THE CHARACTERISTICS OF THE PRIMES

Anthony J. Browne

April 22, 2016

ABSTRACT

Summing characteristic equations to find forms of theoretical functions in number theory will be discussed. Forms of many number theoretic functions will be derived. Although many may not be efficient in a computing sense for large numbers, the aim in this paper will simply be to explore what these forms are and show relationships between expressions.

For a modular congruence $f \equiv mod\ g$, it is a simple question of whether or not $g|f$. A question about primes is a question about division. This question can be turned into an indicator function and summed to find the number of possible solutions where $g|f$ up to a chosen point.

Theorem:

$$\left\lfloor\frac{f}{g}\right\rfloor - \left\lfloor\frac{f-1}{g}\right\rfloor = \begin{cases} 1 \ if \ g|f \\ 0 \ otherwise \end{cases} \quad g > 0$$

Proof:

If $g|f$, than $\frac{f}{g} = a$ for some constant a making the expression $\lfloor a \rfloor - \left\lfloor a - \frac{1}{g}\right\rfloor$. Since $\frac{1}{g} < 1$, $a - 1 < a - \frac{1}{g} < a$. Therefore by the definition of the floor function, the expression will yield $a - (a-1) = 1$. If $g \nmid f$, then the expression will be of the form $\left\lfloor a + \frac{x}{y}\right\rfloor - \left\lfloor a + \frac{gx-y}{gy}\right\rfloor$. Since $\frac{x}{y} < 1$, this must imply that $x < y$ and so $gx - y < gy$. So, by the definition of the floor function, the expression would yield $a - a = 0$.
End Proof

This indicator function, or any other, can be used to count solutions of even divisions by summing them. As an obvious example the divisor function, τ(n) has the form

$\tau(n) = \sum_{j=1}^{n} \left\lfloor\frac{n}{j}\right\rfloor - \left\lfloor\frac{n-1}{j}\right\rfloor$. This is clearly counting the number of times j divides n. This is nothing new. The question here is what else can be derived using this form and of what use is it. Well, let's start with the first question.

Theorem:

$$n + 1 - \sum_{j=1}^{n} \left\lfloor \frac{j!^n}{n} \right\rfloor - \left\lfloor \frac{j!^n - 1}{n} \right\rfloor = Greatest\ prime\ factor\ of\ n, GPF(n)$$

Proof:

The indicator function will be 1 if the denominator divides the numerator evenly and 0 otherwise. Division of course can only occur if the prime factors in the denominator are a subset of the factors of the numerator and if the exponents in the prime factorization of the numerator are equal or larger than those in the denominator. The exponents here are clearly not an issue because the numerator is raised to the power of the denominator so only the factors matter. But, $j!$ contains every prime $p \leq j$ as a factor. So, n cannot divide evenly until j reaches the greatest prime factor of n. Every term after that will also divide. Therefore, summing this indicator function to n will give $n + 1 - GPF(n)$. Subtracting this from n+1 gives the desired result.

End Proof

Theorem:

$$\sum_{j=1}^{n} \left\lfloor \frac{n^j}{j} \right\rfloor - \left\lfloor \frac{n^j - 1}{j} \right\rfloor = the\ \#\ of\ \#'s \leq n\ whose\ prime\ factorization\ is\ a\ subset\ of\ n's$$

Proof:

As mentioned in the previous proof, the numerator is raised to the power of the denominator so this does not affect even division. So, j can only divide evenly if it's prime factorization is a subset of n's. Summing these even divisions using the indicator function gives the desired result.

End Proof

That is the theoretical function used by Andrew Granville in his work on the ABC Conjecture.

Theorem:

The smallest prime coprime to n, $spc(n)$ at the value $n!$ is equal to the nextprime function, $nextprime(n)$.

Proof:

Any factorial contains a factor of all primes$\leq n$, therefore the smallest prime that will not divide $n!$ must be the prime after n. By definition, this is $nextprime(n)$.

Theorem:

$$1+\sum_{j=1}^{n}\left\lfloor\frac{n^j}{j!}\right\rfloor-\left\lfloor\frac{n^j-1}{j!}\right\rfloor = \text{smallest prime coprime to n}, spc(n)$$

Proof:

If you expand the equation you will have,

$1+\left(\left\lfloor\frac{n^1}{1!}\right\rfloor-\left\lfloor\frac{n^1-1}{1!}\right\rfloor\right)+\left(\left\lfloor\frac{n^2}{2!}\right\rfloor-\left\lfloor\frac{n^2-1}{2!}\right\rfloor\right)+\cdots+\left(\left\lfloor\frac{n^n}{n!}\right\rfloor-\left\lfloor\frac{n^n-1}{n!}\right\rfloor\right).$
The first term in parenthesis can be simplified as, $\lfloor n\rfloor-\lfloor n-1\rfloor=1$. Thus, the expansion becomes $2+\left(\left\lfloor\frac{n^2}{2!}\right\rfloor-\left\lfloor\frac{n^2-1}{2!}\right\rfloor\right)+\left(\left\lfloor\frac{n^3}{3!}\right\rfloor-\left\lfloor\frac{n^3-1}{3!}\right\rfloor\right)+\cdots+\left(\left\lfloor\frac{n^n}{n!}\right\rfloor-\left\lfloor\frac{n^n-1}{n!}\right\rfloor\right).$

Consider $\frac{n^2}{2}$, the denominator will only divide the numerator evenly when the exponents of the factorization are larger or equal in the numerator and it contains a factor of 2. Since the exponents are clearly larger in the numerator, this will occur when n is a multiple of 2. Now consider both $\frac{n^3}{6}$ and $\frac{n^4}{24}$ and notice that there is an added factor of 3 in the denominator of both fractions. So, it will divide evenly when n is a multiple of $2\cdot 3$ or simply 6. Now consider both $\frac{n^5}{120}$ and $\frac{n^6}{720}$ and note that there is now an added factor of 5 in the denominators of both fractions, so even division will occur when n is a multiple of $2\cdot 3\cdot 5$ or simply 30. Continuing this, it is clear that even division occurs at primorial numbers and a new factor is introduced when j is a prime. The distance until the next introduced factor is just the distance between primes, which is just the difference between consecutive primes. Thus, the expansion of the equation may be written as,

$2+(p_2-p_1)\left(\left\lfloor\frac{n}{p\#_1}\right\rfloor-\left\lfloor\frac{n-1}{p\#_1}\right\rfloor\right)+\cdots+(p_{n+1}-p_n)\left(\left\lfloor\frac{n}{p\#_n}\right\rfloor-\left\lfloor\frac{n-1}{p\#_n}\right\rfloor\right)$. Which may be simplified as

$$2+\sum_{j=1}^{n}(p_{j+1}-p_j)\left(\left\lfloor\frac{n}{p\#_j}\right\rfloor-\left\lfloor\frac{n-1}{p\#_j}\right\rfloor\right).$$

Therefore,

$$1+\sum_{j=1}^{n}\left\lfloor\frac{n^j}{j!}\right\rfloor-\left\lfloor\frac{n^j-1}{j!}\right\rfloor=2+\sum_{j=1}^{n}(p_{j+1}-p_j)\left(\left\lfloor\frac{n}{p\#_j}\right\rfloor-\left\lfloor\frac{n-1}{p\#_j}\right\rfloor\right).$$

The right side of the equation at the value of $n!$ is,

$$2+\sum_{j=1}^{n}(p_{j+1}-p_j)\left(\left\lfloor\frac{n!}{p\#_j}\right\rfloor-\left\lfloor\frac{n!-1}{p\#_j}\right\rfloor\right).$$

This contains a characteristic equation that will be 1 when n! is divisible by a primorial evenly and 0 otherwise. Therefore, the equation is only summing the differences between primes and adding 2, which is in fact the prime number sequence. But, the number of terms does not increase until n reaches a prime number, at which point the next primorial may divide, giving the next prime number at exactly the previous one and staying at that value until the next prime number is reached by n. By definition, this is the nextprime function. This means that the left side of the equation must also be the nextprime function when evaluated at n! and the original equation is in fact the smallest prime coprime to n. End Proof

Theorem:

A proper upper bound for the nextprime function in the form $1+\sum_{j=1}^{n}\left\lfloor\frac{n!^j}{j!}\right\rfloor-\left\lfloor\frac{n!^j-1}{j!}\right\rfloor$ is 2n.

Proof:

The formula is only adding 1's and equates to the next prime, yet the next prime is larger than n for all n in the formula. Therefore, you must sum past n. This implies the need for an upper bound. Since the formula is to equal the next prime, we need an upper bound for this. It has been shown by Bertrand's Postulate that $2p_n > p_{n+1}$. If we define $nextprime(n)$ to be p_{n+1}, then because n will always be ≥ the previous prime p_n and $2p_n > p_{n+1}$, this shows 2n is a proper upper bound. Therefore,

$$1+\sum_{j=1}^{2n}\left\lfloor\frac{n!^j}{j!}\right\rfloor-\left\lfloor\frac{n!^j-1}{j!}\right\rfloor=nextprime(n).$$

End Proof

Just by making n equal to the n^{th} prime number p_n in the above equation we have a formula for p_{n+1} which is,

$$1+\sum_{j=1}^{2p_n-1}\left\lfloor\frac{p_n!^j}{j!}\right\rfloor-\left\lfloor\frac{p_n!^j-1}{j!}\right\rfloor=p_{n+1}.$$

Theorem:

$$\sum_{j=1}^{n}\left\lfloor\frac{j^n}{n}\right\rfloor-\left\lfloor\frac{j^n-1}{n}\right\rfloor=\frac{n}{rad(n)}$$

Proof:

Consider $\frac{2^n}{n}$, just as before, the exponents play no role in division so n must be a power of 2 to divide evenly. Put another way, the largest square-free divisor of n must be 2. Now consider $\frac{3^n}{n}$.

In this case, the largest square-free divisor of n must be 3 to divide evenly. Continuing this, it is clear that the indicator function is counting the number of multiples of $rad(n) \leq n$. But this is just $\frac{n}{rad(n)}$. End Proof

There are countless other functions that can be derived. I will list some here without proof.

$$\sum_{j=1}^{n} \left\lfloor \frac{aj}{n} \right\rfloor - \left\lfloor \frac{aj-1}{n} \right\rfloor = \gcd(a,n)$$

$$n + 1 - \sum_{j=1}^{n} \left\lfloor \frac{j!^{n-1}}{(n-1)!} \right\rfloor - \left\lfloor \frac{j!^{n-1}-1}{(n-1)!} \right\rfloor = prevprime(n)$$

$$\sum_{j=1}^{2^{n+1}-1} \left\lfloor \frac{(n+1)^{\frac{1}{n+1}}}{(1+\pi(j))^{\frac{1}{n+1}}} \right\rfloor - \left\lfloor \frac{(n+1)^{\frac{1}{n+1}}-1}{(1+\pi(j))^{\frac{1}{n+1}}} \right\rfloor = g_n$$

$$\sum_{j=1}^{n} \left\lfloor \frac{j^{n-1}-1}{n} \right\rfloor - \left\lfloor \frac{j^{n-1}-2}{n} \right\rfloor = \prod_{p|n} \gcd(p-1, n-1)$$

$$\left\lfloor \frac{1}{\sum_{j=1}^{n} \left\lfloor \frac{j^n}{n} \right\rfloor - \left\lfloor \frac{j^n-1}{n} \right\rfloor} \right\rfloor \prod_{j=1}^{n} 1 - 2\chi_p(j) \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) = \mu(n)$$

$$\sum_{j=1}^{n} \left(\left\lfloor \frac{j^n}{n} \right\rfloor - \left\lfloor \frac{j^n-1}{n} \right\rfloor \right) \left(\left\lfloor \frac{n}{j} \right\rfloor - \left\lfloor \frac{n-1}{j} \right\rfloor \right) = Product\ of\ exponents\ in\ prime\ factorization\ of\ n = \tau\left(\frac{n}{rad(n)}\right)$$

The list goes on and on. Now this answers the first question in the beginning of the paper, what can we derive with indicator functions? The answer is just about anything. The second question was, of what use is it? Well this is harder to answer. None of these equations have any efficient computing power. For example, I can derive the smallest billion digit prime number here it is.

$$1 + \sum_{j=1}^{2\cdot 10^{999,999,999}} \left\lfloor \frac{(10^{999,999,999})!^j}{j!} \right\rfloor - \left\lfloor \frac{(10^{999,999,999})!^j - 1}{j!} \right\rfloor = smallest\ billion\ digit\ prime$$

Good luck calculating it! But there is potential to derive new and unknown relationships using different forms of indicator functions. For example:

Theorem:

$$\left\lfloor 1 - \frac{sin^2(\pi \frac{f}{g})}{(\pi \frac{f}{g})^2} \right\rfloor = \begin{cases} 1 \; if \; g|f \\ 0 \; otherwise \end{cases}$$

Proof:

For n>0, $f(n) = \frac{sin^2(\pi n)}{(\pi n)^2}$ is clearly 0 if n is an integer and $0 < f(n) < 1$ otherwise.

Therefore, $1 - f(n)$ is 1 if n is an integer and $0 < f(n) < 1$ if n is not an integer. So by the property of the floor function, $\left\lfloor 1 - \frac{sin^2(\pi n)}{\pi n^2} \right\rfloor = \begin{cases} 1 \; if \; n \; is \; an \; integer \\ 0 \; othewise \end{cases}$ end proof

It is well known that $\lfloor x + n \rfloor = n + \lfloor x \rfloor$ for an integer n. So, $\left\lfloor 1 - \frac{sin^2(\pi \frac{f}{g})}{(\pi \frac{f}{g})^2} \right\rfloor = 1 + \left\lfloor - \frac{sin^2(\pi \frac{f}{g})}{(\pi \frac{f}{g})^2} \right\rfloor$.

By Wilson's theorem, for a natural number n >1, $(n-1)! + 1 \equiv mod \; n \;\; iff \; n \; is \; prime.$

So by the proof of the previous theorem, $1 + \left\lfloor - \frac{sin^2(\pi \frac{(n-1)!+1}{n})}{(\pi \frac{(n-1)!+1}{n})^2} \right\rfloor = \begin{cases} 1 \; if \; n \; is \; prime \\ 0 \; otherwise \end{cases} \;\; for \; n > 1.$

This is clearly the characteristic equation of the primes.

Theorem:

$$\sum_{j=1}^{n-1} (1 + \left\lfloor - \frac{sin^2(\pi \frac{(2n-2-j)!+1}{2n-1-j})}{(\pi \frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor)(1 + \left\lfloor - \frac{sin^2(\pi \frac{j!+1}{j+1})}{(\pi \frac{j!+1}{j+1})^2} \right\rfloor) = \widetilde{p_{2n}}$$

where $\widetilde{p_{2n}}$ is the number of unordered partitions of 2n into two primes.

Proof:

For an even number>2, I will use 16 in this example, the numbers from 1 to 2n may be written in order and then one may write the numbers backwards, offset by 1, directly above as follows,

16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

It is clear to see that each column will be equal to 2n, 16 in this example. Where the top and bottom both have primes, this is a solution for 2n of the Goldbach Conjecture. To avoid counting a solution twice and noting that 1 is not in the solution set, it is clear to see that it is only necessary to count solutions between 2n-2 and n inclusive. The primes may be counted backwards using the characteristic equation of the primes at the value 2n-1-j and ranging j from 1 to n-1, the primes may be counted forward by using the characteristic equation of the primes at the value j+1 and ranging j from 1 to n-1. This will include any possible solution. A characteristic equation can only be 1 or 0, so multiplying these two characteristic equations together ensures that both must be one for the product to be 1, otherwise it will be 0. In this case, multiplying these two characteristic equations together and summing will count only primes which sum to 2n and thus,

The form of the number of partitions of 2n into two primes, which I will denote as $\widetilde{p_{2n}}$,

$$\sum_{j=1}^{n-1}(1+\left\lfloor-\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2}\right\rfloor)(1+\left\lfloor-\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor)=\widetilde{p_{2n}}$$

End Proof

Theorem:

$$\widetilde{p_{2n}}\ +\widetilde{c_{2n}}+\widetilde{cp_{2n}}+1=n$$

Where $\widetilde{p_{2n}}$ is the number of unordered partitions of 2n into two primes, $\widetilde{c_{2n}}$ is the number of unordered partitions of 2n into two composites and $\widetilde{cp_{2n}}$ is the number of unordered partitions of 2n into a prime and a composite.

Proof:

The total number of unordered partitions ranges from 1+(n-1) to n+n, for example for 6 the partitions would be 1+5,2+4,3+3. Except for 1+(n-1), each partition must fall into one of the three categories mentioned in the theorem and because we are ranging from 1 to n, there are clearly n unordered partitions total. Therefore summing the three categories, $\widetilde{p_{2n}}\ +\widetilde{c_{2n}}+\widetilde{cp_{2n}}$ and adding 1 to account for 1+(n-1) will clearly be equal to n.
End Proof

Theorem:

$$\widetilde{p_{2n}}=\ \widetilde{c_{2n}}+\Pi(n)+\big(\Pi(2n-2)-\Pi(n-1)\big)-n+1$$

Proof:

As I have shown, $\sum_{j=1}^{n-1}(1+\left\lfloor-\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2}\right\rfloor)(1+\left\lfloor-\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor)=\widetilde{p_{2n}}$

Expanding the sum,

$$\widetilde{p_{2n}}=\sum_{j=1}^{n-1}1+\sum_{j=1}^{n-1}\left\lfloor-\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor+\sum_{j=1}^{n-1}\left\lfloor-\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2}\right\rfloor+\sum_{j=1}^{n-1}\left\lfloor-\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor\times\left\lfloor-\frac{sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2}\right\rfloor$$

The first of these sums is clearly n-1. In the second, $\sum_{j=1}^{n-1}\left\lfloor-\frac{sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor$ the 1 that was added to the floor function has been omitted, so the sum has been subtracted from n-1. The sum with 1 added to the floor function would have been $\pi(n)$, so this is simply the number of composites less than or equal to n minus 1 which is $n - \pi(n) - 1$. With this and noting that the sum is now negative in its form, $\sum_{j=1}^{n-1} \left\lfloor -\frac{\sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor = \pi(n) - n + 1.$

The second sum in the theorem, $\sum_{j=1}^{n-1} \left\lfloor -\frac{\sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor$ using the same approach as before would have been $\pi(2n-2) - \pi(n-1)$ if 1 was still added to the floor function, because the characteristic equation would have been counting primes in this interval. So this sum is the negative of $n - 1 - (\pi(2n-2) - \pi(n-1))$.

Therefore, $\sum_{j=1}^{n-1} \left\lfloor -\frac{\sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor = 1 - n + \pi(2n-2) - \pi(n-1).$

The third sum in the theorem, $\sum_{j=1}^{n-1} \left\lfloor -\frac{\sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2} \right\rfloor \times \left\lfloor -\frac{\sin^2(\pi\frac{(2n-2-j)!+1}{2n-1-j})}{(\pi\frac{(2n-2-j)!+1}{2n-1-j})^2} \right\rfloor$ is still counting partitions of 2n, but now it is clearly counting composites. So this sum is equal to the number of unordered partitions of 2n into two composites, $\widetilde{c_{2n}}$. This sum is positive because the characteristic equations are multiplied together.

Adding the sums together therefore gives,
$\widetilde{p_{2n}} = \widetilde{c_{2n}} + \Pi(n) + \big(\Pi(2n-2) - \Pi(n-1)\big) - n + 1$ End Proof

Theorem:

$2\widetilde{p_{2n}} + \widetilde{cp_{2n}} = \pi(n) + (\pi(2n-2) - \pi(n-1))$ n>1

Proof:

From the previous proofs, $\widetilde{p_{2n}} = \widetilde{c_{2n}} + \Pi(n) + \big(\Pi(2n-2) - \Pi(n-1)\big) - n + 1$ and

$\widetilde{p_{2n}} + \widetilde{c_{2n}} + \widetilde{cp_{2n}} + 1 = n$. Using this information, we can replace n in the first equation above with its equality in the second equation above. So,

$\widetilde{p_{2n}} = \widetilde{c_{2n}} + \Pi(n) + \big(\Pi(2n-2) - \Pi(n-1)\big) - (\widetilde{p_{2n}} + \widetilde{c_{2n}} + \widetilde{cp_{2n}} + 1) + 1$ n>1.

Simplifying gives the desired result as shown.
$2\widetilde{p_{2n}} + \widetilde{cp_{2n}} = \pi(n) + (\pi(2n-2) - \pi(n-1))$ n>1 End Proof

While these may not be the key to proving this conjecture, they are in fact new and interesting relationships. Using this same indicator function and logic on the twin prime counting function,

Theorem:

$$\sum_{j=2}^{n}(1+\left\lfloor-\frac{sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor)(1+\left\lfloor-\frac{sin^2\left(\pi\frac{(j+1)!+1}{j+2}\right)}{\left(\pi\frac{(j+1)!+1}{j+2}\right)^2}\right\rfloor)=\pi_2(n)$$

where $\pi_2(n)$ is the twin prime counting function.

Proof:

For any number n, I will use 16 in this example, the numbers from 1 to n may be written in order and then one may write the numbers again, offset by 2, directly above as follows,

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

It is clear to see that each column will differ by 2. Where the top and bottom both have primes, these are twin primes. The primes may be counted using the characteristic equation of the primes at the value j and ranging j from 2 to n, the primes may be counted using the characteristic equation of the primes at the value j+2 and ranging j from 2 to n. This will include any possible solution. A characteristic equation can only be 1 or 0, so multiplying these two characteristic equations together ensures that both must be one for the product to be 1, otherwise it will be 0. In this case, multiplying these two characteristic equations together and summing will count only primes $\leq n$ which differ by 2 and thus,

The form of the twin prime counting function is,

$$\sum_{j=2}^{n}(1+\left\lfloor-\frac{sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor)(1+\left\lfloor-\frac{sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor)=\pi_2(n)$$ End Proof

Theorem:

$\pi_2(n)+f(n)+g(n)+1=n$

Where $\pi_2(n)$ is the number of twin primes$\leq n$, $f(n)$ is the number of twin composites$\leq n$ and $g(n)$ is the number of prime and composite pairs that differ by $2 \leq n$.

Proof:

With the exception of 1 and 3, the pairs of numbers that differ by 2 must fall into one of the three categories mentioned in the theorem. Because we are ranging from 1 to n, there are clearly n total. Therefore summing the three categories, $\pi_2(n)+f(n)+g(n)$ and adding 1 to account for 1 and 3 will clearly be equal to n. End Proof

Theorem:

$\pi_2(n)= f(n) +\pi(n)+\pi(n+2)-n-1$

Proof:

As I have shown, $\sum_{j=2}^{n}(1+\left\lfloor -\frac{\sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor)(1+\left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor) = \pi_2(n)$

Expanding the sum,

$$\pi_2(n) = \sum_{j=2}^{n} 1 + \sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor + \sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor + \sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor \times \left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor$$

The first of these sums is clearly n-1. In the second, $\sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor$ the 1 that was added to the floor function has been omitted, so the sum has been subtracted from n. The sum with 1 added to the floor function would have been $\pi(n)$, so this is simply the number of composites less than or equal to n minus 1 because the sum begins at j=2, which is $n-\pi(n)-1$. With this and noting that the sum is now negative in its form, $\sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{j!+1}{j+1})}{(\pi\frac{j!+1}{j+1})^2}\right\rfloor = \pi(n)-n+1$.

The second sum in the theorem, $\sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor$ using the same approach as before would have been $\pi(n+2)$ if 1 was still added to the floor function, because the sum would have been counting the characteristic equation of the primes at n+2. So this sum is the negative number of composites $\leq n+2$, which is $-(n+1-\pi(n+2))$.

Therefore, $\sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor = \pi(n+2)-n-1$.

The third sum in the theorem, $\sum_{j=2}^{n}\left\lfloor -\frac{\sin^2(\pi\frac{(j-1)!+1}{j})}{(\pi\frac{(j-1)!+1}{j})^2}\right\rfloor \times \left\lfloor -\frac{\sin^2(\pi\frac{(j+1)!+1}{j+2})}{(\pi\frac{(j+1)!+1}{j+2})^2}\right\rfloor$ is still counting twins, but now it is clearly counting composites. So this sum is equal to the number of twin composites $\leq n$, $f(n)$. This sum is positive because the characteristic equations are multiplied together.

Adding the sums together therefore gives,

$\pi_2(n) = f(n) + \pi(n) + \pi(n+2) - n - 1$ End Proof

Theorem:

$$2\pi_2(n) + g(n) = \pi(n) + \pi(n+2) - 2$$

Proof:

From the previous proofs, $\pi_2(n) = f(n) + \pi(n) + \pi(n+2) - n - 1$ and

$\pi_2(n) + f(n) + g(n) + 1 = n$. Using this information, we can replace n in the first equation above with its equality in the second equation above. So,

$\pi_2(n) = f(n) + \pi(n) + \pi(n+2) - (\pi_2(n) + f(n) + g(n) + 1) - 1.$

Simplifying gives the desired result as shown.
$2\pi_2(n) + g(n) = \pi(n) + \pi(n+2) - 2$ End Proof

Once again, this may not be the piece needed to prove the twin prime conjecture, but every piece helps. There is certainly potential in this area for new discovery. I have found, which I will present without proof, that in particular the Liouville function ʎ(n) and the Mobius function **μ**(n) can be shown to have this relationship,

$$-\left\lfloor \frac{rad(n)}{n} \right\rfloor \frac{i^{\tau(n^2-1)}}{(-1)^{\tau\left(\frac{n}{rad(n)}\right)}} = \mu(n).$$

Where $ʎ(n) = i^{\tau(n^2)-1}$ and the expression involving the floor function is clearly the characteristic equation of the square-free numbers.

In fact, $1 + \sum_{j=2}^{n} \left\lfloor \frac{2\Omega(n)}{\tau(n^2-1)} \right\rfloor = number\ of\ prime\ powers \leq n.$

In conclusion, while sums of characteristic equations seem useless on basis of their computing efficiency, their manipulation may hold new ideas and promise for unrealized relationships between expressions. More work and study should be done in this field.