The Extra-terrestrial pendulum in the gravity field

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April 19, 2016

Abstract

The motion of the mathematical extra-terrestrial pendulum is considered in the spherical gravitational field. The potential energy of the pendulum bob is approximated by the linear term mgh and additional quadratical term in h, where h is height of the pendulum bob over the reference point. The nonlinear equation of motion of pendulum is solved by the Landau-Migdal method to obtain the frequency of motion and the swing amplitude. While the Foucault pendulum bob moves over the sand surface, our pendulum bob moves in ionosphere. It is not excluded that the pendulum project will be the integral part of the NASA cosmical physics.

Key words. Mathematical pendulum, Newton gravity potential, nonlinear equation, Landau-Migdal iteration method.

The cosmical pendulum is the extra-terrestrial mathematical pendulum, the pendant of which moves along a geostationary orbit, where the geostationary Earth orbit or geosynchronous equatorial orbit (GEO) is a circular orbit 35,786 kilometres above the Earth equator and following the direction of the Earth rotation. An object in such an orbit has an orbital period equal to the Earth rotational period.

The length of the (carbon tube) fibre of the pendulum is considered sufficiently long to detect the sphericity of the globe gravity.

The mathematics of pendulums are in general quite complicated problem. However, there are some simplifying assumptions, which allows the equations of motion to be solved analytically for small-angle oscillations. We consider here the pendulum bob motion in the nonuniform gravity field expressed by the Newton gravitational formula. The parameters of pendulum is as follows: the swing fibre is massless, inextensible and always remains of the constant length, the moving bob with mass m is a point mass, motion occurs only in two dimensions, i.e. the bob trajectory is an arc, the motion does not lose energy to friction or air resistance, and finally, the pendant point does not move (the pendant point is on the geo-stationary satellite).

In order to get the differential equation of motion, we first derive the gravitational potential energy generated by the point with mass M and with the mass of the bob m.

The force acting on the point mass in such gravity field is

$$\mathbf{F} = -\kappa \frac{Mm}{r^3}\mathbf{r}. \tag{1}$$

The performed work by this force is defined by the following formula

$$W = -\int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{F} \cdot d\mathbf{r} = \kappa Mm \int_{\mathbf{r_1}}^{\mathbf{r_2}} \frac{\mathbf{r} \cdot d\mathbf{r}}{r^3}, \tag{2}$$

where the sign (-) in front of integral formula denotes the negative work performed by the gravitational field. Using $\mathbf{r} \cdot d\mathbf{r} = |\mathbf{r}||d\mathbf{r}| \cos\alpha = rdr$, we get

$$W = \kappa Mm \int_{r_1}^{r_2} \frac{rdr}{r^3} = -\kappa Mm \left(\frac{1}{r_2} - \frac{1}{r_1} \right). \tag{3}$$

Now, let us introduce the general point r by $r_2 \to r$ and reference point R by , $r_1 \to R$ then the potential energy is of the form

$$E_p = -\kappa Mm \left(\frac{1}{r} - \frac{1}{R} \right). \tag{4}$$

The potential energy of a point m at the vicinity of the reference point R, i.e. at point $R + h$, where $h \ll R$ is then

$$E_p(R+h) = -\frac{\kappa Mm}{R} \left(\frac{1}{1+h/R} - 1 \right) \approx \frac{\kappa Mmh}{R^2} (1 - h/R). \tag{5}$$

With regard to the fact that the local acceleration at point R is

$$g = \frac{\kappa M}{R^2}, \tag{6}$$

we write the potential energy in the simple form:

$$E_p = mgh \left(1 - \frac{h}{R} \right) = mgh - \frac{mg}{R} h^2, \tag{7}$$

which is the suitable formula of energy with spherical gravity correction in order to construct the differential equation for motion of pendulum.

Let us consider the pendulum with the equilibrium z-coordinate at point $z = R$ and the support coordinate (the pendant point) is at $z = R + l$, where l is the length of our pendulum. Then, the standard expression for the h-coordinate is

$$h = 2l \sin^2(\varphi/2) \approx \frac{l}{2}\varphi^2; \varphi \to 0, \tag{8}$$

where φ is the standard deflection angle of the swing with regard to the z-coordinate.

The total energy of the pendulum is

$$E = \frac{1}{2}mv^2 + mgh - \frac{mg}{R}h^2. \tag{9}$$

We obtain for small φ that $\lim_{\varphi \to 0}(dh/dt) \approx l\varphi(d\varphi/dt)$ and from equation $dE/dt = 0$, we get following equation

$$\ddot{\varphi} + \frac{g}{l}\varphi - \frac{gl}{R}\varphi^3 = 0, \tag{10}$$

or,

$$\ddot{\varphi} + \omega_0^2\varphi = \lambda\varphi^3; \quad \omega_0^2 = g/l; \quad \lambda = gl/R. \tag{11}$$

The next step is to solve the last differential equation (11) by the appropriate approximate method.

We will solve the eq. (11) by iteration. In order to avoid the resonance solution, we use the method described for instance in the Migdal special book on special mathematical methods in quantum mechanics (Migdal, 1975). This method was also used by author at solwing the Gross-Pitaevskii equation for the superfluid medium (Pardy, 1989) with the goal to detect the gravity waves by the superfluid system (instead of LIGO and eLISA).

The first step is that we rewrite eq. (11) as follows:

$$\ddot{\varphi} + \omega^2\varphi = \lambda\varphi^3 + (\omega^2 - \omega_0^2)\varphi; \quad \omega_0^2 = g/l; \quad \lambda = gl/R, \tag{12}$$

where the fundamental solution of the left side is $\varphi_0 = C\sin\omega t$, where constant C must be determined from the initial conditions. Then, the equation for the first iteration is as follows

$$\ddot{\varphi}_1 + \omega^2\varphi_1 = \frac{1}{4}\lambda C^3(3\sin\omega t - \sin 3\omega t) + C(\omega^2 - \omega_0^2)\sin\omega t. \tag{13}$$

The mathematical consistency demands the coefficient with $\sin\omega t$, must be zero from which follows that

$$\omega = \left(\omega_0^2 - \frac{3}{4}\lambda C^2\right)^{1/2}; \quad \omega_0^2 = g/l; \quad \lambda = gl/R. \tag{14}$$

Now, we must solve the equation

$$\ddot{\varphi}_1 + \omega^2\varphi_1 = -\frac{\lambda}{4}C^3(\sin 3\omega t). \tag{15}$$

The partial solution of eq. (15) is $\varphi_p = A\sin 3\omega t$, which gives after insertion this function in eq. (15), that

$$A = \frac{\lambda}{32}\frac{1}{\omega^2}C^3 \tag{16}$$

and it means that the first iteration solution of eq. (12) is

$$\varphi_1 = C\sin\omega t + \left(\frac{\lambda}{32}\frac{1}{\omega^2}C^3\right)\sin 3\omega t; \quad \omega = \left(\omega_0^2 - \frac{3}{4}\lambda C^2\right)^{1/2}. \tag{17}$$

We have seen how to solve, in approximation, the mathematical pendulum moving in the spherical gravity field. We considered here the large extra-terrestrial pendulum with the long fibre. The pendant point is considered on the geo-stationary satellite. While the bob of the Foucalt pendulum moves in sand, our pendulum bob moves in ionosphere.

The problem of the cosmical pendulum can be generalized to the situation where Earth is rotated. If we use the Minkowski element

$$ds^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \tag{18}$$

and the nonrelativistic transformation to the rotation system (Landau et al., 2005)

$$x' = x\cos\Omega t - y\sin\Omega t, \quad y' = x\sin\Omega t + y\cos\Omega t, \quad z = z', \tag{19}$$

then we get in general $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, or in our case,

$$ds^2 = [-c^2 + \Omega^2(x^2 + y^2)]dt^2 + dx^2 + dy^2 + dz^2 - 2\Omega y dx dt + 2\Omega x dy dt, \tag{20}$$

which is not relativistically invariant.

In this situation we are forced to replace the original electromagnetic Lorentz equation

$$mc\frac{dv^\mu}{ds} = \frac{e}{c}F^{\mu\nu}v_\nu, \tag{21}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, by the general-relativistic equation, where the standard derivative are replaced by the covariant in order to get the general relativistic equation for the motion of a massive particle moving in the rotating system and gravity (Landau et al., 2005):

$$mc\left(\frac{dv^\mu}{ds} + \Gamma^\mu_{\alpha\beta}v^\alpha v^\beta\right) = \frac{e}{c}G^{\mu\nu}v_\nu, \tag{22}$$

where the Christofel symbols $\Gamma^\mu_{\alpha\beta}$ are defined by the formula

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\alpha}}{\partial x^\beta} + \frac{\partial g_{\lambda\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\lambda}\right) \tag{23}$$

and symbol $G^{\mu\nu}$ is the tensor force, which corresponds to the general-relativistic tensor forces acting on the pendulum. To our knowledge, this (Nobel) problem was not solved in the scintific journals and monographs.

Our large pendulum can also be used for the detection of the gravitational waves if we generalize the tensor $G^{\mu\nu}$ by one, involving the forces due to gravitational waves. It is not excluded that such detection of gravitational waves can form the new deal of the future NASA gravity physics and general relativity.

Let us remark that our pendulum can be applied at the commercial area where the pendulum bob will form the non-magnetic sphere fulfilled by the scientific instruments of all world laboratories (Bell laboratories, Oxford instruments, CERN laboratories and so on). The extra-terrestrial pendulum project is cheaper than the laser project by Hawking and Milner, or LIGO, or eLISA.

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