A Relational Formulation of Special Relativity

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This article presents a relational formulation of special relativity whose kinematic and dynamic quantities are invariant under transformations between inertial reference frames. In addition, a new universal force is presented.

Introduction

From an auxiliary massive particle (called auxiliary-point) can be obtained kinematic quantities (such as relational time, relational position, etc.) that are invariant under transformations between inertial reference frames.

An auxiliary-point is an arbitrary massive particle free of external forces (or that the net force acting on it is zero)

The relational time $(\,\bar{t}\,)$, the relational position $(\,\bar{\mathbf{r}}\,)$, the relational velocity $(\,\bar{\mathbf{v}}\,)$ and the relational acceleration $(\,\bar{\mathbf{a}}\,)$ of a (massive or non-massive) particle relative to an inertial reference frame S are given by:

$$\bar{t} \doteq \gamma \left(t - \frac{\mathbf{r} \cdot \boldsymbol{\psi}}{c^2} \right)$$

$$\bar{\mathbf{r}} \doteq \left[\mathbf{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\mathbf{r} \cdot \boldsymbol{\psi})\, \boldsymbol{\psi}}{c^2} - \gamma\, \boldsymbol{\psi}\, t \right]$$

$$\bar{\mathbf{v}} \doteq \left[\mathbf{v} + \frac{\gamma^2}{\gamma + 1} \frac{(\mathbf{v} \cdot \boldsymbol{\psi})\, \boldsymbol{\psi}}{c^2} - \gamma\, \boldsymbol{\psi} \right] \frac{1}{\gamma\, (1 - \frac{\mathbf{v} \cdot \boldsymbol{\psi}}{c^2})}$$

$$\bar{\mathbf{a}} \doteq \left[\mathbf{a} - \frac{\gamma}{\gamma + 1} \frac{(\mathbf{a} \cdot \boldsymbol{\psi})\, \boldsymbol{\psi}}{c^2} + \frac{(\mathbf{a} \times \mathbf{v}) \times \boldsymbol{\psi}}{c^2} \right] \frac{1}{\gamma^2\, (1 - \frac{\mathbf{v} \cdot \boldsymbol{\psi}}{c^2})^3}$$

where $(\,t,\, \mathbf{r},\, \mathbf{v},\, \mathbf{a}\,)$ are the time, the position, the velocity and the acceleration of the particle relative to the inertial reference frame S, $(\,\boldsymbol{\psi}\,)$ is the velocity of the auxiliary-point relative to the inertial reference frame S and $(\,c\,)$ is the speed of light in vacuum. $(\,\boldsymbol{\psi}\,)$ is a constant. $\gamma = (1 - \boldsymbol{\psi} \cdot \boldsymbol{\psi}/c^2)^{-1/2}$

The relational frequency ($\bar{\nu}$) of a non-massive particle relative to an inertial reference frame S is given by:

$$\bar{\nu} \doteq \nu \, \frac{\left(1 - \dfrac{\mathbf{c} \cdot \boldsymbol{\psi}}{c^2}\right)}{\sqrt{1 - \dfrac{\boldsymbol{\psi} \cdot \boldsymbol{\psi}}{c^2}}}$$

where (ν) is the frequency of the non-massive particle relative to the inertial reference frame S, ($\mathbf{c}$) is the velocity of the non-massive particle relative to the inertial reference frame S, ($\boldsymbol{\psi}$) is the velocity of the auxiliary-point relative to the inertial reference frame S and (c) is the speed of light in vacuum.

The relational mass (m) of a massive particle is given by:

$$m \doteq \frac{m_o}{\sqrt{1 - \dfrac{\bar{\mathbf{v}} \cdot \bar{\mathbf{v}}}{c^2}}}$$

where (m_o) is the rest mass of the massive particle, ($\bar{\mathbf{v}}$) is the relational velocity of the massive particle and (c) is the speed of light in vacuum.

The relational mass (m) of a non-massive particle is given by:

$$m \doteq \frac{h\,\bar{\nu}}{c^2}$$

where (h) is the Planck constant, ($\bar{\nu}$) is the relational frequency of the non-massive particle and (c) is the speed of light in vacuum.

Observations

§ In arbitrary inertial reference frames ($\bar{t}_\alpha \neq \tau_\alpha$ or $\bar{\mathbf{r}}_\alpha \neq 0$) (α = auxiliary-point) a constant must be add in the definition of relational time such that the relational time and the proper time of the auxiliary-point are the same ($\bar{t}_\alpha = \tau_\alpha$) and another constant must be add in the definition of relational position such that the relational position of the auxiliary-point is zero ($\bar{\mathbf{r}}_\alpha = 0$)

§ In the particular case of an isolated system of (massive or non-massive) particles, inertial observers should preferably use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ($\sum_z m_z \, \bar{\mathbf{v}}_z = 0$)

The Relational Dynamics

If we consider a (massive or non-massive) particle with relational mass m then the linear momentum $\mathbf{P}$ of the particle, the angular momentum $\mathbf{L}$ of the particle, the net force $\mathbf{F}$ acting on the particle, the work W done by the net force acting on the particle, and the kinetic energy K of the particle, for an inertial reference frame, are given by:

$$\mathbf{P} \doteq m\,\bar{\mathbf{v}}$$

$$\mathbf{L} \doteq \mathbf{P} \times \bar{\mathbf{r}} = m\,\bar{\mathbf{v}} \times \bar{\mathbf{r}}$$

$$\mathbf{F} = \frac{d\mathbf{P}}{d\bar{t}} = m\,\bar{\mathbf{a}} + \frac{dm}{d\bar{t}}\,\bar{\mathbf{v}}$$

$$\mathrm{W} \doteq \int_1^2 \mathbf{F} \cdot d\bar{\mathbf{r}} = \int_1^2 \frac{d\mathbf{P}}{d\bar{t}} \cdot d\bar{\mathbf{r}} = \Delta\,\mathrm{K}$$

$$\mathrm{K} \doteq m\,c^2$$

where $(\,\bar{t},\ \bar{\mathbf{r}},\ \bar{\mathbf{v}},\ \bar{\mathbf{a}}\,)$ are the relational time, the relational position, the relational velocity and the relational acceleration of the particle relative to the inertial reference frame and $(\,c\,)$ is the speed of light in vacuum. The kinetic energy $(\,\mathrm{K}_o\,)$ of a massive particle at rest is $(\,m_o\,c^2\,)$

Forces and fields must be expressed only with relational quantities (the Lorentz force must be expressed with the relational velocity $\bar{\mathbf{v}}$, the electric field must be expressed with the relational position $\bar{\mathbf{r}}$, etc.)

Conclusions

§ In this article, the quantities $(\,\bar{t}, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \bar{\nu}, m, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}\,)$ are invariant under transformations between inertial reference frames.

§ However, this article considers (1) that it would also be possible to obtain kinematic and dynamic quantities $(\,\bar{t}, \bar{\mathbf{r}}, \bar{\mathbf{v}}, \bar{\mathbf{a}}, \bar{\nu}, m, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}\,)$ that would be invariant under transformations between inertial and non-inertial reference frames and (2) that the dynamic quantities $(\,m, \mathbf{P}, \mathbf{L}, \mathbf{F}, \mathrm{W}, \mathrm{K}\,)$ would also be given by the equations of this article.

The Kinetic Force

In an isolated system of (massive or non-massive) particles, the kinetic force $\mathbf{K}_{ij}$ exerted on a particle i with relational mass m_i by another particle j with relational mass m_j is given by:

$$\mathbf{K}_{ij} \;=\; -\,\frac{d}{d\bar{t}_i}\left[\,\frac{m_i\,m_j}{\mathrm{M}}\,(\,\bar{\mathbf{v}}_i - \bar{\mathbf{v}}_j\,)\,\right]$$

where $\bar{t}_i$ is the relational time of the particle i, $\bar{\mathbf{v}}_i$ is the relational velocity of the particle i, $\bar{\mathbf{v}}_j$ is the relational velocity of the particle j and M ($= \sum_z m_z$) is the relational mass of the isolated system of particles.

From the above equation it follows that the net kinetic force $\mathbf{K}_i$ ($= \sum_z \mathbf{K}_{iz}$) acting on the particle i is given by:

$$\mathbf{K}_i \;=\; -\,\frac{d}{d\bar{t}_i}\left[\,m_i\,\bar{\mathbf{v}}_i\,\right]$$

where $\bar{t}_i$ is the relational time of the particle i, m_i is the relational mass of the particle i and $\bar{\mathbf{v}}_i$ is the relational velocity of the particle i.

Now, substituting ($\mathbf{F}_i = d(m_i\,\bar{\mathbf{v}}_i)/d\bar{t}_i$) and rearranging, we obtain:

$$\mathbf{T}_i \;\doteq\; \mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Therefore, in an isolated system of (massive or non-massive) particles, the total force $\mathbf{T}_i$ acting on a particle i is always zero.

Inertial observers must use an auxiliary-point such that the linear momentum of the isolated system of particles is zero ($\sum_z m_z\,\bar{\mathbf{v}}_z = 0$)

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Appendix I

Generalized Lorentz Transformations

The time (t'), the position ($\mathbf{r}'$), the velocity ($\mathbf{v}'$) and the acceleration ($\mathbf{a}'$) of a (massive or non-massive) particle relative to an inertial reference frame S' are given by:

$$
\begin{aligned}
t' &= \gamma \left(t - \frac{\mathbf{r} \cdot \mathbf{V}}{c^2} \right) \\
\mathbf{r}' &= \left[\mathbf{r} + \frac{\gamma^2}{\gamma + 1} \frac{(\mathbf{r} \cdot \mathbf{V})\, \mathbf{V}}{c^2} - \gamma\, \mathbf{V}\, t \right] \\
\mathbf{v}' &= \left[\mathbf{v} + \frac{\gamma^2}{\gamma + 1} \frac{(\mathbf{v} \cdot \mathbf{V})\, \mathbf{V}}{c^2} - \gamma\, \mathbf{V} \right] \frac{1}{\gamma\, (1 - \frac{\mathbf{v} \cdot \mathbf{V}}{c^2})} \\
\mathbf{a}' &= \left[\mathbf{a} - \frac{\gamma}{\gamma + 1} \frac{(\mathbf{a} \cdot \mathbf{V})\, \mathbf{V}}{c^2} + \frac{(\mathbf{a} \times \mathbf{v}) \times \mathbf{V}}{c^2} \right] \frac{1}{\gamma^2\, (1 - \frac{\mathbf{v} \cdot \mathbf{V}}{c^2})^3}
\end{aligned}
$$

where (t, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$) are the time, the position, the velocity and the acceleration of the particle relative to an inertial reference frame S, ($\mathbf{V}$) is the velocity of the inertial reference frame S' relative to the inertial reference frame S and (c) is the speed of light in vacuum. ($\mathbf{V}$) is a constant. $\gamma = (1 - \mathbf{V} \cdot \mathbf{V}/c^2)^{-1/2}$

Transformation of Frequency

The frequency (ν') of a non-massive particle relative to an inertial reference frame S' is given by:

$$
\nu' = \nu \, \frac{\left(1 - \dfrac{\mathbf{c} \cdot \mathbf{V}}{c^2} \right)}{\sqrt{1 - \dfrac{\mathbf{V} \cdot \mathbf{V}}{c^2}}}
$$

where (ν) is the frequency of the non-massive particle relative to an inertial reference frame S, ($\mathbf{c}$) is the velocity of the non-massive particle relative to the inertial reference frame S, ($\mathbf{V}$) is the velocity of the inertial reference frame S' relative to the inertial reference frame S and (c) is the speed of light in vacuum.

Appendix II

System of Equations

$$
\begin{tikzcd}
 & \boxed{[1]} \arrow[d, "d\bar{t}"] & \\
\boxed{[4]} \arrow[d, "d\bar{t}"] & \boxed{[2]} \arrow[l, "\times \bar{\mathbf{r}}"'] \arrow[d, "d\bar{t}"] & \\
\boxed{[5]} & \boxed{[3]} \arrow[l, "\times \bar{\mathbf{r}}"'] \arrow[r, "\int d\bar{\mathbf{r}}"] & \boxed{[6]}
\end{tikzcd}
$$

$$[1] \qquad \frac{1}{\mu}\left[\int \mathbf{P}\, d\bar{t} - \int\!\!\int \mathbf{F}\, d\bar{t}\, d\bar{t}\right] = 0$$

$$[2] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\, d\bar{t}\right] = 0$$

$$[3] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{d\bar{t}} - \mathbf{F}\right] = 0$$

$$[4] \qquad \frac{1}{\mu}\left[\mathbf{P} - \int \mathbf{F}\, d\bar{t}\right] \times \bar{\mathbf{r}} = 0$$

$$[5] \qquad \frac{1}{\mu}\left[\frac{d\mathbf{P}}{d\bar{t}} - \mathbf{F}\right] \times \bar{\mathbf{r}} = 0$$

$$[6] \qquad \frac{1}{\mu}\left[\int \frac{d\mathbf{P}}{d\bar{t}} \cdot d\bar{\mathbf{r}} - \int \mathbf{F} \cdot d\bar{\mathbf{r}}\right] = 0$$

$[\,\mu\,]$ is an arbitrary (universal) constant with dimension of mass.

Appendix III

System of Equations

$$
\begin{tikzcd}
 & \boxed{[1]} \arrow[d, "d\bar{t}"] & \\
\boxed{[4]} \arrow[d, "d\bar{t}"] & \boxed{[2]} \arrow[l, "\times \bar{\mathbf{r}}"'] \arrow[d, "d\bar{t}"] & \\
\boxed{[5]} & \boxed{[3]} \arrow[l, "\times \bar{\mathbf{r}}"'] \arrow[r, "\int d\bar{\mathbf{r}}"] & \boxed{[6]}
\end{tikzcd}
$$

$$[1] \qquad \frac{1}{\mu}\left[\int m\,\bar{\mathbf{v}}\,d\bar{t} - \iint \mathbf{F}\,d\bar{t}\,d\bar{t}\right] = 0$$

$$[2] \qquad \frac{1}{\mu}\left[m\,\bar{\mathbf{v}} - \int \mathbf{F}\,d\bar{t}\right] = 0$$

$$[3] \qquad \frac{1}{\mu}\left[m\,\bar{\mathbf{a}} + \frac{dm}{d\bar{t}}\,\bar{\mathbf{v}} - \mathbf{F}\right] = 0$$

$$[4] \qquad \frac{1}{\mu}\left[m\,\bar{\mathbf{v}} - \int \mathbf{F}\,d\bar{t}\right] \times \bar{\mathbf{r}} = 0$$

$$[5] \qquad \frac{1}{\mu}\left[m\,\bar{\mathbf{a}} + \frac{dm}{d\bar{t}}\,\bar{\mathbf{v}} - \mathbf{F}\right] \times \bar{\mathbf{r}} = 0$$

$$[6] \qquad \frac{1}{\mu}\left[m\,c^2 - \int \mathbf{F}\cdot d\bar{\mathbf{r}}\right] = 0$$

$[\,\mu\,]$ is an arbitrary (universal) constant with dimension of mass.