A consistent descriptive logographic onomatology of algebraic systems

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Abstract

There are two presently common onamastic (onomatological) methods of logographically naming and thus concisely describing an algebraic system; both methods are often used simultaneously. According to one method, an algebraic system is equivocally denoted by an atomic logographic symbol that originally denotes a certain underlying set of elements, which is regarded as the principal one, while all other objects of the algebraic system, properly named, are kept in mind and are regarded as implicit properties of that set or of its separate elements. That is to say, according to this method, an algebraic system is its principal underlying set of elements together with all its properties, which are implied and are not mentioned explicitly. According to the other method, an algebraic system is regarded as an ordered multiple, whose coordinates properly denote the defining objects of the algebraic system, and consequently the ordered multiple name is equivocally used as a proper name of the algebraic system. Thus, in this case, the togetherness of all constituents of the algebraic system is expressed by the pertinent ordered multiple name in terms of its coordinate names. In my recent article Iosilevskii [2016b]¸ I have demonstrated that both above onomastic methods are inconsistent. Therefore, in that article and also in my earlier article Iosilevskii [2015], I suggested and used another onomastic method of logographically naming the pertinent algebraic systems, namely that employing, as a name of an algebraic system, a complex logographic name the union of all *explicit* constituent sets of the system, namely, *the underlying sets of elements, the surjective binary composition functions*, and *the bijective singulary inversion functions*; a function is a set (class) of ordered pairs. In the present article, the latter onomastic method is substantiated and generalized in two respects. First, the set of *explicit* constituent sets of an algebraic system is now extended to include the *injective choice*, or *selection*, *functions of all additive and multiplicative identity elements* of the algebraic

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system, belonging to its underlying sets, so that all those elements are now mentioned by the logographic name of the system. A general definition of an algebraic system is elaborated in such a way so as to make the new onomastic method universally applicable to any algebraic system.

1. Introduction

In my recent articles Iosilevskii [2016a and 2016b], I have suggested contextually and used the new convenient consistent method of forming proper logographic names of the pertinent algebraic systems, according to which any given algebraic system is properly distinguished by a complex logographic name of the union of all sets (regular, or small, classes) of objects, which are *explicitly* dealt with by the algebraic system and which are therefore regarded as the interrelated constituent parts of the latter. In this case, the set of interrelated constituent sets of an algebraic system has been supposed to include all pertinent sets of the following three kinds (classes): *the underlying sets of elements, the surjective binary composition functions*, and *the bijective singulary inversion functions*; a function is a set (class) of ordered pairs. Names of the pertinent *distinguished elements* (if exist) of an algebraic system, i.e. names of the pertinent *additive* or *multiplicative identities*, – such names, e.g., as ‘0’, ‘1’, or ‘$\hat{0}$’, – were not explicitly included in a complex logographic name of the algebraic system. If exists, a distinguished element of an algebraic system belongs to a certain one of its underlying sets of elements, and therefore it has been implied by its complex logographic name.

In this exposition, the above logographic onomatology of algebraic systems is modified so as the set of constituent sets of an algebraic system is supposed to include the *injective choice*, or *selection*, *function of any given identity element* of the algebraic system. Accordingly, the logographic name of the union of all *explicit* constituent sets of an algebraic system, which serves as a name of the algebraic system, includes now the names of injective choice (selection) functions of all existing distinguished elements of the system. Consequently, the latter elements are explicitly mentioned by the above modified union name. Also, the new onomastic method is supplemented by a general definition, which makes it to be universally applicable to any algebraic system.

In this exposition, the main nomenclature (logographic notation and wordy terminology) of the article Iosilevskii [2016b] is retained. Nevertheless, for the reader's convenience, I shall start with recalling some relevant fundamental notions of class and set theories in that notation.

1) A set is a class, but a class is not necessarily a set. I call a class "*a regular class*" if it is a set and "*an irregular class*" if it is not a set. In the contemporary literature on logic and mathematics, an irregular class is called a *proper class*, whereas a regular class, i.e. a set, is sometimes called a *small class* (see, e.g., Fraenkel et al [1973, p. 128, DEFINITION VII] for the former term or the article **class** in Wikipedia for both terms). The difference between an irregular class and a set (regular class) is discussed in detail in Iosilevskii [2016a, subsection I.9.3.2]). For instance, taxons (taxa, taxonomic classes) of any biological taxonomy of bionts (BTB) are *irregular*, or *proper*, *classes*, i.e. *classes* that are *not sets*. Particularly, *the species* (*specific class*) *of men*, that is formally called '*Homo sapiens*' and informally "*man*", exists as an irregular class but the set of all men does not exist in the sense that the expression "the set of all men" has no denotatum. By contrasts, in mathematics, a well-defined class of numbers as the class of natural (natural integer) numbers, the class of rational numbers, the class of real numbers, or the class of complex numbers is a regular class, i.e. a set.

2) $\overset{\rightarrow}{=}$, $\overset{\leftarrow}{=}$, and $\overset{\leftrightarrow}{=}$ are *equality signs by definition*, a *rightward one*, *a rightward one*, and a *two-sided one* respectively, which are rigorously defined in Iosilevskii [2015, 2016a, and 2016b].

3) 'ω_0' *denotes*, i.e. ω_0 *is*, the set of all *natural numbers* from 0 to infinity. Given $n \in \omega_0$, 'ω_1', 'ω_2', etc *denote* the sets of natural numbers from 1, 2, etc respectively to infinity. Given $m \in \omega_0$, given $n \in \omega_m$, '$\omega_{m,n}$' *denotes* the set of natural numbers from a given number *m* to another given number *n* subject to $n \geq m$.

4) A symbol of the form '$\{\boldsymbol{x}|\boldsymbol{P}(\boldsymbol{x})\}$', called a *class-builder* (or particularly *set-builder*), which is designed to convert a given *relation* (*condition*) $\boldsymbol{P}(\boldsymbol{x})$ into a certain *constant* or *variable class-valued* (or correspondingly) *term* ('$\boldsymbol{P}$' and '$\boldsymbol{x}$' are atomic placeholders having the appropriate ranges).

5) The *unordered pair of two different* (*distinct*) *objects x and y* is the set $\{x, y\}$ of those objects, such that

$$\{x, y\} \overset{\rightarrow}{=} \{z | z = x \text{ or } z = y\}.$$

subject to $x \neq y$ (cf. Halmos [1960, p. 10]). If $x = y$ then the set $\{x\}$ such that $\{x\} = \{x, x\}$, having *x* as its only member, is called *the singleton of x* or less explicitly (more generally) *a singleton.*

6) The *ordered pair* (x, y) *of two objects x and y, different or not*, – particularly that of two different or same elements *x* and *y* of two different or same sets (or in general classes) *X* and *Y* respectively. – is conventionally defined as:

$$(x, y) \stackrel{\rightarrow}{=} \{\{x\}, \{x, y\}\}$$

(see, e.g., Halmos [1960, pp. 22–25]). Therefore, by *Axiom of extension* (*ibid.* p. 2), for any four objects *x*, *y*, *x*', and *y*,.

$$(x, y) = (x', y') \text{ if and only if } x=x' \text{ and } y=y'.$$

The set $X \times Y$, defined as:

$$X \times Y \stackrel{\rightarrow}{=} \{z | z = (x, y) \text{ for some } x \in X \text{ and for some } y \in Y\},$$

is called the *Cartesian*, or *direct*, *product of X* and *Y* (*ibid.* p. 24). Here and throughout this exposition, $\stackrel{\rightarrow}{=}$ is the rightward sign of equality by definition, which, along with $\stackrel{\leftarrow}{=}$ and $\stackrel{\leftrightarrow}{=}$, is rigorously defined, e.g., in Iosilevskii [2015, 2016a, and 2016b].

7) Given $n \in \omega_2$, an *ordered n-tuple* of objects $x_1, x_2, ..., x_{n-1}, x_n$ is defined as a *repeated*, (*n*–1)-*fold ordered pair* thus:

$$\bar{x}_{[1,n]} \stackrel{\rightarrow}{=} (x_i)_{i \in \omega_{1,n}} \stackrel{\rightarrow}{=} (x_1, x_2, ..., x_{n-1}, x_n)$$
$$\stackrel{\rightarrow}{=} (\bar{x}_{[1,n-1]}, x_n) = (\underbrace{((..((}_{n-1} x_1, x_2), x_3), ..., x_{n-1}), x_n).$$

More specifically, an ordered *n*-tuple that is defined by the above formula is called *the left-associated repeated* (or *reiterative*) (*n*–1)-*fold* (or (*n*–1)-*ary*) *ordered pair of* x_1, x_2, …, x_n *in that order*. Accordingly, for any 2*n* objects x_1, x_2,…, x_n, x'_1, x'_2, …, x'_n,

$$(x_1, x_2, ..., x_n) = (x'_1, x'_2, ..., x'_n) \text{ if and only if } x_1 = x'_1, x_2 = x'_2, ..., x_n = x'_n.$$

8) An ordered *n*-tuple with any $n \in \omega_2$ is indiscriminately called an *ordered multiple.* It is worthy of recalling that, in contrast to an ordered multiple, an *ordered set* is a set that serves as a *domain of definition of the liner order relation* (*predicate*) ≤. An ordered irregular class does not exist.

9) It is useful for making some general statements to introduce a *one-component univalent holor* – a conceptual object, which is denoted by ‘ $\bar{x}_{[1,1]}$ ’ or ‘ (x_1) ’ and which can therefore be also called an *ordered one-tuple*, or *ordered single*, the understanding being that such an object *is distinct from a scalar* (*nilvalent holor*) and that it *can have a scalar as its only component*. Therefore, without loss of generality, $\bar{x}_{[1,1]}$ or (x_1) can be identified with the singleton $\{x_1\}$ – the set having x_1 as its only member (element), so that

$$\bar{x}_{[1,1]} \doteqdot (x_1) \doteqdot \{x_1\}.$$

At the same time, a set of n elements with $n \in \omega_2$ can alternatively be called an *unordered n-tuple*. Therefore, (x_1) as defined above,) can be regarded as *an ordered one-tuple and as an unordered one-tuple simultaneously*. Thus, for any $n \in \omega_1 \doteqdot \{1,2,...\}$, an ordered *n*-tuple, i.e. an *n*-component univalent holor, is a nonempty set and is *not a nonempty individual*. A definition of the term “holor” can be found, e.g., in Moon and Spencer [1965, pp. 1, 14]), and also Iosilevskii [2016b, sub-subsection 2.3.1].

10) If x_1 and x_2 are real numbers then the symbol ‘ (x_1, x_2) ’ is ambiguous, for it may stand either for the ordered pair of those numbers in that order or for the open interval (x_1, x_2). Therefore, in denoting ordered pairs and ordered multiples, I use round brackets and angle brackets interchangeably, while preference is given to the latter in all doubtful cases.

11) In accordance with Definition 2.12 of Iosilevskii [2016b], given $m \in \omega_2$, let ξ_1, ..., ξ_m be any m objects to which a binary operation $*$, denoted by the placeholder ‘ $*$ ’, applies repeatedly (iteratively) *m*–1 times in the successive order starting from ξ_1 and ξ_2. Then

$$\ast(\xi_1, \xi_2, ..., \xi_{m-1}, \xi_m) \doteqdot \mathop{\ast}_{i=1}^{m} \xi_i \doteqdot \xi_1 * \xi_2 * ... * \xi_{m-1} * \xi_m \doteqdot \left[\mathop{\ast}_{i=1}^{m-1} \xi_i\right] * \xi_m$$
$$\doteqdot \underbrace{[[...[}_{m-2}[\xi_1 * \xi_2] * \xi_3] * ... * \xi_{m-2}] * \xi_{m-1}] * \xi_m$$

and in general

$$\ast(\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{m-1}}, \xi_{j_m}) \doteqdot \mathop{\ast}_{i=1}^{m} \xi_{j_i} \doteqdot \xi_{j_1} * \xi_{j_2} * ... * \xi_{j_{m-1}} * \xi_{j_m} \doteqdot \left[\mathop{\ast}_{i=1}^{m-1} \xi_{j_i}\right] * \xi_{j_m}$$
$$\doteqdot \underbrace{[[...[}_{m-2}[\xi_{j_1} * \xi_{j_2}] * \xi_{j_3}] * ... * \xi_{j_{m-2}}] * \xi_{j_{m-1}}] * \xi_{j_m},$$

where the sequence $(j_1, j_2,..., j_{m-1}, j_m)$ is any permutation of the sequence $(1,2,...,m-1,m)$; the symbols '$\mathop{\ast}\limits_{i=1}^{m}$' and '$\ast_{i=1}^{i=m}$', e.g., can be used interchangeably. In this case, $\ast$' and '$\ast$' is a pair of *proportional* (*homolographic*) *placeholders*, which should be replaced by a pair of proportional tokens of the respective sizes of any desired binary functional constant as '×','+', '·', '×', '∩', '∪', etc and also as '×′', '+′', $\hat{+}$, '$\hat{+}'$', '+′', '·′', '$\hat{\cdot}$', '$\hat{\cdot}'$', etc. Thus, if an initial binary functional constant **'$\ast$'** is furnished with some labels then **'$\ast$'** should be furnished with the same labels. Particularly, in accordance with Comment 2.3 of Iosilevskii [2016b], if the symbol '+', e.g., is provided with some labels (as one or more primes, a caret, an overbar, a tilde, etc) then the symbol '$+$' is provided with the same labels. It is therefore understood that if the convention of equivocal use of the sign '+' instead of each one of the plus signs such as '+′', $\hat{+}$, '$\hat{+}'$', etc is adopted, tacitly or explicitly, then the sign '$+$' should be used instead of any one of the signs '$+'$', '$\hat{+}$', '$\hat{+}'$', etc. In this case, the denotatum of the operator '$+$' depends on the type of its summand (operatum). It is also understood that if the conventional symbol '$\sum$' is employed instead of '$+$' then the symbols '$\hat{\sum}$', '$\sum'$', and '$\hat{\sum}'$ should be employed instead of '$\hat{+}$', '$+'$', '$\hat{+}'$ respectively; and similarly with '$\prod$' and '·' in place of '$\sum$' and '$+$'. Thus, the conventional symbol '$\sum$' or '$\prod$' is equivocal, so that for avoidance of confusion it should be provided with additional labels to connote the functional constant, which denotes the binary addition or multiplication operation, underlying the sequence of repeated binary addition or multiplication operations equivocally denoted by '$\sum$' or '$\prod$' respectively. Under the above condition, the symbols '$+$' and '$\sum$' or '·' and '$\prod$' can be used interchangeably.

12) A binary operation $\ast$ is said to be:

a) *associative* if and only if for any two ordered triples (repeated ordered pairs) of objects $((\xi \ast \eta) \ast \zeta)$ and $(\xi \ast (\eta \ast \zeta))$ in the domain of its definition it satisfies the *basic law of associativity*: $((\xi \ast \eta) \ast \zeta) = (\xi \ast (\eta \ast \zeta))$;

a) *commutative* if and only if for any two ordered pairs of objects $(\xi * \eta)$ and $(\eta * \xi)$ in the domain of its definition it satisfies the *basic law of commutativity*: $(\xi * \eta) = (\eta * \xi)$.

It has been rigorously proved in Iosilevskii [2016c, Essay 9] that a binary operation $*$ satisfies the *generalized associativity law* for any number of appropriate objects if it is associative and it has also been rigorously proved that $*$ satisfies the *generalized commutativity law* for any number of appropriate objects if it is associative and commutative. In the latter case, it particularly follows from the above item that

$$\ast(\xi_{j_1}, \xi_{j_2}, ..., \xi_{j_{m-1}}, \xi_{j_m}) = \mathop{\ast}_{i=1}^{m} \xi_{j_i} = \xi_{j_1} * \xi_{j_2} * ... * \xi_{j_{m-1}} * \xi_{j_m}$$
$$= \xi_1 * \xi_2 + ... * \xi_{m-1} * \xi_m = \mathop{\ast}_{i=1}^{m} \xi_i = \ast(\xi_1, \xi_2, ..., \xi_{m-1}, \xi_m).$$

13) From the above items 6 and 7, it particularly follows that, given $n \in \omega_2$, given *n* classes X_1, $X_2, \ldots,$ X_n, the class of ordered *n*-tuples defined as:

$$\times(X_1, X_2, ..., X_{n-1}, X_n) \risingdotseq \mathop{\times}_{i=1}^{n} X_i \risingdotseq \times_{i \in \omega_{1,n}} X_i \risingdotseq X_1 \times X_2 \times ... \times X_{n-1} \times X_n$$
$$\risingdotseq \left[\mathop{\times}_{i=1}^{n-1} X_i\right] \times X_n \risingdotseq \underbrace{[[...[}_{n-2}[X_1 \times X_2] \times X_3] \times ...] \times X_{n-1}] \times X_n$$
$$\risingdotseq \{(x_1, x_2, ..., x_{n-1}, x_n) | x_1 \in X_1, x_2 \in X_2, ..., x_{n-1} \in X_{n-1}, x_n \in X_n\}$$

is called *the left-associated repeated* (or *reiterative*) (*n*–1)-*fold* (or (*n*–1)-*ary*) *Cartesian*, or *direct*, *product of* X_1, $X_2, \ldots,$ X_n *in that order*. The operation × is *neither associative nor commutative*.

14) Given $n \in \omega_1$, given a set *X*, if $X_1 = X_2 = ... = X_n = X$, the set of ordered *n*-tuples defined as:

$$X^{n\times} \risingdotseq \underbrace{X \times X \times ... \times X \times X}_{n \text{ times } X}$$
$$\risingdotseq X^{(n-1)\times} \times X \risingdotseq \underbrace{[[...[}_{n-2}[X \times X] \times X] \times ...] \times X] \times X$$
$$\risingdotseq \{(x_1, x_2, ..., x_{n-1}, x_n) | x_1 \in X, x_2 \in X, ..., x_{n-1} \in X, x_n \in X\},$$

i.e. *the left-associated repeated* (or *reiterative*) (*n*–1)-*fold* (or (*n*–1)-*ary*) *direct* (or *Cartesian*) *product of X by itself*, is called *the left-associated n*th *direct* (or *Cartesian*) *power of X*, the understanding being that

$$X^{1\times} \doteqdot \{(x_1)|x_1 \in X\} = \{\{x_1\}|x_1 \in X\} \neq X \ .$$

15) The binary operation of union $\cup$ for sets or generally for classes is *both associative and commutative* (see, e.g., Halmos [1960, p. 13] and also Iosilevskii [2016a, Appendix 5]), so that

$$\bigcup(X_{j_1}, X_{j_2}, ..., X_{j_{n-1}}, X_{j_n}) \doteqdot \bigcup_{i=1}^{n} X_{j_i} \doteqdot X_{j_1} \cup X_{j_2} \cup ... \cup X_{j_{n-1}} \cup X_{j_n} \doteqdot \left[\bigcup_{i=1}^{n-1} X_{j_i}\right] \cup X_{j_n}$$

$$\doteqdot \underbrace{[[...[}_{n-2}[X_{j_1} \cup X_{j_2}] \cup X_{j_3}] \cup ... \cup X_{j_{n-2}}] \cup X_{j_{n-1}}] \cup X_{j_n}$$

$$= \underbrace{[[...[}_{n-2}[X_1 \cup X_2] \cup X_3] \cup ... \cup X_{n-2}] \cup X_{n-1}] \cup X_n$$

$$\doteqdot \left[\bigcup_{i=1}^{n-1} X_i\right] \cup X_n \doteqdot X_1 \cup X_2 \cup ... \cup X_{n-1} \cup X_n \doteqdot \bigcup_{i=1}^{n} X_i \doteqdot \bigcup(X_1, X_2, ..., X_{n-1}, X_n),$$

where the sequence $(j_1, j_2, ..., j_{n-1}, j_n)$ is any permutation of the sequence $(1,2,...,n-1,n)$. In this case, the arrangement of *n*–2 pairs of square brakets in either one of the two final (middle) definientia can be arbitrary, and not necessarily with the association to the left.

16) Given $n \in \omega_2$, an *unordered n-tuple of n different objects* $x_1, x_2, ..., x_{n-1}, x_n$ is defined thus:

$$\{x_i\}_{i \in \omega_{1,n}} \doteqdot \{x_1, x_2, ..., x_{n-1}, x_n\} \doteqdot \{x_1, x_2, ..., x_{n-1}, x_{n-1}\} \cup \{x_n\}$$
$$\doteqdot \{x_1\} \cup \{x_2\} \cup ... \cup \{x_{n-1}\} \cup \{x_n\}$$

(see, e.g., Halmos, [1960, p. 14]). If $x_{n-1} = x_n$ (e.g.) then $\{x_1, x_2, ..., x_{n-1}, x_n\}$ turns into $\{x_1, x_2, ..., x_{n-1}\}$.

17) In the Clairaut-Euler placeholders '$f(x_1, x_2)$' and '$f(x_1, x_2, ..., x_{n-1}, x_n)$' of functional forms, '$(x_1, x_2)$' is a placeholder for an ordered pair, whereas '$(x_1, x_2, ..., x_{n-1}, x_n)$' is a placeholder for an ordered *n*-tuple of elements.

In order to justify the final general definition of the latest, more explicit version of the new subsistent *logographic onamastics* (*onomatology*) of algebraic systems, I shall, in the next three sections, develop it for the most fundamental *specific* classes of algebraic systems, namely for an *abstract group*, for an *abstract field*, along with an *abstract commutative ring* and an *abstract integral domain*, being its two successive predecessors, and also for an *abstract affine additive group*.

2. An abstract group

Definition 2.1: ***An abstract group.*** 1) Let G be a set of a finite or infinite number of elements and let each of the letters 'x', 'y', 'z', alone or furnished with any one of the subscripts $_1$, $_2$, etc or with any number of primes or with both, be a variable having G as its range. A group $\boldsymbol{G}$ is a set G *together* with a *surjective binary composition function* $\phi: G \times G \to G$, an *injective choice*, or *selection*, *function of an identity element e*, $\varepsilon: \{G\} \to \{e\} \subset G$, and a *bijective singulary inversion function* $\alpha: G \to G$, which satisfy the following axioms, called the *Group Axioms* or briefly *GA's*.

GA1: The closure law. For each $(x, y) \in G \times G$ there is exactly one $z \in G$ such that $z = \phi(x, y)$.

GA2:The associative law. For each $((x, y), z) \in [G \times G] \times G$,

$$\phi(x, \phi(y, z)) = \phi(\phi(x, y), z). \tag{2.1}$$

GA3: The identity law. There exists a unique element $e = \varepsilon(G) \in G$, which is called *the identity element of G*, such that for each $x \in G$,

$$\phi(e, x) = \phi(x, e) = x . \tag{2.2}$$

The function ε is called the *choice*, or *selection, function of e in G*, because $\varepsilon \doteqdot \{(G, e)\}$, i.e. ε is by definition the *singleton of the ordered pair* (G, e). Accordingly, 'ε' is, mnemonically, the first letter of either one of the Greek nouns "εκλογή" \eklogí\, meaning *choice* or *selection*, or "επιλογή" \epilogí\, meaning *selection.*

GA4: The inverse law. For each $x \in G$, there exists exactly one element $\alpha(x) \in G$, which is called *the inverse*, or *reciprocal*, *of* x, such that

$$\phi(\alpha(x), x) = \phi(x, \alpha(x)) = e ; \tag{2.3}$$

mnemonically, 'α' is the first letter of the Greek noun "αντιστροφή" \antistrofí\ meaning *inversion.*

2) The functions ϕ and α are respectively called *the composition function* (or *operation*) *of* $\boldsymbol{G}$ and *the inversion function of* $\boldsymbol{G}$. Depending on my alternating mental attitude towards the logographs '$\boldsymbol{G}$' and 'G', «*togetherness*» *of G with* ϕ, ε, *and* α is understood in one of the following two ways:

a) $\boldsymbol{G}$ equals G, i.e. $\boldsymbol{G} \stackrel{\rightarrow}{=} G$, subject to GA1–GA4 and hence also subject to all theorems that can be proved from GA1–GA4.

b) $\boldsymbol{G} \stackrel{\rightarrow}{=} \bigcup(G, \phi, \varepsilon, \alpha) \stackrel{\rightarrow}{=} G \cup [\phi \cup \varepsilon \cup \alpha]$ subject to the same properties.

Here and throughout this exposition, $\stackrel{\rightarrow}{=}$ is the rightward sign of equality by definition, which, along with $\stackrel{\leftarrow}{=}$ and $\stackrel{\leftrightarrow}{=}$, is rigorously defined in Iosilevskii [2015 and 2016b]. In the case a, ‘G’ is called (a') a *group* interchangeably with ‘$\boldsymbol{G}$’ and it is also equivocally called (a") *the underlying set [of elements] of the group* **G** once it is mentally freed of the *connotative* properties GA1–GA4 that are assigned to it and is hence freed of the functions ϕ, ε, and α defined on it. In the case b, the latter mental attitude is expressed explicitly (denotatively). Under either definition a or b, $x \in \boldsymbol{G}$ if $x \in G$.•

Comment 2.1. 1) A mathematical structure, e.g. an ordered set or an algebraic system, is always defined as a set *together* with some relations (particularly functions) and perhaps *together* with some other given sets. Conventionally, this togetherness is formally expressed by defining the mathematical structure as an *ordered multiple*, whose coordinates are the pertinent sets and relations (see, e.g., MacLane and Birkhoff [1967, pp. 61, 63, 118, etc]). In accordance with this tacit convention, a group $\boldsymbol{G}$ should be defined, e.g., thus:

$$\boldsymbol{G} \stackrel{\rightarrow}{=} (G, \phi, \varepsilon, \alpha) \stackrel{\rightarrow}{=} (((G, \phi), \varepsilon), \alpha). \tag{2.4}$$

This definition is however inconsistent – like the similar definition of any other algebraic system, as was demonstrated in Iosilevskii [2016b]. For instance, under definition (2.4), $\neg[x \in \boldsymbol{G}]$, which is an absurd. Therefore, I do not adopt such definitions.

2) In accordance with Definition 2.1,

$$\begin{gathered} D_{\mathrm{df}}(\phi) = G \times G,\, D_{\mathrm{df}}(\varepsilon) = \{G\},\, D_{\mathrm{df}}(\alpha) = G, \\ D_{\mathrm{v}}(\phi) = D_{\mathrm{v}}(\alpha) = G,\, D_{\mathrm{v}}(\varepsilon) = \{e\} \subseteq G. \end{gathered} \tag{2.5}$$

In general, $D_{\mathrm{df}}(f)$ is the *domain of definition* of the function f and $D_{\mathrm{v}}(f)$ is the *domain of variation*, or domain of values, of the function f, the understanding being that $D_{\mathrm{v}}(f) \subseteq D_{\mathrm{a}}(f)$, where $D_{\mathrm{a}}(f)$ is the *domain of arrival* of the function f. Also, in an axiomatic set theory, a binary relation in intension R from a set, X, to (onto or into) a set, Y, – a functional relation, i.e. a function f or not (e.g. a partial order relation $\leq$ in X if exists), – is conventionally treated as a certain *set of ordered pairs* (x, y) *of an element x of X* i.e. as a certain *subset of the direct product*

$X\times Y$. Hence, $R\subseteq X\times Y$, $D_{\mathrm{df}}(R)\subseteq X$, $D_{\mathrm{v}}(R)\subseteq Y$, $D_{\mathrm{a}}(R)=Y$, and similarly with '*f*' in place of '*R*'. Particularly, it follows from (2.5) that

$$D_{\mathrm{df}}(\phi)\cap D_{\mathrm{df}}(\alpha)=D_{\mathrm{df}}(\phi)\cap D_{\mathrm{df}}(\varepsilon)=D_{\mathrm{df}}(\varepsilon)\cap D_{\mathrm{df}}(\alpha)=\varnothing\,. \tag{2.6}$$

Also, in accordance with the above said, the set σ defined as

$$\sigma \doteqdot \phi\cup\varepsilon\cup\alpha \doteqdot [\phi\cup\varepsilon]\cup\alpha=\phi\cup[\varepsilon\cup\alpha] \tag{2.7}$$

is a function such that

$$D_{\mathrm{df}}(\sigma)=[G\times G]\cup\{G\}\cup G,\, D_{\mathrm{a}}(\sigma)=G\,. \tag{2.8}$$

Accordingly, ϕ, ε, and α are restrictions of σ such that

$$\begin{gathered}\phi(\xi)=\sigma(\xi)\text{ for each }\xi\in G\times G,\, e=\varepsilon(\xi)=\sigma(\xi)\text{ for }\xi=G,\\ \alpha(\xi)=\sigma(\xi)\text{ for each }\xi\in G\times G.\end{gathered} \tag{2.9}$$

By (2.7)–(2.9), Definition 2.1 can be restated as the following one.•

Definition 2.1a: A group ***G*** is a set *G* *together* with a function σ, which satisfies the following *Group Axioms* (*GA's*.).

GA1a: The closure law. For each $(x,y)\in G\times G$, there is exactly one $z\in G$ such that $z=\sigma(x,y)$.

*GA2a: The associative law***.** For each $((x,y),z)\in[G\times G]\times G$,

$$\sigma(x,\sigma(y,z))=\sigma(\sigma(x,y),z). \tag{2.1a}$$

GA3a: The identity law. There exists a unique element $e=\sigma(G)\in G$, which is called *the identity element of G*, such that for each $x\in G$,

$$\sigma(e,x)=\sigma(x,e)=x\,. \tag{2.2a}$$

GA4a: The inverse law. For each $x\in G$, there is exactly one element $\sigma(x)\in G$, which is called *the inverse*, or *reciprocal*, *of* x, such that

$$\sigma(\sigma(x),x)=\sigma(x,\sigma(x))=e\,. \tag{2.3a}$$

2) The function σ is called *the unified*, or *synthesized, composition and inversion function* (or *operation*) *of* ***G***. Just as in Definition 2.1(2), «*togetherness*» *of G with* σ is understood in one of the following two ways depending on my mental attitude towards the logographs '***G***' and '*G*':

a) ***G*** equals *G*, i.e. ***G***$\doteqdot$*G*, subject to GA1a–GA4a and hence also subject to all theorems that can be proved from GA1a–GA4a.

b) $\boldsymbol{G} \stackrel{\rightharpoonup}{\equiv} \bigcup(G,\sigma) \stackrel{\rightharpoonup}{\equiv} G \cup \sigma$ subject to the same properties.

The comment that has been made in Definition 2.1(2) on the items a and b applies to the above items a and b with "GA1a–GA4a" and "function σ" in place of "GA1–GA4" and "functions ϕ, ε, and α" respectively.•

Comment 2.2. In accordance with definition (2.7), the items 2a and 2b of Definition 2.1a are identical with the respective items of Definition 2.1. At the same time, definition (2.4) should in this case be replaced with the following completely different definition:

$$\boldsymbol{G} \stackrel{\rightharpoonup}{\equiv} (G,\sigma) = (G,\phi \cup \varepsilon \cup \alpha), \tag{2.4a}$$

which is however inconsistent just as definition (2.4). Accordingly, as was already said earlier, I shall not define an algebraic system as an *ordered multiple* of its underlying sets and its intrinsic functions.•

Convention 2.1. In accordance with Definitions 2.1 and 2.1a, in the subsequent *definitions* of various algebraic systems, I shall distinguish, either informally, after the manner of the item 1 or 2a of Definition 2.1 or 2.1a, or formally, after the manner of the item 2b of Definition 2.1 or 2b, between an algebraic system, which will be denoted by the appropriate bold-faced logographic symbol, and its principal (major) underlying set, which will be denoted by the appropriate light-faced logographic symbol, not necessarily being a token of the former. Accordingly, in statements following the definition of an algebraic system, I may use the light-faced logographic symbol of the principal underlying set of the algebraic system equivocally, after the manner of the item 2a of Definition 2.1 or 2.1a, for mentioning, i.e. as a name of, the entire algebraic system, – in agreement with the common practice. •

Comment 2.3: Definition 2.1 or 2.1a determines a *class of algebraic systems*, namely *the class of groups*, and not an individual member of the class. Particularly, Definition 2.1 (e.g) determines neither the underlying set G nor the composition function ϕ of a concrete group. In other words, Definition 2.1 is a *categorial definition of a group*, i.e. *classifying and not particularizing* (*not individualizing, not identifying*) one. *The class of groups* is alternative denoted by the count noun "*group*" (without any limiting modifier as the indefinite or definite article), whereas a common (general) member of that class, i.e. an object that has that class as its classifying property, is called "*a group*". Any two groups, i.e. any two objects, each of which is, in accordance with Definition 2.1, entitled to be called ‘a group’, are *indistinguishable* in the framework of Definition 2.1, apart from the verbal (phonographic) or logographic names of the groups and their attributes (as ‘$\boldsymbol{G}$’ ,‘G’, ’ϕ’, ‘α’, or ‘e’). In accordance with the item 1 of section 1, a set is a class, but a class is not necessarily a set. From the standpoint of cognitive processes, the most natural restriction of the class of groups (e.g.) is not the set of all groups, but rather *a unique abstract object* (*substance*), which is defined by Definition 2.1 and which is called “*an abstract group*”. In order to distinguish a group from any other group, the former should be provided with a certain *additional distinguishing property*, called “*difference*” or “*differentia*” (pl. “*differentiae*”). If the differentia of a group is its *strictly typifying* (*not individualizing*) property then the pertinent definition of the group through a genus (as that specified in Definition 2.1) and the differentia is another classifying definition that defines the corresponding narrower class of groups or, equivalently, an abstract group of the corresponding narrower class (as an abstract commutative group, an abstract cyclic group, an abstract continuous group, etc.). If the differentia of a group is its individualizing property then the pertinent definition of the group through a genus and the differentia is an individualizing definition that defines the corresponding particular (concrete) group (as the symmetry group of a single crystal of NaCl).

The above remarks apply, *mutatis mutandis*, to any class of algebraic systems that will be introduced in the sequel. In this connection, I shall make explicit the following general convention, which usually remains implicit.•

Convention 2.2: Whenever I treat of two *different* abstract or concrete, congeneric or conspecific, algebraic systems (as groups, fields, vector spaces, or affine spaces) that satisfy a given classifying definition, I tacitly assume that there are *certain*, i.e. *specific but unspecified*, *differentiae* by which the two systems can be distinguished from each other.•

Theorem 2.1. The conjunction of GA1-GA4 is redundant. In fact, the axioms GA3 and GA4 are *theorems* that are provable from GA1 and GA2 and from the following two axioms (cf. Hall [1963, sec 1.3]).

GA3′: The left identity law. There exists an element $e \in G$, which is called a *left identity element*, suc
h that for each $x \in G$,

$$\phi(e, x) = x. \tag{2.10}$$

GA4′: The left inverse law. For each $x \in G$: there is an element $\alpha(x) \in G$, which is called a *left inverse*, or *left reciprocal*, *of* x, such that

$$\phi(\alpha(x), x) = e. \tag{2.11}$$

Proof: 1) Let

$$x' \doteqdot \alpha(x),\ x'' \doteqdot (x')' \doteqdot \alpha(\alpha(x)). \tag{2.12}$$

Let, in accordance with GA3′ and GA4′, for some $x \in G$,

$$\phi(x', x) = e,\ \phi(x'', x') = e, \tag{2.13}$$

subject to (2.12) By (2.13) and by the pertinent variants of (2.1), (2.10), and (2.11), it follows that

$$\begin{aligned} \phi(x, x') &= \phi(e, \phi(x, x')) = \phi(\phi(x'', x'), \phi(x, x')) = \phi(x'', \phi(x', \phi(x, x'))) \\ &= \phi(x'', \phi(\phi(x', x), x')) = \phi(x'', \phi(e, x')) = \phi(x'', x') = e, \end{aligned} \tag{2.14}$$

i.e. a left inverse x' of x is also its right inverse. Likewise,

$$x = \phi(e, x) = \phi(\phi(x, x'), x) = \phi(x, \phi(x', x)) = \phi(x, e), \tag{2.15}$$

i.e. a left identity element e is also a right identity element.

2) Assume now that there are two different *symmetric* (*two-sided*) *identity elements* e_1 and e_2, so that $\phi(e_1, x) = \phi(x, e_1) = x$ for each $x \in G$, and $\phi(e_2, y) = \phi(y, e_2) = x$ for each $y \in G$, in accordance with GA3′. At $x \doteqdot e_2$ and $y \doteqdot e_1$, the above two equations yield $e_1 = \phi(e_1, e_2) = \phi(e_2, e_1) = e_2$, i.e. there is exactly one identity element. Thus, GA3 is established.

3) Likewise, let us assume that a certain element x has two different inverse elements x_1' and x_2', so that $\phi(x, x_1') = \phi(x_1', x) = \phi(x, x_2') = \phi(x_2', x) = e$. Hence, by the pertinent variants of (2.2) and (2.1), it follows that

$$x_1' = \phi(e, x_1') = \phi(\phi(x_2', x), x') = \phi(x_2', \phi(x, x_1')) = \phi(x_2', e) = x_2', \tag{2.16}$$

i.e. each element of G has exactly one inverse. Thus, GA4 is also established.•

Theorem 2.2.

$$\alpha(e) = e. \tag{2.17}$$

Proof: From (2.2) at $x \triangleright \alpha(e)$ and (2.3) at $x \triangleright e$, it follows that $\alpha(e) = \phi(\alpha(e), e) = \phi(e, \alpha(e)) = e$. QED.•

Theorem 2.3. For each $x \in G$,

$$\alpha(\alpha(x)) = x. \tag{2.18}$$

Proof: Under definitions (2.12), it follows from Definition 2.1 that

$$x'' = \phi(x'', e) = \phi(x'', \phi(x', x)) = \phi(\phi(x'', x'), x) = \phi(e, x) = x\,, \tag{2.19}$$

where use of the following equations in that order has been made: (i) the variant of (2.2) with 'x''' in place of 'x'; (ii) (2.3) in the form

$$\phi(x', x) = \phi(x, x') = e\,; \tag{2.20}$$

(iii) the variant of (2.1) with 'x''', 'x'', and 'x' in place of 'x', 'y', and 'z', respectively; (iv) the variant of (2.20) with 'x''' in place 'x'' and 'x'' in place 'x'; (v) (2.2). QED.•

Definition 2.2: ***An abstract commutative group.*** A group ***G*** is called a *commutative*, or *Abelian, group* if the following additional axiom holds.

GA5: The commutative law. For each $(x, y) \in G \times G$.

$$\phi(x, y) = \phi(y, x)\,. \tag{2.21}$$

In accordance with Definition 2.1(2), a commutative, or Abelian, group ***G*** is more precisely defined in one of the following two alternative ways:

a) ***G*** equals G, i.e. $\boldsymbol{G} \stackrel{=}{\equiv} G$, subject to GA1–GA5 and hence also subject to all theorems that can be proved from GA1–GA5.

b) $\boldsymbol{G} \stackrel{=}{\equiv} \bigcup(G, \phi, \varepsilon, \alpha) \stackrel{=}{\equiv} G \cup [\phi \cup \varepsilon \cup \alpha]$ subject to GA1–GA5.•

Comment 2.4. 1) Definition 2.2 is a typifying differentia, which, along with Definition 2.1, is a classifying definition of *an abstract commutative* (*Abelian*) *group*. Once GA1–GA4 are supplemented by GA5, the function ϕ, which has been introduced in Definition 2.1, changes so that the new, homonymous function ϕ contains, e.g., an ordered pair $((y, x), \phi(x, y))$ besides then ordered pair $((x, y), \phi(x, y))$ Therefore, a function ϕ that satisfies the conjunction of GA1–GA4 and a function ϕ that satisfies the conjunction of GA1–GA5 should in principle be denoted differently, – say, as 'ϕ' and as 'ϕ^{c}', respectively. Alternatively, Definition 2.2 should be supplemented by a statement such as: «The symbol 'ϕ' which has been introduced in Definition 2.1, is now freed of its previous denotatum (meaning) and it denotes a function that satisfies the conjunction of GA1–GA5.» In stating Definition 2.2, this statement is tacitly omitted. A similar implicit procedure will tacitly be followed in the sequel every time when an initial conjunction of axioms that was imposed on a function earlier in the course of defining an algebraic system is then supplemented by some additional axioms to include the function into the corresponding new algebraic system.•

Definition 2.3. For each $(x, y) \in G \times G$,

$$x\phi y \doteqdot [x\phi y] \doteqdot \phi(x, y). \tag{2.22}$$

A *functional form schema* '$\phi(x, y)$' is said to be one in the *Clairaut-Euler,* or *inhomogeneous, representation,* while either *denotatively concurrent* functional form schema '$x\phi y$' or '[$x\phi y$]' is said to be one in the *bilinear*, or *homogeneous*, or *algebraic, representation*.•

Definition 2.4. 1) If the placeholder 'ϕ' is specified by any one of the conventional *signs of multiplication*, e.g. if $\phi \rhd \cdot$, then the group ***G*** is said to be *multiplicative*. In this case,

$$\phi(x, y) = x\phi y \rhd x \cdot y,\ \varepsilon \rhd \varepsilon_1,\ \alpha(x) \rhd x^{-1},\ e \rhd 1. \tag{2.23}$$

Consequently,

a) $x \cdot y$ is called *the product of x and y*, or *the element obtained from multiplication of x by y*,

b) 1 and is called *the multiplicative identity element*,

c) x^{-1} is called *the multiplicative inverse of x*.

It is generally accepted to abbreviate '$x \cdot y$' as 'xy', but I shall not follow this practice here.

2) If the placeholder 'ϕ' is specified by any one of the conventional *signs of addition*, e.g. if $\phi \rhd +$, then the group ***G*** is said to be *additive*. In this case,

$$\phi(x, y) = x\phi y \rhd x + y,\ \varepsilon \rhd \varepsilon_0,\ \alpha(x) \rhd -x,\ e \rhd 0. \tag{2.24}$$

Consequently,

a) $x + y$ is called *the sum of x and y*, or *the element obtained by addition of y to x*;

b) 0 is called *the additive identity element*, or *the null element*,

c) $-x$ is called *the additive inverse of x* or *the opposite of x*.

Comment 2.5. For convenience in further references, Definitions 2.1 and 2.2 and Theorems 2.2 and 2.3 are restated below so as to explicitly define an abstract multiplicative group, an abstract commutative multiplicative group, an abstract additive group, and an abstract commutative additive group in the notations, which are most easily adjustable to further applications.•

Definition 2.5: ***An abstract multiplicative group.*** 1) A multiplicative group ***M*** is a set *S* of elements *together* with a *surjective binary multiplication function* $\cdot : S\times S\to S$, an *injective choice* (*selection*) *function of the unity* (*multiplicative identity*) *element*, $\varepsilon_1 : \{S\}\to\{1\}\subset S$, and a *bijective singulary multiplicative inversion function* $^{-1}: S\to S$, which satisfy the following axioms, called the *Multiplicative Group Axioms* (*MGA's*). The elements of ***M*** are denoted by the variables '*a*', '*b*', and '*c*' that are taken alone or furnished with any one of the subscripts $_1$, $_2$, etc or with any number of primes or with both.

MGA1: The closure law. For each $(a,b)\in S\times S$, there is exactly one $c\in S$ such that $c=a\cdot b$.

MGA2: The associative law. For each $((a,b),c)\in[S\times S]\times S$,

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c. \tag{2.25}$$

MGA3: The identity law. There exists a unique element $1=\varepsilon_1(S)\in S$, which is called *the unity*, or *multiplicative identity*, *of S*, such that for each $a\in S$,

$$1\cdot a=a\cdot 1=a. \tag{2.26}$$

MGA4: The inverse law. For each $a\in S$: there is exactly one element $a^{-1}\in S$, which is called *the multiplicative inverse*, *multiplicative reciprocal*, *of a*, such that

$$a^{-1}\cdot a=a\cdot a^{-1}=1. \tag{2.27}$$

2) Formally,

$$\boldsymbol{M}\doteqdot\bigcup\left(S,\cdot,\varepsilon_1,{}^{-1}\right)\doteqdot\left[S\cup\cdot\cup\varepsilon_1\cup{}^{-1}\right]\doteqdot S\cup\Pi, \tag{2.28}$$

where

$$\Pi\doteqdot\bigcup\left(\cdot,\varepsilon_1,{}^{-1}\right)\doteqdot\left[\cdot\cup\varepsilon_1\cup{}^{-1}\right] \tag{2.29}$$

subject to MGA1–MGA4. The binary function $\cdot$ is called *the multiplication function* (or *operation*) *of **M*** or *on S×S*, whereas *the singulary function* $^{-1}$ *is called the multiplicative inversion function* (or *operation*) *of **M*** or *on S*.•

Corollary 2.1.

$$1^{-1}=1. \tag{2.30}$$

$$\left(a^{-1}\right)^{-1}=a \text{ for each } a\in S. \tag{2.31}$$

Proof: The corollary is the instance of Theorems 2.2 and 2.3 subject to (2.23).•

Definition 2.6: ***An abstract commutative multiplicative group.*** A multiplicative group ***M*** is called a *commutative*, or *Abelian*, *multiplicative group* if and only if the following additional axiom holds.

MGA5: The commutative multiplication law. For each $(a,b)\in S\times S$,

$$a\cdot b = b\cdot a\,. \tag{2.32}$$

Thus, a commutative (Abelian) multiplicative group ***M*** is defined by (2.28) and (2.29) subject to MGA1–MGA5.•

Definition 2.7. In accordance with MGA1 and MGA5 with 'y^{-1}' in place of 'y', there is a *composite binary function* $/ : S\times S\to S$, denoted also by '—', such that

$$\frac{a}{b} \doteqdot a/b = a\cdot b^{-1} = b^{-1}\cdot a \text{ for each } (a,b)\in S\times S\,, \tag{2.33}$$

The binary function / is called *the division operation of **M*** or *on S×S*, whereas the element *a*/*b* is called *the quotient of a by b*, or *the element obtained from division of a by b*.•

Definition 2.8: ***An abstract vector additive group.*** 1) A *vector additive group* $\hat{\boldsymbol{A}}$ is a set $\hat{E}$ *together* with a *surjective binary addition function* $\hat{+} : \hat{E}\times\hat{E}\to\hat{E}$, an *injective choice* (*selection*) *function of the null* (*additive identity*) *element*, $\varepsilon_{\hat{0}} : \{\hat{E}\}\to\{\hat{0}\}\subset\hat{E}$, and a *bijective singulary additive inversion function* $\hat{-} : \hat{E}\to\hat{E}$, which satisfy the following axioms, called the *Additive Group Axioms* (*AGA's*). The elements of $\hat{E}$, , called *vectors*, are denoted by the careted variables '$\hat{x}$', '$\hat{y}$', and '$\hat{z}$' that are taken alone or furnished with any one of the subscripts $_1$, $_2$, etc or with any number of primes or with both.

AGA1: The closure law. For each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$, there is exactly one $\hat{z}\in\hat{E}$ such that $\hat{z}=\hat{x}\,\hat{+}\,\hat{y}$.

AGA2: The associative law. For each $((\hat{x},\hat{y}),\hat{z})\in[\hat{E}\times\hat{E}]\times\hat{E}$,

$$\hat{x}\,\hat{+}\,(\hat{y}\,\hat{+}\,\hat{z}) = (\hat{x}\,\hat{+}\,\hat{y})\,\hat{+}\,\hat{z}\,. \tag{2.34}$$

AGA3: The identity law. There exists a unique element $\hat{0}=\varepsilon_{\hat{0}}(\hat{E})\in\hat{E}$, which is called *the null*, or *additive identity, of* $\hat{E}$, such that for $\hat{x}\in\hat{E}$,

$$\hat{0}\,\hat{+}\,\hat{x} = \hat{x}\,\hat{+}\,\hat{0} = \hat{x}\,. \tag{2.35}$$

*AGA4: The additive inverse law***.** For each $\hat{x} \in \hat{E}$: there is exactly one element $\hat{-}\hat{x} \in \hat{E}$, which is called *the additive inverse*, or *additive reciprocal*, or *opposite of* $\hat{x}$, such that

$$\hat{x} \hat{+} (\hat{-}\hat{x}) = (\hat{-}\hat{x}) \hat{+} \hat{x} = \hat{0}\,. \tag{2.36}$$

2) Formally,

$$\hat{\boldsymbol{A}} \dot{\equiv} \bigcup\left(\hat{E}, \hat{+}, \varepsilon_{\hat{0}}, \hat{-}\right) \dot{\equiv} \left[\hat{E} \cup \hat{+} \cup \varepsilon_{\hat{0}} \cup \hat{-}\right] \dot{\equiv} \left[\hat{E} \cup \hat{\Sigma}\right], \tag{2.37}$$

where

$$\hat{\Sigma} \dot{\equiv} \bigcup\left(\hat{+}, \varepsilon_{\hat{0}}, \hat{-}\right) \dot{\equiv} \left[\hat{+} \cup \varepsilon_{\hat{0}} \cup \hat{-}\right] \tag{2.38}$$

subject to AGA1–AGA4. The binary function $\hat{+}$ is called *the addition function* (or *operation*) *of* $\hat{\boldsymbol{A}}$ or *on* $\hat{E} \times \hat{E}$, whereas the singulary function $\hat{-}$ is called *the additive inversion function* (or *operation*) *of* $\hat{\boldsymbol{A}}$ or *in* (and also *of*) $\hat{E}$.•

Comment 2.6. The notation that is used in Definition 2.8 is the variant of the notation that has been introduced by the specifications (substitutions) (2.24) with '$\hat{+}$', '$\hat{-}$', '$\hat{0}$', '$\hat{x}$', '$\hat{y}$', '$\hat{z}$' in place of '+', '-', '0', '*x*', '*y*', '*z*', respectively. Conversely, the variant of Definition 2.8, in which all carets are omitted, is a valid alternative definition of an additive group that will, more specifically. be called a *scalar additive group*, while its elements are called *scalars*.

The carets have been introduced in Definition 2.8 in order to avoid in the sequel confusion between the operations on vector elements of a vector additive group and the similar operations on scalar elements of a filed in the cases where these two algebraic systems are used as constituent parts of a single whole algebraic system such as a *vector* (or *linear*) *space* or such as an *affine space*. The two variants of the definition of an additive group are conveniently incorporated without confusion into the entire system of notation that has been developed for the latter spaces in Iosilevskii [2016b] (see also the next section of this article).•

Corollary 2.2.

$$\hat{-}\hat{0} = \hat{0}\,. \tag{2.39}$$

$$\hat{-}(\hat{-}\hat{x}) = \hat{x} \text{ for each } \hat{x} \in \hat{E}\,. \tag{2.40}$$

Proof: The corollary is the instance of Theorems 2.2 and 2.3 subject to the pertinent variant of (2.24) (see Comment 2.6).•

Definition 2.9: ***A commutative additive vector group.*** An additive group $\hat{\boldsymbol{A}}$ is called a *commutative* (or *Abelian*) *additive vector* (or *linear*) *group* (*CAVG*) , or briefly *commutative additive group* (*CAG*), if and only if the following additional axiom holds.

AGA5: The commutative addition law. For each $(\hat{x},\hat{y})\in\hat{E}\times\hat{E}$,

$$\hat{x}\hat{+}\hat{y}=\hat{y}\hat{+}\hat{x}. \qquad (2.41)\bullet$$

Thus, an abstract commutative (Abelian) additive group $\hat{\boldsymbol{A}}$ is defined by (2.37) and (2.38) subject to AGA1–AGA5.•

Definition 2.10. In accordance with AGA1 and AGA5 with '$\hat{-}\hat{y}$' in place of '$\hat{y}$', there is *a composite binary function* $\hat{\div}:\hat{E}\times\hat{E}\to\hat{E}$ such that

$$\hat{x}\hat{\div}\hat{y}\stackrel{=}{=}\hat{x}\hat{+}(\hat{-}\hat{y})=(\hat{-}\hat{y})\hat{+}\hat{x} \text{ for each } (\hat{x},\hat{y})\in\hat{E}\times\hat{E}. \qquad (2.42)$$

The *binary* function $\hat{\div}$ is called *the subtraction operation of* $\hat{\boldsymbol{A}}$ or on $\hat{E}\times\hat{E}$, whereas the element $\hat{x}\hat{\div}\hat{y}$ is called *the difference* $\hat{x}$ *minus* $\hat{y}$ or *the element obtained by subtraction of the element* $\hat{y}$ *from the element* $\hat{x}$.•

Comment 2.7: It is commonly accepted to denote a binary subtraction operation by the same sign as that denoting the singulary operation of additive inversion. For avoidance of confusion, I do not, however, follow this practice.•

3. An abstract field and relevant algebraic systems

Definition 3.1. A field $\boldsymbol{S}$ is a set S of elements *together* with the following functions: a *surjective binary addition function* $+:S\times S\to S$, a *surjective binary multiplication function* $\cdot:S\times S\to S$, an *injective choice* (*selection*) *function of the null* (*additive identity*) *element*, $\varepsilon_0:\{S\}\to\{0\}\subset S$, an *injective choice* (*selection*) *function the unity* (*multiplicative identity*) *element*, $\varepsilon_1:\{S\}\to\{1\}\subset S$, a *bijective singulary additive inversion function* $-:S\to S$, and a *bijective singulary multiplicative inversion function* $^{-1}:S-\{0\}\to S-\{0\}$ – the functions that satisfy the following axioms, called the *Field Axioms* (*FA's*). The elements of $\boldsymbol{S}$ are denoted by the variables '*a*', '*b*', '*c*', '*d*' that are taken alone or furnished with any one of the subscripts $_1$, $_2$, etc or with any number of primes or with both (cf. Definition 2.5).

FA1: The closure laws. For each $(a,b) \in S \times S$, there is (i) exactly one $c \in S$ such that $c = a + b$ and (ii) exactly one $d \in S$ such that $d = a \cdot b$.

FA2: The associative laws. For each $((a,b),c) \in [S \times S] \times S$,

$$a + (b + c) = (a + b) + c, \tag{3.1}$$

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c. \tag{3.2}$$

FA3: The null element and unity element laws. S contains *the null element* $0 = \varepsilon_0(S) \in S$ and *the unity element* $1 = \varepsilon_1(S) \in S$, such that

$$1 \neq 0. \tag{3.3}$$

and for each $a \in S$,

$$a + 0 = a, \tag{3.4}$$

$$a \cdot 1 = a. \tag{3.5}$$

FA4: The additive and multiplicative inverse element laws.

i) For each $a \in S$ there is exactly one element $-a \in S$, called *the additive inverse of a* or *the opposite of a,* such that

$$a + (-a) = (-a) + a = 0. \tag{3.6}$$

ii) For each $a \in S - \{0\}$ (so that $a \neq 0$) there is a unique element $a^{-1} \in S - \{0\}$, which is called *the multiplicative inverse of a,* such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1. \tag{3.7}$$

FA5: The commutative laws. For each $(a,b) \in S \times S$,

$$a + b = b + a, \tag{3.8}$$

$$a \cdot b = b \cdot a. \tag{3.9}$$

FA6: The distributive law for over +. For each $((a,b),c) \in [S \times S] \times S$,

$$a \cdot (b + c) = a \cdot b + a \cdot c. \tag{3.10}$$ •

Corollary 3.1. An algebraic system ***A*** such that

$\boldsymbol{S}^{(0)} \text{–>} \boldsymbol{A}$

$$\boldsymbol{A} \doteq \bigcup(S,+,\varepsilon_0,-) \doteq [S \cup + \cup \varepsilon_0 \cup -] \doteq [S \cup \Sigma] \tag{3.11}$$

(cf. (2.37) and (2.38)), whose functions satisfy FA1(i) and the conjunction of equations (3.1), (3.4), (3.6), and (3.8), bound by the respective typified universal quantifiers, is an *abstract commutative additive group*.

Proof: FA1(i) and the conjuncts of FA2-FA5 which contain equations (3.1), (3.4), (3.6), and (3.8) as operata are the variants AGA1-AGA5 of Definitions 2.8 and 2.9 with

$$\text{‘}S\text{’, ‘}+\text{’, ‘}\varepsilon_0\text{’, ‘}0\text{’, ‘}-\text{’, ‘}a\text{’, ‘}b\text{’, ‘}c\text{’} \tag{3.12}$$

in place of

$$\text{‘}\hat{E}\text{’, ‘}\hat{+}\text{’, ‘}\varepsilon_{\hat{0}}\text{’, ‘}\hat{0}\text{’, ‘}\hat{-}\text{’, ‘}\hat{x}\text{’, ‘}\hat{y}\text{’, ‘}\hat{z}\text{’} \tag{3.13}$$

in that order.•

Corollary 3.2.

$$-0 = 0. \tag{3.14}$$

$$-(-a) = a \text{ for each } a \in S. \tag{3.15}$$

Proof: In accordance with Corollary 3.1, the corollary is the pertinent variant of Corollary 2.2.•

Corollary 3.3.

$$1^{-1} = 1. \tag{3.16}$$

$$\left(a^{-1}\right)^{-1} = a \text{ for each } a \in S - \{0\}. \tag{3.17}$$

Proof: (3.16) is the same as (2.30) and it follows straightforwardly from (3.5) and (3.7) for $a \triangleright 1$. (3.17) is the instance of (2.31) with ‘$S - \{0\}$’ in place of ‘S’ and it immediately follows from FA4(ii) with ‘$\left(a^{-1}\right)^{-1}$’ in place of ‘a’.•

Definition 3.2. In accordance with FA1(i) and equation (3.8) of FA5 with ‘$-b$’ in place of ‘b’, there is *a composite binary function* $-: S \times S \to S$ such that

$$a - b \doteqdot a + (-b) = (-b) + a \text{ for each } (a,b) \in S \times S. \tag{3.18}$$

The *binary* function $-$ is called *the subtraction operation of **S*** or *on S×S*, and also *of **A***, whereas the element $a - b$ is called *the difference a minus b* or *the element obtained by subtraction of the element b from the element a*. The whole of the above definition is the pertinent variant of Definition 2.10.•

Definition 3.3. In accordance with FA4(ii), the range of the functional form 'b^{-1}' equals $S - \{0\}$, i.e. the element b^{-1} exists for each $b \in S - \{0\}$. Consequently, it follows from FA4(ii) and equation (3.9) of FA5 with 'b^{-1}' in place of 'b' that there is a *composite binary function* $/ : S \times [S - \{0\}] \to S$, denoted also by '—', such that

$$\frac{a}{b} \doteq a/b = a \cdot b^{-1} = a^{-1} \cdot b \text{ for each } (a,b) \in S \times [S - \{0\}], \tag{3.19}$$

The binary function / is called *the division operation of* ***S*** or *on* $S \times [S - \{0\}]$, whereas the element a/b is called *the quotient of a by b*, or *the element obtained from division of a by b*. The whole of the above definition is, *mutatis mutandis*, the same as Definition 1.7.•

Theorem 3.1.

$$a \cdot 0 = 0 \cdot a = 0 \text{ for each } a \in S. \tag{3.20}$$

Proof: Multiply both sided of the equation $a + 0 = a$, (3.4), by a to get $a \cdot (a + 0) = a \cdot a$. Now, $a \cdot (a + 0) = a \cdot a + a \cdot 0$, by (3.10), while $a \cdot a = a \cdot a + 0$, by (3.4). Hence, $a \cdot a + a \cdot 0 = a \cdot a + 0$. Addition of -($a{\cdot}a$) to, i.e. subtraction of $a{\cdot}a$ from, both sides of the above equation yields $a \cdot 0 = 0$, by the pertinent variants of (3.1) and (3.6). Use of (3.9) at $b \triangleright 0$ completes the proof of the theorem.•

Comment 3.1. If $0 = 1$ then $a = a \cdot 1 = a \cdot 0 = 0$ for each $a \in S$, by (3.5) and (3.20). This result explains the necessity in axiom (3.3).•

Comment 3.2. All rules of calculation, which are familiar for rational or real numbers, can be deduced from Definitions 3.1–3.3 in analogy with the rules stated in Corollaries 3.2 and 3.3 and in Theorem 3.1. In the sequel, I shall use all these rules without any comments. In this case, Comment 2.7 applies also to the binary subtraction operation introduced in Definition 3.2.•

Definition 3.4. An algebraic system $\boldsymbol{S}^{(1)}$ such that

$$\boldsymbol{S}^{(1)} \doteq \boldsymbol{A} \cup [\cdot \cup \varepsilon_1] = \bigcup(S, +, \varepsilon_{\hat{0}}, -, \cdot, \varepsilon_1) = S \cup \Xi^{(1)} \tag{3.21}$$

subject to

$$\Xi^{(1)} \doteq \bigcup(+, \varepsilon_0, -, \cdot, \varepsilon_1) = [+ \cup \varepsilon_0 \cup - \cup \cdot \cup \varepsilon_1] \tag{3.22}$$

is called an *abstract commutative ring* if the functions included into $\Sigma^{(1)}$ satisfy FA1–FA3, FA4(i), FA5, and FA6,. Accordingly, these axioms are alternatively called the *Commutative Ring Axioms* or briefly and more specifically *CRA1-CRA6* in that order.•

Corollary 3.4.

$$\boldsymbol{S} \doteqdot \boldsymbol{S}^{(1)} \cup {}^{-1} = \bigcup\left(S,+,\varepsilon_0,-,\cdot,\varepsilon_1,{}^{-1}\right) = S \cup \Xi \tag{3.23}$$

subject to

$$\Xi \doteqdot \bigcup\left(+,\varepsilon_0,-,\cdot,\varepsilon_1,{}^{-1}\right) = \left[+\cup\varepsilon_0\cup - \cup \cdot \cup \varepsilon_1 \cup {}^{-1}\right], \tag{3.24}$$

i.e. in words, a field is a commutative ring together with FA4(i) as an additional axiom.

Proof: The corollary follows from Definitions 3.1 and 3.4.•

Comment 3.3: 1) In ordinary non-technical English the copula '*to be*' and especially its third person singular form '*is*', followed by a predicative either with an article '*a*' or '*the*' or without, is *a common denominative* (*generic*, *class*) *name* which *equivocally* denotes many different *relations in intension* such as those of *class-membership*, *class-inclusion*, *identity*, *entailment*, and some others. Particularly, in the sentence of Corollary 3.4, following the train of equalities (3.23), the occurrence of the link-verb 'is' is a verbal counterpart of the occurrence of the sign of equality by definition, '$\doteqdot$', in (3.23). Therefore, in this case, 'is' is used as a name of the corresponding identity relation in intension.

2) At the same time, one may also assert that a commutative ring *is a* commutative additive group and that a field *is* both *a* commutative additive group and *a* commutative ring. This means that among three classes of algebraic systems, namely the [class of] groups, the [class of] commutative rings, and the [class of] fields, the first is the most inclusive one and the last is the least inclusive one. Analogously, the proposition that *Socrates is a man* means that Socrates is a member of the class of men (*Homo sapiens*) or that the singleton of Socrates, i.e. the class consisting of Socrates as its only member, is a subclass of the class of men.

3) On the other hand, it follows from (3.21) and (3.23) that

$$\boldsymbol{A} \subset \boldsymbol{S}^{(1)} \subset \boldsymbol{S}\,. \tag{3.25}$$

These inclusion relations for the sets, being members of the corresponding ones of the above three classes of algebraic systems, are opposite to the inclusion relations for the classes themselves, indicated in the item 2.•

Definition 3.5. 1) A *commutative ring* $\boldsymbol{S}^{(1)}$ is called an *integral domain* if and only if it satisfies the following additional axiom, called *the cancellation law*:

For each $(a,b) \in S \times S$, for each $c \in S - \{0\}$,

$$[c \cdot a = c \cdot b] \Leftrightarrow [a = b]. \tag{3.26}$$

2) In agreement with Definition 3.4, CRA1–CRA6, i.e. FA1–FA3, FA4(i), FA5, and FA6, along with the above cancellation law are alternatively called the *Integral Domain Axioms* or briefly and more specifically *IDA1–IDA7* in that order.•

Comment 3.4. Let '*N*', '*I*', '*Q*', '*R*', and '*C*' denote the sets of *natural numbers* (including the *natural null*, i.e. the *empty set*, ∅), *natural integers* (strictly positive, strictly negative, and null), and *rational*, *real*, and *complex numbers* respectively. Under the conventional definition of a *semigroup* (see, e.g., MacLane and Birkhoff [1967, p. 61), the algebraic system ***N*,** defined as:

$$\boldsymbol{N} \doteqdot \bigcup(N,+,\varepsilon_0,\cdot,\varepsilon_1) = N \cup [+ \cup \varepsilon_0 \cup \cdot \cup \varepsilon_1], \tag{3.27}$$

is a *specific additive and multiplicative semigroup* simultaneously. The set *N* contains neither negative numbers nor fractional numbers, so that neither an additive inversion function nor a multiplicative inversion function can be defined on it. Therefore, ***N*** is not a group. At the same time, the algebraic system ***I***, defined by (3.21) with '***I***' and '*I*' in place of '$\boldsymbol{S}^{(1)}$' and '*S*' respectively, i.e. as:

$$\boldsymbol{I} \doteqdot I \cup \Sigma^{(1)}, \tag{3.28}$$

is a *specific commutative ring*. In this case, the algebraic system $\boldsymbol{Z}(\sqrt{2})$, defined as

$$\boldsymbol{Z}(\sqrt{2}) \doteqdot Z(\sqrt{2}) \cup \Sigma^{(1)} \tag{3.29}$$

subject to

$$Z(\sqrt{2}) \doteqdot \{m + n \cdot \sqrt{2} \mid m \in I \text{ and } n \in I\}, \tag{3.30}$$

is another *specific* commutative ring. The field of rational, real, and complex numbers, denoted by '***Q***', '***R***', or '***C***' respectively, is defined by the instance of definition (3.23) and (3.24) with '***Q***' and '*Q*', or '***R***' and '*R*', or '***C***' and '*C*', in place of '***S***' and '*S*', respectively, so that

$$\boldsymbol{Q} \doteqdot Q \cup \Xi, \boldsymbol{R} \doteqdot R \cup \Xi, \boldsymbol{C} \doteqdot C \cup \Xi, \tag{3.31}$$

subject to (3.24), where now '0' and '1' equivocally denote the respective rational, real, or complex integers. In this case, the set *C* can be conventionally be defined as:

$$C \doteqdot \{x + y \cdot \sqrt{-1} \mid x \in R \text{ and } y \in R\}. \tag{3.32}$$

Under the definition

$$K(\sqrt{2}) \doteqdot \{p + q \cdot \sqrt{2} \mid p \in Q \text{ and } q \in Q\}, \tag{3.33}$$

which is analogous to both (3.30) and (3.32), the instance of (3.21) with '$\boldsymbol{K}(\sqrt{2})$' and '$K(\sqrt{2})$' in place of '$\boldsymbol{S}$' and 'S' respectively subject to the instance of (3.22) with '0' and '1', denoting the respective rational numbers, in place of '0' and '1', defines the specific field $\boldsymbol{K}(\sqrt{2})$.•

4. An abstract affine additive group

Peliminary Remark 4.1. The subject matter of this section coincides with an initial portion of section 4 of Iosilevskii [2016b], which is cosmetically modified as follows. First, I employ the symbols '$\hat{\boldsymbol{A}}$' and '$\dot{\boldsymbol{A}}$' in place of '$\hat{\boldsymbol{E}}^{\mathrm{g}}$' and '$\dot{\boldsymbol{E}}^{\mathrm{g}}$' respectively that are employed in the latter article. In this way, I just demonstrate how all distinguished (identity) elements of an algebraic system can consistently be mentioned explicitly in the logographic name of the system. The logographic names of vector and affine spaces of various classes that have been introduced and used in the above article can be modified likewise.•

Definition 4.1. 1) Let $\hat{\boldsymbol{A}}$ be a commutative additive group (CAG) defined by Definitions 2.8 and 2.9 and let $\hat{E}$ be as before the underlying set of its elements called vectors; $\hat{E}$ may sometimes be identified with $\hat{\boldsymbol{A}}$. In accordance with Definition 2.8, the binary composition operation of addition and the singulary operation of additive inversion *in* $\hat{\boldsymbol{A}}$ (or respectively *on* $\hat{E}\times\hat{E}$ and *on* $\hat{E}$) are denoted by '$\hat{+}$' and '$\hat{-}$' respectively. The latter operation is defined relative the *additive identity element of* $\hat{\boldsymbol{A}}$ (or *of*, and also *in*, $\hat{E}$) that is denoted by '$\hat{0}$' and is called the *null vector*. Elements (vectors) of $\hat{E}$ are denoted by the *variables* '$\hat{x}$', '$\hat{y}$', and '$\hat{z}$', which can be furnished with some appropriate labels as Arabic numeral subscripts '$_1$', '$_2$', etc or as primes.

2) An *affine additive group* (*AAG*) $\dot{\boldsymbol{A}}$ is an algebraic system that consists of a certain *underlying set of points* $\dot{E}$, called its *affine additive group manifold* (*AAGM*), and of a certain *vector group* $\hat{\boldsymbol{A}}$ whose underlying set $\hat{E}$ of elements, called *vectors*, is related to $\dot{E}$ by a *binary surjection*

$$\hat{V}:\dot{E}\times\dot{E}\to\hat{E}\,, \tag{4.1}$$

which satisfies the following two *AAGM axioms* (*AAGMA's*).

AAGMA1: *The law of composition of vectors from ordered pairs of points – The set of bijections between* $\dot{E}$ *and* $\hat{E}$. For each $(\dot{x},\dot{y}) \in \dot{E} \times \dot{E}$, there is exactly one $\hat{z} \in \hat{E}$ such that

$$\hat{z} = \hat{V}_{\dot{x}}(\dot{y}) \doteqdot \hat{V}(\dot{x},\dot{y}) \tag{4.2}$$

and conversely for each $(\hat{z},\dot{x}) \in \hat{E} \times \dot{E}$, there is exactly one $\dot{y} \in \dot{E}$ such that (4.2) holds, i.e.

$$\dot{y} = \hat{V}_{\dot{x}}^{-1}(\hat{z}). \tag{4.3}$$

That is to say, given $\dot{x} \in \dot{E}$, the singular functions $\hat{V}_{\dot{x}}: \dot{E} \to \hat{E}$ and $\hat{V}_{\dot{x}}^{-1}: \hat{E} \to \dot{E}$, as defined in terms of the binary function (4.1) by (4.2) and (4.3), are two mutually inverse *bijections*

AAGMA2: The Chasle, or *triangle*, *law*. For each $(\dot{x},\dot{y},\dot{z}) \in \dot{E}^{3\times}$,

$$\hat{V}(\dot{x},\dot{y}) \hat{+} \hat{V}(\dot{y},\dot{z}) \hat{+} \hat{V}(\dot{z},\dot{x}) = \hat{0}. \tag{4.4}$$

3) The commutative additive group (CAG) $\hat{\boldsymbol{A}}$ and its underlying vector set $\hat{E}$ are said to be *adjoint* of the AAG $\dot{\boldsymbol{A}}$ and of its underlying point set (AAGM) $\dot{E}$ respectively. «*Togetherness*» of $\dot{E}$, $\hat{\boldsymbol{A}}$, and $\hat{V}$ as constituent parts forming a single whole algebraic system $\dot{\boldsymbol{A}}$ can be expressed by the following formal definition of the latter

$$\dot{\boldsymbol{A}} \doteqdot \bigcup\left(\dot{E},\hat{\boldsymbol{A}},\hat{V}\right) \doteqdot \dot{E} \cup \hat{\boldsymbol{A}} \cup \hat{V} = \dot{E} \cup \left[\hat{E} \cup \hat{+} \cup \varepsilon_{\hat{0}} \cup \hat{-}\right] \cup \hat{V} \doteqdot \bigcup\left(\dot{E},\hat{E},\hat{+},\varepsilon_{\hat{0}},\hat{-},\hat{V}\right) \tag{4.5}$$

subject to (2.37).•

Comment 4.1: Definition 4.1 has been made with the purpose to introduce specifically the notions of an affine *additive* group and of an affine *additive* group manifold for convenience in further references. At the same time, Definition 4.1 can obviously be altered to introduce the like notions with "*multiplicative*" instead of "*additive*" or in general without either qualifier. With the help of the appropriate substitutions, all corollaries that are deduced below from Definition 4.1 can be restated so as to become corollaries of the respective modified definition.•

Corollary 4.1: ***The identity law for*** $\hat{V}$**.** For each $\dot{x} \in \dot{E}$.

$$\hat{V}(\dot{x},\dot{x}) = \hat{0} \tag{4.6}$$

and hence

$$\hat{V}_{\dot{x}}(\dot{x}) = \hat{0}, \tag{4.7}$$

$$\hat{V}_{\dot{x}}^{-1}(\hat{0}) = \dot{x}. \tag{4.8}$$

Proof: (4.6) follows from (4.4) at $\dot{z} \dot{\equiv} \dot{y} \dot{\equiv} \dot{x}$. (4.7) follows from (4.2) at $\dot{y} \dot{\equiv} \dot{x}$, by (4.6). (4.8) follows from (4.3) at $\hat{z} \hat{\equiv} \hat{0}$, by (4.7).•

Corollary 4.2: ***The basic inversion law for*** $\hat{V}$ **.** For each $(\dot{x}, \dot{y}) \in \dot{E} \times \dot{E}$,

$$\hat{V}(\dot{y}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{y}), \tag{4.9}$$

where $\hat{-}\hat{V}(\dot{x}, \dot{y})$ is *the additive inverse of* $\hat{V}(\dot{x}, \dot{y})$. That is to say, $\hat{V}(\dot{y}, \dot{x})$ and $\hat{V}(\dot{x}, \dot{y})$ are *the additive inverse of each other*

Proof: By the variant of (4.6) with '$\dot{y}$' or '$\dot{z}$' in place of '$\dot{x}$', it follows from (4.4) at $\dot{z} = \dot{y}$ that

$$\hat{V}(\dot{x}, \dot{y}) \hat{+} \hat{V}(\dot{y}, \dot{x}) \hat{\equiv} \hat{0} \text{ for each } (\dot{x}, \dot{y}) \in \dot{E} \times \dot{E} . \tag{4.10}$$

The corollary immediately follows from (4.10) by the item CAGA4 of Definition 2.4.•

Corollary 4.3: *A modified triangle law*. For each $(\dot{x}, \dot{y}, \dot{z}) \in \dot{E}^{3\times}$,

$$\hat{V}(\dot{x}, \dot{y}) \hat{+} \hat{V}(\dot{y}, \dot{z}) = \hat{V}(\dot{x}, \dot{z}). \tag{4.11}$$

Proof: By the equation $\hat{V}(\dot{z}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{z})$, which is the variant of (4.9) with '$\dot{z}$' in place of '$\dot{y}$', and also by the item CAGA4 of Definition 2.4, equation (4.11) is equivalent to (4.4).•

Corollary 4.4. The binary surjection $\hat{V}$, (4.1), has the property that for each $(\dot{x}, \dot{y}) \in \dot{E} \times \dot{E}$, there is exactly one $\hat{z} \in \hat{E}$ such that

$$\hat{z} = \hat{V}_{\dot{x}}(\dot{y}) \hat{\equiv} \hat{V}(\dot{x}, \dot{y}) = \hat{-}\hat{V}(\dot{y}, \dot{x}) \hat{\equiv} \hat{-}\hat{V}_{\dot{y}}(\dot{x}) \tag{4.12}$$

and conversely for each $(\hat{z}, \dot{y}) \in \hat{E} \times \dot{E}$, there is exactly one $\dot{x} \in \dot{E}$ such that both (4.12) and hence (4.3) hold and in addition

$$\dot{x} = \hat{V}_{\dot{y}}^{-1}(\hat{-}\,\hat{z}). \tag{4.13}$$

That is to say, in accordance with AAGMA1, relation (4.3) is the inverse of relation (4.12) at $\dot{x}$ held constant, whereas relation (4.13) is the inverse of relation (4.12) at $\dot{y}$ held constant. At the same time, relations (4.3) and (4.13) are mutually inverses at $\hat{z}$ held constant.

Proof: The train of equations (4.12) is the train (4.1), which is developed by supplementing it by equation (4.9) and also by the variant with '$\dot{x}$' and '$\dot{y}$' exchanged of the definition occurring in (4.1). The train (4.12) is equivalent to this one:

$$\hat{-}\,\hat{z} = \hat{V}_{\dot{y}}(\dot{x}) \hat{\equiv} \hat{V}(\dot{y}, \dot{x}) = \hat{-}\hat{V}(\dot{x}, \dot{y}) \hat{\equiv} \hat{-}\hat{V}_{\dot{x}}(\dot{y}), \tag{4.12$_1$}$$

while (4.13) is equivalent to the first equation in (4.12_1). QED.•

Comment 4.2. By Corollary 4.1, at $\hat{z} \mathrel{\dot{\equiv}} \hat{0}$ and $\dot{x} \mathrel{\dot{\equiv}} \dot{y}$, the conjunction of equations (4.12) and (4.13) reduces to the conjunction of the variants of equations (4.6)–(4.8) with ‘ $\dot{y}$ ’ in place of ‘ $\dot{x}$ ’.•

Theorem 4.1. There is a binary composition *surjection*

$$\dot{P}\colon \dot{E} \times \hat{E} \to \dot{E} \ , \tag{4.14}$$

such that for each $\hat{z} \in \hat{E}$: (a) for each $\dot{x} \in \dot{E}$, there is exactly one $\dot{y} \in \dot{E}$ such that

$$\dot{y} = \dot{P}_{\hat{z}}(\dot{x}) \mathrel{\dot{\equiv}} \dot{P}(\dot{x}, \hat{z}) \mathrel{\dot{\equiv}} \hat{V}_{\dot{x}}^{-1}(\hat{z}) \, , \tag{4.15}$$

and conversely (b) for each $\dot{y} \in \dot{E}$, there is exactly one $\dot{x} \in \dot{E}$ such that

$$\dot{x} = \dot{P}_{\hat{z}}^{-1}(\dot{y}) = \dot{P}_{\dot{-}\hat{z}}(\dot{y}) \mathrel{\dot{\equiv}} \hat{V}_{\dot{y}}^{-1}(\dot{-}\,\hat{z}) \, . \tag{4.16}$$

By (4.16), for each $\hat{z} \in \hat{E}$,

$$\dot{P}_{\hat{z}}^{-1} = \dot{P}_{\dot{-}\hat{z}} \, , \tag{4.17}$$

the understanding being that the singular functions

$$\dot{P}_{\hat{z}}\colon \dot{E} \to \dot{E} \ \text{ and } \ \dot{P}_{\hat{z}}^{-1}\colon \dot{E} \to \dot{E} \, , \tag{4.18}$$

which are defined in terms of the binary function (4.1) by (4.15) and (4.16), are two mutually inverse *bijections*.

Proof: The final definientia of the trains of definitions (4.15) and (4.16) are given by equations (4.3) and (4.13) respectively, which are, by Corollary 4.4, mutually inverses at $\hat{z}$ held constant. At the same time, the relation ‘ $\dot{x} = \dot{P}_{\hat{z}}^{-1}(\dot{y})$’, occurring in (4.16), is the inverse of the relation ‘ $\dot{y} = \dot{P}_{\hat{z}}(\dot{x})$’, occurring in (4.15), while the definition $\dot{P}_{\dot{-}\hat{z}}(\dot{y}) \mathrel{\dot{\equiv}} \hat{V}_{\dot{y}}^{-1}(\dot{-}\,\hat{z})$, occurring in (4.16), is the variant of the definition $\dot{P}(\dot{x}, \hat{z}) \mathrel{\dot{\equiv}} \hat{V}_{\dot{x}}^{-1}(\hat{z})$, occurring in (4.15), with ‘ $\dot{y}$ ’ in place of ‘ $\dot{x}$ ’ and ‘ $\dot{-}\,\hat{z}$ ’ in place of ‘ $\hat{z}$ ’.•

Comment 4.3. It should be recalled that the function $\hat{V}_{\dot{x}}^{-1}$, e.g., is the inverse of $\hat{V}_{\dot{x}}$ at $\dot{x}$ held constant. At the same time, the function $\dot{P}_{\hat{z}}^{-1}$ is the inverse of $\dot{P}_{\hat{z}}$ at $\hat{z}$ held constant. Therefore, the equations ‘ $\dot{P}_{\hat{z}}(\dot{x}) = \hat{V}_{\dot{x}}^{-1}(\hat{z})$ ’ and $\dot{P}_{\dot{-}\hat{z}}(\dot{y}) \mathrel{\dot{\equiv}} \hat{V}_{\dot{y}}^{-1}(\dot{-}\,\hat{z})$, e.g., which occur in (4.14) and

(4.15), cannot be rewritten as ‘ $\dot{P}_{\hat{z}}^{-1}(\dot{x})=\hat{V}_{\dot{x}}(\hat{z})$ ’ and ‘ $\dot{P}_{\hat{-}\hat{z}}^{-1}(\dot{y})\mathrel{\dot{=}}\hat{V}_{\dot{y}}(\hat{-}\hat{z})$ ’ respectively. The former two equations are true by definition, whereas the latter two are false.•

Definition 4.2. 1) The surjection (4.1) is called *the first*, or *basic, surjection of the affine additive group manifold* $\dot{E}$ and also *the vectorization of the set* $\dot{E}\times\dot{E}$.

2) The surjection (4.13) is called *the second surjection of the affine additive group manifold* $\dot{E}$ and also *the pointillage of the set* $\dot{E}\times\hat{E}$.

3) Given $\dot{x}\in\dot{E}$, the bijection $\hat{V}_{\dot{x}}$ as defined by (4.2) is called *the vectorization of the point set* $\dot{E}$ *relative to the point* $\dot{x}$, whereas the inverse bijection $\hat{V}_{\dot{x}}^{-1}$ is called *the pointillage of the vector set* $\hat{E}$ *relative to the point* $\dot{x}$.

4) Given $\hat{z}\in\hat{E}$, the bijection $\dot{P}_{\hat{z}}$ as defined by (4.14) and having the property (4.16) is called *the translation of the affine additive group manifold* $\dot{E}$ *over the vector* $\hat{z}$. In this case, the inverse bijection $\dot{P}_{\hat{z}}^{-1}$ is, by (4.16), *the translation of the affine additive group manifold* $\dot{E}$ *over the vector* $\hat{-}\hat{z}$.•

Corollary 4.5.

$$\dot{P}_{\hat{0}}^{-1}(\dot{x})=\dot{P}_{\hat{0}}(\dot{x})\mathrel{\dot{=}}\dot{P}(\hat{0},\dot{x})\mathrel{\dot{=}}\hat{V}_{\dot{x}}^{-1}(\hat{0})=\dot{x}\ \text{ for each }\ \dot{x}\in\dot{E}\,, \tag{4.19}$$

whence

$$\dot{P}_{\hat{0}}^{-1}=\dot{P}_{\hat{0}}\mathrel{\dot{=}}\dot{P}(\hat{0})=I_{\dot{E}}\,, \tag{4.20}$$

where $I_{\dot{E}}$ is the identity function from $\dot{E}$ onto $\dot{E}$.•

Proof: The corollary follows from (4.15)–(4.17) by (4.8).•

Definition 4.3. 1) For each $(\dot{x},\dot{y})\in\dot{E}\times\dot{E}$: the ordered pair $(\dot{x},\dot{y})$ is called *the position group-vector of the point* $\dot{y}$ *relative to the point* $\dot{x}$. The point $\dot{x}$ is called *the base*, or *tail*, *of the position group-vector* $\langle\dot{x},\dot{y}\rangle$, whereas the point $\dot{y}$ is called *the head*, or *terminal, of the position group-vector* $(\dot{x},\dot{y})$.

2) In contrast to a position group-vector, which belongs to the set $\dot{E}\times\dot{E}$, a group-vector, which belongs to the set $\hat{E}$ is called a *free group-vector*.•

Comment 4.4. The term “*position group-vector*” (“*of a point relative to a point*”) as specified in Definition 4.3 should not be confused with the term ‘*group-vector*’ without the qualifier '*position*'. By AAGMA1, to each ordered pair of points $\dot{x}$ and $\dot{y}$ in $\dot{E}$, different or not, there corresponds a unique group-vector $\hat{z} = \hat{V}(\dot{x}, \dot{y})$ in $\hat{E}$. Since $\hat{V}$ is a surjection, therefore any group-vector $\hat{z} \in \hat{E}$ is *a class of equivalence of ordered pairs* $(\dot{x}, \dot{y}) \in \dot{E} \times \dot{E}$ *of points relative to the surjection* $\hat{V}$. In this case, this class is a regular one, i.e. a set, so that

$$\hat{z} \doteqdot \{(\dot{x}, \dot{y}) | (\dot{x}, \dot{y}) \in \dot{E} \times \dot{E} \text{ and } \hat{V}(\dot{x}, \dot{y}) = \hat{z}\} \text{ for each } \hat{z} \in \hat{E} \quad (4.21)$$

and particularly

$$\hat{0} = \{(\dot{x}, \dot{x}) | \dot{x} \in \dot{E} \text{ and } \hat{V}(\dot{x}, \dot{x}) = \hat{0}\} \in \hat{E}. \quad (4.22)$$

These relations are of course *tautologies*, but they demonstrate that any attempt to treat the vector as an *arrow* that has certain end points, i.e. a certain tail (base) point and a certain head (terminal) point, is inconsistent. Therefore, the term "position *group-vector*" should not mislead the reader. Either of these terms is just a synonym of the term "ordered pair of points".

2) Incidentally, if a vector group $\hat{\boldsymbol{A}}$ is treated as an *autonomous* algebraic system in no connection with any affine group $\dot{\boldsymbol{A}}$ then a group-vector in $\hat{\boldsymbol{A}}$ can be regarded as an insensible *nonempty individual*. A point of $\dot{\boldsymbol{A}}$ is also an insensible *nonempty individual*. If, however, $\hat{\boldsymbol{A}}$ is treated as the adjoint vector group of a certain affine group $\dot{\boldsymbol{A}}$ then, a group-vector of $\hat{\boldsymbol{A}}$ including the null group-vector becomes, as explicated in the previous item, a *set* (*regular class*, *small class*) *of equivalence* of ordered pairs of points of $\dot{\boldsymbol{A}}$ and therefore it ceases to be a nonempty individual. At the same time, a separate ordered pair $(\dot{x}, \dot{y}) \in \dot{E} \times \dot{E}$, i.e. a separate position group-vector, is a set, namely $(\dot{x}, \dot{y}) = \{\dot{x}, \{\dot{x}, \dot{y}\}\}$, and therefore it is not a nonempty individual either.

3) In the general case, a single point in $\dot{E}$ is not a group-vector in $\hat{E}$, except a certain special case to be explicated by Theorem 4.2 below in subsection 4.3.•

5. A abstract general algebraic system

The instances of algebraic systems that have been discussed in the previous sections and also vector and affine spaces of various classes that have been introduced and used in Iosilevskii [2016b] (cf. Preliminary Remark 4.1) can be generalized as follows.

Definition 5.1. 1) Most generally, an algebraic system $\mathcal{S}$ is a system (complex) of a certain number $m\geq1$ of *underlying sets of elements*, $\mathcal{S}_1$, $\mathcal{S}_2$, …, $\mathcal{S}_m$, and of a certain number $n\geq1$ of *intrinsic functions* f_1, f_2, …, f_n, that interrelate elements of those sets in accordance with certain defining axioms of $\mathcal{S}$. Each intrinsic function of $\mathcal{S}$ is a set (regular class) of ordered pairs. Therefore, $\mathcal{S}$ can most generally be logographically defined as:

$$\mathcal{S} \doteqdot \left[\bigcup_{i=1}^{m} \mathcal{S}_i\right] \cup \left[\bigcup_{j=1}^{n} f_j\right]. \tag{2.1}$$

2) Except a *semigroup*, being a simplest algebraic system that has a single underlying set of elements and a single surjective binary composition function (operation), and also except a *monoid*, being a semigroup that is augmented with a choice function of an identity element (see, e.g., MacLane and Birkhoff [1967, pp. 61–64]), an algebraic system of any higher rank, starting from a group, has at least three intrinsic functions:

i) *a surjective binary composition function* ϕ;

ii) *an injective choice, or selection, function* ε *of a certain identity (distinguished) element e, a unity or a null*;

iii) *a singulary additive or multiplicative inverse function* α.

Thus, except a semigroup, for which $n=1$, the number n of intrinsic functions of any higher algebraic system, including a monoid, satisfies the condition $n\geq2$.

3) In the general case, a surjective binary composition function ϕ can be one of the following two subclasses.

a) For some $\kappa \in \omega_{1,m}$, there exists ϕ_κ such that $\phi_\kappa : S_\kappa \times S_\kappa \to S_\kappa$:

b) For some $\kappa \in \omega_{1,m}$ and some $\lambda \in \omega_{1,m}$, there exists $\phi_{\kappa,\lambda}$ such that $\phi_{\kappa,\lambda} \doteqdot \vec{\phi}_{\kappa,\lambda} \cup \overleftarrow{\phi}_{\kappa,\lambda}$ subject to $\vec{\phi}_{\kappa,\lambda} : S_\kappa \times S_\lambda \to S_\lambda$ and $\overleftarrow{\phi}_{\kappa,\lambda} : S_\lambda \times S_\kappa \to S_\lambda$.

For instance, in the case of an abstract vector (linear) space $\hat{\boldsymbol{E}}(\boldsymbol{R})$ over the field $\boldsymbol{R}$ of *real* numbers, there is a single function of the class b that has been denoted as '$\hat{\cdot} : [R \times \hat{E}] \cup [\hat{E} \times R] \to \hat{E}$' in Definition 2.6 of Iosilevskii [2016b]. At the same time, given $\mu \in \omega_{1,m}$, there can exist none one or at most two selection functions in S_μ: $\varepsilon_{0_\mu} : S_\mu \to 0_\mu$ and $\varepsilon_{1_\mu} : S_\mu \to 1_\mu$, i.e. $\varepsilon_{0_\mu} \doteqdot \{(S_\mu, 0_\mu)\}$ and $\varepsilon_{1_\mu} \doteqdot \{(S_\mu, 1_\mu)\}$.

4) If $m \geq 2$ then all underlying sets *of elements*, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$, of $\mathcal{S}$ are mutually disjoint, i.e.

$$\mathcal{S}_i \cap \mathcal{S}_j = \varnothing \text{ if } i \neq j. \tag{5.2}$$

At the same time, if $n \geq 2$ then, most typically, the intrinsic functions $f_1, f_2, \ldots, f_n$, of $\mathcal{S}$ are also mutually disjoined, i.e. $f_k \cap f_l = \varnothing$ if $k \neq l$. However, in contrast to the similar former condition (5.2), the latter one is not the must. For instance, in the field of rational or real numbers, $2+2=2 \cdot 2=4$, so that $[\cdot \cap +] \neq \varnothing$. Therefore, the formal definition (5.1) of $\mathcal{S}$ after the manner of Definition 1.1(2a) is valid without any exception. •

Comment 5.1. In contrast to the *formal* Definition 5.1, if $m \geq 2$ then a certain one of the underlying sets of elements, $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$, of $\mathcal{S}$, say $\mathcal{S}_1$, is often mentally put forward as the *principal*, or *major*, *underlying set*, which is therefore equivocally and *informally* identified with $\mathcal{S}$, i.e. $\mathcal{S} \stackrel{\cdot}{=} \mathcal{S}_1$ in analogy with Definition 1.1(2a), while the other underlying sets are mentally put backward as the *minor underlying sets of* $\mathcal{S}$. Along with the intrinsic functions of $\boldsymbol{S}$, elements of the minor sets are used in stating defining axioms of $\boldsymbol{S}$, which are informally regarded as properties of elements of $\mathcal{S}_1$. These are kept in mind and are not mentioned explicitly,•

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