# HARMONY

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ABSTRACT

Use of the harmonic numbers to create congruencies is discussed. Interesting relations to known congruencies are shown.

The congruence $\binom{2n-1}{n-1} \equiv 1\ (mod\ n)$ is well known and has been shown to have solutions in primes, squares of odd primes and cubes of primes $\geq 5$. This congruence also has some composite and non-prime power solutions as well. To the authors best knowledge, what may not be known is its relationship to the denominator of the harmonic numbers. If you divide the denominator of the $(n+1)^{th}$ harmonic number $H_{n+1}$ by the denominator of the $n^{th}$ harmonic number $H_n$, you will find a sequence that looks like this,

$$\frac{denominator(H_{n+1})}{denominator(H_n)} = 2, 3, 2, 5, \frac{1}{3}, 7, 2, 9, 1, 11, 1, 13, 1, 1, 2, 17, \frac{1}{3}, 19, \frac{1}{5}, \frac{1}{3}, 1, 23, 3, 25, \cdots$$

If you study the behavior of this sequence for higher and higher numbers you can find that it takes on values of 1, $p^2$ for odd primes, $p^3$ for $p \geq 5$ and the inverses of these, $\frac{1}{p}, \frac{1}{p^2}, \frac{1}{p^3}$. It takes on the prime and prime power solutions to the above congruence at n+1. For example if you check the value at $p^2 + 1$, you will get $p^2$. All other values are <n+1 so dividing this fraction by n+1 gives,

$$\frac{denominator(H_{n+1})}{(n+1)denominator(H_n)} = 1, 1, \frac{1}{2}, 1, \frac{1}{18}, 1, \frac{1}{4}, 1, \frac{1}{10}, 1, \frac{1}{12}, 1, \frac{1}{14}, \frac{1}{15}, \frac{1}{8}, 1, \frac{1}{54}, 1, \frac{1}{100}, \cdots$$

What is interesting is that this is 1 for all the prime and prime power solutions mentioned in the above congruence and $< 1$ otherwise, but does not seem to have any of the composite solutions. At the composite solutions of the congruence, this fraction is $< 1$. So, taking the floor of this creates a characteristic equation.

$$\left\lfloor\frac{denominator(H_{n+1})}{(n+1)denominator(H_n)}\right\rfloor = \begin{cases} 1\ if\ n+1 = p, p^2\ for\ p > 2, p^3\ for\ p \geq 5 \\ 0\ otherwise \end{cases}$$

The Mersenne numbers have the form $2^n - 1$ which clearly cannot take the form $p^2$ or $p^3$ so the statement can be made,

A number $2^n - 1$ is a Mersenne Prime $iff$ $\left\lfloor\frac{denominator(H_{2^n-1})}{(2^n-1)denominator(H_{2^n-2})}\right\rfloor = 1$

Otherwise, the expression would yield 0. And of course we can count Mersenne primes as well.

$$\sum_{j=1}^{n}\left\lfloor\frac{denominator(H_{2^j-1})}{(2^j-1)denominator(H_{2^j-2})}\right\rfloor = \#\ of\ Mersenne\ Primes\ M_p \leq n+1$$

Although, not very efficiently. This will hold for any function that cannot be a square or a cube. For example,

$$\sum_{j=1}^{n}\left\lfloor\frac{denominator(H_{j^2+1})}{(j^2+1)denominator(H_{j^2})}\right\rfloor = \#\ of\ Primes\ p\ of\ the\ form\ (m^2+1) \leq n+1$$

Interestingly, the partial sums of $\binom{2n-1}{n-1} mod\ n = 1$ are $n^2 + 1$.

The numerators to the harmonic numbers show the same property.

$$\left\lfloor\frac{denominator(H_{n+1})}{(n+1)denominator(H_n)}\right\rfloor = \left\lfloor\frac{numerator(H_{n+1})}{(n+1)numerator(H_n)}\right\rfloor$$

It follows that,

$$\frac{denominator(H_{p_n})}{denominator(H_{p_n-1})} = p_n$$

And for different orders of harmonic numbers,

$$\sum_{j=1}^{n}\left\lfloor\frac{denominator(H_{n+1,m})}{(j+1)^m denominator(H_{n,m})}\right\rfloor = \pi(n+1) \qquad m > 1$$