Brief introduction in non-archimedean analysis on the extended hyperreal line ${}^*\mathbb{R}_{\mathbf{d}}$. Euler's proof revisited.

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Abstract. In this paper possible completion of the Robinson non-archimedean field ${}^*\mathbb{R}$ constructed by Dedekind sections. As interesting example I show how, a few simple ideas from non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ gives a short clear nonstandard reconstruction for the Euler's original proof of the Goldbach-Euler theorem.

Contents

Introduction.

Nonstandard analysis, in its early period of development, shortly after having been established by A. Robinson [1],[4],[5] dealt mainly with nonstandard extensions of some traditional mathematical structures. The system of its foundations, referred to as "model-theoretic foundations" was proposed by Robinson and E. Zakon [12]. Their approach was based on the type-theoretic concept of superstructure $V(S)$ over some set of individuals S and its nonstandard extension (enlargement) ${}^*V(S)$, usually constructed as a (bounded) ultrapower of the "standard" superstructure $V(S)$. They formulated few principles concerning the elementary embedding $V(S) \mapsto {}^*V(S)$, enabling the use of methods of nonstandard analysis without paying much attention to details of construction of the particular nonstandard extension.

In classical Robinsonian nonstandard analysis we usualy deal only with completely internal objects which can defined by internal set theory **IST** introduced by E.Nelson [11]. It is known that **IST** is a conservative extension of ZFC. In **IST** all the classical infinite sets, e.g., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, acquire new, nonstandard elements (like *"infinite" natural numbers* or *"infinitesimal" reals*). At the same time, the families ${}^\sigma\mathbb{N} = \{x \in \mathbb{N} : \mathbf{st}(x)\}$ or ${}^\sigma\mathbb{R} = \{x \in \mathbb{R} : \mathbf{st}(x)\}$ of all standard,i.e., "true," natural numbers or reals, respectively, are not sets in **IST** at all. Thus, for a traditional mathematician inclined to ascribe to mathematical objects a certain kind of objective existence or reality, accepting **IST** would mean confessing that everybody has lived in confusion, mistakenly having regarded as, e.g., the set $\mathbb{N}$ just its tiny part ${}^\sigma\mathbb{N}$ (which is not even a set) and overlooked the rest. Edvard Nelson and Karel Hrbček have improved this lack by introducing several "nonstandard" set theories dealing with standard, internal and *external sets* [13]. Note that in contrast with early period of development of the nonstandard analysis in latest period many mathematicians dealing with external and internal set simultaneously,for example see [14],[15],[16],[17].

Many properties of the standard reals $x \in \mathbb{R}$ suitably reinterpreted, can be transfered to the internal hyperreal number system. For example, we have seen that ${}^*\mathbb{R}$, like $\mathbb{R}$, is a totally ordered field. Also, jast $\mathbb{R}$ contain the natural number $\mathbb{N}$ as a discrete subset with its own characteristic properties, ${}^*\mathbb{R}$ contains the hypernaturals ${}^*\mathbb{N}$ as the corresponding discrete subset with analogous properties.For example, the standard archimedean property $\forall x_{x\in\mathbb{R}} \forall y_{y\in\mathbb{R}} \exists n_{n\in\mathbb{N}}[(|x| < |y|) \dashrightarrow n|x| \geq |y|]$ is preserved in non-archimedean field ${}^*\mathbb{R}$ in respect hypernaturals ${}^*\mathbb{N}$, i.e. the next property is satisfied $\forall x_{x\in\mathbb{R}} \forall y_{y\in\mathbb{R}} \exists n_{n\in\mathbb{N}}[(|x| < |y|) \dashrightarrow n|x| \geq |y|]$. However, there are many fundamental properties of $\mathbb{R}$ do not transfered to ${}^*\mathbb{R}$.

I. This is the case one of the fundamental *supremum property* of the standard totally ordered field $\mathbb{R}$. It is easy to see that it apper bound property does not necesarily holds by considering, for example, the (external) set $\mathbb{R}$ itself which we ragard as canonically imbedded into hyperreals ${}^*\mathbb{R}$. This is a non-empty set which is bounded above (by any of the infinite member in ${}^*\mathbb{R}$) but does not have a least apper bound in ${}^*\mathbb{R}$. However by using transfer one obtain the next statement [18] :

Weak supremum property for ${}^*\mathbb{R}$

Every non-empty *internal* subset $A \subsetneqq {}^*\mathbb{R}$ which has an apper bound in ${}^*\mathbb{R}$ has a least apper bound in ${}^*\mathbb{R}$.

This is a problem, because any advanced variant of the analysis on the field ${}^*\mathbb{R}$ is needed more strong fundamental supremum property. At first sight one can improve this lack by using corresponding external constructions which known as Dedekind sections and Dedekind completion (see section **I.3.**).We denote corresponding Dedekind completion by symbol ${}^*\mathbb{R}_{\mathbf{d}}$. It is clear that ${}^*\mathbb{R}_{\mathbf{d}}$ is completely external object. But unfortunately ${}^*\mathbb{R}_{\mathbf{d}}$ is not iven a non-archimedean ring but non-archimedean *pseudo-ring* only. However this lack does not make greater difficulties because non-archimedean pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ contains non-archimedean subfield $\Re_{\mathbf{c}} \subset {}^*\mathbb{R}_{\mathbf{d}}$ such that $\Re_{\mathbf{c}} \approx {}^*\mathbb{R}_{\mathbf{c}}$. Here ${}^*\mathbb{R}_{\mathbf{c}}$ this is a Cauchy completion of the non-archimedean field ${}^*\mathbb{R}$ (see section **I.4.**).

II. This is the case two of the fundamental *Peano's induction property*

$$\forall B[[(1 \in B) \wedge \forall x(x \in B \Rightarrow x+1 \in B)] \Rightarrow B = \mathbb{N}] \tag{1}$$

does not necesarily holds for arbitrary subset $B \subset {}^*\mathbb{N}$. Therefore (1) is true for ${}^*\mathbb{N}$ when interpreted in ${}^*\mathbb{N}$ i.e.,

$$\forall^{\mathbf{int}} B\left[[(1 \in B) \wedge \forall x(x \in B \Rightarrow x+1 \in B)] \Rightarrow B = {}^*\mathbb{N}\right] \tag{2}$$

true for ${}^*\mathbb{N}$ provided that we read "$\forall B$" as "for each internal subset B of ${}^*\mathbb{N}$", i.e. as $\forall^{\mathbf{int}} B$. In general the importance of internal versus external entities rests on the fact that each statement that is true for $\mathbb{R}$ is true for ${}^*\mathbb{R}$ provided its quantifiers are restricted to the internal entities (subset) of ${}^*\mathbb{R}$ only [18].This is a problem, because any advanced variant of the analysis on the field ${}^*\mathbb{R}$ is needed more strong induction property than property (2).In this paper I have improved this lack by using external construction two different types for operation of exteral summation: $Ext-\sum_{n\in\mathbb{N}} q_n,\ \#Ext-\sum_{n\in\mathbb{N}} q_n^{\#}$ and two different types for operation of exteral multiplication: $Ext-\prod_{n\in\mathbb{N}} q_n, \#Ext-\prod_{n\in\mathbb{N}} q_n^{\#}$ for arbitrary countable sequences such as $q_n : \mathbb{N} \to \mathbb{R}$ and $q_n^{\#} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{d}}$.

As interesting example I show how, this external constructions from non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ gives a short and clear nonstandard reconstruction for the Euler's original proof of the Goldbach-Euler theorem.

I.The classical hyperreals numbers.

I.1.The construction non-archimedean field ${}^*\mathbb{R}$.

Let $\Re$ denote the ring of real valued sequences with the usual pointwise operations.If x is a real number we let s_x denote the constant sequence,$\mathbf{s}_x = x$ for all n. The function sending x to $\mathbf{s}_x$ is a one-to-one ring homomorphism,providing an embedding of $\mathbb{R}$ into $\Re$. In the following, wherever it is not too confusing we will not distinguish between $x \in \mathbb{R}$ and the constant function $\mathbf{s}_x$,leaving the reader to derive intent from context. The ring $\Re$ has additive identity 0 and multiplicative identity 1. $\Re$ is not a field because if r is any sequence having 0 in its range it can have no multiplicative inverse. There are lots of zero divisors in $\Re$.

We need several definitions now. Generally, for any set S, $\mathbf{P}(S)$ denotes the set of all subsets of S. It is called the power set of S. Also, a subset of $\mathbb{N}$ will be called cofinite if it contains all but finitely many members of $\mathbb{N}$. The symbol $\varnothing$ denotes the empty set. A partition of a set S is a decomposition of S into a union of sets, any pair of which have no elements in common.

Definition.1.1.1. An ultrafilter **H** over $\mathbb{N}$ is a family of sets for which:

(i) $\varnothing \notin \mathbf{H} \subset \mathbf{P}(\mathbb{N}), \mathbb{N} \in \mathbf{H}$.

(ii) Any intersection of finitely many members of **H** is in **H**.

(iii) $A \subset \mathbb{N}, B \in \mathbf{H} \Rightarrow A \cup B \in \mathbf{H}$.

(iv) If $V_1, \dots, V_n$ is any finite partition of $\mathbb{N}$ then **H** contains exactly one of the V_i.

If, further,

(v) **H** contains every cofinite subset of $\mathbb{N}$.

the ultrafilter is called *free.*

If an ultrafilter on $\mathbb{N}$ contains a finite set then it contains a one-point set, and is nothing more than the family of all subsets of $\mathbb{N}$ containing that point. So if an ultrafilter is not free it must be of this type, and is called a *principal ultrafilter.*

The existence of a free ultrafilter containing any given infinite subset of $\mathbb{N}$ is implied by the Axiom of Choice.

Remark 1.1.1. Suppose that $x \in X$. An ultrafilter denoted $\mathbf{prin}_X(x) \subseteq X$ consisting of all subsets $S \subseteq X$ which contain x, and called the *principal ultrafilter* generated by x.

Proposition 1.1.1. If an ultrafilter $\mathcal{F}$ on X contains a finite set $S \subseteq X$, then $\mathcal{F}$ is principal.

Proof: It is enough to show $\mathcal{F}$ contains $\{x\}$ for some $x \in S$. If not, then $\mathcal{F}$ contains the complement $X\backslash\{x\}$ for every $x \in S$, and therefore also the finite intersection $\mathcal{F} \ni \bigcap_{x \in S} X\backslash\{x\} = X\backslash S$, which contradicts the fact that $S \in \mathcal{F}$.

It follows that nonprincipal ultrafilters can exist only on infinite sets X, and that every cofinite subset of X (complement of a finite set) belongs to such an ultrafilter.

Remark 1.1.2. Our construction below depends on the use of a free-not a principal-ultrafilter.

We are going to be using conditions on sequences and sets to define subsets of $\mathbb{N}$. We introduce a convenient shorthand for the usual “set builder” notation. If P is a property that can be true or false for natural numbers we use $[[P]]$ to denote $\{n \in \mathbb{N} | P(n)$ is true $\}$. This notation will only be employed during a discussion to decide if the set of natural numbers defined by P is in **H**, or not. For example, if s, t is a pair of sequences in $\Re$ we define three sets of integers For example, if s, t is a pair of sequences in S we define three sets of integers

$$[[s < t]],\ \ [[s = t]],\ [[s > t]]. \tag{1.1}$$

Since these three sets partition $\mathbb{N}$, exactly one of them is in **H**, and we declare $s \equiv t$ when $[[s = t]] \ \in$ **H**.

Lemma 1.1.1. $\equiv$ is an equivalence relation on $\Re$. We denote the equivalence class of any sequence s under this relation by $[s]$. Define for each $r \in \Re$ the sequence $\tilde{r}$ by

$$\tilde{r} = \left\{ \begin{array}{l} 0 \quad \textbf{iff}\ r_n = 0 \\ r_n^{-1}\ \textbf{iff}\ r_n \neq 0 \end{array} \right\}. \tag{1.2}$$

Lemma 1.1.2. (**a**) There is at most one constant sequence in any class $[r]$.
(**b**) $[0]$ is an ideal in $\Re$ so $\Re/[0]$ is a commutative ring with identity [1].
(**c**) Consequently $[r] = r + [0] = \{r + t | t \in [0]\}$ for all $r \in \Re$.
(**d**) If $[r] \neq [0]$ then $[\tilde{r}] \cdot [r] = [1]$. So $[r]^{-1} = [\tilde{r}]$.
From Lemma 1.1.2., we conclude that ${}^*\mathbb{R}$, defined to be $\Re/[0]$, is a field containing an embedded image of as a subfield. $[0]$ is a maximal ideal in $\Re$.
Definition.1.1.2. This quotient ring is called the field ${}^*\mathbb{R}$ of **classical hyperreal numbers**.
We declare $[s] < [t]$ provided $[[s < t]] \ \in$ **H**.
Recall that any field with a linear order $<$ is called an ordered field provided
(i) $x + y > 0$ whenever $x,y > 0$
(ii) $x \cdot y > 0$ whenever $x,y > 0$
(iii) $x + z > y + z$ whenever $x > y$
Theorem 1.1.3. (**a**) The relation given above is a linear order on ${}^*\mathbb{R}$, and makes ${}^*\mathbb{R}$ into an ordered field. As with any ordered field, we define $|x|$ for $x \in {}^*\mathbb{R}$ to be x or $-x$, whichever is nonnegative.
(b) If x,y are real then $x \leq y$ if and only if $[x] \leq [y]$. So the ring morphism of $\mathbb{R}$ into ${}^*\mathbb{R}$ is also an order isomorphism onto its image in ${}^*\mathbb{R}$.

Because of this last theorem and the essential uniqueness of the real numbers it is common to identify the embedded image of $\mathbb{R}$ in ${}^*\mathbb{R}$ with $\mathbb{R}$ itself. Though obviously circular, one does something similar when identifying $\mathbb{Q}$ with its isomorphic image in ${}^*\mathbb{R}$, and $\mathbb{N}$ itself with the corresponding subset of $\mathbb{Q}$. This kind of notational simplification usually does not cause problems.

Now we get to the ideas that prompted the construction. Define the sequence r by $r_n = (n+1)^{-1}$. For every positive integer $k, [[r < k^{-1}]] \ \in$ **H**. So $0 < [r] < 1/k.$ We have found a positive hyperreal smaller than (the embedded image of) any real number. This is our first nontrivial infinitesimal number. The sequence $\tilde{r}$ is given by $\tilde{r}_n = n + 1.$ So $[r]^{-1} = [\tilde{r}] > k$ for every positive integer $k. [r]^{-1}$ is a hyperreal larger than any real number.

I.2.The brief nonstandard vocabulary.

Definition.1.2.1. We call a member $x \in {}^*\mathbb{R}$ $\mathbb{R}$-*limited* if there are members $a,b \in \mathbb{R}$ with $a < x < b$.
We will use $\mathbf{L}_*$= $\mathbf{L}({}^*\mathbb{R})$ to indicate the limited members of ${}^*\mathbb{R}$. x is called $\mathbb{R}$-*unlimited* if it not $\mathbb{R}$-limited.
These terms are preferred to “finite” and “infinite,”
which are reserved for concepts related to cardinality.
Definition.1.2.2. If $x,y \in {}^*\mathbb{R}$ and $x < y$ we use ${}^*[x,y]$ to denote $\{t \in {}^*\mathbb{R} | x \leq t \leq y\}$.
This set is called a *closed hyperinterval.* Open and half-open hyperintervals are defined and denoted similarly.
Definition.1.2.3. A set $S \subset {}^*\mathbb{R}$ is called *hyperbounded* if there are members x,y of ${}^*\mathbb{R}$ for which S is a subset of the hyperinterval ${}^*[x,y]$.
Abusing standard vocabulary for ordered sets, S is called bounded if x and y can be chosen to be limited members of ${}^*\mathbb{R}$. x and y could, in fact, be chosen to be real if S is bounded.
The vocabulary of bounded or hyperbounded above and below can be used.
Definition.1.2.3. We call a member $x \in^* \mathbb{R}$ infinitesimal if $|x| < a$ for every positive $a \in \mathbb{R}$. We write $x \approx 0$ iff x is infinitesimal.
The only real infinitesimal is obviously 0.
We will use $\mathbf{I}_* = \mathbf{I}({}^*\mathbb{R})$ to indicate the infinitesimal members of $\mathbf{I}_*$.
Definition.1.2.4. A member $x \in {}^*\mathbb{R}$ is called *appreciable* if it is limited but not infinitesimal.
Definition.1.2.5. Hyperreals x and y are said to have *appreciable separation* if $|x-y|$ is appreciable.
We will be working with various subsets S of ${}^*\mathbb{R}$ and adopt the following convention: $S_\infty = S \backslash \mathbf{L}_* = \{x \in S | x \notin \mathbf{L}_*\}$. These are the unlimited members of S, if any.
Definition.1.2.6. (**a**) We say two hyperreals x,y are infinitesimally close or have infinitesimal separation if $|x-y| \in \mathbf{I}_*$.
We use the notation $x \approx y$ to indicate that x and y are infinitesimally close.
(**b**) They have limited separation if $|x-y| \in \mathbf{L}_*$.
(**c**) Otherwise they are said to have unlimited separation.
We define the halo of x by $\mathbf{halo}(x) = x + \mathbf{I}_*$. There can be at most one real number in any halo. Whenever $\mathbf{halo}(x) \cap \mathbb{R}$ is nonempty we define the shadow of x, denoted $\mathbf{shad}(x)$, to be that unique real number.
The galaxy of x is defined to be $\mathbf{gal}(x) = x + \mathbf{L}_*$. $\mathbf{gal}(x)$ is the set of hyperreal numbers a limited distance away from x. So if x is limited $\mathbf{gal}(x) = \mathbf{L}_*$.

If n is any fixed positive integer we define ${}^{*}\mathbb{R}^{n}$ to be the set of equivalence classes of sequences in $\mathbb{R}^{n}$ under the equivalence relation $x \equiv y$ exactly when $[[x = y]] \in$ **H**.

Definition.1.2.7. We call ${}^{*}\mathbb{N}$ the set of *classical or A. Robinson's hypernatural numbers,* ${}^{*}\mathbb{N}_{\infty}$ the set of *classical or A. Robinson's infinite hypernatural numbers,* ${}^{*}\mathbb{Z}$ the set of *classical or A. Robinson's hyperintegers*, and ${}^{*}\mathbb{Q}$ the set of *classical or A. Robinson's hyperrational numbers.*

Theorem 1.2.1. ${}^{*}\mathbb{R}$ **is not Dedekind complete**.

(hint: $\mathbb{N}$ is bounded above by the member $[\mathbf{t}] \in {}^{*}\mathbb{N}_{\infty}$, where **t** is the sequence given by $t_n = n$ for all $n \in \mathbb{N}$. But $\mathbb{N}$ can have no least upper bound: if $n \leq c$ for all $n \in \mathbb{N}$ then $n \leq c - 1$ for all $n \in \mathbb{N}$.

As another example consider $\mathbf{I}_{*}$. This set is (very) bounded, but has no least upper bound.)

I.3.The construction non-archimedean pseudoring ${}^{*}\mathbb{R}_{\mathbf{d}}$.

From **Theorem 1.2.1.** above we knov that: ${}^{*}\mathbb{R}$ **is not Dedekind complete**.

Possible completion of the field ${}^{*}\mathbb{R}$ can be constructed by Dedekind sections. More general construction well known from topoi theory [10].

Definition 1.3.1. A *Dedekind hyperreal* $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ is a pair $(U, V) \in \mathbf{P}({}^{*}\mathbb{Q}) \times \mathbf{P}({}^{*}\mathbb{Q})$ satisfying the next conditions:

1.$\exists x \exists y (x \in U \wedge y \in V)$.

2. $U \cap V = \varnothing$.

3.$\forall x (x \in U \Leftrightarrow \exists y (y \in V \wedge x < y))$.

4. $\forall x (x \in V \Leftrightarrow \exists y (y \in V \wedge y < x))$.

5. $\forall x \forall y (x < y \Rightarrow x \in U \vee y \in V)$.

Let A be a subset of ${}^{*}\mathbb{R}_{\mathbf{d}}$ is bounded above then $\sup(A)$ exists in ${}^{*}\mathbb{R}_{\mathbf{d}}$.
For example $\sup(\mathbb{R}_{+}) = \inf\left({}^{*}\mathbb{R}_{+}\right) \in {}^{*}\mathbb{R}_{\mathbf{d}}$.

We shall very briefly remind a way Dedekind's constructions of a field ${}^{*}\mathbb{R}_{\mathbf{d}}$.

Definition 1.3.2. A Dedekind cut α on ${}^{*}\mathbb{Q}$ is a subset $\alpha \subset {}^{*}\mathbb{Q}$ of the hyperrational numbers ${}^{*}\mathbb{Q}$ that satisfies these properties:

1. α is not empty.

2. ${}^{*}\mathbb{Q} \backslash \alpha$ is not empty.

3. α contains no greatest element

4. For $x, y \in {}^{*}\mathbb{Q}$, if $x \in \alpha$ and $y < x$, then $y \in \alpha$ as well.

Definition 1.3.3. A Dedekind hyperreal number $\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$ is a Dedekind cut α on ${}^*\mathbb{Q}$. We denote the set of all Dedekind hyperreal numbers by ${}^*\mathbb{R}_{\mathbf{d}}$ and we order them by set-theoretic inclusion, that is to say, for any $\alpha,\beta \in {}^*\mathbb{R}_{\mathbf{d}}$, $\alpha < \beta$ if and only if $\alpha \subsetneqq \beta$ where the inclusion is strict. We further define $\alpha = \beta$ as real numbers if and are equal as sets. As usual, we write $\alpha \leqslant \beta$ if $\alpha < \beta$ or $\alpha = \beta$.

Definition 1.3.4. A hyperreal number α is said to be *Dedekind hyperirrational* if ${}^*\mathbb{Q}\backslash\alpha$ contains no least element.

Theorem 1.3.1. Every nonempty subset $A \subsetneqq {}^*\mathbb{R}_{\mathbf{d}}$ of Dedekind hyperreal numbers that is bounded above has a least upper bound.

Proof. Let A be a nonempty set of hyperreal numbers, such that for every $\alpha \in A$ we have that $\alpha \leqslant \gamma$ for some real number $\gamma \in {}^*\mathbb{R}_{\mathbf{d}}$. Now define the set $\sup A = \bigcup_{\alpha\in A} \alpha$. We must show that this set is a Dedekind hyperreal number. This amounts to checking the four conditions of a Dedekind cut. $\sup A$ is clearly not empty, for it is the nonempty union of nonempty sets. Because γ is a Dedekind hyperreal number, there is some hyperrational $x \in {}^*\mathbb{Q}$ that is not in γ. Since every $\alpha \in A$ is a subset of γ, x is not in any α, so $x \notin \sup A$ either. Thus, ${}^*\mathbb{Q}\backslash\sup A$ is nonempty. If $\sup A$ had a greatest element $g \in {}^*\mathbb{Q}$, then $g \in \alpha$ for some $\alpha \in A$. Then g would be a greatest element of α, but α is a Dedekind hyperreal number, so by contrapositive law, $\sup A$ has no greatest element. Lastly, if $x \in {}^*\mathbb{Q}$ and $x \in \sup A$, then $x \in \alpha$ for some α, so given any $y \in {}^*\mathbb{Q}$, $y < x$ because α is a Dedekind hyperreal number $y \in \alpha$ whence $y \in \sup A$. Thus $\sup A$, is a Dedekind hyperreal number.

Trivially,$\sup A$ is an upper bound of A, for every $\alpha \in A$, $\alpha \subseteqq \sup A$. It now suffices to prove that $\sup A \leqslant \gamma$, because was an arbitrary upper bound. But this is easy, because every $x \in \sup A, x \in {}^*\mathbb{Q}$ is an element of α for some $\alpha \in A$, so because $\alpha \subseteq \gamma, x \in \gamma$. Thus, $\sup A$ is the least upper bound of A.

Definition 1.3.5. Given two Dedekind hyperreal numbers α and β we define

1.The additive identity $0_{{}^*\mathbb{R}_{\mathbf{d}}}$, denoted 0, is
$0 \triangleq \left\{x \in {}^*\mathbb{Q} | \; x < 0\right\}$.

2.The multiplicative identity $1_{{}^*\mathbb{R}_{\mathbf{d}}}$, denoted 1, is
$1 \triangleq \left\{x \in {}^*\mathbb{Q} | \; x < 1\right\}$.

3. Addition $\alpha +_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ of α and β denoted $\alpha + \beta$ is
$\alpha + \beta \triangleq \left\{x + y | \; x \in \alpha, y \in \beta\right\}$.

4.The opposite $-_{{}^*\mathbb{R}_{\mathbf{d}}} \alpha$ of α, denoted $-\alpha$, is
$-\alpha \triangleq \left\{x \in {}^*\mathbb{Q} | \; -x \notin \alpha, -x \text{ is not the least element of } {}^*\mathbb{Q}\backslash\alpha\right\}$.

5.If $\alpha, \beta > 0$ then multiplication $\alpha \times_{{}^*\mathbb{R}_{\mathbf{d}}} \beta$ of α and β denoted $\alpha \times \beta$ is
$\alpha \times \beta \triangleq \left\{z \in {}^*\mathbb{Q} | \; z = x \times y \text{ for some } x \in \alpha, y \in \beta \text{ with } x, y > 0\right\}$.

In general, $\alpha \times \beta = \mathbf{0}$ if $\alpha = \mathbf{0}$ or $\beta = \mathbf{0}$,

$\alpha \times \beta \triangleq |\alpha| \times |\beta|$ if $\alpha > \mathbf{0}, \beta > \mathbf{0}$ or $\alpha < \mathbf{0}, \beta < \mathbf{0}$,

Definition 1.3.6. Let $\mathbf{S}_X$ denote the group of permutations of the set X and $\mathbf{H}_X$ denote ultrafilter on the set X. Permutation $\sigma \in \mathbf{S}_X$ is *admissible* iff σ preserv $\mathbf{H}_X$, i.e. for any $A \in \mathbf{H}_X$ the next condition is satisfied: $\sigma(A) \in \mathbf{H}_X$.
Below we denote by $\widehat{\mathbf{S}}_{X,\mathbf{H}_X}$ the subgroup $\widehat{\mathbf{S}}_{X,\mathbf{H}_X} \subsetneqq \mathbf{S}_X$ of the all admissible permutations.
Definition 1.3.7. Let us consider countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$, such that (**a**) $\forall n(\mathbf{s}_n \geq 0)$ or (**b**) $\forall n(\mathbf{s}_n < 0)$ and hyperreal number denoted $[\mathbf{s}_n]$ which formed from sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ by the law

$$[\mathbf{s}_n] = \left(\mathbf{s}_0, \mathbf{s}_0 + \mathbf{s}_1, \mathbf{s}_0 + \mathbf{s}_1 + \mathbf{s}_2, \ldots, \textstyle\sum_0^i \mathbf{s}_i, \ldots\right) \in {}^*\mathbb{R}. \tag{1.3.1}$$

Then external sum of the countable sequence $\mathbf{s}_n$ denoted

$$Ext\text{-}\sum_{n\in\mathbb{N}} \mathbf{s}_n \tag{1.3.2}$$

is

$$\begin{aligned} &(a) : Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \triangleq \inf\left\{[\mathbf{s}_{\sigma(n)}] | \, \sigma \in \widehat{\mathbf{S}}_{\mathbb{N},\mathbf{H}_\mathbb{N}}\right\}, \\ &(b) : Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \triangleq \sup\left\{[\mathbf{s}_{\sigma(n)}] | \, \sigma \in \widehat{\mathbf{S}}_{\mathbb{N},\mathbf{H}_\mathbb{N}}\right\} \end{aligned} \tag{1.3.3}$$

accordingly.

Example 1.3.1. Let us consider countable sequence $\{\mathbf{1}_n\}_{n\in\mathbb{N}}$ such that: $\forall n(\mathbf{1}_n = 1)$. Hence $[\mathbf{1}_n] = (1,2,3,\ldots,i,\ldots) = \varpi \in {}^*\mathbb{R}$ and using Eq.(1.3.3) one obtain

$$Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_n = \varpi \in {}^*\mathbb{R}. \tag{1.3.4}$$

Example 1.3.2. Let us consider countable sequence $\{\mathbf{1}_n^{\blacktriangledown}\}_{n\in\mathbb{N}}$ such that: $\{n|\mathbf{1}_n^{\blacktriangledown} = 1\} \in \mathbf{H}_{\mathbb{N}}$. Hence $[\mathbf{1}_n^{\blacktriangledown}] = (1,2,3,\ldots,i,\ldots)(\mathrm{mod}\mathbf{H}_{\mathbb{N}}) = \varpi \in {}^*\mathbb{R}$ and using Eq.(1.3.3) one obtain

$$Ext-\sum_{n\in\mathbb{N}} \mathbf{1}_n^{\blacktriangledown} = \varpi \in {}^*\mathbb{R}. \tag{1.3.5}$$

Example 1.3.3. (**Euler's infinite number** $E^{\#}$). Let us consider countable sequence $\mathbf{h}_n = n^{-1}$. Hence

$$[\mathbf{h}_n] = \left(1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3},\ldots, 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{i},\ldots\right) \in {}^*\mathbb{R}$$

and using Eq.(1.3.3) one obtain

$$Ext-\sum_{n=1}^{\infty} \mathbf{h}_n = E^{\#} \in {}^*\mathbb{R}_{\mathbf{d}}. \tag{1.3.6}$$

Definition 1.3.8. Let us consider countable sequence $\mathbf{s}_n : \mathbb{N} \to \mathbb{R}$ and two subsequences denoted $\mathbf{s}_n^+ : \mathbb{N} \to \mathbb{R}, \mathbf{s}_n^- : \mathbb{N} \to \mathbb{R}$ which formed from sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ by the law

$$\begin{gathered}\mathbf{s}_n^+ = \mathbf{s}_n \Leftrightarrow \mathbf{s}_n \geq 0,\\ \mathbf{s}_n^+ = 0 \Leftrightarrow \mathbf{s}_n < 0\end{gathered} \tag{1.3.7}$$

and accordingly by the law

$$\begin{gathered}\mathbf{s}_n^- = \mathbf{s}_n \Leftrightarrow \mathbf{s}_n < 0,\\ \mathbf{s}_n^- = 0 \Leftrightarrow \mathbf{s}_n \geq 0\end{gathered} \tag{1.3.8}$$

Hence $\{\mathbf{s}_n\}_{n\in\mathbb{N}} = \{\mathbf{s}_n^+ + \mathbf{s}_n^-\}_{n\in\mathbb{N}}$.

Example 1.3.4. Let us consider countable sequence

$$\{\mathbf{1}_n^{\pm}\}_{n\in\mathbb{N}} = \{1,-1,1,-1,\ldots,1,-1,\ldots\}. \qquad (1.3.9)$$

Hence $\{\mathbf{1}_n^{\pm}\}_{n\in\mathbb{N}} = \{\mathbf{1}_n^{+} + \mathbf{1}_n^{-}\}_{n\in\mathbb{N}}$ where

$$\begin{aligned} \{\mathbf{1}_n^{+}\}_{n\in\mathbb{N}} &= \{1,0,1,0,\ldots,1,0,\ldots\} \\ \{\mathbf{1}_n^{-}\}_{n\in\mathbb{N}} &= \{0,-1,0,-1,\ldots,0,-1,\ldots\}. \end{aligned} \qquad (1.3.10)$$

Definition 1.3.9. The external sum of the arbitrary countable sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ denoted

$$Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \qquad (1.3.12)$$

is

$$Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n \triangleq \left(Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{+}\right) + \left(Ext - \sum_{n\in\mathbb{N}} \mathbf{s}_n^{-}\right). \qquad (1.3.13)$$

Example 1.3.5. Let us consider countable sequence (1.3.9) Using Eq.(1.3.3),Eq.(1.3.13) and Eq.(1.3.5) one obtain

$$\begin{aligned} Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_{n n\in\mathbb{N}}^{\pm} &= \left(Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_n^{+}\right) + \left(Ext - \sum_{n\in\mathbb{N}} \mathbf{1}_n^{-}\right) = \\ &= \varpi - \varpi = 0. \end{aligned} \qquad (1.3.14)$$

Definition 1.3.10. Let us consider countable sequence $\mathbf{s}_n^{\#} : \mathbb{N} \to {}^*\mathbb{R}_{\mathbf{c}}$, such that
(**a**) $\forall n(\mathbf{s}_n^{\#} \geq 0)$ or (**b**) $\forall n(\mathbf{s}_n^{\#} < 0)$.

Then external sum of the countable sequence $\mathbf{s}_n^{\#}$ denoted

$$\#Ext\text{-}\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#} \qquad (1.3.15)$$

is

$$\begin{aligned}(a)&: \#Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#} \triangleq \sup_{k\in\mathbb{N}}\left\{\sum_{n\le k}\mathbf{s}_{\sigma(n)}^{\#}\,\middle|\,\sigma\in\mathbf{S}_{\mathbb{N}}\right\},\\ (b)&: \#Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#} \triangleq \inf_{k\in\mathbb{N}}\left\{\sum_{n\le k}\mathbf{s}_{\sigma(n)}^{\#}|\,\middle|\,\sigma\in\mathbf{S}_{\mathbb{N}}\right\}.\end{aligned} \qquad (1.3.16)$$

Definition 1.3.11. Let us consider countable sequence $\mathbf{s}_n^{\#}:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{c}}$ and two subsequences denoted ${}^{\#}\mathbf{s}_n^{+}:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{c}},{}^{\#}\mathbf{s}_n^{-}:\mathbb{N}\to{}^*\mathbb{R}_{\mathbf{c}}$ which formed from sequence $\{\mathbf{s}_n^{\#}\}_{n\in\mathbb{N}}$ by the law

$$\begin{aligned}{}^{\#}\mathbf{s}_n^{+} &= \mathbf{s}_n \Leftrightarrow \mathbf{s}_n^{\#}\ge 0,\\ {}^{\#}\mathbf{s}_n^{+} &= 0 \Leftrightarrow \mathbf{s}_n^{\#}<0\end{aligned} \qquad (1.3.17)$$

and accordingly by the law

$$\begin{aligned}{}^{\#}\mathbf{s}_n^{-} &= \mathbf{s}_n^{\#} \Leftrightarrow \mathbf{s}_n^{\#}<0,\\ {}^{\#}\mathbf{s}_n^{-} &= 0 \Leftrightarrow \mathbf{s}_n^{\#}\ge 0\end{aligned} \qquad (1.3.18)$$

Hence $\{\mathbf{s}_n^{\#}\}_{n\in\mathbb{N}} = \left\{{}^{\#}\mathbf{s}_n^{+} + {}^{\#}\mathbf{s}_n^{-}\right\}_{n\in\mathbb{N}}$.

Definition 1.3.12. The external sum of the arbitrary countable sequence $\{\mathbf{s}_n\}_{n\in\mathbb{N}}$ denoted

$$\#Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#} \quad (1.3.19)$$

is

$$\#Ext-\sum_{n\in\mathbb{N}}\mathbf{s}_n^{\#} \triangleq \left(\#Ext-\sum_{n\in\mathbb{N}}({}^{\#}\mathbf{s}_n^{+})\right)+\left(\#Ext-\sum_{n\in\mathbb{N}}({}^{\#}\mathbf{s}_n^{-})\right). \quad (1.3.20)$$

I.4.The construction non-archimedean field ${}^{*}\mathbb{R}_{\mathbf{c}}$.

Definition 1.4.1. A hypersequence $\mathbf{s_n} : {}^{*}\mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}, \mathbf{n} \in {}^{*}\mathbb{N}$ tends to a $*$-limit α ($\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}$) in ${}^{*}\mathbb{R}_{\mathbf{d}}$ iff

$$\exists\alpha\left(\alpha \in {}^{*}\mathbb{R}_{\mathbf{d}}\right)\forall\varepsilon_{\varepsilon>0}\left(\varepsilon \in {}^{*}\mathbb{R}_{\mathbf{d}}\right)\exists\mathbf{n}_0\left(\mathbf{n}_0 \in {}^{*}\mathbb{N}_{\infty}\right)\forall\mathbf{n}[\mathbf{n} \geqslant \mathbf{n}_0 \Rightarrow |\alpha-\mathbf{s}_{\mathbf{n}_0}| < \varepsilon]. \quad (1.4.1)$$

We write $*\text{-}\lim_{\mathbf{n}\to{}^{*}\infty}\mathbf{s_n} = \alpha$ or $\underset{\mathbf{n}\to{}^{*}\infty}{*\text{-}\lim}\,\mathbf{s_n} = \alpha$ iff condition (1.4.1) is satisfied.

Definition 1.4.2. A hypersequence $\mathbf{s_n} : {}^{*}\mathbb{N} \to {}^{*}\mathbb{R}$ is divergent in ${}^{*}\mathbb{R}$, or tends to ${}^{*}\infty$ iff

$$\forall r_{r>0}\left(r \in {}^{*}\mathbb{R}\right)\exists\mathbf{n}_0\left(\mathbf{n}_0 \in {}^{*}\mathbb{N}_{\infty}\right)\forall\mathbf{n}[\mathbf{n} \geqslant \mathbf{n}_0 \Rightarrow |\mathbf{s_n}| > r]. \quad (1.4.2)$$

Definition 1.4.3. A *Cauchy hypersequence in* ${}^{*}\mathbb{R}_{\mathbf{d}}$ is a sequence $\mathbf{s_n} :{}^{*}\mathbb{N} \to {}^{*}\mathbb{R}_{\mathbf{d}}$ with the following property: for every $\varepsilon \in {}^{*}\mathbb{R}_{\mathbf{d}}$ such that $\varepsilon > 0$, there exists an $\mathbf{n}_0 \in {}^{*}\mathbb{N}_{\infty}$ such that $\mathbf{m},\mathbf{n} \geqslant n_0$ implies $|\mathbf{s_m}-\mathbf{s_n}| < \varepsilon$, i.e.

$$\forall\varepsilon_{(\varepsilon\in{}^{*}\mathbb{R}_{\mathbf{d}})}(\varepsilon > 0)\exists\mathbf{n}_0\left(\mathbf{n}_0 \in {}^{*}\mathbb{N}_{\infty}\right)[\mathbf{m},\mathbf{n} \geqslant n_0 \Rightarrow |\mathbf{s_m}-\mathbf{s_n}| < \varepsilon] \quad (1.4.3)$$

Definition 1.4.4. Cauchy hypersequences $(x_{\mathbf{n}})_{\mathbf{n}\in{}^{*}\mathbb{N}}$ and $(y_{\mathbf{n}})_{\mathbf{n}\in{}^{*}\mathbb{N}}$, can be

added, multiplied and compared as follows:
(a) $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} + (y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} = (x_{\mathbf{n}} + y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$,
(b) $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} = (x_{\mathbf{n}} \times y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$,
(c) $\dfrac{(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}}{(y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}} = \left(\dfrac{x_{\mathbf{n}}}{y_{\mathbf{n}}}\right)_{\mathbf{n}\in {}^*\mathbb{N}}$ iff $\forall \mathbf{n}\left(\mathbf{n} \in {}^*\mathbb{N}\right) [y_n \neq 0]$,
(d) $(x_{\mathbf{n}})^{-1}_{\mathbf{n}\in {}^*\mathbb{N}} = (x^{-1}_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ iff $\forall \mathbf{n} \in {}^*\mathbb{N}(y_{\mathbf{n}} \neq 0)$,
(e) $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \geq (y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ if and only if for every $\epsilon > 0, \epsilon \in {}^*\mathbb{Q}$ there exists an integer $\mathbf{n}_0$ such that $x_{\mathbf{n}} \geq y_{\mathbf{n}} - \epsilon$ for all $\mathbf{n} > \mathbf{n}_0$.

Definition 1.4.5. Two Cauchy hypersequences $(x_{\mathbf{n}})$ and $(y_{\mathbf{n}})$ are called equivalent: $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (y_n)_{\mathbf{n}\in {}^*\mathbb{N}}$ if the hypersequence $(x_{\mathbf{n}} - y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ has $*$-limit zero, i.e. $*\text{-}\lim_{\mathbf{n}\to {}^*\infty}(x_{\mathbf{n}} - y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} = 0$.
Lemma 1.4.1. If $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x'_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ and $(y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (y'_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$, are two pairs of equivalent Cauchy hypersequences, then:
(a) hypersequence $(x_{\mathbf{n}} + y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$(x_{\mathbf{n}} + y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x'_{\mathbf{n}} + y'_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \tag{1.4.4}$$

(b) hypersequence $(x_{\mathbf{n}} - y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$(x_{\mathbf{n}} - y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x'_{\mathbf{n}} - y'_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \tag{1.4.5}$$

(c) hypersequence $(x_{\mathbf{n}} \times y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$(x_{\mathbf{n}} \times y_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x'_{\mathbf{n}} \times y'_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \tag{1.4.6}$$

(d) hypersequence $\left(\dfrac{x_{\mathbf{n}}}{y_{\mathbf{n}}}\right)_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$\left(\frac{x_{\mathbf{n}}}{y_{\mathbf{n}}}\right)_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} \left(\frac{x'_{\mathbf{n}}}{y'_{\mathbf{n}}}\right)_{\mathbf{n}\in {}^*\mathbb{N}} \tag{1.4.7}$$

iff $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})}[(y_{\mathbf{n}} \not\equiv 0) \wedge (y'_{\mathbf{n}} \not\equiv 0) \wedge (y_{\mathbf{n}} \not\approx_{\mathbf{c}} 0)]$,
(e) hypersequence $(x_{\mathbf{n}} + 0_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ where $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})}[0_{\mathbf{n}} = 0]$ is Cauchy and

$$(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} + (0_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \tag{1.4.8}$$

here $(0_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is a *null hypersequence,*

(f) hypersequence $(x_{\mathbf{n}} \times 1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ where $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})}[1_{\mathbf{n}} = 1]$ is Cauchy and

$$(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}, \tag{1.4.9}$$

here $(1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}}$ is a unit hypersequence.
(g) hypersequence $(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (x_{\mathbf{n}})^{-1}_{\mathbf{n}\in {}^*\mathbb{N}}$ is Cauchy and

$$(x_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \times (x_{\mathbf{n}})^{-1}_{\mathbf{n}\in {}^*\mathbb{N}} \approx_{\mathbf{c}} (1_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}} \tag{1.4.10}$$

iff $\forall \mathbf{n}_{(\mathbf{n}\in {}^*\mathbb{N})}[(x_{\mathbf{n}} \not\equiv 0) \wedge (x_{\mathbf{n}} \not\approx_{\mathbf{c}} (0_{\mathbf{n}})_{\mathbf{n}\in {}^*\mathbb{N}})]$.
Proof. (a) From definition of the Cauchy hypersequences one obtain:

$$\exists \varepsilon_1 \exists \mathbf{m}_{(\mathbf{m}\in {}^*\mathbb{N}_\infty)} \forall \mathbf{k}(\mathbf{k} \geqslant \mathbf{m}) \forall \mathbf{l}(\mathbf{l} \geqslant \mathbf{m})[(|x_{\mathbf{k}} - x_{\mathbf{l}}| < \varepsilon_1) \wedge (|y_{\mathbf{k}} - y_{\mathbf{l}}| < \varepsilon_1)]. \tag{1.4.11}$$

Suppose $\varepsilon_1 = \varepsilon/2$, then from formula above we can to choose $\mathbf{m} = \mathbf{m}(\varepsilon_1)$ such that for all $\mathbf{k} \geqslant \mathbf{m}, \mathbf{l} \geqslant \mathbf{m}$ valid the next inequalities:

$$\begin{gathered}
|(x_{\mathbf{k}} + y_{\mathbf{k}}) - (x_{\mathbf{l}} + y_{\mathbf{l}})| = |(x_{\mathbf{k}} - x_{\mathbf{l}}) + (y_{\mathbf{k}} - y_{\mathbf{l}})| \leqslant \\
\leqslant |(x_{\mathbf{k}} - x_{\mathbf{l}})| + |(y_{\mathbf{k}} - y_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \\
|(x'_{\mathbf{k}} + y'_{\mathbf{k}}) - (x'_{\mathbf{l}} + y'_{\mathbf{l}})| = |(x'_{\mathbf{k}} - x'_{\mathbf{l}}) + (y'_{\mathbf{k}} - y'_{\mathbf{l}})| \leqslant \\
\leqslant |(x'_{\mathbf{k}} - x'_{\mathbf{l}})| + |(y'_{\mathbf{k}} - y'_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{gathered} \tag{1.4.12}$$

From **Definition 1.4.5** and inequalities (1.4.12) we have
the statement (**a**).

(b) Similarly proof the statement (a) we have the next inequalities:

$$\begin{aligned} &|(x_{\mathbf{k}} - y_{\mathbf{k}}) - (x_{\mathbf{l}} - y_{\mathbf{l}})| = |(x_{\mathbf{k}} - x_{\mathbf{l}}) + (y_{\mathbf{k}} - y_{\mathbf{l}})| \leqslant \\ &\leqslant |(x_{\mathbf{k}} - x_{\mathbf{l}})| + |(y_{\mathbf{k}} - y_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \\ &|(x'_{\mathbf{k}} - y'_{\mathbf{k}}) - (x'_{\mathbf{l}} - y'_{\mathbf{l}})| = |(x'_{\mathbf{k}} - x'_{\mathbf{l}}) - (y'_{\mathbf{k}} - y'_{\mathbf{l}})| \leqslant \\ &\leqslant |(x'_{\mathbf{k}} - x'_{\mathbf{l}})| + |(y'_{\mathbf{k}} - y'_{\mathbf{l}})| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \tag{1.4.13}$$

From **Definition 1.4.5** and inequalities (1.4.13) we have the statement (**b**).
(c) $\forall\mathbf{k}(\mathbf{k} \geqslant \mathbf{m})$ and $\forall\mathbf{l}(\mathbf{l} \geqslant \mathbf{m})$ we have the next inequalities:

$$\begin{aligned} &|x_{\mathbf{k}} \cdot y_{\mathbf{k}} - x_{\mathbf{l}} \cdot y_{\mathbf{l}}| = |(x_{\mathbf{k}} \cdot y_{\mathbf{k}} - x_{\mathbf{l}} \cdot y_{\mathbf{k}}) + (x_{\mathbf{l}} \cdot y_{\mathbf{k}} - x_{\mathbf{l}} \cdot y_{\mathbf{l}})| \leqslant \\ &\leqslant |x_{\mathbf{k}} - x_{\mathbf{l}}| \cdot |y_{\mathbf{l}}| + |y_{\mathbf{k}} - y_{\mathbf{l}}| \cdot |x_{\mathbf{l}}|, \\ &|x'_{\mathbf{k}} \cdot y'_{\mathbf{k}} - x'_{\mathbf{l}} \cdot y'_{\mathbf{l}}| = |(x'_{\mathbf{k}} \cdot y'_{\mathbf{k}} - x'_{\mathbf{l}} \cdot y'_{\mathbf{k}}) + (x'_{\mathbf{l}} \cdot y'_{\mathbf{k}} - x'_{\mathbf{l}} \cdot y'_{\mathbf{l}})| \leqslant \\ &\leqslant |x'_{\mathbf{k}} - x'_{\mathbf{l}}| \cdot |y'_{\mathbf{l}}| + |y'_{\mathbf{k}} - y'_{\mathbf{l}}| \cdot |x'_{\mathbf{l}}|, \\ &|x_{\mathbf{k}} \cdot y_{\mathbf{k}} - x'_{\mathbf{k}} \cdot y'_{\mathbf{k}}| = |(x_{\mathbf{k}} \cdot y_{\mathbf{k}} - x_{\mathbf{k}} \cdot y'_{\mathbf{k}}) + (x_{\mathbf{k}} \cdot y'_{\mathbf{k}} - x'_{\mathbf{k}} \cdot y'_{\mathbf{k}})| \leqslant \\ &\leqslant |x_{\mathbf{k}} - x'_{\mathbf{k}}| \cdot |y'_{\mathbf{k}}| + |y_{\mathbf{k}} - y'_{\mathbf{k}}| \cdot |x_{\mathbf{k}}|. \end{aligned} \tag{1.4.14}$$

From definition Cauchy hypersequences one obtain $\exists c \forall \mathbf{k} : |x_{\mathbf{k}}| \leqslant c, |y_{\mathbf{k}}| \leqslant c$, $|x'_{\mathbf{k}}| \leqslant c, |y'_{\mathbf{k}}| \leqslant c$. From **Definition 1.4.5** and inequalities (1.4.14) we have the statement (**c**).
Let $\mathfrak{R}^*_{\mathbf{c}}$ denote the set of the all equivalence classes $\{(x_n)_{\mathbf{n} \in {}^*\mathbb{N}}\} \in \mathfrak{R}^*_{\mathbf{c}}$
Using Lemma 1.4.1. one can define an equivalence relation $\approx_{\mathbf{c}}$,which is compatible with the operations defined above, and the set ${}^*\mathbb{R}_{\mathbf{c}} = \mathfrak{R}^*_{\mathbf{c}} / \approx_{\mathbf{c}}$ is satisfy of the all usual field axioms of the hyperreal numbers.
Lemma 1.4.2. Suppose that $\{(x_{\mathbf{n}})_{\mathbf{n} \in {}^*\mathbb{N}}\}, \{(y_{\mathbf{n}})_{\mathbf{n} \in {}^*\mathbb{N}}\}, \{(z_{\mathbf{n}})_{\mathbf{n} \in {}^*\mathbb{N}}\} \in {}^*\mathbb{R}_{\mathbf{c}}$, then:

$$\text{(a)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} = \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\},$$

$$\text{(b)}\quad [\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}] + \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} =$$

$$= \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + [\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}],$$

$$\text{(c)}\quad \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times [\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}] =$$

$$= \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\},$$

$$\text{(d)}\quad [\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}] \times \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} =$$

$$= \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times [\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}],$$

$$\text{(e)}\quad [\{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}] \times \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} = \qquad (1.4.2)$$

$$= \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times [\{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}],$$

$$\text{(f)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} + \{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} = \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\},$$

$$\text{(g)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \bullet \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}^{-1} = \{(1_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\},$$

$$\text{(i)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} = \{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\},$$

$$\text{(j)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(1_{\mathbf{n}})\} = \{(x_{\mathbf{n}})\},$$

$$\text{(k)}\quad \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} < \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \wedge \{(0_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} < \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \Rightarrow$$

$$\Rightarrow \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(x_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} < \{(z_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\} \times \{(y_{\mathbf{n}})_{\mathbf{n}\in{}^*\mathbb{N}}\}.$$

Proof. Statements (a),(b),(c),(d),(e),(f),(g),(i) (j) and (k) is evidently from **Lemma.1.4.1** and definition of the equivalence relation $\approx_{\mathbf{c}}$.

II.Euler’s proofs by using non-archimedean analysis on the pseudo-ring ${}^*\mathbb{R}_{\mathbf{d}}$ revisited.

II.1.Euler’s original proof of the Goldbach-Euler Theorem revisited.

Euler’s paper of 1737 “Variae Observationes Circa Series Infinitas,” is Euler’s first paper that closely follows the modern Theorem-Proof format. There are no definitions in the paper, or it would probably follow the Definition-Theorem-Proof format. After an introductory paragraph in which Euler tells part of the story of the problem, Euler gives us a theorem and a "proof". Euler’s "proof" begins with an 18-th century step that treats *infinity as a number.* Such steps became unpopular among rigorous mathematicians about a hundred years later. He takes x to be the "sum" of the harmonic series:

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots + \frac{1}{n} + \ldots \qquad (2.4.0)$$

The Euler’s original proofs is one of those examples of *completely misuse* of divergent series to obtain *completely correct results* so frequent during the seventeenth and eighteenth centuries.The acceptance of Euler’s proofs seems to lie in the fact that,at the time,Euler (and most of his contemporaries) actually manipulated a model of real numbers which included infinitely large and infinitely small numbers. A model that much later Bolzano would try to build on solid grounds and that today is called “nonstandard” after A.Robinson definitely established it in the 1960’s [1],[2],[3],[4],[5]. This last approach, though, is completely in tune with Euler’s proof [7] Nevertheless using ideas borrowed from modern nonstandard analysis the same reconstruction rigorous by modern Robinsonian standards *is not found.* In particular "nonstandard" proof proposed in paper [7] is not completely nonstandard becourse authors use the solution Catalan’s conjecture [9]

Unfortunately completely correct proofs of the Goldbach-Euler Theorem, was presented many authors as rational reconstruction only in terms which could be considered rigorous by modern Weierstrassian standards.

In this last section we show how, a few simple ideas from non-archimedean analysis on the pseudoring ${}^*\mathbb{R}_{\mathbf{d}}$, vindicate Euler’s work.

Theorem 1.(Euler [6],[8]) Consider the following series, infinitely continued,

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\ldots \tag{2.4.1}$$

whose denominators, increased by one, are all the numbers which are powers of the integers, either squares or any other higher degree.Thus each term may be expressed by the formula

$$\frac{1}{m^n-1} \tag{2.4.2}$$

where m and n are integers greater than one. The sum of this series is 1.
Proof. Let

$$\mathbf{h}=\mathbf{cl}\left(1+\frac{1}{2},1+\frac{1}{2}+\frac{1}{3},1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4},1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5},\ldots\right) \tag{2.4.3}$$

from Eq.(2.4.3), as we have

$$1=\mathbf{cl}\left(\frac{1}{2},\frac{1}{2}+\frac{1}{4},\frac{1}{2}+\frac{1}{4}+\frac{1}{8},\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16},\right.$$

$$\left.\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32},\ldots,\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{2^i},\ldots\right)-\varepsilon_1, \tag{2.4.4}$$

$$\varepsilon_1\approx 0,$$

$$\varepsilon_1=\mathbf{cl}\left(\frac{1}{2^M},\frac{1}{2^{M+1}},\ldots,\frac{1}{2^{M+i}},\ldots\right)$$

we obtain

$$\mathbf{h}-1=\mathbf{cl}\left(1,1+\frac{1}{3},1+\frac{1}{3}+\frac{1}{5},1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6},1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7},\right.$$

$$\left.1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10},\ldots\right)-\varepsilon_1 \tag{1.4.5}$$

from Eq.(2.4.5), as we have

$$\frac{1}{2} = \mathbf{cl}\left(\frac{1}{3}, \frac{1}{3} + \frac{1}{9}, \frac{1}{3} + \frac{1}{9} + \frac{1}{27}, \ldots, \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{3^i}, \ldots\right) - \varepsilon_2,$$

$$\varepsilon_2 \approx 0, \tag{2.4.6}$$

$$\varepsilon_2 = \frac{1}{2}\mathbf{cl}\left(\frac{1}{3^M}, \frac{1}{3^{M+1}}, \ldots, \frac{1}{3^{M+i}}, \ldots\right)$$

we obtain

$$\mathbf{h} - \left(1 + \frac{1}{2}\right) = \mathbf{cl}\left(1, 1 + \frac{1}{5}, 1 + \frac{1}{5} + \frac{1}{6}, 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7},\right.$$
$$\left. 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11}, \ldots\right) - (\varepsilon_1 + \varepsilon_2). \tag{2.4.7}$$

from Eq.(2.4.7), as we have

$$\frac{1}{4} = \mathbf{cl}\left(\frac{1}{5}, \frac{1}{5} + \frac{1}{25}, \frac{1}{5} + \frac{1}{25} + \frac{1}{125}, \ldots, \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots + \frac{1}{5^i}, \ldots\right) - \varepsilon_3,$$

$$\varepsilon_3 \approx 0, \tag{2.4.8}$$

$$\varepsilon_3 = \frac{1}{4}\mathbf{cl}\left(\frac{1}{5^M}, \frac{1}{5^{M+1}}, \ldots, \frac{1}{5^{M+i}}, \ldots\right)$$

we obtain

$$\mathbf{h} - \left(1 + \frac{1}{2} + \frac{1}{4}\right) =$$
$$\mathbf{cl}\left(1 + \frac{1}{6}, 1 + \frac{1}{6} + \frac{1}{7}, 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10}, \ldots\right) - (\varepsilon_1 + \varepsilon_2 + \varepsilon_3). \tag{2.4.9}$$

Proceeding similarly, i.e. deleting all the all terms that remain,we get

$$\mathbf{h}-[\Im_n] =$$

$$= \mathbf{cl}\left(1+\frac{1}{5},\ldots,1+\frac{1}{m(n')},1+\frac{1}{m(n')}+\ldots,\ldots\right)- \quad (2.4.10)$$

$$-\left(\#Ext-\sum\nolimits_{n\in\mathbb{N}}\varepsilon_n\right),$$

$$m > n'(n)$$

where

$$\Im_n = \left(1,\frac{1}{2},\frac{1}{4},\frac{1}{5},\ldots,\frac{1}{n'(n)},\ldots\right) \quad (2.4.11)$$

whose denominators, increased by one, are all the numbers which are not powers. From Eq.(2.4.10) we obtain

$$\mathbf{h}-[\Im_n] =$$

$$1+\left(\#Ext-\sum\nolimits_{n\in\mathbb{N}}\varepsilon_n\right) = \quad (2.4.12)$$
$$= 1+\epsilon$$

$$\epsilon = \#Ext-\sum\nolimits_{n\in\mathbb{N}}\varepsilon_n \approx 0.$$

Thus we obtain

$$\mathbf{h}-[\Im_n] = 1+\epsilon, \quad (2.4.13)$$

Substitution Eq.(2.4.3) into Eq.(2.4.13) gives

$$1+\epsilon = \frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\ldots \quad (2.4.14)$$

$$\epsilon \approx 0$$

series whose denominators, increased by one, are all the powers of the integers and whose sum is one.

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