The Multitude behind the Buddhabrot

AGGOTT HÖNSCH István
www.apeirography.com

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1 Introduction

I have found an endless city with an infinitude of denizens long-hidden in plain sight upon the complex plane. I hereby offer the terminological framework and algorithms that led me there.

While an effort has been made to give credit where credit is known to me to be due, the focus of my mathematical competence is uncommonly pragmatic and there doubtless yet remain citations that would be proper but are presently missing on account of being unknown to me. This, along with other shortcomings as may become obvious, shall be addressed in future revisions.

1.1 Notation

The mathematical notation used throughout the article is generally in line with mathematical literature covering like topics, with two exceptions: the use of iteral notation introduced in Salov (2012) for iterations and Strachey brackets for multisets as illustrated in Wooldridge (2000).

The symbol И, pronounced /i/ or /iʒe/[1], denotes the iteral of a function, the result arrived at after a specified number of iterations. Analogously to Σ notation, the initial input into the function is specifed by subscripting and the limit by superscripting. Thus given $f(x) = x^2$: $\text{И}_2^0 f(x) = 2$, $\text{И}_2^1 f(x) = 4$, $\text{И}_2^2 f(x) = 16$, $\text{И}_2^3 f(x) = 256$.

In seeking to put Salov's notation to full utility, it has been extended to enable the specification of iteral sequences, by way of specifying not a single value

1. This is not the authentic Russian pronunciation of /'iz̨ɨ/, but rather an approximation better suited for broad European pronunciation. Pseudo-phonetically *ee-zheh* for English speakers and *ijè* for Francophones.

but a range in the superscript. In combination with the use of the plain inclusion Strachney bracket notation for multisets, instead of the more traditional *unique values + multiplicity* notation, given $g(y) = 2y \bmod 5$ we can write equations like $\text{И}_{y=1}^{1..6}\, g(y) = [\![2, 4, 1, 3, 0, 2]\!]$.

2 Definitions

Definition 1. The **Mandelbrot superset** $\mathbb{M}$ is the set of all points on the complex plane that fall within the origin-centred disk of radius 2, the smallest possible disk containing all points of the Mandelbrot set (Pražák 2000).

$$\mathbb{M} = \{c \in \mathbb{C} : |c| \leq 2\}$$

Equation 1: The Mandelbrot superset

Definition 2. The **Fatou-Julia function** $f_c(z)$ is the quadratic function used to determine membership both within the Mandelbrot set and the Fatou-Julia sets.

$$\begin{aligned} f_c(z) : \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z^2 + c \end{aligned}$$

Equation 2: The Fatou-Julia function

Definition 2.1. The **seed** is the initial input of the iteration, namely z_0. (Avalos-Bock 2009)

Definition 2.2. The value z is the **iterand** of the equation and c is the **constant** or the **addend** thereof. (Avalos-Bock 2009)

Definition 2.3. The series of values arising from the iteration of a function, $\text{И}_{z=0}^{1..\infty}\, f_c(z)$, are called the **orbit** of the iteration.

Definition 2.4. An orbit with a seed of 0 is termed a **critical orbit**. In such cases, the seed is rendered impotent and is superfluous in the first iteration, therefore c is also called the **governing seed** of the iteration.[2] In turn, the critical orbit that has c as its governing seed is c's **governed orbit** and the corresponding iteration set is called c's **governed iteration set**.

Definition 2.5. A specific point of iteration within an orbit is called a **singularity**, multiple discrete ones are a **constellation**; a range of continuous points comprise an **arc**, and a discontinuous range containing at least one arc is termed a **diversity**.

2. Given $f_c(z) = z^2 + c$, the iteral equations $\text{И}_0^0\, f_c(z)$, $\text{И}_0^1\, f_c(z)$, $\text{И}_0^2\, f_c(z)$, and $\text{И}_0^3\, f_c(z)$ can be rewritten to eliminate z altogether as 0, c, $c^2 + c$, and $c^4 + 2c^3 + c^2 + c$ respectively.

Definition 3. The **Mandelbrot set** $\mathcal{M}$ is the set of all points on the complex plane, whose governed iteration of $f_c(z)$ fails to cause the orbit to leave $\mathbb{M}$ and thereby diverge towards infinity.

$$\mathcal{M} = \{c \in \mathbb{C} : \forall n \in \mathbb{N} : |\mathop{\text{И}}_{z=0}^{n} f_c(z)| \in \mathbb{M}\}$$

Equation 3: The Mandelbrot set

Definition 4. Points outside $\mathcal{M}$ but still within $\mathbb{M}$ are **ectocopial** $\mathcal{M}'$, whereas those within $\mathcal{M}$ are **endocopial**.

$$\mathcal{M}' = \mathbb{M} \cap \mathcal{M}^c$$

Equation 4: Ectocopial points of the Mandelbrot superset

Definition 5. The **depth** of an ectocopial point is the number of times its governed iteration can be repeated before the orbit leaves $\mathbb{M}$.

Sets of ectocopial points may be categorized in a number ways according to the correlative depths of their members:

Definition 5.1. **Homobathic** sets contain points of the same depth; whereas **heterobathic** sets contain points of different depths. Additionally, subsets of the latter include **homoiobathic** sets contain points of different but similar/like depths[3], and **juxtabathic** sets contain points of different but contiguously adjacent depths.

Definition 5.2. **Omnibathic** or **orthobathic** sets contain points having a uniform distribution of all depths within their depth range; whereas **oligobathic** sets contain points of non-uniform distribution of depths within their depth range.

Definition 5.3. **Heretobathic** sets contain points of a specific subset of depth(s) within their depth range. **Monobathic** sets contain points of only a single specific depth; whereas **polybathic** sets contain points of multiple specific depths.

Definition 6. The **Apeiropolis multiset** $\mathbb{A}$ is the multiset of complex points that comprise the critical orbits of the Fatou-Julia function iterated with the ectocopial points of the Mandelbrot superset used as governing seeds.

$$\mathbb{A} = [\![k : k \in \mathbb{M}]\!] \subseteq [\![\mathop{\text{И}}_{z=0}^{1..\infty} f_c(z) : c \in \mathcal{M}']\!]$$

Equation 5: The Apeiropolis multiset

3. e.g.: all even depths within a range or all prime depths within a range

Definition 7. An **anthropobrot multiset** $\mathcal{A}_q$ is a subset of $\mathbb{A}$, derived by way of restricting membership to a specific subset of the critical orbits.

$$\mathcal{A}_q = [\![k : \prod_{z=0}^{q} f_c(z) = k : q \in \mathbb{N}_1 : c \in \mathcal{M}']\!] \subseteq \mathbb{A}$$

Equation 6: The Anthropobrot multiset of the q^{th} orbital singularity

$$\mathcal{A}_{p,q,r} = [\![\mathcal{A}_p \cup \mathcal{A}_q \cup \mathcal{A}_r]\!]$$

Equation 7: The Anthropobrot multiset of the p, q, r orbital constellation

$$\mathcal{A}_{p..q} = [\![\mathcal{A}_p \cup .. \cup \mathcal{A}_q]\!]$$

Equation 8: The Anthropobrot multiset of the $p..q$ orbital arc

$$\mathcal{A}_{p..q,r} = [\![\mathcal{A}_p \cup .. \cup \mathcal{A}_q \cup \mathcal{A}_r]\!]$$

Equation 9: The Anthropobrot multiset of the $p..q, r$ orbital diversity

Less formally, anthropobrot multisets, may be referred to as anthropobrots of the relevant numerical designators and the orbital subset type, yielding the terms **singularity anthropobrot**, **constellation anthropobrot**, **arc anthropbrot**, and **diversity anthropobrot**.

e.g.: The 1^{st} singularity anthropobrot is $\mathcal{A}_1$; the *perfect-numbers-under-a-thousand* or $6, 28, 496$ constellation anthropobrot is $\mathcal{A}_{6,28,496}$, the 29-31 arc anthropobrot is $\mathcal{A}_{29..31}$; and the *primes-under-twenty* or $2, 3, 5, 7, 11, 13, 17$, and 19 diversity anthropobrot is $\mathcal{A}_{2..3,5,7,11,13,17,19}$.

If obvious from the numerical designator, the orbital subset type need not be explicitly specified.

e.g.: Anthropobrot 7 is the 7^{th} singularity anthropobrot multiset ($\mathcal{A}_7$); Anthropobrot 6-12 is the 6-12 arc anthropobrot multiset ($\mathcal{A}_{6..12}$); etc.

3 Observations

1. The ectocopial points of the Mandelbrot superset are comprised of an infinite and concentric series of increasingly more accurate monobathic approximations of the outline of the Mandelbrot set.

2. The Aperiopolis multiset is the mathematical object underlying not just the anthropobrot multisets, but also the Buddhabrot fractal.

(a) The Buddhabrot fractal is generated via the orbit density visualization of a positionally random but depth-blind subset of ectocopial governing seeds, which heavily favour shallow points.

As a consequence, traditionally generated Buddhabrot fractals are comprised of a large number of overlaid singularity anthropobrots, with low-numbered ones contributing most to the output, and higher number ones having only limited influence due to their being increasingly less defined for lack of sufficiently numerous deep governing seeds.

3. The first singularity anthropobrot is an approximation of the outline of the Mandelbrot set, by virtue of being identical to the set of its governing seeds. Subsequent singularity anthropobrots arise from the stretching and folding of said outline.

4. Since there are an infinite number of governing seeds for any anthropobrot—the criteria for inclusion simply being for a given governing seed to be deeper than the orbital subset chosen—necessarily finite visualizations of anthropobrots are obviously but approximations. The accuracy of these approximations, like with the ectocopial set, is a function of depth; the deeper the governing seeds used, the more accurate the approximation.

4 Nomenclature

After studying the inhabitants of the Apeiropolis for some time, I conceived of the idea of classifying them. After having generated hundreds, I decided to start naming them. What follows is a short summation of my efforts to date.

1. Singularity anthropobrots are all unique, but can be most readily classified based on the following visually apparent characteristics (described in a $+90^\circ$ rotation context):

   (a) All odd-numbered singularity anthropobrots have their period-2 bulbs north of the main cardioid, giving the pareidolic impression of the head of a standing figure. I classify these simply **anthropobrots** or **upright anthropobrots**.

   (b) All even-numbered singularity anthropobrots have their period-2 bulbs folded down atop the main cardioid, giving the pareidolic impression of a standing figure in a deep bow. I classify these **anthropobrots in piety**.

   (c) All singularity anthropobrots whose numbers' division by four yields a remainder of one or two have their period-4 bulbs pointing away from their period-2 bulbs, giving the pareidolic impression of a crown atop the figures head. I classify such anthropobrots as **royal**.

   (d) All singularity anthropobrots whose numbers' division by four yields a remainder of three or zero have their period-4 bulbs folded back onto their period-2 bulbs, lacking the pareidolic impression of a crown. I classify such anthropobrots as **priestly**.

2. In naming singularity anthropobrots, noting the aforementioned regularities, I decided to name *upright* and *in piety* pairs together—excepting the first twenty, which have been named as per the system described below.

3. The first anthropobrot and subsequent even numbered anthropobrots up to and including 20, I have dubbed **mnemeiobrots**—memorial fractals.

   (a) The first anthropobrot is named the **Mandelbrot mnemeiobrot**, in honour of Benoît B. Mandelbrot.

   (b) Even numbered anthropobrots in piety from two to twenty are mnemeiobrots named after other significant figures who contributed to the eventual discovery and understanding of fractals: Pierre Fatou & Gaston Julia, Lewis Fry Richardson, Paul Pierre Lévy, Felix Hausdorff, Wacław Franciszek Sierpiński, Niels Fabian Helge von Koch, Felix Christian Klein & Jules Henri Poincaré, Georg Ferdinand Ludwig Philipp Cantor, Karl Theodor Wilhelm Weierstraß, and Gottfried Wilhelm Leibniz.

4. Anthropobrots three, five, seven, and nine are named after mythological figures: Cyclops (Greek), the Parson of Andreas (Manx), Cangjie (Chinese), and Jorōgumo (Japanese).

5. Odd numbered singularity anthropobrots 11 to 19 are named after individuals notable for their involvement with the Buddhabrot fractal: Melinda Green, Linas Vepstas, the as-yet anonymous colleague of Ms. Green who first named the pareidolic figure *Ganesha*, Lori Gardi, and Alex Boswell.

5 Remarks

This manuscript has long been in the works, going through numerous iterations; some more ambitious than the present version. In limiting the article's goals to the introduction of the Apeiropolis and Anthropobrot multisets, and the terminological framework through which I have discovered them, my intent was to enable the informal publication and publicization of my work sooner than otherwise could have been managed.

Timeliness seemed to me a sound priority for two reasons. Firstly, because a fractal treasure trove as rich the anthropobrots are, ought not be the private domain of but one man. Secondly, because it is my hope that by giving insight into the mathematically—as opposed to programmatically—grounded approach that led me to my discovery, perhaps other researchers will be encouraged to adopt like approaches that lead to their own revelations.

And so, I encourage the reader to boldly go forward and build upon the foundations presented herein. On a frontier as fraught with infinities as the complex plane, there must yet remain further wonders to be discovered.

AGGOTT HÖNSCH István

References

Avalos-Bock, Stephanie. 2009. "Fractal Geometry: The Mandelbrot and Julia Sets." Accessed April 15, 2015. http://www.math.uchicago.edu/~may/VIGRE/VIGRE2009/REUPapers/Avalos-Bock.pdf.

Pražák, Dalibor. 2000. "The Mandelbrot Set." Accessed April 15, 2015. http://www.karlin.mff.cuni.cz/~prazak/doc/mandel.pdf.

Salov, Valerii. 2012. "Notation for Iteration of Functions, Iteral." Accessed January 16, 2015. http://arxiv.org/pdf/1207.0152.pdf.

Wooldridge, Mike. 2000. "Lecture 14: Bags (Multisets)." Accessed January 18, 2015. http://www.cs.ox.ac.uk/people/michael.wooldridge/teaching/soft-eng/lect14.pdf.

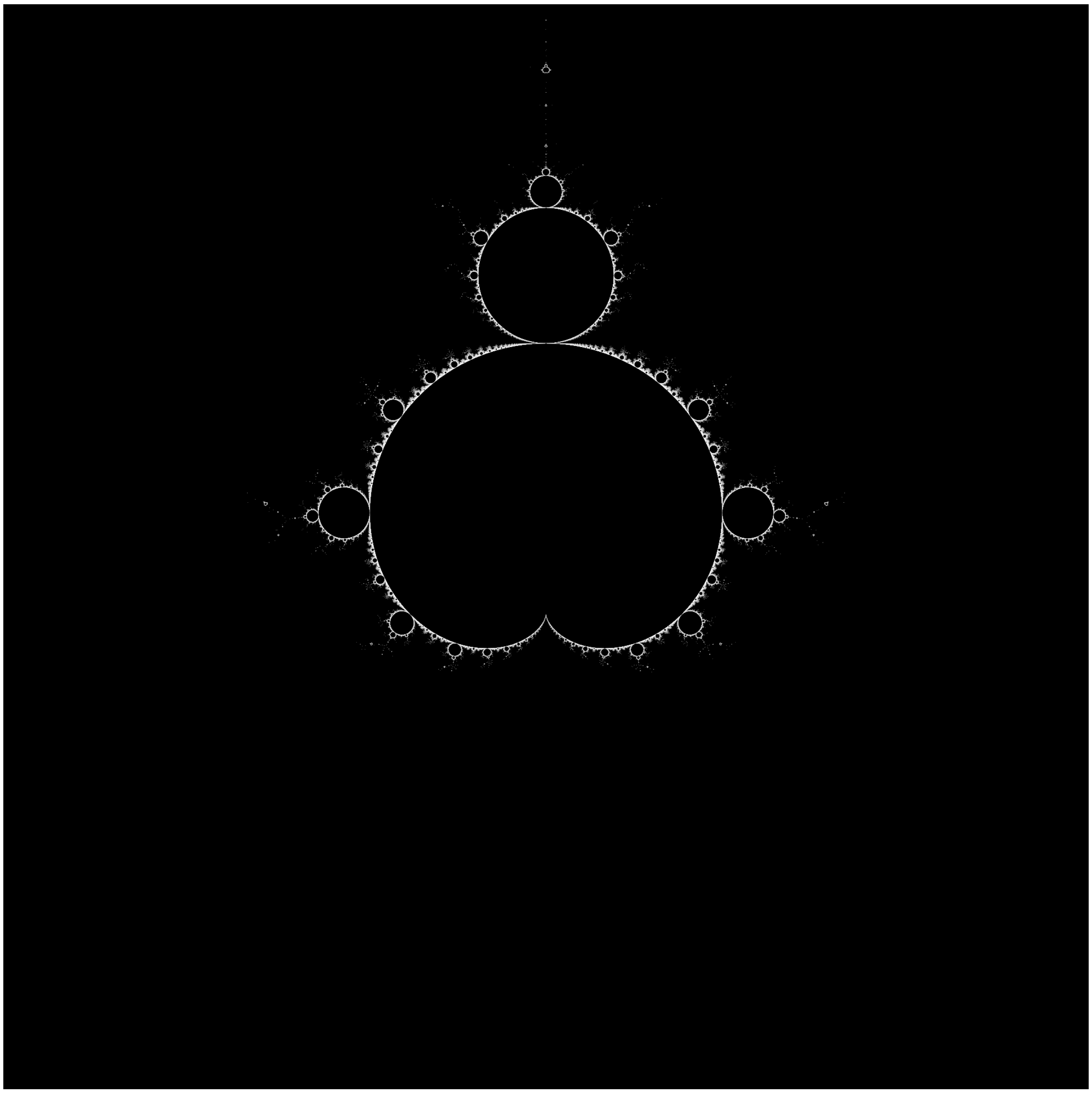
Addendum: Reference Code & Visualizations

```
package main

import (
    "flag"
    "fmt"
    "image"
    "image/color"
    "image/png"
    "math/cmplx"
    "math/rand"
    "os"
    "strconv"
    "time"
)

func main() {

    fmt.Println("\nSingularity Anthropobrot Visualizer v0.1")
    fmt.Println("(C) AGGOTT HÖNSCH István (apeirography.com) 2016")
    fmt.Println("Call with -h for list of possible arguments.")

    rand.Seed(time.Now().UTC().UnixNano())

    var singularity int
    var dimension int
    var minlimit, maxlimit int
    var passes int
    var updates int

    flag.IntVar(&singularity, "s", 1, "the number of the [s]
        ingularity anthropobrot to be generated")
    flag.IntVar(&dimension, "d", 513, "the width and height [d]
        imension of the image to be generated")
    flag.IntVar(&minlimit, "min", 0, "the [min]imum depth of
        governing seeds to be used; min > s")
    flag.IntVar(&maxlimit, "max", 0, "the [max]imum depth of
        governing seeds to be used; max > min")
    flag.IntVar(&passes, "pass", 5, "the number of [pass]es worth
        of seeds should be used")
    flag.IntVar(&updates, "updates", 100, "the number of progress
        updates to provide from beginning to end")
    flag.Parse()

    if singularity < 1 {
        singularity = 1
    }

    if dimension < 10 {
        dimension = 10
    }

    if minlimit < singularity {
        minlimit = singularity * 32
    }
```

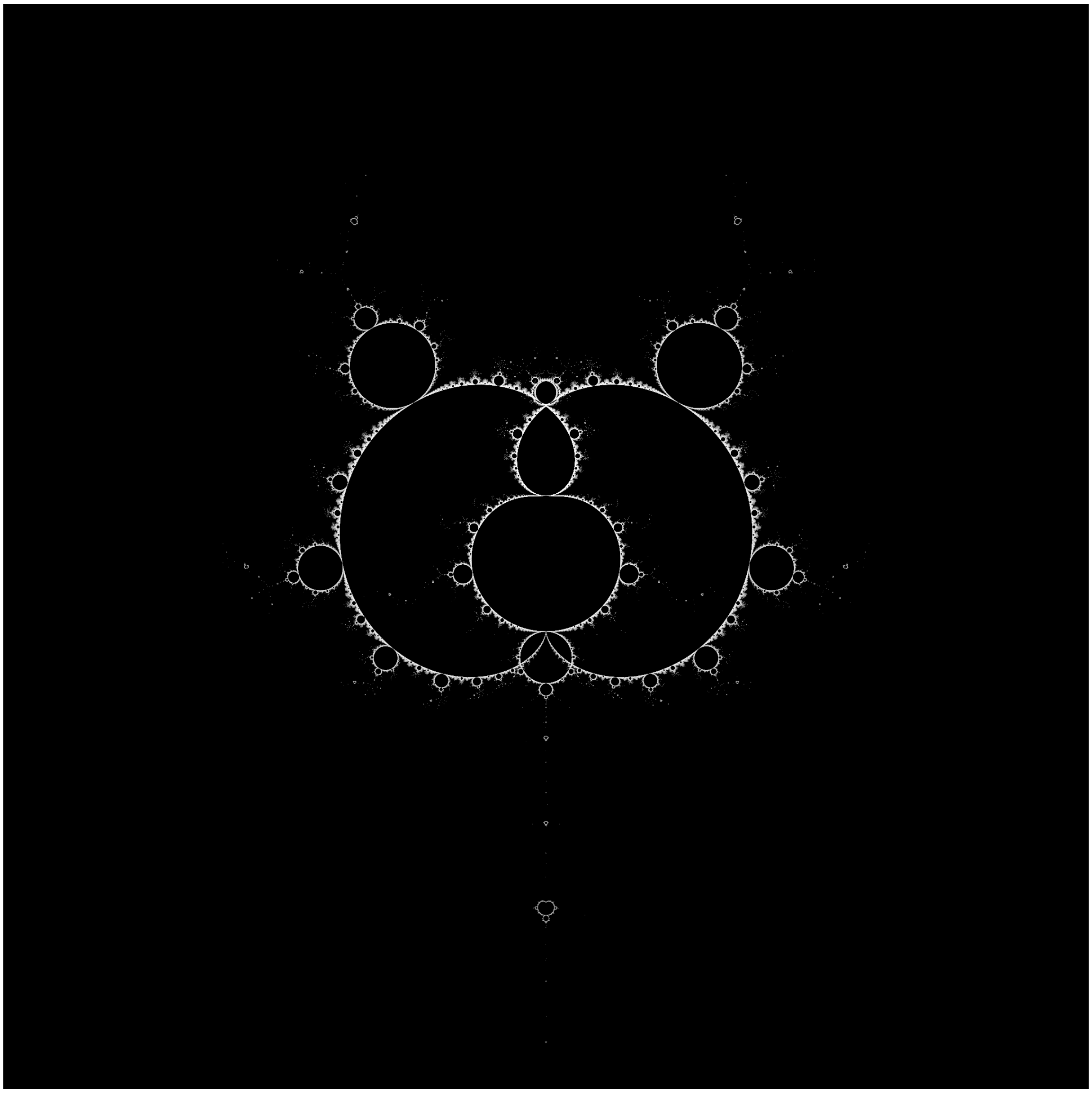
```
if maxlimit < minlimit {
    maxlimit = minlimit * 2
}

if updates < 1 {
    updates = 1
}

width, height := dimension, dimension

fmt.Println("\n\tpreparing to generate: anthropobrot",
    singularity)

fmt.Println("\tcreating", width, "by", height, "image")
img := image.NewRGBA(image.Rect(0, 0, width, height))

for y := 0; y < height; y++ {
    for x := 0; x < width; x++ {
        img.Set(x, y, color.RGBA{R: 0, G: 0, B: 0, A: 255})
    }
}

fmt.Println("\trestricting governing seed depth from", minlimit
    , "to", maxlimit)

soughtseeds := width * height * passes

fmt.Println("\tseeking", soughtseeds, "or", passes, "passes
    worth of governing seeds\n")

usedseeds := 0 // Number of seeds used so far.
for usedseeds < soughtseeds {
    var z, c complex128
    depth := -1
    for depth < minlimit {
        c = complex(rand.Float64()*4.00-2.00, rand.Float64()
            *4.00-2.00)
        depth = CheckDepth(c, maxlimit)
    }

    iters := 0
    z = complex(0.00, 0.00)
    for iters < singularity {
        z = z*z + c
        iters++
        if iters == singularity {
            x := int((real(z) + 2.00) * (float64(width) / 4.00)
                )
            y := int((imag(z) + 2.00) * (float64(height) /
                4.00))
            r, _, _, _ := img.At(x, y).RGBA()
            gray := uint8(r / 256)
            if gray < 255 {
                gray++
            }
            img.Set(x, y, color.RGBA{R: gray, G: gray, B: gray,
                 A: 255})
```

```
            }
        }
        usedseeds++
        if usedseeds%(soughtseeds/updates) == 0 {
            fmt.Printf("\t%.2f percent complete\n", float64(
                usedseeds)/float64(soughtseeds)*100)
        }
    }

    filename := "anthropobrot_" + strconv.Itoa(singularity)
    outfile, _ := os.Create(filename + ".png")
    png.Encode(outfile, img)
    outfile.Close()

}

func CheckDepth(c complex128, limit int) int {
    z := complex(0, 0)
    var counter int
    for counter = 1; counter < limit && cmplx.Abs(z) < 2.00;
        counter++ {
        z = z*z + c
    }
    if counter == limit {
        counter = -1
    }
    return counter
}
```

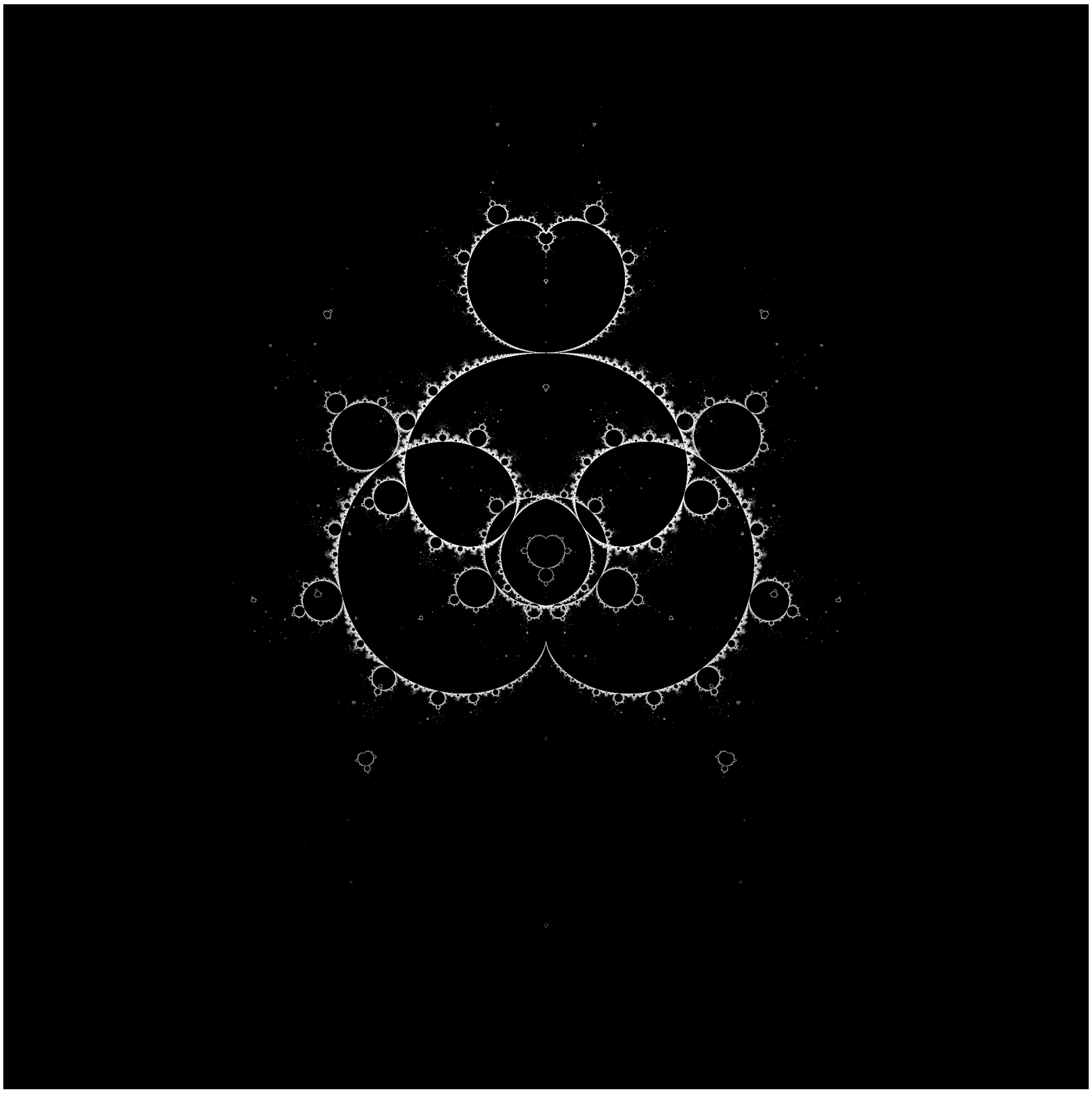
anthropobrot/main.go (Go language Singularity Anthropobrot Visualizer)



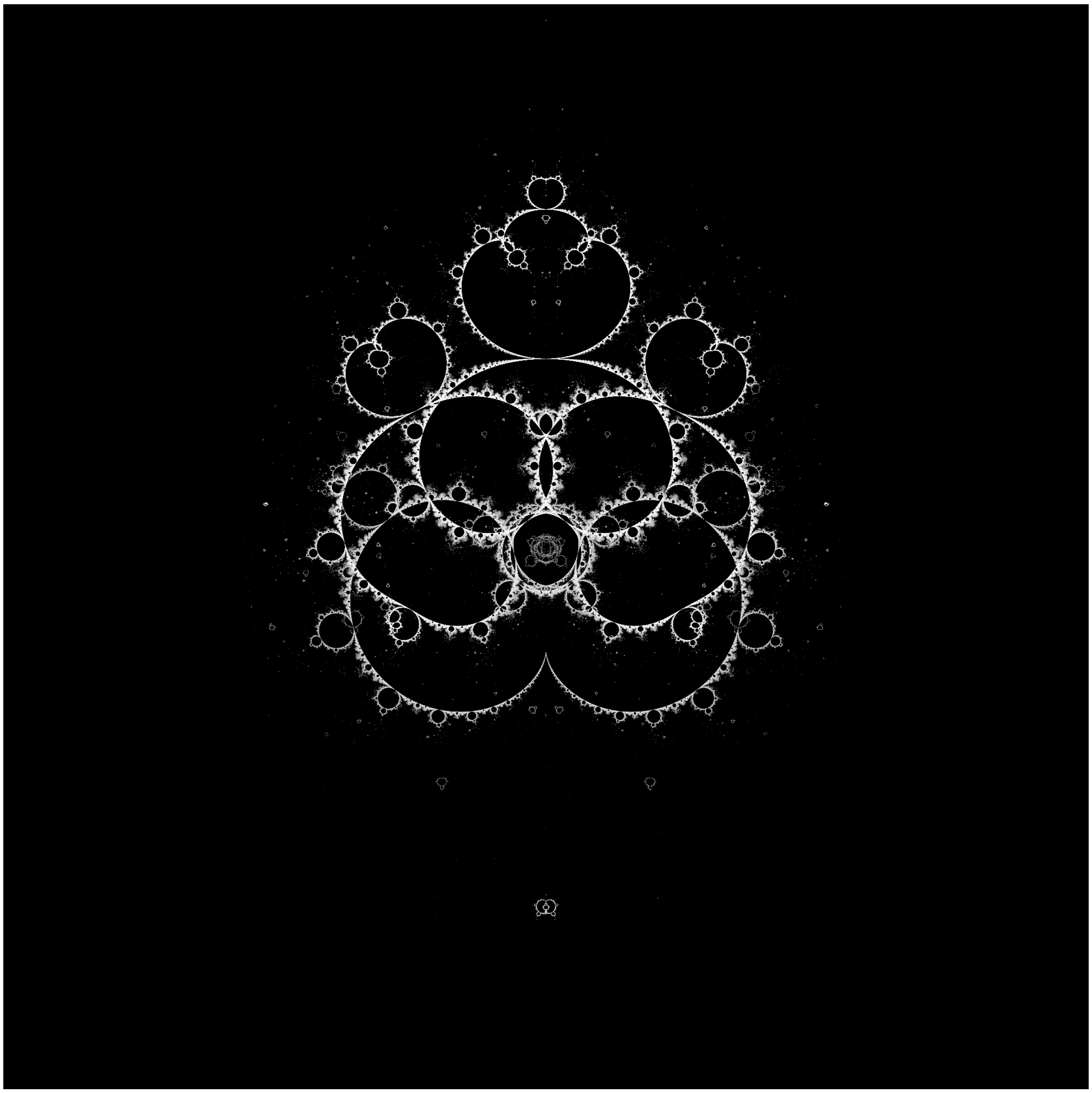
$\mathcal{A}_1$: Mandelbrot mnemeiobrot



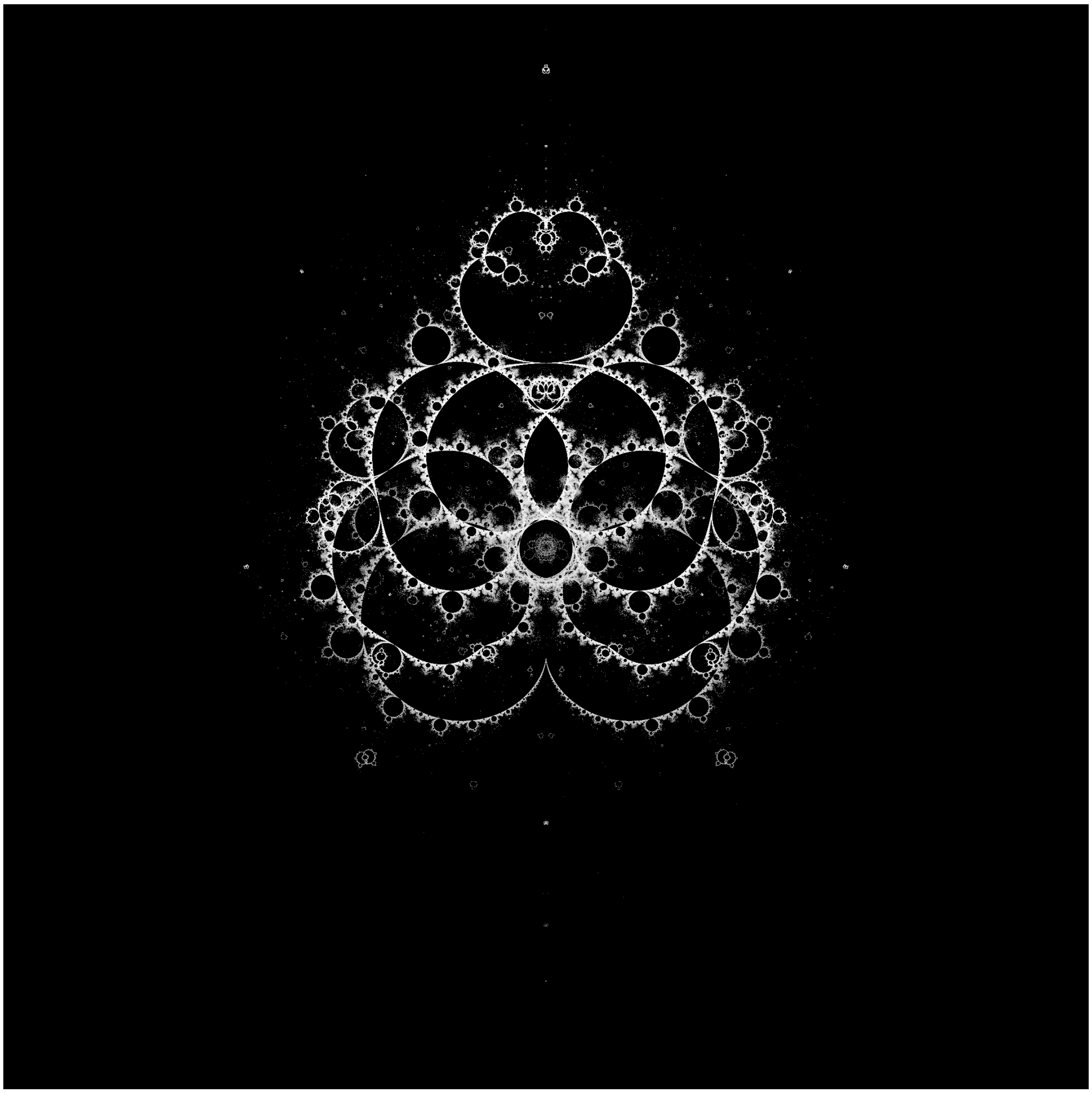
$\mathcal{A}_2$: Fatou-Julia mnemeiobrot



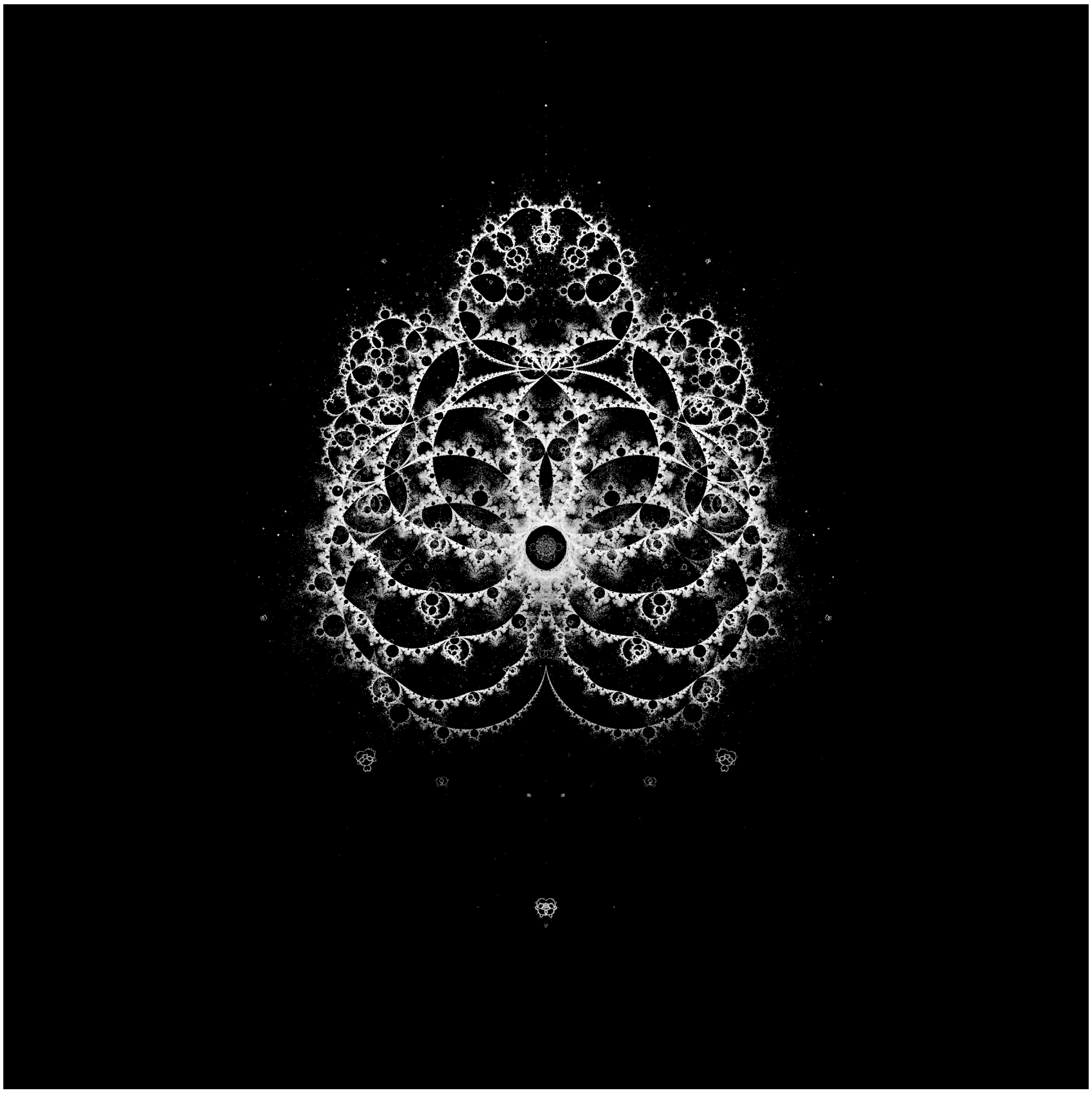
$\mathcal{A}_3$: Cyclops, priestly anthropobrot



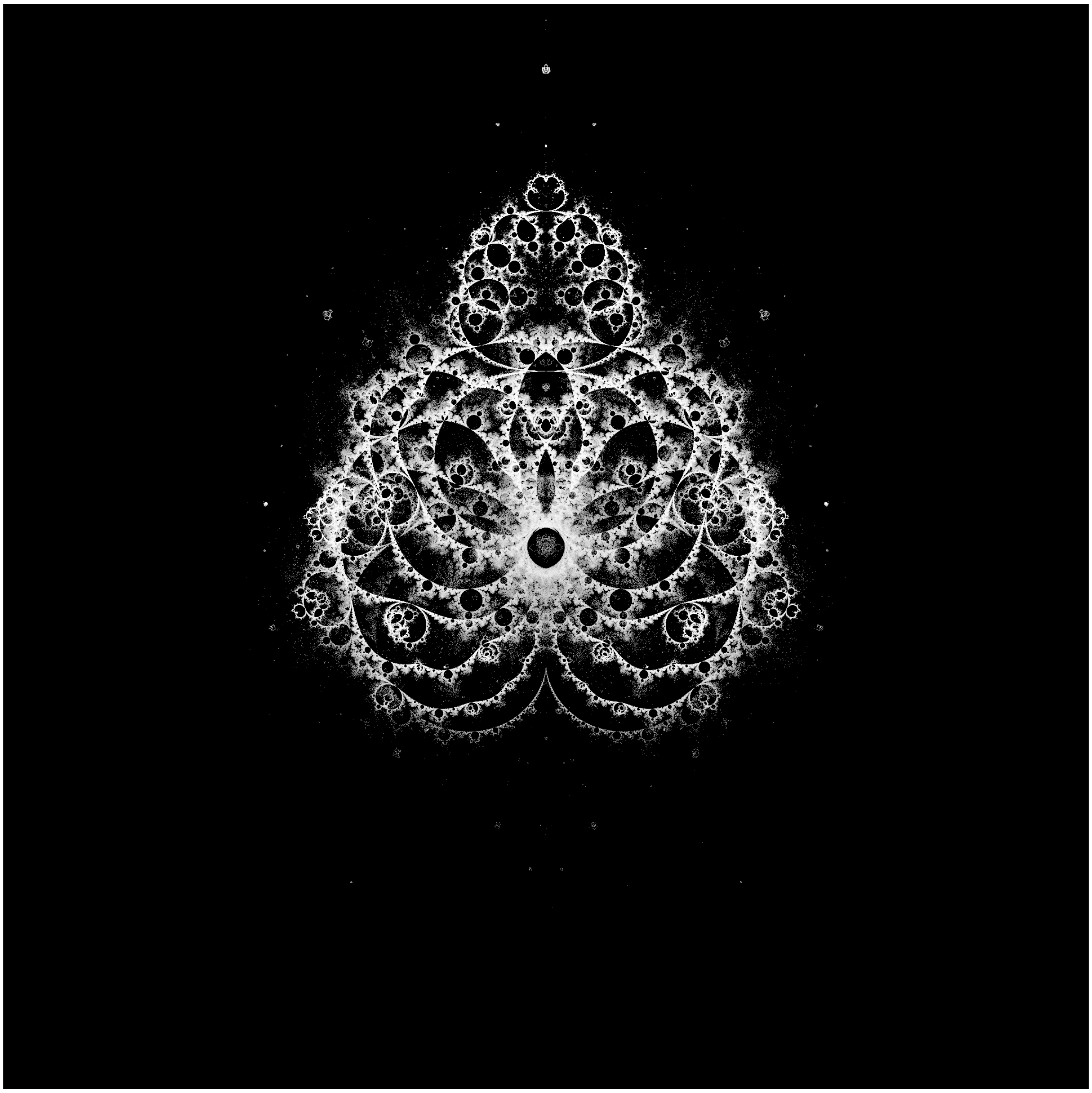
$\mathcal{A}_5$: Parson of Andreas, royal anthropobrot



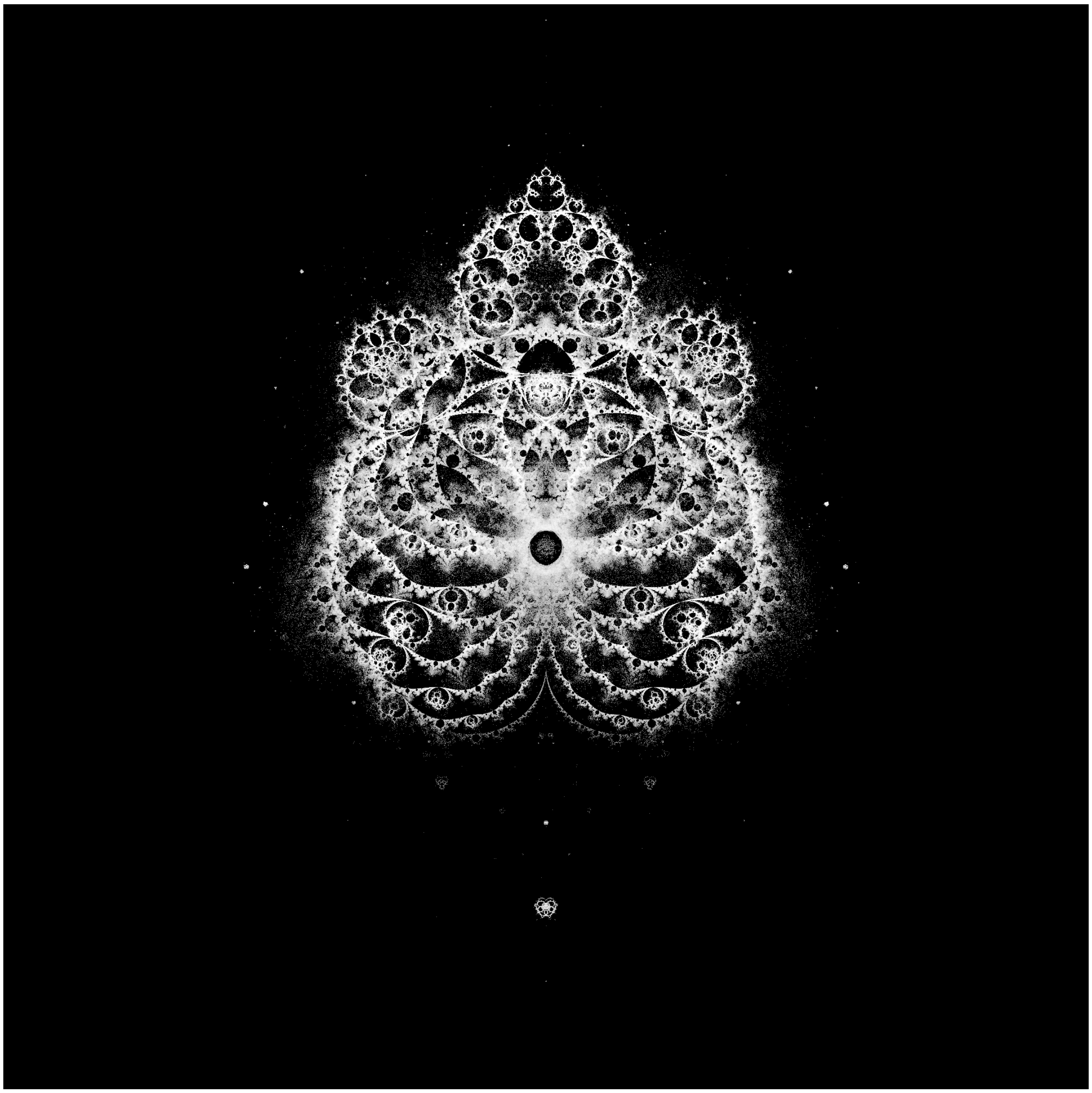
$\mathcal{A}_7$: Cangjie, priestly anthropobrot



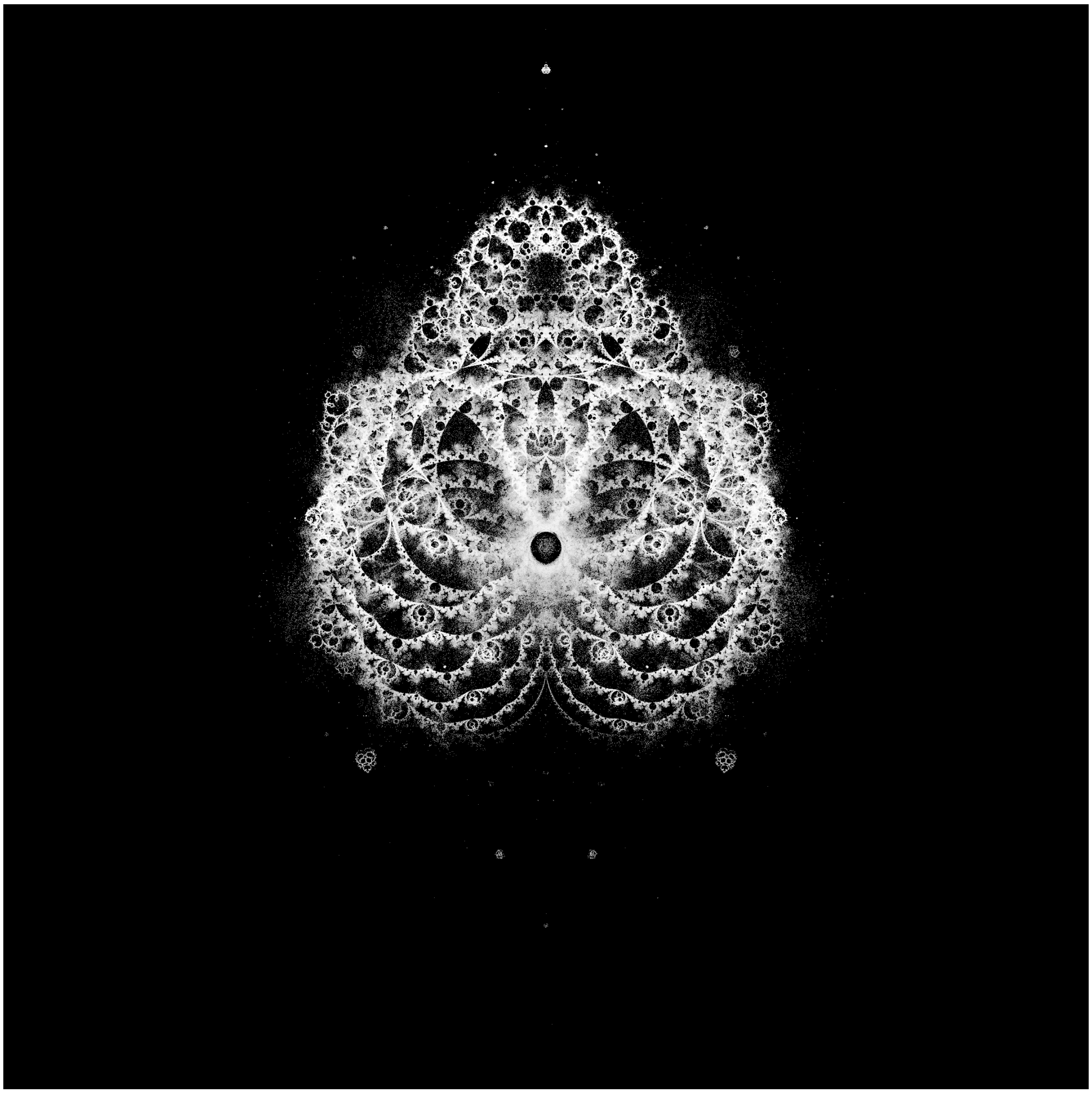
$\mathcal{A}_{11}$: Green, priestly anthropobrot



$\mathcal{A}_{13}$: Vepstas, royal anthropobrot



$\mathcal{A}_{17}$: Gardi, royal anthropobrot



$\mathcal{A}_{19}$: Boswell, priestly anthropobrot