Kochen-Specker theorem as a precondition for quantum computing

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(Dated: May 1, 2016)

We study the relation between the Kochen-Specker theorem (the KS theorem) and quantum computing. The KS theorem rules out a realistic theory of the KS type. We consider the realistic theory of the KS type that the results of measurements are either $+1$ or -1. We discuss an inconsistency between the realistic theory of the KS type and the controllability of quantum computing. We have to give up the controllability if we accept the realistic theory of the KS type. We discuss an inconsistency between the realistic theory of the KS type and the observability of quantum computing. We discuss the inconsistency by using the double-slit experiment as the most basic experiment in quantum mechanics. This experiment can be an easy detector to a Pauli observable. We cannot accept the realistic theory of the KS type to simulate the double-slit experiment in a significant specific case. The realistic theory of the KS type can not depicture quantum detector. In short, we have to give up both the observability and the controllability if we accept the realistic theory of the KS type. Therefore the KS theorem is a precondition for quantum computing, i.e., the realistic theory of the KS type should be ruled out.

PACS numbers: 03.67.Lx(Quantum computer), 03.65.Ud(Quantum nonlocality), 03.65.Ca(Formalism)

I. INTRODUCTION

Quantum mechanics (cf. [1–6]) gives approximate and at times remarkably accurate numerical predictions. Much experimental data approximately fits to the quantum predictions for the past some 100 years. We do not doubt the correctness of quantum mechanics. Quantum mechanics also says new science with respect to information theory. The science is called the quantum information theory [6]. Therefore, quantum mechanics gives us very useful another theory in order to create new information science and to explain the handling of raw experimental data in our physical world.

On the other hand, from the incompleteness argument of Einstein, Podolsky, and Rosen (EPR) [7], a hidden-variable interpretation of the quantum theory has been an attractive topic of research [3, 4]. One is the Bell-EPR theorem [8]. This theorem says that some quantum predictions violate the inequality following from the EPR-locality condition. The condition tells that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system.

The other is the no-hidden-variables theorem of Kochen and Specker (the KS theorem) [9]. The original KS theorem says the non-existence of a real-valued function which is multiplicative and linear on commuting operators. The quantum theory does not accept the KS type of hidden-variable theory. The proof of the original KS theorem relies on intricate geometric argument. Greenberger, Horne, and Zeilinger discover [10, 11] the so-called GHZ theorem for four-partite GHZ state. And, the Bell-KS theorem becomes very simple form (see also Refs. [12–16]).

More recently, Leggett-type non-local variables theory [17] is experimentally investigated [18–20]. The experiments report that the quantum theory does not accept Leggett-type non-local variables interpretation. However there are debates for the conclusions of the experiments. See Refs. [21–23].

As for the applications of the quantum theory, implementation of a quantum algorithm to solve Deutsch's problem [24] on a nuclear magnetic resonance quantum computer is reported firstly [25]. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [26]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [27]. Single-photon Bell states are prepared and measured [28]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [29]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [30]. In 1993, the Bernstein-Vazirani algorithm was reported [31]. It can be considered as an extended Deutsch-Jozsa algorithm. In 1994, Simon's algorithm was reported [32]. Implementation of a quantum algorithm to solve the Bernstein-Vazirani parity problem without entanglement on an ensemble quantum computer is reported [33]. Fiber-optics implementation of the Deutsch-Jozsa and Bernstein-Vazirani quantum algorithms with three qubits is discussed [34]. A quantum algorithm for approximating the influences of Boolean functions and its applications is recently reported [35].

Given the fundamental studies and the application reports, we consider why quantum computer is faster than classical counterpart. Some measurement outcome admits a realistic theory of the Bell type [36]. No measurement outcome admits a realistic theory of the Kochen-Specker (KS) type [37]. The KS type is stronger than the Bell type and the KS type is equivalent to the classical theory [38]. In 2010, it was discussed that implementation of the Deutsch-Jozsa algorithm violates the realistic theory of the KS type [39]. It is essential to study the general relation between the KS theorem and quantum computing to investigate the quantum computation problem more and more. So we address studying the relation between the KS theorem and quantum computing here again.

We study the relation between the KS theorem and quantum computing. The possible values of the predetermined result of measurements are ± 1 (in $\hbar/2$ unit). The assumption was used in the "original" realistic theory of the KS type.

The reference frames are necessary to control a quantum state. We need the controllability of quantum computing. Let us consider the controllability of quantum computing. We derive some quantum proposition concerning a quantum expected value under an assumption about the existence of the orientation of reference frames in N spin-1/2 systems. However, the realistic theory of the KS type violates the proposition with a magnitude that grows exponentially with the number of particles. To derive the inconsistency, we rely on the maximum value of the square of the KS realistic theoretical expected value. Therefore, we have to give up either the existence of the reference frames or the realistic theory of the KS type. The realistic theory of the KS type does not depicture physical phenomena using reference frames with a violation factor that grows exponentially with the number of particles.

We assume an implementation of the double-slit experiment [40]. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes through another slit, then the value of the result of measurement is -1. This is an easy detector model to a Pauli observable.

We consider whether the realistic theory of KS type meets the easy detector model to the Pauli observable. We assume that a source of spin-carrying particles emits them in a state, which can be described as an eigenvector of the Pauli observable σ_z. We consider a single expected value of the Pauli observable σ_x in the double-slit experiment. A wave function analysis says that the quantum expected value of it is zero. However, the realistic theory of KS type can predict different value to the expected value of $\langle \sigma_x \rangle = 0$. To derive the inconsistency, we use the maximum value of the square of an expected value. Hence, the realistic theory of KS type does not meet the easy detector model as the whole.

Our paper is organized as follows.

In Sec. II, we discuss the fact that the realistic theory of the KS type does not meet the reference frames. The realistic theory of the KS type does not meet the controllability of quantum computing.

In Sec. III, we discuss the relation between the double-slit experiment and the realistic theory of the KS type. The realistic theory of the KS type does not meet the observability of quantum computing.

Section IV concludes this paper.

II. THE REALISTIC THEORY OF THE KS TYPE DOES NOT MEET THE CONTROLLABILITY

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is a spin-1/2 pure state lying in the x-y plane. Let us assume that one source of N uncorrelated spin-carrying particles emits them in a state, which can be described as a multi spin-1/2 pure uncorrelated state. Let us parameterize the settings of the jth observer with a unit vector $\vec{n}_j$ (its direction along which the spin component is measured) with $j = 1, \ldots, N$. One can introduce the 'realistic' correlation function, which is the average of the product of the hidden results of measurement

$$E_{\mathrm{HV}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) \rangle_{\mathrm{avg}}, \qquad (1)$$

where r is the hidden result. We assume the value of r is ± 1 (in $(\hbar/2)^N$ unit), which is obtained if the measurement directions are set at $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Also one can introduce a quantum correlation function with the system in such a pure uncorrelated state

$$E_{\mathrm{QM}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \mathrm{tr}[\rho \vec{n}_1 \cdot \vec{\sigma} \otimes \vec{n}_2 \cdot \vec{\sigma} \otimes \cdots \otimes \vec{n}_N \cdot \vec{\sigma}] \qquad (2)$$

where $\otimes$ denotes the tensor product, $\cdot$ the scalar product in $\mathbf{R}^2$, $\vec{\sigma} = (\sigma_x, \sigma_y)$ is a vector of two Pauli operators, and ρ is the pure uncorrelated state,

$$\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_N \qquad (3)$$

with $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane.

One can write the observable (unit) vector $\vec{n}_j$ in a plane coordinate system as follows:

$$\vec{n}_j(\theta_j^{k_j}) = \cos\theta_j^{k_j} \vec{x}_j^{(1)} + \sin\theta_j^{k_j} \vec{x}_j^{(2)}, \qquad (4)$$

where $\vec{x}_j^{(1)} = \vec{x}$ and $\vec{x}_j^{(2)} = \vec{y}$ are the Cartesian axes. Here, the angle $\theta_j^{k_j}$ takes two values (two-setting model):

$$\theta_j^1 = 0, \ \theta_j^2 = \frac{\pi}{2}. \qquad (5)$$

We derive a necessary condition to be satisfied by the quantum correlation function with the system in a pure uncorrelated state given in (2). In more detail, we derive the maximum value of the product of the quantum correlation function, $E_{\rm QM}$ given in (2), i.e., $\|E_{\rm QM}\|^2_{\rm max}$. We use the decomposition (4). We introduce simplified notations as

$$T_{i_1 i_2 \ldots i_N} = {\rm tr}[\rho \vec{x}_1^{(i_1)} \cdot \vec{\sigma} \otimes \vec{x}_2^{(i_2)} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{x}_N^{(i_N)} \cdot \vec{\sigma}] \quad (6)$$

and

$$\vec{c}_j = (c_j^1, c_j^2) = (\cos\theta_j^{k_j}, \sin\theta_j^{k_j}). \quad (7)$$

Then, we have

$$\begin{aligned} &\|E_{\rm QM}\|^2 \\ &= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \left(\sum_{i_1,\ldots,i_N=1}^{2} T_{i_1 \ldots i_N} c_1^{i_1} \cdots c_N^{i_N} \right)^2 \\ &= \sum_{i_1,\ldots,i_N=1}^{2} T^2_{i_1 \ldots i_N} \le 1, \end{aligned} \quad (8)$$

where we use the orthogonality relation $\sum_{k_j=1}^2 c_j^\alpha c_j^\beta = \delta_{\alpha,\beta}$. The value of $\sum_{i_1,\ldots,i_N=1}^2 T^2_{i_1\ldots i_N}$ is bounded as $\sum_{i_1,\ldots,i_N=1}^2 T^2_{i_1\ldots i_N} \le 1$. We have

$$\prod_{j=1}^{N} \sum_{i_j=1}^{2} ({\rm tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \le 1. \quad (9)$$

From the convex argument, all quantum separable states must satisfy the inequality (8). Therefore, it is a separability inequality. It is important that the separability inequality (8) is saturated iff ρ is a multi spin-1/2 pure uncorrelated state such that, for every j, $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane. The reason of the inequality (8) is due to the following quantum inequality

$$\sum_{i_j=1}^{2} ({\rm tr}[\rho_j \vec{x}_j^{(i_j)} \cdot \vec{\sigma}])^2 \le 1. \quad (10)$$

The inequality (10) is saturated iff $\rho_j = |\Psi_j\rangle\langle\Psi_j|$ and $|\Psi_j\rangle$ is a spin-1/2 pure state lying in the x-y plane. The inequality (8) is saturated iff the inequality (10) is saturated for every j. Thus we have the maximum possible value of the scalar product as a quantum proposition concerning the reference frames

$$\|E_{\rm QM}\|^2_{\rm max} = 1 \quad (11)$$

when the system is in such a multi spin-1/2 pure uncorrelated state.

On the other hand, a correlation function satisfies the realistic theory of the KS type if it can be written as

$$E_{\rm HV}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^m r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m} \quad (12)$$

where l denotes some hidden variable and r is the hidden result of measurement of the dichotomic observables parameterized by the directions of $\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N$.

Assume the quantum correlation function with the system in a pure uncorrelated state given in (2) admits the realistic theory of the KS type. One has the following proposition concerning the realistic theory of the KS type

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l=1}^m r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)}{m}. \quad (13)$$

In what follows, we show that we cannot assign the truth value "1" for the proposition (13) concerning the realistic theory of the KS type. We rely on the maximum value of the square of an expected value. Assume the proposition (13) is true. By changing the hidden variable l into l', we have the same quantum expected value as follows

$$E_{\rm QM}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \lim_{m\to\infty} \frac{\sum_{l'=1}^m r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')}{m}. \quad (14)$$

An important note here is that the value of the right-hand-side of (13) is equal to the value of the right-hand-side of (14) because we only change the hidden variable.

We abbreviate $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l)$ to $r(l)$ and $r(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N, l')$ to $r(l')$.

We introduce an assumption that Sum rule and Product rule commute with each other [41]. We do not pursue the details of the assumption. To pursue the details is an interesting point. It is suitable to the next step of researches. Then we have

$$\begin{aligned} &\|E_{\rm QM}\|^2 \\ &= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \\ &\left(\lim_{m\to\infty} \frac{\sum_{l=1}^m r(l)}{m} \times \lim_{m\to\infty} \frac{\sum_{l'=1}^m r(l')}{m} \right) \\ &\le \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \\ &\left(\lim_{m\to\infty} \frac{\sum_{l=1}^m}{m} \times \lim_{m\to\infty} \frac{\sum_{l'=1}^m}{m} |r(l) r(l')| \right) \\ &= \sum_{k_1=1}^{2} \cdots \sum_{k_N=1}^{2} \\ &\left(\lim_{m\to\infty} \frac{\sum_{l=1}^m}{m} \times \lim_{m\to\infty} \frac{\sum_{l'=1}^m}{m} \right) = 2^N. \end{aligned} \quad (15)$$

Here we use the fact

$$|r(l) r(l')| = 1 \quad (16)$$

since the possible values of $r(l)$ are ± 1. The above inequality can be saturated because we have

$$\begin{aligned} &\|\{l | r(l) = 1\}\| = \|\{l' | r(l') = 1\}\| \\ &\|\{l | r(l) = -1\}\| = \|\{l' | r(l') = -1\}\|. \end{aligned} \quad (17)$$

Hence we derive the following proposition if we assign the truth value "1" for the realistic theory of the KS type

$$\|E_{\rm QM}\|^2_{\rm max} = 2^N. \tag{18}$$

Clearly, we cannot assign the truth value "1" for two propositions (11) (concerning the reference frames) and (18) (concerning the realistic theory of the KS type), simultaneously, when the system is in a multiparticle pure uncorrelated state. Of course, each of them is a spin-1/2 pure state lying in the x-y plane. Therefore, we are in the KS contradiction when the system is in such a multiparticle pure uncorrelated state. Thus, we cannot accept the validity of the proposition (13) (concerning the realistic theory of the KS type) if we assign the truth value "1" for the proposition (11) (concerning the reference frames). In other words, the realistic theory of the KS type does not reveal physical phenomena using reference frames. The reference frames are necessary to control a quantum state. Thus, the realistic theory of the KS type does not reveal physical phenomena controlling a quantum state.

III. THE REALISTIC THEORY OF THE KS TYPE DOES NOT MEET THE OBSERVABILITY

We consider the relation between the double-slit experiment and the realistic theory of the KS type. We assume an implementation of the double-slit experiment. There is a detector just after each slit. Thus interference figure does not appear, and we do not consider such a pattern. The possible values of the result of measurements are ± 1 (in $\hbar/2$ unit). If a particle passes one side slit, then the value of the result of measurement is $+1$. If a particle passes through another slit, then the value of the result of measurement is -1.

A. A wave function analysis

Let (σ_z, σ_x) be a Pauli vector. We assume that a source of spin-carrying particles emits them in a state $|\psi\rangle$, which can be described as an eigenvector of the Pauli observable σ_z. We consider a quantum expected value $\langle\sigma_x\rangle$ as

$$\langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle = 0. \tag{19}$$

The above quantum expected value is zero if we consider only a wave function analysis.

We derive a necessary condition for the quantum expected value for the system in the pure spin-1/2 state $|\psi\rangle$ given in (19). We derive the possible value of the product $\langle\sigma_x\rangle \times \langle\sigma_x\rangle = \langle\sigma_x\rangle^2$. $\langle\sigma_x\rangle$ is the quantum expected value given in (19). We have

$$\langle\sigma_x\rangle^2 = 0. \tag{20}$$

Thus,

$$\langle\sigma_x\rangle^2 \le 0. \tag{21}$$

We derive the following proposition

$$(\langle\sigma_x\rangle^2)_{\rm max} = 0. \tag{22}$$

B. The realistic theory of the KS type

On the other hand, a mean value E admits the realistic theory of the KS type if it can be written as

$$E = \frac{\sum_{l=1}^m r_l(\sigma_x)}{m} \tag{23}$$

where l denotes some hidden variable and r is the hidden result of measurement of the Pauli observable σ_x. We assume the value of r is ± 1 (in $\hbar/2$ unit).

Assume the quantum mean value with the system in an eigenvector ($|\psi\rangle$) of the Pauli observable σ_z given in (19) admits the realistic theory of the KS type. One has the following proposition concerning the realistic theory of the KS type

$$\langle\sigma_x\rangle(m) = \frac{\sum_{l=1}^m r_l(\sigma_x)}{m}. \tag{24}$$

We can assume as follows by Strong Law of Large Numbers,

$$\langle\sigma_x\rangle(+\infty) = \langle\sigma_x\rangle = \langle\psi|\sigma_x|\psi\rangle. \tag{25}$$

In what follows, we show that we cannot assign the truth value "1" for the proposition (24) concerning the realistic theory of the KS type. We rely on the maximum value of the square of a mean value.

Assume the proposition (24) is true. By changing the hidden variable l into l', we have the same quantum mean value as follows

$$\langle\sigma_x\rangle(m) = \frac{\sum_{l'=1}^m r_{l'}(\sigma_x)}{m}. \tag{26}$$

An important note here is that the value of the right-hand-side of (24) is equal to the value of the right-hand-side of (26) because we only change the hidden variable. We have

$$\begin{aligned}
&\langle\sigma_x\rangle(m) \times \langle\sigma_x\rangle(m) \\
&= \frac{\sum_{l=1}^m r_l(\sigma_x)}{m} \times \frac{\sum_{l'=1}^m r_{l'}(\sigma_x)}{m} \\
&\le \frac{\sum_{l=1}^m}{m} \times \frac{\sum_{l'=1}^m}{m} |r_l(\sigma_x) r_{l'}(\sigma_x)| \\
&= \frac{\sum_{l=1}^m}{m} \times \frac{\sum_{l'=1}^m}{m} = 1.
\end{aligned} \tag{27}$$

Here we use the fact

$$|r_l(\sigma_x) r_{l'}(\sigma_x)| = 1 \tag{28}$$

since the possible values of $r_l(\sigma_x)$ are ± 1. The above inequality can be saturated because we have

$$\begin{aligned}
&\|\{l|r_l(\sigma_x) = 1\}\| = \|\{l'|r_{l'}(\sigma_x) = 1\}\| \\
&\|\{l|r_l(\sigma_x) = -1\}\| = \|\{l'|r_{l'}(\sigma_x) = -1\}\|.
\end{aligned} \tag{29}$$

Hence we derive the following proposition if we assign the truth value "1" for the realistic theory of the KS type

$$(\langle \sigma_x \rangle(m) \times \langle \sigma_x \rangle(m))_{\max} = 1. \tag{30}$$

From Strong Law of Large Numbers, we have

$$(\langle \sigma_x \rangle \times \langle \sigma_x \rangle)_{\max} = 1. \tag{31}$$

Hence we derive the following proposition concerning the realistic theory of the KS type

$$(\langle \sigma_x \rangle^2)_{\max} = 1. \tag{32}$$

We do not assign the truth value "1" for two propositions (22) (concerning a wave function analysis) and (32) (concerning the realistic theory of the KS type), simultaneously. We are in the KS contradiction.

We cannot accept the validity of the proposition (24) (concerning the realistic theory of the KS type) if we assign the truth value "1" for the proposition (22) (concerning a wave function analysis). In other words, we cannot accept the realistic theory of the KS type to simulate the detector model for the spin observable σ_x.

IV. CONCLUSIONS

In conclusion, we have studied the relation between the KS theorem and quantum computation. The possible values of the pre-determined result of measurements have been ± 1 (in $\hbar/2$ unit).

The reference frames have been necessary to control a quantum state. We have derived some proposition concerning a quantum expected value under an assumption about the existence of the orientation of reference frames in N spin-1/2 systems. However, the realistic theory of the KS type has violated the proposition with a magnitude that grows exponentially with the number of particles. Therefore, we have had to give up either the existence of the reference frames or the realistic theory of the KS type. The realistic theory of the KS type does not have depictured physical phenomena using reference frames with a violation factor that grows exponentially with the number of particles.

Also we have discussed the fact that the realistic theory of the KS type do not meet an easy detector model to a single Pauli observable. The realistic theory of the KS type can not have depictured quantum detector.

In short, we have had to give up both the observability and the controllability if we accept the realistic theory of the KS type. Therefore the KS theorem has been a precondition for quantum computing, i.e., the realistic theory of the KS type should be ruled out.

ACKNOWLEDGMENTS

We thank Professor Weinstein for valuable discussions.

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