The mathematical foundations of quantum indeterminacy

Self-referential transformations underlying logical independence and quantum randomness

Steve Faulkner

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Abstract In 2008, Tomasz Paterek et al published ingenious research, proving that quantum randomness is the *output* of measurement experiments, whose *input* commands a logically independent response. Following up on that work, this paper develops a full mathematical theory of quantum indeterminacy. I explain how, the Paterek experiments imply, that the measurement of *pure* eigenstates, and the measurement of *mixed* states, cannot both be isomorphically and faithfully represented by the same single operator. Specifically, unitary representation of pure states is contradicted by the Paterek experiments. Profoundly, this denies the *axiomatic* status of *Quantum Postulates,* that state, symmetries are unitary, and observables Hermitian. Here, I show how indeterminacy is the information of transition, from pure states to mixed. I show that the machinery of that transition is unpreventable, logically circular, unitary-generating self-reference: all logically independent. Profoundly, this indeterminate system becomes apparent, as a visible feature of the mathematics, when unitarity — imposed *by Postulate* — is given up and abandoned.

Keywords foundations of quantum theory, quantum mechanics, quantum randomness, quantum indeterminacy, quantum information, prepared state, measured state, pure states, mixed states, unitary, redundant unitarity, orthogonal, scalar product, inner product, mathematical logic, logical independence, self-reference, logical circularity, mathematical undecidability.

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Main Edits

- ▷ The self-reference proof is now corrected. This had been unfinished.
- ▷ Analysis of the Paterek research thesis is given enhanced explanation.
- ▷ *Permanent axioms*, introduced earlier:
  - † are explained as being empirically and logically necessary in isolating the specific logical independence, that has been linked to quantum randomness.
  - † are matched by a corresponding rule for the conduct of experiments.
- ▷ Explanations are generally improved.
- ▷ Subtitle added.

1 Introduction

In Mathematical Physics, *validity* of a formula stems from its *provability* — from Principles and Postulates (Axioms). From that standpoint, Mathematical Physics is a collection of mathematical systems, under a regime of *logical dependence.*

But, over the past decade, researchers are taking seriously, *logical independence,* as having impact and significance in Physical Theory [16]. Research includes experimental evidence showing that quantum randomness is rooted in logically independent, mathematical information [11, 12, 13]. Logical independence refers to the null logical connectivity that exists between mathematical formulae, that neither prove nor disprove one another.

For certain theories, Mathematical Physics must embrace that independence. In the normal way, the system of mathematics should include all provable formulae, derivable from Axioms (whatever they may be); but in addition, the system must include the class of formulae which are *not disprovable.* This is because the set of non-provable, non-disprovable formulae is not empty; and formulae it contains do not contradict, but comply with Axioms. Together, both classes of formulae form a single, consistent theory. Interpretationally, such a system comprises formulae expressing *cause & effect,* and others expressing *effect by non-prevention.*

For a fundamental example, well-known to Mathematical Logicians [15], I refer to logical independence of the imaginary unit.
Consider the mathematical system we know as Elementary Algebra. This is ordinary school-algebra; the abstracted arithmetic of scalars. In relation to this algebra,

Logical Independence in Physics — Information flow & self-reference in Elementary Algebra.
 E-mail: StevieFaulkner@googlemail.com

the statement:

$$\exists x \mid x^2 = -1$$

is logically independent; it can neither be proved nor disproved by the algebra's axioms [6]. And yet, existence of any rational number is logically dependent.

Critically, quantum mathematics rests on a foundation of Elementary Algebra. Understanding the origins the of imaginary information, its entry into quantum mathematics and its logical relationship with Quantum Postulates is fundamental and crucial, in revealing Quantum Indeterminacy, as mathematical theory.

In anticipation of readers believing, beyond question, that the imaginary unit is foundationally axiomatic, I refer to a related article, by this author, providing one example contradicting that view. That paper shows that symmetry underpinning wave mechanics of the free particle — the homogeneity symmetry and the Canonical Commutation Relation deriving from it — has the imaginary unit inserted by the mathematician, for other reasons, unrelated to homogeneity [7].

This present paper shows the imaginary unit is not axiomatic, because quantum mathematics of pure states does not require it. And also, this paper shows that entry of the imaginary unit originates, as a requirement, in allowing orthogonality between complimentary systems, necessary in the formation of mixed states.

Quantum indeterminacy is a theoretical concept which must be seen in the context of empirical quantum randomness; conceived as underlying ontology, explaining quantum randomness. This is ontology, associated with single quantum systems, whose existence we *infer,* the evidence for which, we *witness* as randomness in statistics of experiments repeated many times over.

This randomness is not an epistemic randomness. It is not due to information, the detail of which is inaccessible. It is genuine randomness that some regard as fundamentally irreducible.

In *classical physics*, experiments of chance, such as coin-tossing and dice-throwing, are *deterministic,* in the sense that, perfect knowledge of the initial conditions would render outcomes perfectly predictable. This 'classical randomness' stems from ignorance of *physical information* in the initial toss or throw.

In diametrical contrast, in the case of *quantum physics*, the theorems of Kocken and Specker [10], the inequalities of John Bell [4], and experimental evidence of Alain Aspect [1,2], all indicate that *quantum randomness* does not stem from any such *physical information.*

As response, Tomasz Paterek et al provide an explanation in *mathematical information.* They demonstrate a link between quantum randomness and *logical independence* in a *formal system* of *Boolean propositions* [11,12,13]. In experiments measuring photon polarisation, Paterek et al demonstrate statistics correlating *predictable* outcomes with logically dependent mathematical propositions, and *random* outcomes with propositions that are logically independent.

Whilst, from the Paterek research, we may reliably infer that the machinery of quantum randomness *does* entail logical independence, the fact that this logical independence is seen in a *Boolean* system, rather obscures any insight. To understand the workings of quantum randomness, theory must be written exhibiting logical independence in context of *standard textbook quantum theory* — specifically, in terms of the Pauli algebra $\mathsf{su}(2)$.

Here, in this paper, I show what the Paterek Boolean information means for the system of Pauli operators. The interesting surprise revealed, is that although every measurement of polarisation is representable by the Pauli algebra $\mathsf{su}(2)$, only the measurement of mixed states *requires* this algebra. Measurement of pure eigenstates does not. For pure states, the unitary component of the Pauli algebra is not involved.

In predictable experiments, where measurement is on pure states, unitarity is shown to be 'redundant' — *possible* but *not necessary.* And in experiments whose outcomes are random, where measurement is on mixed states, unitarity is shown unavoidably *necessary.* My conclusion is that there is a *unitary switch-on* in passing from pure states to mixed and a *unitary switch-off* in passing from mixed to pure.

Logically, this regime can be viewed in two ways. It can be viewed as a system that is always unitary, but where unitarity switches between possible and necessary: such a *possible / necessary* system constitutes a *modal logic.* Or otherwise, it can be seen as a complete switch between different symmetries, where unitarity is new, *logically independent*, extra information required for the transition. To adequately describe the transition between pure and mixed states, either modal logic is needed, or logical independence. The classical logic of *true* and *false* is not an option.

The question of where the newly formed unitary information comes from is solved. I show that it has origins in uncaused, unprevented, logically circular *self-reference*. By uncaused and unprevented, I mean that no information already present in the system implies nor denies the logically circular self-reference.

In experiments measuring mixed states, whose outcomes are random; in the usual way, the system symmetry is isomorphically and faithfully represented, one-one, by the (unitary) Pauli matrices:

$$\sigma_{\mathsf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_{\mathsf{y}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

But for measurements on pure states, whose outcomes are predictable, the Paterek findings prove the Pauli operators do not offer isomorphic, faithful representation. Measurement on pure states, in the Paterek experiments, is faithfully represented by this set of non-unitary, matrices:

$$\mathsf{s}_{\mathsf{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{s}_{\mathsf{y}}(\zeta,\eta) = \begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix} \qquad \mathsf{s}_{\mathsf{z}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

$\begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ \eta\,(\lambda-\zeta) \end{bmatrix}$

with eigenvalues $\lambda = \pm\sqrt{\zeta^2+1}$

where ζ and η are a scalars of any value. It can be seen that σ_{y} is particular value of $\mathsf{s}_{\mathsf{y}}(\zeta,\eta)$. The crucial distinction between (1) and (2) is that, whereas, in the three Pauli matrices (1) there is *3-way orthogonality* – all are mutually orthogonal – in the non-unitary matrices (2), there is orthogonality, only between s_{x} and s_{z} except in the accidental coincidence of $\zeta = 0$ and $\eta = \pm i$.

In the case I argue, I reason that logical independence, identified by Paterek as the origin of quantum randomness, is traceable to the Pauli algebra — then demonstrate that only in representing the *mixed states* is the Pauli algebra *faithfully* isomorphic. And further, demonstrate that, separating the pure states from mixed states, there is an information-gap, bridgeable by a mechanism of unitary-generating, self-referential mathematical transformations, which is quantitatively indefinite and logically non-causative.

Sections 2 – 5 expounds the Paterek thesis and method. The Paterek approach treats measurement experiments like computer hardware, whose input and output is machine binary. The machine 'zeros' and 'ones' register involutory and orthogonal items of hardware information. This is related to *separated* involutory and orthogonal items of information, extracted from the Pauli algebra — as opposed to the unseparated Pauli algebra itself. Ingress of logical independence enters measurement information as hardware interacts with the photon density matrix.

Section 6 shows how the Pauli algebra consists of 6 logically independent items of algebraic information – 3 involutory and 3 orthogonal.

Section 7 shows that all polarisation states need involutory information. And that only mixed states need the 3 orthogonal items of algebraic information.

Section 8 takes the non-unitary, algebraic system[1] (2), and hypothesises certain quantitative coincidences which accidentally permit logically circular self-reference. The resultant is the unitary Pauli system.

2 Information and logic

In Mathematical Logic, a *formal system* is a system of mathematical formulae, treated as propositions, where focus in on *provability* and *non-provability.*

A formal system comprises: a precise language, rules for writing formulae, and further rules of deduction. Within such a formal system, any two propositions are **either** *logically dependent* — in which case, one proves, or disproves the other — **or otherwise** they are *logically independent*, in which case, neither proves, nor disproves the other.

A helpful perspective on this is the viewpoint of Gregory Chaitin's information-theoretic formulation [5]. In that, logical independence is seen in terms of information content. If a proposition contains information, not contained in some given set of axioms[2], then those axioms can neither prove nor disprove the proposition.

[1] The algebraic system (2) does not form a Lie algebra.

[2] *Axioms* are propositions presupposed to be 'true' and adopted *a priori.*

Edward Russell Stabler explains logical independence in the following terms. A formal system is a postulate-theorem structure; the term postulate being synonymous with axiom. In this structure, there is discrimination, separating assumed from provable statements. Any statement labelled as a postulate which is capable of being proved from other postulates should be relabelled as a theorem. And if retained as a postulate, it is logically superfluous and redundant [15]. If incapable of being proved or disproved from other postulates, it is logically independent.

Central to the formal system used in the Paterek et al research are these Boolean functions of a binary argument:

$$x \in \{0,1\} \mapsto f(x) \in \{0,1\}$$

Typical propositions, stemming from those functions, are these:

$$\begin{array}{ccc} f(0)=0 & f(1)=0 & f(0)=f(1) \\ f(0)=1 & f(1)=1 & f(0)\neq f(1) \end{array} \tag{3}$$

Such propositions are items of information, taken as being openly true or openly false. Our interest lies, not so much, in their truth or falsity, but in, which statements prove which, which disprove which, and which do neither. In other words, which are logically dependent and which are logically independent.

As illustration, if $f(0)=0$ were considered to be true, the statement $f(0)=1$ would be proved false. More simply, we could say: $f(0)=0$ disproves $f(0)=1$, and accordingly, $f(0)=1$ is *logically dependent* on $f(0)=0$.

On the other hand, again, if $f(0)=0$ were considered to be true, that would not prove, or disprove $f(1)=0$. We could say: $f(0)=0$ neither proves, nor disprove $f(1)=0$, and accordingly, $f(0)=0$ and $f(1)=0$ are *logically independent.*

Over and above the propositions in (3), I introduce *permanent axioms*, which Paterek et al take for granted, but do not state. They are:

$$f(0)=0 \;\Rightarrow f(1)=1 \qquad\qquad f(1)=0 \;\Rightarrow f(0)=1 \tag{4}$$

These prohibit the combination $f(0)=0$, $f(1)=0$. More is said about this in Section 5.

3 The Paterek et al experiments

The Paterek et al research involves polarised photons as information carriers through measurement experiments. The experiment hardware consists of a sequence of three segments, which I denote: State preparation, Black box and Measurement. These *prepare*, then *transform*, then *measure* polarisation states. The orientational configuration of the three segments is the experiment's input data. This is read from an X–Y–Z reference system fixed to the hardware. Outcome states of polarisation are the experiment's output data. Experiments were performed, very many times, and statistics of outcomes gathered. The configuration input, is related to whether the experiment's output is random or predictable.

1. State preparation
   Photons prepared, either as $|z+\rangle$, $|x+\rangle$ or $|y+\rangle$ eigenstates, by filtering, directly after one of these Pauli transformations:
   (a) σ_{z}, aligned with the Z axis.
   (b) σ_{x}, aligned with the X axis.
   (c) σ_{y}, aligned with the Y axis.
2. Black box
   The prepared eigenstates are altered through one of these Pauli transformations:
   (a) σ_{z}, aligning states with the Z axis,
   (b) σ_{x} aligning states with the X axis,
   (c) σ_{y} aligning states with the Y axis.
3. Measurement
   Measurement is performed, by detecting photon capture, directly after one of these Pauli transformations:
   (a) σ_{z}, aligned with the Z axis.
   (b) σ_{x}, aligned with the X axis.

(c) σ_{y}, aligned with the Y axis.

Thus, there are 27 possible experiments. In practice, nine are necessary. Results are sufficiently demonstrated by always keeping the State preparation orientation, set at the same alignment as the Measurement orientation. The fact that Measurement copies the State preparation orientation means the full hardware configuration can be encoded, taking orientations of the Black box and Measurement segments, only. These encodings come in the form of Boolean *'4-sequences'* and '*quad-products'* introduced below.

Within experiments, there exist two classes of orientational information. The more obvious is *segment alignment*; this is the orientation of individual hardware segments with respect to the X–Y–Z reference system. Normally, in standard theory, segment alignment would be represented as Pauli information, through the σ_{x}, σ_{y}, σ_{z} operators. In the Paterek et al research, alignment information is fully conveyed in two bits, through three *Boolean pairs* — $(0,1)$, $(1,0)$, $(1,1)$.

The less obvious class of information, I refer to as *orthogonality index.* This is the degree of orthogonality between one hardware segment and the next — either orthogonal, or not orthogonal. Orthogonality index is conveyed through the experiment, as information propagated in the *density matrix.*

4 Boolean pairs and 4-sequences

In their treatment of the mathematics, Paterek et al represent their experiment configurations, using sequences of the three *Boolean pairs* — $(0,1)$, $(1,0)$, $(1,1)$. Information held in these pairs is taken directly from the indices, in the product $\sigma_{\mathsf{x}}^{i}\sigma_{\mathsf{z}}^{j}$, where i and j are interpreted as integers, modulo 2. Thus:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{1} \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{0} \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{1} \tag{5}$$

By way of these three formulae, Boolean pairs $(0,1)$, $(1,0)$, $(1,1)$ are *linked* to the operators: σ_{z}, σ_{x}, σ_{y}.

The idea is that the whole information of any Pauli operator can be encoded through different arrangements of σ_{x} and σ_{z} — just two of the Pauli operators. The precise structure of that encoding is key to accessing and revealing the information that constitutes indeterminacy.

Stringing together sequences of Pauli operators, to form '*quad-products',* invokes corresponding Boolean *'4-sequences'.* These represent orientational information, linking two consecutive segments of the experiment hardware. Examples are:

$$\sigma_{\mathsf{z}}\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{1}\sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{1} \rightarrow (0,1)\,(0,1) \tag{6}$$
$$\sigma_{\mathsf{x}}\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{0}\sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{1} \rightarrow (1,0)\,(0,1) \tag{7}$$
$$-i\sigma_{\mathsf{y}}\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{1}\sigma_{\mathsf{x}}^{0}\sigma_{\mathsf{z}}^{1} \rightarrow (1,1)\,(0,1) \tag{8}$$

These can be used to represent the action of the State preparation followed by the action of the Black box; OR, the action of the Black box followed by the action of the Measurement.

Consider a specific experiment where the action of the State preparation is encoded thus:

$$\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n} \rightarrow (m,n)$$

where the action of the Black box is encoded thus:

$$\sigma_{\mathsf{x}}^{f(0)}\sigma_{\mathsf{z}}^{f(1)} \rightarrow (f\,(0)\,,f\,(1)) \tag{9}$$

where $f\,(0)$ and $f\,(1)$ are the Boolean functions relating to propositions written in (3); and the action of the Measurement is encoded thus:

Variables p and q are not used by Paterek et al. I introduce them for the sake of completeness.

$$\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q} \rightarrow (p,q)$$

In this experiment, the joint action for the State preparation and Black box is encoded in the *quad-product* and *4-sequence*:

$$\sigma_{\mathsf{x}}^{f_{\mathrm{in}}(0)}\sigma_{\mathsf{z}}^{f_{\mathrm{in}}(1)}\sigma_{\mathsf{x}}^{m}\sigma_{\mathsf{z}}^{n} \rightarrow (f_{\mathrm{in}}\,(0)\,,f_{\mathrm{in}}\,(1))\,(m,n)$$

And the joint action for the Black box and Measurement is encoded in the *quad-product* and *4-sequence*:

$$\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q}\sigma_{\mathsf{x}}^{f_{\mathrm{ex}}(0)}\sigma_{\mathsf{z}}^{f_{\mathrm{ex}}(1)} \rightarrow (p,q)\,(f_{\mathrm{ex}}\,(0)\,,f_{\mathrm{ex}}\,(1))$$

5 Logical independence from the Boolean viewpoint

This section charts the progress of logical dependence and logical independence through the experiment hardware. Information flow is considered in two stages. In ***Stage 1*** the ingress and egress of information is considered, through the Black box; and in ***Stage 2,*** the reading of information by the Measurement hardware.

In the Paterek paper, the approach is to enquire whether propositions agree or disagree, to serve as a *test* for the presence or absence of logical independence. Here, in this paper, emphasis is on tracing lines of dependency, and independency, to reveal the point where logical independence enters. That is of interest because the 'anomaly' at that point will shed light on the workings of indeterminacy.

Before tracing those lines of dependency and independency, I comment on *permanent axioms,* mentioned in Section 2, written again here:

$$f(0) = 0 \Rightarrow f(1) = 1 \qquad\qquad f(1) = 0 \Rightarrow f(0) = 1$$

These do not feature in the Paterek research, but they are taken for granted. Their purpose here, is to preclude any 'null Black box' from being included in the analysis; that is, any Black box performing an identity transformation. The reason I make a special point of this, is that a 'null Black box' introduces a type of logical independence, unrelated and irrelevant to the substance of the analysis, and which superficially makes a lie of the reasoning and conclusion.

But it is important that these axioms correspond to rules of conduct for the experiments; they cannot be simply imposed by stating them! There are two classes of configuration for the Black box, resulting in non-random, predictable outcomes. One is where the Black box imparts the same polarisation alignment, as does the State preparation. In the other, the Black box imparts no alteration, and photons exit the Black box, in whatever state they entered from State preparation.

So as to eliminate this second type of predictable outcome, I propose an extra duty for the experiment operator. Under the assumption that the operator is kept in ignorance of the Black box configuration settings, she should test for *zero change in predictability* as the Black box alignment is rotated. If this is done, or we simply guarantee that 'null Black box' alignments never occur, we can guarantee legitimacy of the *permanent axioms,* added here.

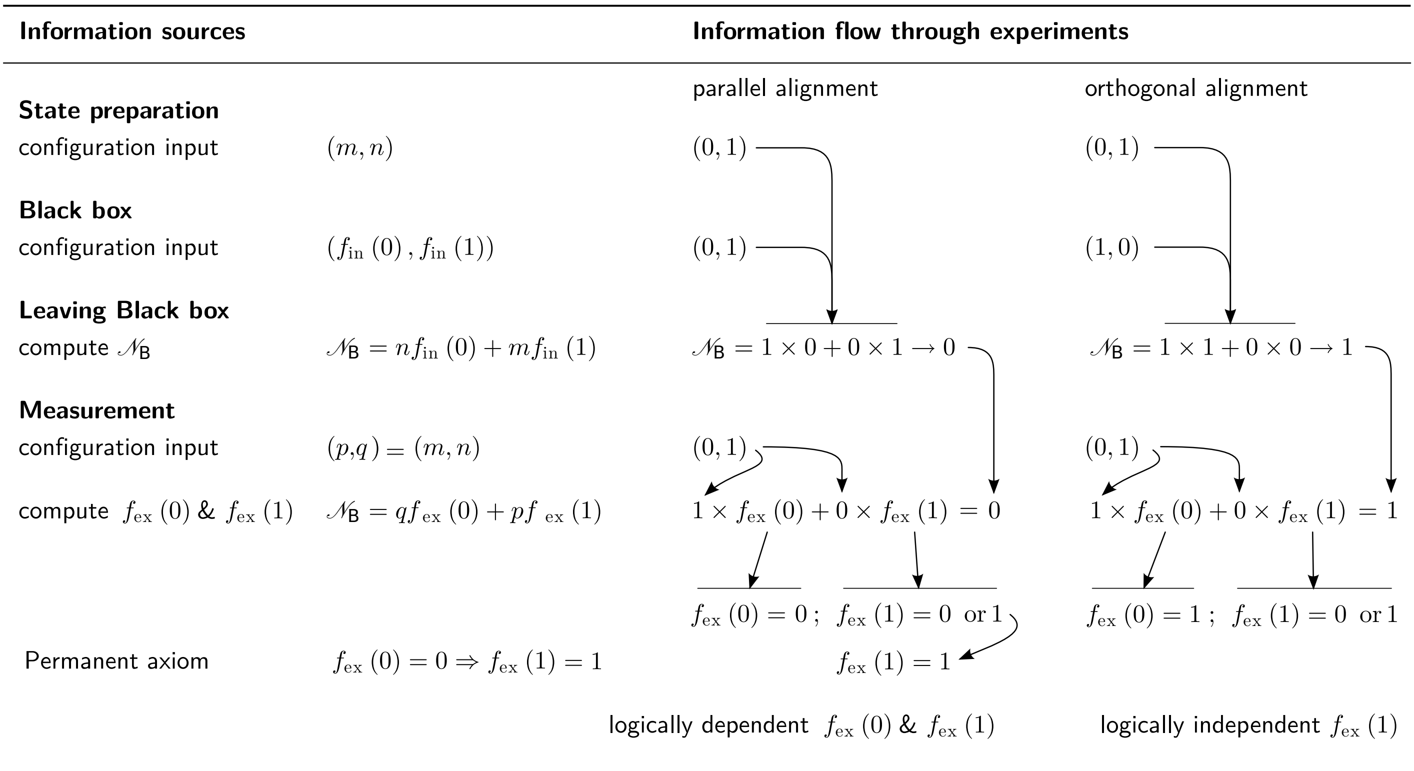

Figure 1 The Paterek research involves polarised photons as information carriers through measurement experiments. Orthogonality index $\mathscr{N}_{\mathsf{B}} = nf(0) + mf(1)$ is a Boolean quantity, conveyed through experiments by the density matrix. For the cases of *parallel* and *orthogonal* aligned measurement experiments, the diagram shows information flow involved. For the orthogonal case only, information conveyed is logically independent.

Stage 1 Boolean pairs, representing X–Y–Z information, from State preparation and Black box, feed into the *density matrix.*

The propagation of information, that encodes, whether states are mixed or pure, is conveyed in the density matrix. On entry into the Black box, the *input* density matrix, is:

$$\rho = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} i^{mn} \sigma_{\mathsf{x}}^{m} \sigma_{\mathsf{z}}^{n}\right]$$

with $\lambda = \pm 1$. Under the action of the Black box the density matrix evolves to:

$$U\rho U^{\dagger} = \frac{1}{2}\left[\mathbb{1} + \lambda_{mn} \left(-1\right)^{nf_{\text{in}}(0)+mf_{\text{in}}(1)} i^{mn} \sigma_{\mathsf{x}}^{m} \sigma_{\mathsf{z}}^{n}\right]$$

The index, on the factor $\left(-1\right)^{nf_{\text{in}}(0)+mf_{\text{in}}(1)}$, I call the *orthogonality index* and give it the label $\mathscr{N}_{\mathsf{B}}$, thus:

$$\mathscr{N}_{\mathsf{B}} = nf_{\text{in}}\left(0\right) + mf_{\text{in}}\left(1\right)$$

The suffix B stands for 'leaving the Black box'. This is just downstream of the Black box; but upstream of any interference from Measurement. Depending on whether the Black box imparts orthogonal information, the value of $\mathscr{N}_{\mathsf{B}}$ is either 0 or 1. All sums are taken modulo 2.

Incidental: when alignment is parallel, $\mathscr{N}_{\mathsf{B}} = 0$ and consequently $\rho = U\rho U^{\dagger}$, so there is no evolved change in ρ.

$\mathscr{N}_{\mathsf{B}} = nf_{\text{in}}\left(0\right) + mf_{\text{in}}\left(1\right) = 0$	zero orthogonality imparted by the Black box
$\mathscr{N}_{\mathsf{B}} = nf_{\text{in}}\left(0\right) + mf_{\text{in}}\left(1\right) = 1$	unit orthogonality imparted by the Black box

Leaving the Black box, $\mathscr{N}_{\mathsf{B}}$ has a definite, *deterministic* value, logically dependently computed from (m, n) and $\left(f_{\text{in}}\left(0\right), f_{\text{in}}\left(1\right)\right)$. That determination can be thought of as an information process where (m, n) and $\left(f_{\text{in}}\left(0\right), f_{\text{in}}\left(1\right)\right)$ are copied from the State preparation and Black box, then given as input to $nf_{\text{in}}\left(0\right) + mf_{\text{in}}\left(1\right)$, from which $\mathscr{N}_{\mathsf{B}}$ is computed, as output.

Stage 2 Measurement attempts to read the Black box X–Y–Z information.

Now comes the interaction between $\mathscr{N}_{\mathsf{B}}$ and the Measurement hardware. By now, values of $f_{\text{in}}\left(0\right)$ and $f_{\text{in}}\left(1\right)$ are either lost or upstream and inaccessible. The density matrix conveys $\mathscr{N}_{\mathsf{B}}$, not $f_{\text{in}}\left(0\right)$ and $f_{\text{in}}\left(1\right)$.

Leaving the Black box, the definite, deterministic value $\mathscr{N}_{\mathsf{B}}$, continues its propagation through the experiment, to be read as input, into the Measurement hardware. But the Measurement hardware will have the awkward job of attempting a computation in the 'backwards' sense, which will present a problem of computability.

Once the Measurement hardware knows the value $\mathscr{N}_{\mathsf{B}}$, given the Measurement orientation, set by

$$\sigma_{\mathsf{x}}^{p}\sigma_{\mathsf{z}}^{q} \to (p, q)$$

the Measurement hardware attempts to compute $f_{\text{out}}\left(0\right)$ and $f_{\text{out}}\left(1\right)$, from

$$\mathscr{N}_{\mathsf{B}} = qf_{\text{ex}}\left(0\right) + pf_{\text{ex}}\left(1\right)$$

However, $f_{\text{ex}}\left(0\right)$ and $f_{\text{ex}}\left(1\right)$ are not both determinable from $\mathscr{N}_{\mathsf{B}}$ and (p, q), because, one or the other of $f_{\text{ex}}\left(0\right)$ and $f_{\text{ex}}\left(1\right)$, will be logically independent.

To demonstrate the above, it is sufficient to set the Measurement configuration (p, q) to the same basis (m, n), set for the State preparation. Figure 1 shows the flow of information schematically, comparing the straight-through, parallel aligned experiment, against the orthogonal experiment configuration.

6 Information content of the Pauli algebra

It is instructive to review the information content of the Pauli algebra, or more significantly, the information implied in the formula: $-i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^{1}\sigma_{\mathsf{z}}^{1}$; or rather more strictly, asserted in this abstract formulae:

$$-i\mathsf{b} = \mathsf{ac} \tag{10}$$

That review means going through the process of constructing (10), from scratch, and noting all information needed. The procedure I give is an adaption of a proof given by W E Baylis, J Huschilt and Jiansu Wei [3].

This proof possibly originates from a paper by David Hestenes [9].

The Pauli algebra is a Lie algebra; and hence, is a linear vector space. Therefore, I begin with information inherited from the vector space axioms, and then add other information peculiar to the Pauli Lie algebra, $\mathsf{su}(2)$.

Closure: For any two vectors u and v, there exists a vector w such that

$$\mathsf{w} = \mathsf{u} + \mathsf{v}$$

Identities: There exist additive and multiplicative identities, $\mathbb{0}$ and $\mathbb{1}$. For any arbitrary vector v:

$$\mathsf{v}\mathbb{1} = \mathbb{1}\mathsf{v} = \mathsf{v} \tag{11}$$

$$\mathsf{v} + \mathbb{0} = \mathbb{0} + \mathsf{v} = \mathsf{v} \tag{12}$$

$$\mathsf{v}\mathbb{0} = \mathbb{0}\mathsf{v} = \mathbb{0} \tag{13}$$

Additive inverse: For any arbitrary vector v, there exists an additive inverse $-\mathsf{v}$ such that

$$(-\mathsf{v}) + \mathsf{v} = \mathbb{0} \tag{14}$$

Scaling: For any arbitrary vector v, and any scalar a, there exists a vector u such that

$$\mathsf{u} = a\mathsf{v} \tag{15}$$

Products: A feature of Lie algebras is that, between any two arbitrary vectors, u and v, there exist products uv and vu. Commutators of these products (Lie brackets) are members of the vector space.

Dimension: Assume a 3 dimensional vector space, with independent basis a, b, c.

The six items of information

Involutory information: Assume all three basis vectors are involutory. Thus:

$$\mathsf{aa} = \mathbb{1} \qquad \mathsf{a}\text{ involutory} \tag{16}$$

$$\mathsf{bb} = \mathbb{1} \qquad \mathsf{b}\text{ involutory} \tag{17}$$

$$\mathsf{cc} = \mathbb{1} \qquad \mathsf{c}\text{ involutory} \tag{18}$$

Orthogonal information: Assume products between basis vectors are orthogonal. Thus:

$$\mathsf{ab} + \mathsf{ba} = \mathbb{0} \qquad \mathsf{ab}\text{ orthogonal} \tag{19}$$

$$\mathsf{bc} + \mathsf{cb} = \mathbb{0} \qquad \mathsf{bc}\text{ orthogonal} \tag{20}$$

$$\mathsf{ca} + \mathsf{ac} = \mathbb{0} \qquad \mathsf{ca}\text{ orthogonal} \tag{21}$$

Bringing items of information together, the Pauli algebra is constructed thus:

$$\begin{aligned} \mathsf{bc} + \mathsf{cb} &= \mathbb{0} && \text{by (20)} \;,\; \mathsf{bc}\text{ orthogonal} \\ \mathsf{b} + \mathsf{cbc} &= \mathbb{0} && \text{by (18)} \;,\; \mathsf{c}\text{ involutory} \\ \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} && \text{by (13)} \end{aligned} \tag{22}$$

And similarly:

$$\begin{aligned} \mathsf{ca} + \mathsf{ac} &= \mathbb{0} && \text{by (21)} \;,\; \mathsf{ca}\text{ orthogonal} \\ \mathsf{cac} + \mathsf{a} &= \mathbb{0} && \text{by (18)} \;,\; \mathsf{c}\text{ involutory} \\ \mathsf{cacb} + \mathsf{ab} &= \mathbb{0} && \text{by (13)} \end{aligned} \tag{23}$$

Adding (23) and (22) gives:

$$\begin{aligned} \mathsf{cacb} + \mathsf{ab} + \mathsf{ba} + \mathsf{cbca} &= \mathbb{0} \\ \mathsf{cacb} + \mathsf{cbca} &= \mathbb{0} && \text{by (19)} \;,\; \mathsf{ab}\text{ orthogonal} \\ \mathsf{acb} + \mathsf{bca} &= \mathbb{0} && \text{by (18)} \;,\; \mathsf{c}\text{ involutory} \\ \mathsf{acba} + \mathsf{bc} &= \mathbb{0} && \text{by (16)} \;,\; \mathsf{a}\text{ involutory} \\ \mathsf{acbac} + \mathsf{b} &= \mathbb{0} && \text{by (18)} \;,\; \mathsf{c}\text{ involutory} \\ \mathsf{acbacb} + \mathbb{1} &= \mathbb{0} && \text{by (17)} \;,\; \mathsf{b}\text{ involutory} \\ (\mathsf{acb})^2 &= -\mathbb{1} && \text{by (14)} \\ (\mathsf{acb})^2 &= (-1)\,\mathbb{1} \\ \mathsf{acb} &= \pm i\mathbb{1} \\ \mathsf{ac} &= \pm i\mathsf{b} && \text{by (17)} \;,\; \mathsf{b}\text{ involutory} \end{aligned} \tag{24}$$

And a couple of extra steps gives the Pauli algebra:

$$\mathsf{ca} = \mp i\mathsf{b} \qquad \text{by (24)} \quad , \qquad \mathsf{a}, \mathsf{b}, \mathsf{c} \text{ involutory} \tag{25}$$

$$\mathsf{ac} - \mathsf{ca} = \pm 2i\mathsf{b} \qquad \text{by (24) \& (25)} \tag{26}$$

The six formulae (16) – (21) constitute six items of logically independent information. They are logically independent because none can be proved nor disproved from the others. All six are needed in proving $\mathsf{ac} = \pm i\mathsf{b}$.

The '3-way orthogonality' resulting from (19), (20) and (21) implies complex unitarity.

7 Logical independence from the viewpoint of symmetry

Quantitatively, standard Pauli theory is superbly successful. But, in terms of representing the logic of experiments, it would seem the Paterek Boolean system is an improvement. Accepting that as fact, the Boolean system must be traced through for information that standard theory misses.

The Paterek research shows that mathematics encoding the measurement of *mixed* states has logically independent structure; and that the measurement of *pure* states does not. And therefore, any mathematical structure *faithfully*[3] representing the measurement of mixed states cannot *faithfully* represent pure eigenstates, also. For the faithful representation of pure, and of mixed states, two structures are needed which are not mutually isomorphic: meaning that no one, single mathematical structure can be isomorphic with every polarisation measurement experiment. This contradicts standard theory, where the Pauli algebra is understood to represent every measurement configuration.

Faithful representation is one-one, isomorphic representation.

Note that $\begin{pmatrix} 0 & \eta^{-1} \\ \eta & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ cannot be isomorphic because only one of them is a member of the unitary group.

Consequently, the Paterek paper establishes, that measurement of arbitrarily prepared polarised photons, cannot, in general, be isomorphically represented by any single, exclusive, mathematical structure. Specifically, the Pauli algebra cannot be relied upon as a general theory, isomorphically representing every configuration of measurement experiment. Instead, measurement aligned parallel to the prepared state – and – measurement aligned orthogonal against it, are separately represented by distinct mathematical structures, not isomorphic with one another.

Having said all the above, *quantitatively*, the Pauli theory *does* work. Resolution to this *quantitative* versus *logical* dichotomy, as will be seen, is in the fact that one of those distinct mathematical structures agrees with the other, but the other does not agree with the one.

The above is helpful news. Of course, we take for granted the fact that individual experiments are independent of one another. But extra and further to that, the above tells us, experiments are independent, to the extent, that algebra for one experiment does not extrapolate to all others. All Pauli experiments do not share one same algebraic environment.

In practice, this means the formula (8) does not confer existence of σ_{y} upon the formulae (6). Nor does (8) confer its value of σ_{z} upon (6). Et cetera. We must regard all such formulae, entailing the Pauli quad-products, as individual constructs of information, in isolation from one another, without passing information between them.

The Paterek findings rely on a *logical isomorphism*, linking the Boolean system with Pauli experiments. That isomorphism is a one – one correspondence that connects the logic of experiments with the logic of the Boolean system. The Paterek paper remarks on this logical isomorphism in its conclusion.

In contrast, the Pauli system lacks that one – one logical correspondence with experiment. The position is that the Pauli system faithfully represents experiments *quantitatively* whilst the Boolean system faithfully represents experiments *logically*. In order that the Pauli system should be logical also, it must connect logically, one – one, with Pauli experiments. That means Pauli experiments must connect logically, one – one, with the Boolean system (as they do); and then in turn, the Boolean system must connect logically, one – one, with the Pauli system. Thus:

$$\text{Pauli system} \rightleftarrows \text{Boolean system} \rightleftarrows \text{Pauli experiments}$$

3

To approach this, we must examine the exact nature of the link relating the Pauli and Boolean systems to see where logical correspondence between them currently fails.

Readers of the Paterek paper might infer that there is one – one correspondence linking the Pauli products with Boolean pairs. The actual picture is one – way. Implication is only directed from the Pauli products, to the Boolean pairs, in the sense of the arrows shown here:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0 \sigma_{\mathsf{z}}^1 \longrightarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^0 \longrightarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^1 \longrightarrow (1,1) \tag{27}$$

If the Pauli system were to connect logically, one – one, with the Boolean system, we would witness a backwards implication, also, in the sense of these reverse arrows:

$$\sigma_{\mathsf{z}} = \sigma_{\mathsf{x}}^0 \sigma_{\mathsf{z}}^1 \longleftarrow (0,1) \qquad \sigma_{\mathsf{x}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^0 \longleftarrow (1,0) \qquad -i\sigma_{\mathsf{y}} = \sigma_{\mathsf{x}}^1 \sigma_{\mathsf{z}}^1 \longleftarrow (1,1) \tag{28}$$

But, as they stand, the formulae in (28) are invalid. Generally, the Boolean pairs do not imply the Pauli operators. They invoke operators that are not necessarily Paulian; they invoke operators belonging to some wider system. They do not form a Lie algebra. The Pauli operators are merely the special case that happens to be unitary. And so, we must either abandon the backwards implication — but this is implicit in the Paterek findings — or accept the replacement of Pauli operators with operators that maintain backwards validity.

The situation is made clearer when all Pauli notation is dropped and replaced by abstract symbols c, a, b. Formulae can then be seen for the information they *assert*, rather than content we *presume*, that stems from meaning we place on the symbols they contain.

Restating (28) abstractly:

$$\mathsf{c} = \mathsf{a}^0\mathsf{c}^1 \longleftarrow (0,1) \qquad \mathsf{a} = \mathsf{a}^1\mathsf{c}^0 \longleftarrow (1,0) \qquad -i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1 \longleftarrow (1,1) \tag{29}$$

The first two of these formulae imply *involutory* information only; whereas the last formula, corresponding to $(1,1)$, implies information that is both *involutory* and *unitary.*

Now consider these Boolean 4-sequences:

$$\mathsf{cc} = \mathsf{a}^0\mathsf{c}^1\mathsf{a}^0\mathsf{c}^1 \longleftarrow (0,1)\,(0,1) \tag{30}$$
$$\mathsf{ac} = \mathsf{a}^1\mathsf{c}^0\mathsf{a}^0\mathsf{c}^1 \longleftarrow (1,0)\,(0,1) \tag{31}$$
$$-i\mathsf{bc} = \mathsf{a}^1\mathsf{c}^1\mathsf{a}^0\mathsf{c}^1 \longleftarrow (1,1)\,(0,1) \tag{32}$$

These express information representing three independent experiments. For the 'straight-through' experiment (30), the equality holds true for values of $\mathsf{a} \neq \sigma_{\mathsf{x}}$.

Involutory matrices:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for } a^2 + bc = 1$$

Cases of interest are:

$$\begin{pmatrix} a & -b \\ b & -a \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for } a^2 - b^2 = 1$$

$$\begin{pmatrix} a & b^{-1} \\ b & -a \end{pmatrix}^2 = \mathbb{1}_2 \qquad \text{for } a^2 + 1 = 1$$

Measurement		**Logio – symmetry properties**		**Algebraic Information**		**Algebra implied by Boolean 4-sequences**				
Random outcomes	state	Unitarity	Circularly Self-referent	Involutory $\mathsf{aa} = \mathbb{1}$ $\mathsf{bb} = \mathbb{1}$ $\mathsf{cc} = \mathbb{1}$	Orthogonal $\mathsf{ab}+\mathsf{ba} = \mathbb{0}$ $\mathsf{bc}+\mathsf{cb} = \mathbb{0}$ $\mathsf{ca}+\mathsf{ac} = \mathbb{0}$	Implied algebra		Implied quad product		Boolean 4-sequence
no	pure	redundant	no	yes	no	$\mathsf{a}^2 = \mathbb{1}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(0,1)(0,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ac} = -i\mathsf{b}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(1,0)(0,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{bc} = +i\mathsf{a}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^0\mathsf{c}^1$	$\leftarrow$	$(1,1)(0,1)$
no	pure	redundant	no	yes	no	$\mathsf{c}^2 = \mathbb{1}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(1,0)(1,0)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ba} = -i\mathsf{c}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(1,1)(1,0)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ca} = +i\mathsf{b}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^0$	$\leftarrow$	$(0,1)(1,0)$
no	pure	redundant	no	yes	no	$(\mathsf{ac})^2 = -\mathbb{1}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(1,1)(1,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{cb} = -i\mathsf{a}$	$\leftarrow$	$\mathsf{a}^0\mathsf{c}^1\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(0,1)(1,1)$
yes	mixed	necessary	yes	yes	yes	$\mathsf{ab} = +i\mathsf{c}$	$\leftarrow$	$\mathsf{a}^1\mathsf{c}^0\,\mathsf{a}^1\mathsf{c}^1$	$\leftarrow$	$(1,0)(1,1)$

Table 1 Comparison of randomness in experiment outcomes, and logical independence in symmetry information, implied by the Paterek Boolean system.

This experiment invokes directly, the formulae $\mathsf{c} = \mathsf{a}^0\mathsf{c}^1$ and indirectly, the formula $\mathsf{a} = \mathsf{a}^1\mathsf{c}^0$ from (29). The 4-sequence $(0,1)\,(0,1)$ implies only that a and c be any *involutory* operator, nothing more; and not that it should be a Pauli operator belonging to the Pauli algebra. No unitary information is implied and any unitarity attributed is redundant.

Considering (31). The right hand side of the equality directly invokes both $\mathsf{c} = \mathsf{a}^0\mathsf{c}^1$ and $\mathsf{a} = \mathsf{a}^1\mathsf{c}^0$ from (29), implying involutory c and a. The left hand side invokes unitarity, indirectly, through $-i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$. As for (32); this implies unitarity, directly through the formula $-i\mathsf{b} = \mathsf{a}^1\mathsf{c}^1$. See Table 1 for the other 4-sequences.

The fact these different experiments invoke different sets of information taken from (29) shows the variables a, b and c should not be regarded as fixed across all experiments. For some experiments they are unitary, others, not.

8 Logical independence from the viewpoint of self-reference

An orthogonal vector space can be thought of as a composite of information – consisting of – information that comprises a general, arbitrary vector space, plus additional information that renders that space orthogonal. More formally we might think of axioms imposing rules for vector spaces with additional axioms imposing orthogonality. However, the information of orthogonality need not originate in axioms or definitions; it can originate through *self-reference* or *logical circularity* [14].

The same theoretical ideas should apply to orthogonal tensor spaces.

This has profound implications for the logical standing of vector spaces used in the representation of quantum states: in particular – the logical standing of pure states, in relation to, the logical standing of mixed states; for, it is this self-reference, that takes place at the interface between pure and mixed states, that is the root of logical independence in quantum systems — and of an *information deficiency* that manifests as quantum randomness. The self-reference constitutes valid and viable computational machinery, in an environment where no axiomatic or system information is capable of preventing the process from running, but which lacks definite quantitative information as input.

This can be compared to a computer program, running in a loop, which needed no bootstrap and cannot be escaped or halted, and which outputs data, when the only input available was ambiguous.

Within Elementary Algebra, self-reference can express Linear Algebraic information, normally conceptualised as axioms. Thus, this self-reference moves Linear Algebra into the arena of Elementary Algebra, meaning that, the Hilbert space mathematics of a quantum theory is expressible as a single algebraic system, rather than a composite amalgamation of Elementary Algebra plus Linear Algebra. And so, instead of information, normally expressed as definitions from Linear Algebra, equivalent information is expressed as self-reference in Elementary Algebra. So instead of the usual *definitional* demarcation that separates the two algebras, there is now *logic* that interfaces them: wholly within Elementary Algebra. Thus, the whole information of the Hilbert space is expressed as a single integrated algebraic system — with logical structure *within*, that replaces definitions that were from *outside*.

Matrices acting on vectors are notation for sets of simultaneous equations, within Elementary Algebra. Self-reference imposes the orthogonal scalar product.

In the case of Pauli systems, before the self-reference may proceed, a triplet of non-orthogonal vector spaces (Banach spaces) forms into a closed system. This self-reference consists of the passing of information, from each vector space to the next, in complete cycles. But the process is capable of sustaining only orthogonal spaces, so acts as a unitary filter. Unitarity is implied in this completely mutual, '3-way orthogonality' [8].

In momentum-position wave mechanics, a dual-pair of spaces forms into a closed system. The reason this is *dual* rather than a *triplet* is that the system algebra:

$$[\mathsf{p}, \mathsf{x}] = -i\mathbb{1}$$

has $\mathbb{1}$ as its third operator. So the third vector space is trivial.

The whole process is possible because its component subprocesses are *logically independent* of axioms; so no information in the system opposes it. Specifically, neither the axioms of Linear Algebra nor Elementary Algebra contradicts it. The incursion of logical independence is marked by the explicit need for the imaginary unit [8]. This number's logical independence is well-known to Mathematical Logic [6]. That logical independence can be regarded as inherited from the self-referential process.

In the derivations that follow, the overall plan is to begin with information faithful to the straight-through experiments – the pure state measurements, then perform self-reference, arriving at the information faithful to mixed states.

I start with the 3 axioms (16), (18) and (21), capable of proving the 3 pure state entries of the implied algebra column of Table 1:

Note: (33) implies $(\mathsf{ac})^2 = -\mathbb{1}$.

$$\mathsf{a}^2 = \mathbb{1} \qquad \mathsf{c}^2 = \mathbb{1} \qquad \mathsf{ac} + \mathsf{ca} = \mathbb{0} \tag{33}$$

but, at the same time, note that the 3 axioms (17), (19) and (20):

$$\mathsf{b}^2 = \mathbb{1} \qquad \mathsf{ab} + \mathsf{ba} = \mathbb{0} \qquad \mathsf{bc} + \mathsf{cb} = \mathbb{0} \tag{34}$$

are not needed for the pure states.

Now write down matrices that faithfully represent the algebraic system, requiring axiom system (33), but for which axioms (34) are extraneous, are not needed, and do not take part:

$$\mathsf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{b}(\zeta,\eta) = \begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix} \qquad \mathsf{c} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{35}$$

and note that matrices faithful to information of all six axioms (33) and(34) are the Pauli matrices of the Lie algebra $\mathsf{su}(2)$:

$$\mathsf{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{b} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{c} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{36}$$

The self-reference takes the step from the non-unitary (35) to the unitary (36), without imposing the axioms from (34).

Overall, ζ and η permit a matrix-switch, facilitating the transition $\mathsf{b}(\zeta,\eta) \to \mathsf{b}$, which precisely matches the Boolean information, gleaned from the Paterek research, and listed in Table 1. Any non-zero ζ prevents $\mathsf{b}(\zeta,\eta)$ itself, from being involutory, as well as blocking orthogonality with a and c. The condition $\zeta = 0$, guarantees involutory $\mathsf{b}(\zeta,\eta)\ \forall\eta$; and for $\eta = \pm i$, permits these orthogonalities.

My reason for choosing the matrix $\begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix}$, in preference to $\begin{pmatrix} \zeta & -\eta \\ \eta & -\zeta \end{pmatrix}$, is to maintain ζ and η as independent variables. Whereas the former matrix is free of ζ,η interdependence – in demanding an involutory condition – the latter imposes the relation: $\zeta^2 - \eta^2 = 1$. And hence the former is involutory, more generally, under quantifiers: $\forall\zeta\forall\eta$.

I now derive (36) from (35), paying particular attention to all assumptions made. Starting with the three matrices of (35), I begin by writing the most general arbitrary transformation of which each of these matrices is capable.

$$\forall\alpha_1\forall\alpha_2\exists\psi_1\exists\psi_2 \;\middle|\; \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{37}$$

$$\forall\zeta\,\forall\eta\,\forall\beta_1\forall\beta_2\exists\phi_1\exists\phi_2 \;\middle|\; \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \tag{38}$$

$$\forall\gamma_1\forall\gamma_2\exists\chi_1\exists\chi_2 \;\middle|\; \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \tag{39}$$

Note that these formulae do not assert equality, they assert existence. I now explore the possibility of (37), (38) and (39) accepting information, circularly, from one another, through a ‘forward’ cyclic mechanism where:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{40}$$

and a ‘backward’ mechanism where:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} \qquad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \qquad \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} \text{feeds off} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{41}$$

These form closed, self-referential flows of information. There is no *cause* implying this self-reference; the idea is that no information, occupying the system, prevents it.

To proceed with the derivation, the strategy followed will be to make a formal assumption, by positing the hypothesis that such self-reference does occur; then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Part One
Hypothesised forward coincidences:

$$\forall A \forall\phi_1\forall\phi_2\exists\alpha_1\exists\alpha_2 \;\Bigg|\; \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = A\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{42}$$

$$\forall B \forall\chi_1\forall\chi_2\exists\beta_1\exists\beta_2 \;\Bigg|\; \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = B\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{43}$$

$$\forall C \forall\psi_1\forall\psi_2\exists\gamma_1\exists\gamma c \;\Bigg|\; \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} = C\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{44}$$

Substitution involving quantifiers

$$\begin{aligned} \forall\beta\underline{\forall\gamma}\exists\alpha \;&|\; \alpha=\beta+\gamma\\ \forall\lambda\underline{\exists\gamma} \;&|\; \gamma=2\lambda\\ \Rightarrow\ \forall\lambda\forall\beta\exists\alpha \;&|\; \alpha=\beta+2\lambda \end{aligned}$$

An *existential* quantifier of one proposition is matched with a *universal* quantifier of the other. Those matched are underlined.

Note: there is no guarantee that any such coincidence should exist. We proceed to investigate.. In this block of manipulations, I begin with the transformation (38), then repeatedly make substitutes, cyclicly.

$$\forall\zeta\,\forall\eta\,\forall\beta_1\forall\beta_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \qquad \text{by (38)}$$

$$\forall B\forall\zeta\,\forall\eta\,\forall\chi_1\forall\chi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix} B\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \qquad \text{by (43)}$$

$$\forall B\forall\zeta\,\forall\eta\,\forall\gamma_1\forall\gamma_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix} B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} \qquad \text{by (39)}$$

$$\forall C\forall B\forall\zeta\,\forall\eta\,\forall\psi_1\forall\psi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix} B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} C\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \qquad \text{by (44)}$$

$$\forall C\forall B\forall\zeta\,\forall\eta\,\forall\alpha_1\forall\alpha_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix} B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} C\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} \qquad \text{by (37)}$$

$$\forall A\forall C\forall B\forall\zeta\,\forall\eta\,\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix} B\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} C\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} A\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \qquad \text{by (42)}$$

In summary, assuming the **Hypothesised forward coincidences,** the overall result is the assertion:

$$\forall X\forall\zeta\,\forall\eta\,\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2 \;\Bigg|\; \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = X\begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{45}$$

Where, for the sake of readability, I define $X = BCA$. I note the ambiguous quantification $\forall\phi_1\forall\phi_2\exists\phi_1\exists\phi_2$, but in some capacity or other, (45) implies the following:

$$\begin{aligned} \forall X\forall\zeta\,\forall\eta \;&|\; X\begin{pmatrix}\zeta & \eta^{-1}\\ \eta & -\zeta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \mathbb{1}\\ \Longrightarrow\ \forall X\forall\zeta\,\forall\eta \;&|\; X\begin{pmatrix}-\eta^{-1} & \zeta\\ \zeta & \eta\end{pmatrix} = \mathbb{1} \end{aligned} \tag{46}$$

The assertion (46) is self-contradictory, because the operator cannot equal the identity for all values of X, ζ and η. This confirms there is something invalid about the **Hypothesised forward coincidences.** Nevertheless, it is important to retain the full information of (46), if valid conditionality is to be revealed.

Part two
Hypothesised backward coincidences:

$$\forall\bar{A}\forall\chi_1\forall\chi_2\exists\alpha_1\exists\alpha_2 \;\Bigg|\; \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \bar{A}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{47}$$

$$\forall\bar{B}\forall\psi_1\forall\psi_2\exists\beta_1\exists\beta_2 \;\Bigg|\; \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} = \bar{B}\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} \tag{48}$$

$$\forall\bar{C}\forall\phi_1\forall\phi_2\exists\gamma_1\exists\gamma c \;\Bigg|\; \begin{bmatrix}\gamma_1\\ \gamma_2\end{bmatrix} = \bar{C}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \tag{49}$$

Note: there is no guarantee that any such coincidence should exist. We proceed to investigate.. In this block of manipulations, I begin with the transformation (37), then repeatedly make substitutes, cyclicly.

$$\forall\alpha_1\forall\alpha_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} \qquad \text{by (37)}$$

$$\forall\bar{A}\forall\chi_1\forall\chi_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{A}\begin{bmatrix}\chi_1\\\chi_2\end{bmatrix} \qquad \text{by (47)}$$

$$\forall\bar{A}\forall\gamma_1\forall\gamma_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{A}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{bmatrix}\gamma_1\\\gamma_2\end{bmatrix} \qquad \text{by (39)}$$

$$\forall\bar{C}\forall\bar{A}\forall\phi_1\forall\phi_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{A}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\bar{C}\begin{bmatrix}\phi_1\\\phi_2\end{bmatrix} \qquad \text{by (49)}$$

$$\forall\bar{C}\forall\bar{A}\forall\zeta\,\forall\eta\,\forall\beta_1\forall\beta_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{A}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\bar{C}\begin{pmatrix}\zeta&\eta^{-1}\\\eta&-\zeta\end{pmatrix}\begin{bmatrix}\beta_1\\\beta_2\end{bmatrix} \qquad \text{by (38)}$$

$$\forall\bar{B}\forall\bar{C}\forall\bar{A}\forall\zeta\,\forall\eta\,\forall\psi_1\forall\psi_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\bar{A}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\bar{C}\begin{pmatrix}\zeta&\eta^{-1}\\\eta&-\zeta\end{pmatrix}\bar{B}\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} \qquad \text{by (48)}$$

In summary, assuming the **Hypothesised backward coincidences**, the overall result is the assertion:

$$\forall Y\forall\zeta\,\forall\eta\,\forall\psi_1\forall\psi_2\exists\psi_1\exists\psi_2 \;\Big|\; \begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} = Y\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\zeta&\eta^{-1}\\\eta&-\zeta\end{pmatrix}\begin{bmatrix}\psi_1\\\psi_2\end{bmatrix} \tag{50}$$

Where, for the sake of readability, I define $Y = \bar{A}\bar{C}\bar{B}$. I note the ambiguous quantification $\forall\psi_1\forall\psi_2\exists\psi_1\exists\psi_2$, but in some capacity or other, (50) implies the following:

$$\forall Y\forall\zeta\,\forall\eta \;\mid\; Y\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\zeta&\eta^{-1}\\\eta&-\zeta\end{pmatrix} = \mathbb{1} \tag{51}$$

$$\Longrightarrow \quad \forall Y\forall\zeta\,\forall\eta \;\mid\; Y\begin{pmatrix}-\eta&\zeta\\\zeta&\eta^{-1}\end{pmatrix} = \mathbb{1} \tag{52}$$

The assertion (52) is self-contradictory, because the operator cannot equal the identity for all values of Y, ζ and η. This confirms there is something invalid about the **Hypothesised backward coincidences.** Nevertheless, it is important to retain the full information of (52), if valid conditionality is to be revealed.

Part three

Noting the forward and backward self-references (46) and (52), both result in the identity, they can be equated:

$$\forall X\forall Y\forall\zeta\,\forall\eta \;\mid\; X\begin{pmatrix}-\eta^{-1}&\zeta\\\zeta&\eta\end{pmatrix} = Y\begin{pmatrix}-\eta&\zeta\\\zeta&\eta^{-1}\end{pmatrix}$$

$$\Longrightarrow \quad \forall X\forall Y\forall\zeta\,\forall\eta \;\mid\; X\begin{pmatrix}-\eta^{-1}&\zeta\\\zeta&\eta\end{pmatrix} - Y\begin{pmatrix}-\eta&\zeta\\\zeta&\eta^{-1}\end{pmatrix} = \mathbb{0}$$

Reading the quantifiers, this holds true for all products $X = BCA$ and all products $Y = \bar{A}\bar{C}\bar{B}$. Hence, for every product Y there exists a negative X:

$$\forall Y\exists X \mid \quad X = -Y$$

$$\Longrightarrow \quad \forall\zeta\,\forall\eta\exists X \;\mid\; X\begin{pmatrix}-\eta^{-1}&\zeta\\\zeta&\eta\end{pmatrix} + X\begin{pmatrix}-\eta&\zeta\\\zeta&\eta^{-1}\end{pmatrix} = \mathbb{0}$$

$$\Longrightarrow \quad \forall\zeta\,\forall\eta\exists X \;\mid\; \begin{pmatrix}-\eta^{-1}&\zeta\\\zeta&\eta\end{pmatrix} + \begin{pmatrix}-\eta&\zeta\\\zeta&\eta^{-1}\end{pmatrix} = \mathbb{0}$$

$$\Longrightarrow \quad \forall\zeta\,\forall\eta\exists X \;\mid\; \begin{pmatrix}-\left(\eta^{-1}+\eta\right)&2\zeta\\2\zeta&\eta^{-1}+\eta\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix} \tag{53}$$

But (53) is contradictory because ζ and η cannot be zero, $\forall\zeta\,\forall\eta$. Nevertheless, replacement of the universal quantifiers $\forall\zeta\,\forall\eta$ by existential quantifiers $\exists\zeta\,\exists\eta$ removes the contradiction, thus:

$$\exists X \exists\zeta\,\exists\eta \;\mid\; \begin{pmatrix} -\left(\eta^{-1}+\eta\right) & 2\zeta \\ 2\zeta & \eta^{-1}+\eta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{54}$$

Hence, conditionality on the assumed **Hypothesised forward coincidence** and **Hypothesised backward coincidences** is as follows:

$$X = -Y \qquad\qquad \zeta = 0 \qquad\qquad \eta^2 = -1 \tag{55}$$

Part four

More conditionality is extractable from the forward and backward self-references, (46) and (52), by multiplying them. They give:

$$\begin{aligned} \forall X \forall Y \forall\zeta\,\forall\eta \;\mid\;& X \begin{pmatrix} -\eta^{-1} & \zeta \\ \zeta & \eta \end{pmatrix} Y \begin{pmatrix} -\eta & \zeta \\ \zeta & \eta^{-1} \end{pmatrix} = \mathbb{1} \\ \forall X \forall Y \forall\zeta\,\forall\eta \;\mid\;& XY \begin{pmatrix} -\eta^{-1} & \zeta \\ \zeta & \eta \end{pmatrix} \begin{pmatrix} -\eta & \zeta \\ \zeta & \eta^{-1} \end{pmatrix} = \mathbb{1} \\ \forall X \forall Y \forall\zeta \;\mid\;& XY \begin{pmatrix} \zeta^2+1 & 0 \\ 0 & \zeta^2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{56}$$

But (56) is contradictory because ζ and η cannot be zero, $\forall\zeta\,\forall\eta$. And the product XY cannot be equal to one, $\forall X \forall Y$. Nevertheless, replacement of all universal quantifiers for existential quantifiers removes the contradiction, thus:

$$\exists X \exists Y \exists\zeta \;\mid\; XY \begin{pmatrix} \zeta^2+1 & 0 \\ 0 & \zeta^2+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{57}$$

This formula (57) is resolved by the conditionality:

$$X = Y^{-1} \qquad\qquad \zeta = 0 \tag{58}$$

Gathering together conditionality from (55) and (58)

$$X = -Y = Y^{-1} \qquad\qquad \zeta = 0 \qquad\qquad \eta^2 = -1 \tag{59}$$

Hence as a result of self-reference:

$$\mathsf{b}(\eta) = \begin{pmatrix} \zeta & \eta^{-1} \\ \eta & -\zeta \end{pmatrix} \quad\longmapsto\quad \mathsf{b} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

9 Discussion – Redundant unitarity in free particle pure states

Another quantum system – that of the *free particle* – mirrors this same unitary logic, between pure and mixed states.

It is instructive to understand the difference between *syntactical information* versus a *semantical information. Syntax* concerns rules used for constructing and transforming formulae – the rules of Elementary Algebra, say. *Semantics*, on the other hand, concerns interpretation. Here, *interpretation* does not refer to *physical* meaning, but to *mathematical* meaning: whether symbols might be understood to mean: complex scalars, real scalars, or rational. Such interpretation has null logical connectivity with the rules of algebra — the syntax. Indeed, typically, the interpretation may be only in the theorist's mind and not asserted by the mathematics, at all.

A most relevant illustration is the comparison of syntax versus semantics in the mathematics representing pure eigenstates, set against mixed states, in the quantum free particle system. Consider the *eigenformulae pair:*

$$\frac{d}{dx}\left[\Phi(\mathsf{k})\exp(+i\mathsf{k}x)\right] = +i\mathsf{k}\left[\Phi(\mathsf{k})\exp(+i\mathsf{k}x)\right] \tag{60}$$

$$\frac{d}{dk}\left[\Psi(\mathsf{x})\exp(-ik\mathsf{x})\right] = -i\mathsf{x}\left[\Psi(\mathsf{x})\exp(-ik\mathsf{x})\right] \tag{61}$$

This pair of formulae is true, irrespective of any interpretation placed on the variable i. But in contrast, the *superposition pair:*

$$\Psi(x) = \int \left[\Phi(\mathsf{k}) \exp(+i\mathsf{k}x)\right] d\mathsf{k} \tag{62}$$

$$\Phi(k) = \int \left[\Psi(\mathsf{x}) \exp(-ik\mathsf{x})\right] d\mathsf{x} \tag{63}$$

is true, only if we interpret i as *pure imaginary.* (And if k is restricted to real or rational k; and if x is restricted to real or rational x.) In the case of the eigenvalue pair (60) & (61) the imaginary interpretation is purely in the mind of the theorist, but for the superposition pair (62) & (63), the imaginary interpretation is implied by the mathematics. Whilst for the superposition pair (62) & (63), specific interpretation is *necessary*, for the eigenvalue pair (60) & (61), interpretation is *possible*, but *not necessary.*

In Mathematical Logic, *'necessary information* versus *possible information'* is recognised as constituting what is known as a 'modal logic'. However, in textbook quantum theory, the distinction separating possible from necessary is not noticeable, nor is it recognised; and this logical distinction between pure states and mixed states is lost. The crucial difference in expressing pure states is that their information derives from pure syntax. The transition in forming mixed states from pure states demands the creation of new information[4]. That creation goes unopposed.

The important point is that the logical status of pure states and mixed is distinct, not only in experiments, but in current Theory too, even though, currently, the fact is not recognised.

The fact is that quantum theory for pure states need not be unitary (or self-adjoint); whereas, for mixed states, unitarity is necessary. The jump between pure states and mixed states represents a logical jump between *possible unitarity* and *necessary unitarity.*

Historically, this distinction between necessary and possible unitarity has not drawn attention, as any point of significance. No doubt, standard quantum theory ignores the fact, for reasons of consistency. But, rewriting (60) – (63) as formulae in *first order logic* overcomes any inconsistency; it conveys the whole information of the mathematics; and it preserves the intrinsic logic, in a single theory. Thus, for pure states:

$$\forall \eta \mid \frac{d}{dx}\left[\Phi(\mathsf{k}) \exp\left(\eta^{+1}\mathsf{k}x\right)\right] = \eta^{+1}\mathsf{k}\left[\Phi(\mathsf{k}) \exp\left(\eta^{+1}\mathsf{k}x\right)\right] \tag{64}$$

$$\forall \eta \mid \frac{d}{dk}\left[\Psi(\mathsf{x}) \exp\left(\eta^{-1}\mathsf{x}k\right)\right] = \eta^{-1}\mathsf{x}\left[\Psi(\mathsf{x}) \exp\left(\eta^{-1}\mathsf{x}k\right)\right] \tag{65}$$

The specific choice of scalars η^{+1} and η^{-1}, over the more instinctive choice of $+\eta$ and $-\eta$, is suggested by theory for the Pauli system, shown above. Also, this choice forces the exact value $\eta = i$ on the Fourier transforms, rather than the restriction merely to imaginary values. That said, this must be made consistent with algebra deriving from the homogeneity symmetry [7].

And for mixed:

$$\exists \eta \mid \Psi(x) = \int \left[\Phi(\mathsf{k}) \exp\left(\eta^{+1}\mathsf{k}x\right)\right] d\mathsf{k} \tag{66}$$

$$\exists \eta \mid \Phi(k) = \int \left[\Psi(\mathsf{x}) \exp\left(\eta^{-1}\mathsf{x}k\right)\right] d\mathsf{x} \tag{67}$$

But having rewritten formulae as (64) – (67), these new formulae are inconsistent with the *Postulates of Quantum Mechanics.* Specifically, (64) & (65) disagree with unitarity (or self-adjointness) – imposed *by Postulate.* Whilst (64) – (67) represent a mathematical system that is logically self-consistent, that conveys the whole information of unitarity; that conveyance of whole information is gained at the expense of textbook quantum theory's most treasured fact — the self-adjointness of operators.

Not to worry. The *Postulated* unitarity (or self-adjointness) is not needed. Unitarity is implied where it is needed – in the mathematics of the mixed states. Elsewhere, unitarity (or self-adjointness) is redundant.

10 Discussion – Self-reference in free particle mixed states

As in the Pauli system, the transition (64) – (67) from pure to mixed states, again involves logical self-reference.

[4] In some way, yet to be understood, this information is lost again during measurement.

Consider the following pair of formulae, asserting existence of general sums over all eigenvectors.

$$\forall\eta\forall x\exists a\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x\mathsf{k}\right) a(\mathsf{k})\right] \tag{68}$$

$$\forall\eta\forall k\exists b\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k\mathsf{x}\right) b(\mathsf{x})\right] \tag{69}$$

In writing these, the san-serif notated k and x are the dummy (bound) variables over the integrals. The italicised variables η, k, x, a, b are all bound variables over the existential quantifier $\exists$ and universal quantifier $\forall$. The ordering of variables is laid out to mirror the convention of repeated dummy indices used in summations of discrete quantities, so as to emphasise the fact that these are transformations.

I use the notation $\int_{\mathsf{k}} f(\mathsf{k}) = \int_{-\infty}^{+\infty} f(\mathsf{k})\, d\mathsf{k}$.

Note that these formulae do not assert equality, they assert existence. Note also; the integrals exist, and the pair of propositions is true, when amplitudes a and b are restricted to the (bounded functions) Banach space[5] L^1.

I now explore the possibility of (68) and (69) accepting information, circularly, from one another, through a mechanism where $a(\mathsf{k})$ feeds off $\Phi(k)$ and $b(\mathsf{x})$ feeds off $\Psi(x)$. There is no *cause* implying this self-reference; the idea is that nothing prevents it. Indeed, the self-referential process is logically independent of all algebraic rules in operation.

To proceed, the strategy followed will be to make a formal assumption, by positing the hypothesis that such self-reference does occur; then investigate for conditionality implied. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised coincidence:

$$\forall\Phi\exists a \mid \quad a = \Phi; \tag{70}$$

$$\forall\Psi\exists b \mid \quad b = \Psi. \tag{71}$$

When these assumptions are substituted into (68) and (69), circular dependency is enabled, via Φ and Ψ, through this pair of formulae:

$$\forall\eta\forall x\exists\Phi\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x\mathsf{k}\right) \Phi(\mathsf{k})\right] \tag{72}$$

$$\forall\eta\forall k\exists\Psi\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k\mathsf{x}\right) \Psi(\mathsf{x})\right] \tag{73}$$

In these, if both Φ and Ψ are in the Banach space L^1, then both integrals exist, and no issue arises. Without making the assumption of Banach space we proceed by making the cross-substitution of Φ and Ψ, and watch out for contradiction. We get:

$$\forall\eta\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x\mathsf{k}\right) \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} \mathsf{k}\mathsf{x}\right) \Psi(\mathsf{x})\right]\right] \tag{74}$$

$$\forall\eta\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k\mathsf{x}\right) \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} \mathsf{x}\mathsf{k}\right) \Phi(\mathsf{k})\right]\right] \tag{75}$$

Taking the integral signs outside and reversing their order, these tidy up to become:

$$\forall\eta\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{x}}\int_{\mathsf{k}} \exp\left[\left(\eta^{+1} x + \eta^{-1}\mathsf{x}\right)\mathsf{k}\right] \Psi(\mathsf{x}) \tag{76}$$

$$\forall\eta\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{k}}\int_{\mathsf{x}} \exp\left[\left(\eta^{-1} k + \eta^{+1}\mathsf{k}\right)\mathsf{x}\right] \Phi(\mathsf{k}) \tag{77}$$

In the first of these two formulae (76), $\Psi(\mathsf{x})$ serves to bound, only the $\int_{\mathsf{x}}$ sum, to finite values. The sum in $\int_{\mathsf{k}}$ is generally unbounded, unless $\eta = i$. And so overall, for arbitrary values of η, the double integral fails. The predicament is precisely similar for the second formulae (77). Hence, (76) and (77) are untrue statements, and hence the hypothesised coincidence (70) & (71) contradicts (68) & (69).

The contradiction is resolved by replacing $\forall\eta$ by $\exists\eta$ in (72) & (73). Thus:

Please note that quantifiers $\forall$ and $\exists$ do not commute. The common use in this paper would be $\forall a\exists b$; where, for each a there exist distinct assignments of a. The other use is seen in (78) & (79); in these, $\exists\eta\forall x$ means there exists a unique η for any and every assignment of x.

$$\exists\eta\forall x\exists\Phi\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{k}} \left[\exp\left(\eta^{+1} x\mathsf{k}\right) \Phi(\mathsf{k})\right] \tag{78}$$

$$\exists\eta\forall k\exists\Psi\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{x}} \left[\exp\left(\eta^{-1} k\mathsf{x}\right) \Psi(\mathsf{x})\right] \tag{79}$$

resulting in

$$\exists\eta\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{x}}\int_{\mathsf{k}} \exp\left[\left(\eta^{+1} x + \eta^{-1}\mathsf{x}\right)\mathsf{k}\right] \Psi(\mathsf{x}) \tag{80}$$

$$\exists\eta\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{k}}\int_{\mathsf{x}} \exp\left[\left(\eta^{-1} k + \eta^{+1}\mathsf{k}\right)\mathsf{x}\right] \Phi(\mathsf{k}) \tag{81}$$

[5] Banach space L^1 consists of bounded functions, ensuring convergence of these integrals

Releasing bound variable η from its quantifier and replacing by particular value $\boldsymbol{\eta}$:

$$\exists\Psi \mid \quad \Psi(\mathsf{x}') = \int_{\mathsf{x}}\int_{\mathsf{k}} \exp\left[\left(\boldsymbol{\eta}^{+1}\mathsf{x}' + \boldsymbol{\eta}^{-1}\mathsf{x}\right)\mathsf{k}\right]\Psi(\mathsf{x}) \tag{82}$$

$$\exists\Phi \mid \quad \Phi(\mathsf{k}') = \int_{\mathsf{k}}\int_{\mathsf{x}} \exp\left[\left(\boldsymbol{\eta}^{-1}\mathsf{k}' + \boldsymbol{\eta}^{+1}\mathsf{k}\right)\mathsf{x}\right]\Phi(\mathsf{k}) \tag{83}$$

These integrals exist only when $\boldsymbol{\eta} = \pm\,\mathsf{i}$. And therefore this pair of propositions is true — with the **Hypothesised coincidence** guaranteed — only for $\boldsymbol{\eta} = \pm\,\mathsf{i}$.

But, up to this point, no imaginary information exists in the system. In order to validate the pair of integrals, new information must be introduced. This information must be assumed. To properly document this assumption, the hypothesis is formally declared, thus:

Hypothesised existence:

$$\exists\boldsymbol{\eta} \mid \quad \boldsymbol{\eta}^2 = -1$$

Setting the *particular* number $\mathsf{i} = \sqrt{\ } - 1$ and also $\boldsymbol{\eta} = \mathsf{i}$:

$$\forall x\exists\Psi \mid \quad \Psi(x) = \int_{\mathsf{x}}\int_{\mathsf{k}} \exp\left[+\mathsf{i}\,(x - \mathsf{x})\,\mathsf{k}\right]\Psi(\mathsf{x}) \tag{84}$$

$$\forall k\exists\Phi \mid \quad \Phi(k) = \int_{\mathsf{k}}\int_{\mathsf{x}} \exp\left[-\mathsf{i}\,(k - \mathsf{k})\,\mathsf{x}\right]\Phi(\mathsf{k}) \tag{85}$$

and in conclusion, claim that this pair of formulae are true, providing they are allowed self-referential information.

It is important to say that, within Elementary Algebra, this number's existence is very well-known, by Mathematical Logicians, to be logically independent [6].

11 Conclusions

Treating an algebra as a system based on *axioms,* those axioms prove (cause) theorems. And those theorems are *logically dependent* information. However, in certain algebras, there are statements and information which axioms do not prove; nor do they disprove (prevent). That information is known as *logically independent.* It might be thought of as having 'null logical connectivity' with axioms.

In this paper, a logically independent mathematical mechanism is derived, matching logical independence, linked empirically to quantum randomness. That mechanism comprises a logically circular, self-referential set of geometric transformations, which is permitted because it does not contradict, but is consistent with system axioms.

Quantum indeterminacy is strictly a phenomenon of *mixed* states. Measurement outcomes from pure eigenstates are never random. That is well-known. In alignment with that, new research of Tomasz Paterek et al shows that *logical independence,* also, is a strict feature of mixed states – pure states being *logically dependent* [12, 13]. And that randomness is the response to logical independence.

That logical dependence and independence is mathematical information. The transition from pure states to mixed is reflected in corresponding mathematical transition stepping from dependence to independence. Because only mixed states include indeterminate randomness, the detail, inner workings of that mathematical transition reveals information about the inner workings of quantum indeterminacy.

To begin, the only information in the algebraic system is the axioms, along with all theorems they prove. The transition begins to get underway following an unpreventable *coincidence* of vectors, from separate vector spaces. These coincidences are new information to the system, which permits the onset of vectorial information being passed, cyclicly, around a set of transformations. The logical circularity is in both cyclic and anticyclic senses. The new information comes with unitary conditionality which transforms a geometrically definite system into one of ambiguous right and left handedness. In effect the self-referent mechanism is one of 'spontaneous symmetry creation' – the converse of 'spontaneous symmetry breaking' – as in the Higgs mechanism.

There is nothing exceptional about the coincidences; they are assumed on a daily basis, without noticing, by mathematicians using orthogonal vector spaces. On the face of it, these coincidences seem like quantitative information, but they are logical information, and for that reason, the creation of Hilbert space from Banach space is a logical matter. Notice that (finite) Banach spaces are non-orthogonal and can be constructed purely from information in Elementary Algebra. This is ordinary school algebra: the algebra of scalars.

Textbook quantum theory demands: Hilbert space, self-adjoint operators and unitary symmetries, as features. From the viewpoint of the transition, none of these are required by pure eigenstates; they are required only by mixed states. A truly faithful, isomorphic theory would need to be *non-unitary* on the pure state side of the transition, and *unitary* on the mixed state side.

Whilst the mathematician might feel free to simply declare a theory *unitary*, by declaring: observable operators should be Hermitian — although such a declaration might seem to impose a purely *quantitative* restriction on variables — that eigenvalues be real, for instance — such declaration includes hidden logical structure. This logical structure sits at the interface between Elementary Algebra and *orthogonal* Linear Algebra. The juxtaposition of these two algebras, in a single environment, is inherent in quantum mathematics, placing that logical structure squarely and unavoidably in the domain of quantum theory.

Unlike (physical) energy or momentum, that self-reference is perfectly free and not subject to any conservation law. There is no resistance to its onset. Self-reference is a spontaneous logical option, neither caused nor prevented (implied nor denied) by any information in the mathematical environment — it is *logically independent* of all information in that mathematical environment.

The effect of the self-reference is to create the consequent existence of a unitary symmetry, along with structures that follow from it: self-adjoint operators and Hilbert space, et cetera – all logically independent of the system axioms. The impact of all this is that unitarity or self-adjointness, imposed – *by Postulate* – is redundant. This is because unitarity and self-adjointness arise freely in the mathematics; they don't need to be 'given' to it. They occur unpreventably in Elementary algebra; they don't need to be taken from linear algebra.

The conclusion of this research is that a quantum theory that adheres strictly to the *faithful* representation of (non-unitary) pure states – *that switches to* – the strict and *faithful* representation of (unitary) mixed states, automatically invokes representation of quantum indeterminacy. Those faithful representations require isomorphisms under two distinct systems: a non-unitary algebra representing pure states, and a unitary symmetry representing mixed. Transition between these is logically self-referential. To allow this logical mechanism to operate, unitarity (and self-adjointness) must be free to switch on and off. But in standard theory, unitarity (or self-adjointness) is imposed – *by Postulate* – and this freedom is blocked.

The most profound conclusion, therefore, is that unitarity and self-adjointness, imposed – *by Postulate* – must be given up; the benefit being a quantum theory that expresses theory and logic of quantum indeterminacy.

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