}(-x)$,
(iii). $\tilde{\mu_A}(x) \leq \tilde{\mu_A}(ax)$ and $\tilde{\gamma_A}(x) \geq \tilde{\gamma_A}(ax)$.

Definition 4.2 An interval-valued intuitionistic fuzzy set A of a hypermodule M is said to be an *interval-valued* $(\in, \in \vee q)$ *intuitionistic fuzzy sub-hypermodule* of M if for all $x, y \in M, a \in R, r, t, s, t_1, s_1 \in (0, 1]$ and the following axioms are satisfied,
(i). $x_{(\tilde{t},\tilde{s})} \in A$ and $y_{(\tilde{t_1},\tilde{s_1})} \in A$ imply $z_{(rmin\{\tilde{t},\tilde{t_1}\}, rmax\{\tilde{s},\tilde{s_1}\})} \in \vee qA$, for all $z \in x + y$,
(ii). $x_{(\tilde{t},\tilde{s})} \in A$ implies $-x_{(\tilde{t},\tilde{s})} \in \vee qA$,
(iii). $x_{(\tilde{t},\tilde{s})} \in A$ implies $ax_{(\tilde{t},\tilde{s})} \in \vee qA$.

The following theorem can be proved

Theorem 4.3 Every interval-valued intuitionistic fuzzy sub-hypermodule of M is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

The following example clarifies the above theorem.

Example 4.4 Let $M = \{e, a, b, c\}, R = \{0, 1, 2, 3\}$ be two non-empty sets with the following hyperoperation tables, we define an intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in X\}$, where $\tilde{\mu_A}(x) = \frac{0.6}{e} + \frac{0.5}{a} + \frac{0.63}{b} + \frac{0.7}{c}$ and $\tilde{\gamma_A}(x) = \frac{0.28}{e} + \frac{0.34}{a} + \frac{0.32}{b} + \frac{0.12}{c}$ for all $x \in X$.

Table 1: Hyper Operation Table for '+'

+	e	a	b	c
e	e	a	e	c
a	a	{a,b}	c	{e,b}
b	e	c	{b,c}	b
c	c	{e,b}	b	c

Here, the intuitionistic fuzzy set A satisfies the conditions of definition 4.1 and 4.2, thus A is an interval-valued intuitionistic fuzzy sub-hypermodule of M as well as A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

But the converse statement of theorem 4.3 need not be true and it is proved by the following example.

Table 2: Hyper Operation Table for '.'

.	e	a	b	c
0	e	a	b	c
1	c	a	b	c
2	e	b	c	c
3	e	e	c	c

Example 4.5 Let $X = \{1, 2, 3\}, R = \{a, b, c\}$ be two non-empty sets, we define an interval-valued intuitionistic fuzzy set $A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in X\}$, where $\tilde{\mu_A}(x) = \frac{0.65}{1} + \frac{0.7}{2} + \frac{0.8}{3}$ and $\tilde{\gamma_A}(x) = \frac{0.31}{1} + \frac{0.25}{2} + \frac{0.18}{3}$ for all $x \in X$ and an interval-valued intuitionistic fuzzy set A is of the form,

$$\tilde{\mu_A}(y) = \begin{cases} [0.6,0.8](\neq [0,0]) & \text{if } y = x, \\ [0,0] & \text{if } y \neq x. \end{cases} \tag{13}$$

and

$$\tilde{\gamma_A}(y) = \begin{cases} [0.45,0.5](\neq [1,1]) & \text{if } y = x, \\ [1,1] & \text{if } y \neq x. \end{cases} \tag{14}$$

The hyperoperation tables are as follows,

Table 3: Hyper Operation Table for '+'

+	1	2	3
1	1	2	2
2	2	3	{2,3}
3	2	{2,3}	3

Table 4: Hyper Operation Table for '.'

.	1	2	3
a	1	2	3
b	2	{1,2}	{1,3}
c	3	{2,3}	1

Here, the interval-valued intuitionistic fuzzy set A satisfies the conditions of definition 4.2, thus A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M and it is not an interval-valued intuitionistic fuzzy sub-hypermodule of M.

Proposition 4.6 The above 3 conditions of definition 4.2, are equivalent to the following corresponding conditions, respectively for all $x, y \in M, a \in R$ and $r \in (0, 1]$,

(1).$rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \leq rinf_{z\in x+y}\tilde{\mu_A}(z)$ and
$rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \geq rsup_{z\in x+y}\tilde{\gamma_A}(z)$,
(2). $rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(-x)$ and
$rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(-x)$,
(3). $rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(ax)$ and
$rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(ax)$.
Proof: Let $x, y \in M, r, t, s, t_1, s_1 \in (0, 1]$ and $a \in R$.
(i) implies (1): Let us assume that $x_{(\tilde{t},\tilde{s})} \in A$ and $y_{(\tilde{t_1},\tilde{s_1})} \in A$ imply $z_{(rmin\{\tilde{t},\tilde{t_1}\},rmax\{\tilde{s},\tilde{s_1}\})} \in \vee qA$, for all $z \in x + y$.
Now, we consider the following two cases,
I) $rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\} < [0.5, 0.5]$ and $rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y)\} > [0.5, 0.5]$
II) $rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\} \geq [0.5, 0.5]$ and $rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y)\} \leq [0.5, 0.5]$
Case: I Assume that there exists $z \in x + y$ such that
$\tilde{\mu_A}(z) < rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\}$
$\Rightarrow \tilde{\mu_A}(z) < rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\}$ by (I)
Choose t such that $\tilde{\mu_A}(z) < \tilde{t} < rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y)\}$
and $\tilde{\gamma_A}(z) > rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\}$
$\Rightarrow \tilde{\gamma_A}(z) > rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y)\}$by (I)
Choose s such that $\tilde{\gamma_A}(z) > \tilde{s} > rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y)\}$. Thus, $z_{(\tilde{t},\tilde{s})}\overline{\in \vee q}A$ which is a contradiction to (i).
Case: II Assume that for some $z \in x + y$, we have
$\tilde{\mu_A}(z) < rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\}$ and $\tilde{\gamma_A}(z) > rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\}$
$\Rightarrow \tilde{\mu_A}(z) < [0.5, 0.5]$ and $\tilde{\gamma_A}(z) > [0.5, 0.5]$ By (II)
Then, $z_{([0.5,0.5],[0.5,0.5])}\overline{\in \vee q}A$ which is a contradiction to (i).
Hence, (1) holds. (ie). $rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \leq rinf_{z\in x+y}\tilde{\mu_A}(z)$
and $rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \geq rsup_{z\in x+y}\tilde{\gamma_A}(z)$.
(ii) implies (2): Let us assume that $x_{(\tilde{t},\tilde{s})} \in A$ implies $-x_{(\tilde{t},\tilde{s})} \in \vee qA$,
Now, we consider the following two cases,
I) $\tilde{\mu_A}(x) < [0.5, 0.5]$ and $\tilde{\gamma_A}(x) > [0.5, 0.5]$
II) $\tilde{\mu_A}(x) \geq [0.5, 0.5]$ and $\tilde{\gamma_A}(x) \leq [0.5, 0.5]$
Case: I Assume that $\tilde{\mu_A}(x) = \tilde{t} < [0.5, 0.5]$ and $\tilde{\gamma_A}(x) = \tilde{s} > [0.5, 0.5]$
Now, $\tilde{\mu_A}(-x) < rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} = \tilde{t} \Rightarrow \tilde{\mu_A}(-x) < \tilde{t}$
and $\tilde{\gamma_A}(-x) > rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} = \tilde{s} \quad \Rightarrow \tilde{\gamma_A}(-x) > \tilde{s}$
Then, $-x_{(\tilde{t},\tilde{s})}\overline{\in \vee q}A$ which is a contradiction to (ii).
Case: II Assume that $\tilde{\mu_A}(x) \geq [0.5, 0.5]$ and $\tilde{\gamma_A}(x) \leq [0.5, 0.5]$
If $\tilde{\mu_A}(-x) < rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} = [0.5, 0.5] \quad \Rightarrow \tilde{\mu_A}(-x) < [0.5, 0.5]$
and $\tilde{\gamma_A}(-x) > rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} = [0.5, 0.5] \quad \Rightarrow \tilde{\gamma_A}(-x) > [0.5, 0.5]$
Then, $-x_{([0.5,0.5],[0.5,0.5])}\overline{\in \vee q}A$ which is a contradiction to (ii).
Hence, (2) holds. (ie) $rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(-x)$ and
$rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(-x)$.
(iii) implies (3): The proof is similar to the proof of **(ii) implies (2)** instead of $-x$ put ax for all $a \in R$.

(1) implies (i): Let us assume that $rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \leq rinf_{z\in x+y}\tilde{\mu_A}(z)$ and $rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \geq rsup_{z\in x+y}\tilde{\gamma_A}(z)$.
Now, let $x_{(\tilde{t},\tilde{s})} \in A$ and $y_{(\tilde{t_1},\tilde{s_1})} \in A$, then $\tilde{\mu_A}(x) \geq \tilde{t}, \tilde{\gamma_A}(x) \leq \tilde{s}$ and $\tilde{\mu_A}(y) \geq \tilde{t_1}, \tilde{\gamma_A}(y) \leq \tilde{s_1}$.
For every $z \in x + y$, we have $\tilde{\mu_A}(z) \geq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\}$
$\Rightarrow \tilde{\mu_A}(z) \geq rmin\{\tilde{t}, \tilde{t_1}, [0.5, 0.5]\}$
If $rmin\{\tilde{t}, \tilde{t_1}\} > [0.5, 0.5]$, then $\tilde{\mu_A}(z) \geq [0.5, 0.5]$
$\Rightarrow \tilde{\mu_A}(z) + rmin\{\tilde{t}, \tilde{t_1}\} > [1, 1]$ ———(a)
If $rmin\{\tilde{t}, \tilde{t_1}\} \leq [0.5, 0.5]$, then $\tilde{\mu_A}(z) \geq rmin\{\tilde{t}, \tilde{t_1}\}$———(b)
and $\tilde{\gamma_A}(z) \leq rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \Rightarrow \tilde{\gamma_A}(z) \leq rmax\{\tilde{s}, \tilde{s_1}, [0.5, 0.5]\}$
If $rmax\{\tilde{s}, \tilde{s_1}\} \leq [0.5, 0.5]$, then $\tilde{\gamma_A}(z) \leq [0.5, 0.5]$
$\Rightarrow \tilde{\gamma_A}(z) + rmax\{\tilde{s}, \tilde{s_1}\} < [1, 1]$ ———(c)
If $rmax\{\tilde{s}, \tilde{s_1}\} > [0.5, 0.5]$, then $\tilde{\gamma_A}(z) \leq rmax\{\tilde{s}, \tilde{s_1}\}$———(d)
From (a), (b), (c) and (d), we get $z_{(rmin\{\tilde{t},\tilde{t_1}\}, rmax\{\tilde{s},\tilde{s_1}\})} \in \vee qA$.
(2) implies (ii): Let us assume that $rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(-x)$
and $rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(-x)$.
Now, let $x_{(\tilde{t},\tilde{s})} \in A$, then $\tilde{\mu_A}(x) \geq \tilde{t}$ and $\tilde{\gamma_A}(x) \leq \tilde{s}$.
we have, $\tilde{\mu_A}(-x) \geq rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \geq rmin\{\tilde{t}, [0.5, 0.5]\}$
$\Rightarrow \tilde{\mu_A}(-x) \geq \tilde{t}$ or $\tilde{\mu_A}(-x) \geq [0.5, 0.5]$———-(a)
also, we have, $\tilde{\gamma_A}(-x) \leq rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \leq rmax\{\tilde{s}, [0.5, 0.5]\}$
$\Rightarrow \tilde{\gamma_A}(-x) \leq \tilde{s}$ or $\tilde{\gamma_A}(-x) \leq [0.5, 0.5]$————(b)
From (a) and (b), we get $-x_{(\tilde{t},\tilde{s})} \in \vee qA$.
(3) implies (iii): The proof is similar to the proof of **(2) implies (ii)** instead of $-x$ put ax for all $a \in R$.
Using definition 4.2 and proposition 4.6, we deduce the following corollary.

Corollary 4.7 An interval-valued intuitionistic fuzzy set A of M is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M if and only if the conditions of proposition 4.6 holds.

Let us characterize the interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M by using their level sub-hypermodules.

Theorem 4.8 Let A be an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M. Then, for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ and $[0.5, 0.5] \leq \tilde{s} < [1, 1]$, the set $A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in M\}$ is an empty set (or) a sub-hypermodule of M. Conversely, if A is an interval-valued intuitionistic fuzzy set of M such that $A(\neq \phi)$is a sub-hypermodule of M, for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ and $[0.5, 0.5] \leq \tilde{s} < [1, 1]$, then A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Proof: Let us assume that A be an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M with for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ and $[0.5, 0.5] \leq \tilde{s} < [1, 1]$

We have to prove that, the set $A = \{\langle x, \tilde{\mu_A}(x), \tilde{\gamma_A}(x)\rangle | x \in M\}$ is an empty set (or) a sub-hypermodule of M.

Let $x, y \in A$, then $\tilde{\mu_A}(x) \geq \tilde{t}, \tilde{\gamma_A}(x) \leq \tilde{s}$ and $\tilde{\mu_A}(y) \geq \tilde{t}, \tilde{\gamma_A}(y) \leq \tilde{s}$.

Since, we have $rinf_{z\in x+y}\tilde{\mu_A}(z) \geq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\}$
$\geq rmin\{\tilde{t}, [0.5, 0.5]\}$
$rinf_{z\in x+y}\tilde{\mu_A}(z) \geq \tilde{t}$ (since,$\tilde{t} \leq [0.5, 0.5]$)
and $rsup_{z\in x+y}\tilde{\gamma_A}(z) \leq rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \geq rmax\{\tilde{s}, [0.5, 0.5]\}$
$rsup_{z\in x+y}\tilde{\gamma_A}(z) \leq \tilde{s}$ (since,$\tilde{s} \geq [0.5, 0.5]$).

Thus, $z \in A$ and so $x + y \subseteq A$.

Now,$\tilde{\mu_A}(-x) \geq rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\}$ and $\tilde{\gamma_A}(-x) \leq rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\}$
$\geq rmin\{\tilde{t}, [0.5, 0.5]\}$ $\leq rmax\{\tilde{s}, [0.5, 0.5]\}$
$\tilde{\mu_A}(-x) \geq \tilde{t}$ $\tilde{\gamma_A}(-x) \leq \tilde{s}$

Thus, $-x \in A$, therefore $(A, +)$ is a sub-hypergroup of $(M, +)$.

For every $x \in A, a \in R$, we have
$\tilde{\mu_A}(ax) \geq rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\}$ and $\tilde{\gamma_A}(ax) \leq rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\}$
$\geq rmin\{\tilde{t}, [0.5, 0.5]\}$ $\leq rmax\{\tilde{s}, [0.5, 0.5]\}$
$\tilde{\mu_A}(ax) \geq \tilde{t}$ $\tilde{\gamma_A}(ax) \leq \tilde{s}$

Thus, $ax \in A$, therefore $(A, +)$ is a sub-hypermodule of $(M, +)$.

Conversely, let A is an interval-valued intuitionistic fuzzy set of M such that $A(\neq \phi)$is a sub-hypermodule of M, for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$ and $[0.5, 0.5] \leq \tilde{s} < [1, 1]$.

Claim: A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

For every $x, y \in M, \tilde{\mu_A}(x) \geq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \Rightarrow \tilde{\mu_A}(x) \geq \tilde{t_0}$ and
$\tilde{\mu_A}(y) \geq rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \Rightarrow \tilde{\mu_A}(y) \geq \tilde{t_0}$
also, $\tilde{\gamma_A}(x) \leq rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \Rightarrow \tilde{\gamma_A}(x) \leq \tilde{s_0}$
and $\tilde{\gamma_A}(y) \leq rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \Rightarrow \tilde{\gamma_A}(y) \leq \tilde{s_0}$.Thus, $x, y \in A$
and $\tilde{\mu_A}(-x) \geq rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \geq \tilde{t_0}$
also, $\tilde{\gamma_A}(-x) \leq rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \leq \tilde{s_0}$.

Therefore, $-x \in A$ and so $x + y \subseteq A$, we get
$rmin\{\tilde{\mu_A}(x), \tilde{\mu_A}(y), [0.5, 0.5]\} \leq rinf_{z\in x+y}\tilde{\mu_A}(z)$ and
$rmax\{\tilde{\gamma_A}(x), \tilde{\gamma_A}(y), [0.5, 0.5]\} \geq rsup_{z\in x+y}\tilde{\gamma_A}(z)$ ———— (1)
$rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(-x)$ and
$rmax\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(-x)$————- (2)

Also, we have $x \in A \Rightarrow ax \in A$ for all $a \in R$, we get
$rmin\{\tilde{\mu_A}(x), [0.5, 0.5]\} \leq \tilde{\mu_A}(ax)$ and
rmax$\{\tilde{\gamma_A}(x), [0.5, 0.5]\} \geq \tilde{\gamma_A}(ax)$————- (3).

From (1), (2) and (3), we get A is an interval-valued $(\in, \in \vee q)$ intuitionistic fuzzy sub-hypermodule of M.

Remark 4.9 The corresponding result can be obtained when A is a sub-hypermodule of M, for all $[0.5, 0.5] < \tilde{t} \leq [1, 1]$ and $[0, 0] \leq \tilde{s} < [0.5, 0.5]$.

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