ON SOME SERIES RELATED TO MÖBIUS FUNCTION AND LAMBERT W-FUNCTION

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Abstract. We derive some new formulas, connecting some series with Möbius function with Sine Integral and Cosine Integral functions, give the formal proof for full version of Stirling's formula with remainder term in form of definite integral of elementary function; investigate the values of new Dirichlet series function at natural numbers $\geqslant 2$ and its behavior at the pole $s = 1$, connecting it with elementary constants.

1 Introduction

It will be later shown in this article that the following formulas are correct:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}(\ln n! - n\ln n + n) = -\frac{1}{\pi}\int_0^{2\pi} \frac{\sin x}{x}dx;$$

$$\sum_{n=1}^{\infty} \mu(n)(H_n - \gamma - \ln n) = 2\gamma + 2\ln 2\pi - 2\int_0^{2\pi} \frac{1-\cos x}{x}dx;$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}\left(\ln G(1+n) - \frac{n^2}{2}\ln n + \frac{3}{4}n^2\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^{2\pi} \frac{\sin x}{x}dx - \frac{1}{2\pi^2}\int_0^{2\pi} \frac{1-\cos x}{x}dx,$$

where G is the Barnes G-function which will be given later.

And some formulas of different type:

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}\left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \frac{\pi^2}{6} - \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^3}\left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \zeta(3) - \frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n^4}\left(1 + n + \frac{n^2}{2!} + ... + \frac{n^n}{n!}\right) = \frac{\pi^4}{90} - \frac{7}{24}$$

Also we are going to introduce an interesting way to derive some well known expressions for $\ln x! = \ln\Gamma(1+x)$ and $H_x = \gamma + \psi(1+x)$:

$$\ln x! = x\ln x - x + \frac{1}{2}\ln 2\pi x + \int_0^{\infty} \frac{2\arctan(\frac{t}{x})}{e^{2\pi t} - 1}dt$$

$$H_x = \ln x + \gamma + \frac{1}{2x} - \int_0^{\infty} \frac{2t}{t^2 + x^2}\frac{dt}{e^{2\pi t} - 1}$$

2 Formal proof of full Stirling's formula

Formal proofs of different formulas are not the mathematical proofs in usual sense. They only could give a clue for the correct formula, which can be proved later by the strict mathematical reasoning. About the following identities we could only say that if they work for every polynomial, we could substitute them by other "natural" functions, to find a rigorous proof for the new identities later.

Let's start from simple formulas:

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!},$$

$$\Gamma(s)\zeta(s) = \int\limits_0^\infty \frac{t^{s-1}}{e^t-1}dt$$

Then

$$\frac{B_{2n}}{(2n)!} = \frac{2(-1)^{n+1}}{(2n-1)!}\int\limits_0^\infty \frac{t^{2n-1}}{e^{2\pi t}-1}dt$$

That's why

$$\frac{x}{e^x-1} = 1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n} = 1-\frac{x}{2}+\int\limits_0^\infty \frac{2x\sin xt}{e^{2\pi t}-1}dt$$

It is important to represent such function as $\frac{x}{e^x-1}$ as an integral of elementary function and now we will explain, why.

Let D be the differential operator $\frac{d}{dx}$. Applying Taylor's theorem, $e^{cD}f(x) = \sum\limits_{n=0}^{\infty}\frac{c^n f^{(n)}(x)}{n!} = f(x+c)$. Let's replace x in integral formula for $\frac{x}{e^x-1}$ by D and notice that formally

$$\frac{D}{e^D-1}\int\limits_x^{x+1} f(t)dt = f(x)$$

Then

$$f(x) = \left(1-\frac{D}{2}+\int\limits_0^\infty \frac{2D\sin Dt}{e^{2\pi t}-1}dt\right)\int\limits_x^{x+1} f(z)dz =$$

$$= \int\limits_x^{x+1} f(z)dz - \frac{f(x+1)-f(x)}{2} + \frac{1}{i}\int\limits_0^\infty \frac{e^{iDt}-e^{-iDt}}{e^{2\pi t}-1}dt(f(x+1)-f(x))$$

And we obtain the full version of trapezoidal rule

$$\int\limits_x^{x+1} f(z)dz = \frac{f(x)+f(x+1)}{2}+$$

$$+i\int\limits_0^{\infty} \frac{f(1+x+it)-f(x+it)-f(1+x-it)+f(x-it)}{e^{2\pi t}-1}dt$$

Using this formula one can obtain Abel-Plana summation formula (which formally works for polynomials too, giving the values of Riemann zeta function at complex points with negative real part):

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int\limits_0^{\infty} f(x)dx + i\int\limits_0^{\infty} \frac{f(it)-f(-it)}{e^{2\pi t}-1}dt$$

But we are not interested in this formula now. Let's try to put $f(x) = \ln\Gamma(x)$ in the trapezoidal rule, remembering Raabe's formula:

$$\int\limits_x^{x+1} \ln\Gamma(t)dt = x\ln x - x + \frac{1}{2}\ln 2\pi$$

Then the equality

$$\ln\Gamma(x) = x\ln x - x + \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln x - i\int\limits_0^{\infty} \frac{\ln(x+it)-\ln(x-it)}{e^{2\pi t}-1}dt$$

holds true. But the complex logarithm could be computed by the formula $\ln(x+iy) = \dfrac{1}{2}\ln(x^2+y^2) + i\arctan\left(\dfrac{y}{x}\right)$, so we finally have

$$\ln x! = x\ln x - x + \frac{1}{2}\ln 2\pi x + \int\limits_0^{\infty} \frac{2\arctan(\frac{t}{x})}{e^{2\pi t}-1}dt$$

Strict proof of this formula could be derived, using the integral representations:

$$\arctan(x) = \int\limits_0^{\infty} \frac{\sin(xt)}{t}e^{-t}dt; \quad \ln x = \int\limits_0^{\infty} \frac{e^{-t}-e^{-xt}}{t}dt;$$

$$\ln x! = \int\limits_0^{\infty} \frac{e^{-xt}-xe^{-t}-1+x}{t(e^t-1)}dt,$$

all of which can be proved in the same manner as the formula for $\dfrac{x}{e^x-1}$, replacing Taylor coefficients by the Γ-function standart integral representation or $\Gamma\zeta$ integral representation multiplyed by the appropriate reciprocal factorials.

3 Derivation of stated results with μ-function

Differentiating the full Stirling's formula, we can now obtain integral representation for generalized Harmonic number H_x and Hurwitz function $\zeta(k+1, x+1)$:

$$H_x = \ln x + \gamma + \frac{1}{2x} - \int_0^\infty \frac{2t}{t^2+x^2} \frac{dt}{e^{2\pi t}-1}$$

$$\zeta(k+1, x+1) = \frac{1}{kx^k} - \frac{1}{2x^{k+1}} + i \int_0^\infty \left(\frac{1}{(x+it)^{k+1}} - \frac{1}{(x-it)^{k+1}} \right) \frac{dt}{e^{2\pi t}-1}$$

Let's change the variable in all integrals $t = xny$ and use the Lambert series formula of Möbius function

$$\sum_{n=1}^\infty \frac{\mu(n)}{e^{ny}-1} = e^{-y} \quad (y>0)$$

with reciprocal zeta function formulas

$$\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0;$$

$$\sum_{n=1}^\infty \frac{\mu(n)\ln n}{n} = -\left.\frac{d}{ds}\frac{s-1}{\zeta(s)(s-1)}\right|_{s=1} = -1$$

and integration by parts technique to derive

$$\sum_{n=1}^\infty \frac{\mu(n)}{n}(\ln\Gamma(1+nx) - (nx)\ln(nx) + nx) = -\frac{1}{2} + \frac{\cos 2\pi x}{\pi}\left(\frac{\pi}{2} - \int_0^{2\pi x} \frac{\sin t}{t} dt\right) +$$

$$+\frac{\sin 2\pi x}{\pi}\left(\gamma + \ln 2\pi x - \int_0^{2\pi x} \frac{1-\cos t}{t} dt\right);$$

$$\sum_{n=1}^\infty \mu(n)(H_{nx} - \gamma - \ln(nx)) = 2\cos 2\pi x\left(\gamma + \ln 2\pi x - \int_0^{2\pi x} \frac{1-\cos t}{t} dt\right) -$$

$$-2\sin 2\pi x\left(\frac{\pi}{2} - \int_0^{2\pi x} \frac{\sin t}{t} dt\right)$$

And not so pretty formulas for Hurwitz zeta function with even and odd parametres:

$$\sum_{n=1}^{\infty}\mu(n)(nx)^{2k-1}\left(\zeta(2k,1+nx)-\frac{1}{(2k-1)(nx)^{2k-1}}\right)=\sum_{m=0}^{k-1}2(-1)^m(2\pi x)^{2m}\frac{(2k-2m-2)!}{(2k-1)!}+$$

$$+\frac{2(-1)^k(2\pi x)^{2k-1}}{(2k-1)!}\cos 2\pi x\left(\frac{\pi}{2}-\int\limits_0^{2\pi x}\frac{\sin t}{t}dt\right)+$$

$$+\frac{2(-1)^k(2\pi x)^{2k-1}}{(2k-1)!}\sin 2\pi x\left(\gamma+\ln 2\pi x-\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt\right);$$

$$\sum_{n=1}^{\infty}\mu(n)(nx)^{2k}\left(\zeta(2k+1,1+nx)-\frac{1}{2k(nx)^{2k}}\right)=\sum_{m=0}^{k-1}2(-1)^m(2\pi x)^{2m}\frac{(2k-2m-1)!}{(2k)!}+$$

$$+\frac{2(2\pi x)^{2k}(-1)^{k+1}}{(2k)!}\cos 2\pi x\left(\gamma+\ln 2\pi x-\int\limits_0^{2\pi x}\frac{1-\cos t}{t}dt\right)+$$

$$+\frac{2(2\pi x)^{2k}(-1)^{k+1}}{(2k)!}\sin 2\pi x\left(\frac{\pi}{2}-\int\limits_0^{2\pi x}\frac{\sin t}{t}dt\right)$$

Setting $x=1$ in all of these formulas we obtain the first two stated results and the following formulas too

$$\sum_{n=1}^{\infty}\mu(n)n\left(\frac{\pi^2}{6}-\left(1+\frac{1}{2^2}+...+\frac{1}{n^2}\right)-\frac{1}{n}\right)=2-2\pi^2+4\pi\int\limits_0^{2\pi}\frac{\sin t}{t}dt;$$

$$\sum_{n=1}^{\infty}\mu(n)n^2\left(\zeta(3)-\left(1+\frac{1}{2^3}+...+\frac{1}{n^3}\right)-\frac{1}{2n^2}\right)=1+4\gamma\pi^2+4\pi^2\ln 2\pi-4\pi^2\int\limits_0^{2\pi}\frac{1-\cos t}{t}dt;$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\ln\binom{2n}{n}-2n\ln 2\right)=\frac{2}{\pi}\int\limits_0^{2\pi}\frac{\sin t}{t}dt-\frac{1}{\pi}\int\limits_0^{4\pi}\frac{\sin t}{t}dt$$

Now the formula with Barnes G-function is left to derive.

The values of Barnes G-function at integer points are given by the equality

$$G(n) = \begin{cases} \prod_{k=0}^{n-2} k!,\ n = 0, 1, ...; \\ \qquad 0,\ n = -1, -2, ... \end{cases}$$

(where empty product is equal to 1) and it can be continued to all complex plane by the same manner as the definition of Γ-function:

$$G(1+z) = (2\pi)^{z/2} e^{-z/2-(1+\gamma)z^2/2} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)}$$

From this definition it could be easily derived the expression

$$\ln G(1+x) = \frac{x(1-x)}{2} + \frac{x}{2} \ln 2\pi + x \ln \Gamma(x) - \int_0^x \ln \Gamma(t) dt$$

So integrating the full Stirling's formula we can derive the identity

$$\ln G(1+x) = \frac{x^2}{2} \ln x - \frac{3}{4} x^2 + \frac{x}{2} \ln 2\pi - \frac{1}{12} \ln x + \int_0^\infty \frac{2t \ln t}{e^{2\pi t} - 1} dt - \int_0^\infty \frac{t \ln(1 + \frac{t^2}{x^2})}{e^{2\pi t} - 1} dt$$

or, using Glaisher-Kinkelin constant A, given by the equality

$$\frac{1}{24} - \frac{1}{2} \ln A = \int_0^\infty \frac{t \ln t}{e^{2\pi t} - 1} dt$$

we could rewrite it more compactly

$$\ln G(1+x) = \frac{x^2}{2} \ln x - \frac{3}{4} x^2 + \frac{x}{2} \ln 2\pi - \frac{1}{12} \ln x + \frac{1}{12} - \ln A - \int_0^\infty \frac{t \ln(1 + \frac{t^2}{x^2})}{e^{2\pi t} - 1} dt$$

Differentiating $\Gamma\zeta$ integral representation we get the equality

$$\zeta'(2) = \frac{\pi^2}{6} (\gamma + \ln 2\pi - 12 \ln A)$$

and so the following formula holds

$$\sum_{n=1}^{\infty} \frac{\mu(n) \ln n}{n^2} = - \left. \frac{d}{ds} \frac{1}{\zeta(s)} \right|_{s=2} = \frac{6}{\pi^2} (\gamma + \ln 2\pi - 12 \ln A)$$

Now we are going to use this formula with the fact that $\sum\limits_{n=1}^{\infty} \dfrac{\mu(n)}{n^2} = \dfrac{6}{\pi^2}$ to get

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\left(\ln G(1+nx)-\frac{(nx)^2}{2}\ln(nx)+\frac{3}{4}(nx)^2\right)=-\frac{1}{2\pi^2}(\gamma+\ln 2\pi x-1)-$$

$$-\int_0^{\infty} t\ln(1+t^2)e^{-2\pi tx}dt=(\gamma+\ln 2\pi x)\left(\frac{x\sin 2\pi x}{\pi}+\frac{\cos 2\pi x}{2\pi^2}-\frac{1}{2\pi^2}\right)+$$

$$+\frac{x\cos 2\pi x}{2}-\frac{\sin 2\pi x}{4\pi}+\left(\frac{\sin 2\pi x}{2\pi^2}-\frac{x\cos 2\pi x}{\pi}\right)\int_0^{2\pi x}\frac{\sin t}{t}dt-$$

$$-\left(\frac{\cos 2\pi x}{2\pi^2}+\frac{x\sin 2\pi x}{\pi}\right)\int_0^{2\pi x}\frac{1-\cos t}{t}dt$$

Setting $x=1$ we obtain the desired result. Similarly using Abel-Plana formula for Hurwitz zeta function at non-integer points we can obtain this type formulas in terms of incomplete Γ-function.

But the Möbius μ-function is not the only non-trivial arithmetic function which have the closed form of Lambert series. Another example is the Euler's totient function φ. This function is appropriate to use it in an analogue of Möbius inversion formula for Γ-function, given by the equality

$$\sum_{n=1}^{\infty}\mu(n)\ln\Gamma\left(1+\frac{x}{n}\right)=x-\ln(1+x)$$

namely

$$\sum_{n=1}^{\infty}\varphi(n)\left(\ln\Gamma\left(1+\frac{x}{n}\right)+\frac{\gamma x}{n}-\frac{\pi^2x^2}{12n^2}\right)=\frac{x}{2}\ln 2\pi-\frac{x+(1+\gamma)x^2}{2}-\ln G(1+x)$$

the validity of which could be seen by expanding both sides into Taylor series as the functions of x.

So let's try to use this function in analogue to the sums with full Stirling's expansion, but firstly deformating it for convergence:

$$\ln x!=x\ln x-x+\frac{1}{2}\ln 2\pi x+\frac{1}{12x}+\int_0^{\infty}\frac{2x(\arctan t-t)}{e^{2\pi tx}-1}dt$$

Now using the Lambert series formula of the totient function

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{e^{ny}-1}=\frac{e^y}{(e^y-1)^2}\quad(y>0)$$

we obtain:

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\left(\ln\Gamma(1+nx)-(nx)\ln(nx)+nx-\frac{1}{2}\ln 2\pi nx-\frac{1}{12nx}\right)=$$

$$=-\frac{1}{\pi}\int\limits_0^{\infty}\frac{t^2}{1+t^2}\frac{dt}{e^{2\pi tx}-1}$$

But because it's $\frac{t^2}{1+t^2}$, not $\frac{t}{1+t^2}$, this result is not representable in terms of generalized Harmonic number. That's why it is not so interesting.

4 Derivation of stated results related to Lambert W-function

Lambert multivalued W-function is given by the equality $W(xe^x)=x$. Then Lagrange inversion theorem implies the formula for coefficients of its principal branch $W_0(x)$ Taylor's expansion, using which it can be derived that

$$\forall x\in[0;1]:\quad\sum_{n=1}^{\infty}\frac{n^{n-1}x^ne^{-nx}}{n!}=x$$

Then integrating both sides of this equality we get

$$\int\limits_0^x tdt=\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}\int\limits_0^x t^ne^{-nt}dt$$

Using integration by parts it is easy to prove that

$$e^{-x}\left(1+x+...+\frac{x^n}{n!}\right)=1-\frac{1}{n!}\int\limits_0^x t^ne^{-t}dt$$

Then

$$\forall x\in[0,1]:\quad\int\limits_0^x tdt=\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}\frac{1}{n^{n+1}}\int\limits_0^{nx}t^ne^{-t}dt=$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-e^{-nx}\left(1+(nx)+...+\frac{(nx)^n}{n!}\right)\right)$$

So the following identity holds

$$\forall x\in[0,1]:\quad\sum_{n=1}^{\infty}\frac{e^{-nx}}{n^2}\left(1+nx+\frac{(nx)^2}{2!}+...+\frac{(nx)^n}{n!}\right)=\frac{\pi^2}{6}-\frac{x^2}{2}$$

Now let's derive the similar formula

$$\forall x \in [0,1]: \quad \sum_{n=1}^{\infty} \frac{e^{-nx}}{n(n+1)} \left(1 + (nx) + ... + \frac{(nx)^n}{n!}\right) = 1 + x - \int\limits_0^x \frac{e^t - 1}{t} dt$$

Changing the variable $t = xe^x$ it becomes obvious that

$$\int W(t)dt = tW(t) - t + \frac{t}{W(t)} + C$$

So the reciprocal of Lambert function's principal branch has the Taylor's series:

$$\frac{x}{W_0(x)} = 1 + x + \sum_{n=1}^{\infty} \frac{(-n)^n}{(n+1)!} x^{n+1}$$

That's why for all x in $[0,1]$ it is true that

$$\frac{e^x - 1}{x} = 1 + \sum_{n=1}^{\infty} \frac{n^n}{(n+1)!} x^n e^{-nx}$$

Integrating this formula we obtain the desired result.

But now we will concentrate on other identities. Let's define

$$M(s) = \sum_{n=1}^{\infty} \frac{e^{-n}}{n^s} \left(1 + n + ... + \frac{n^n}{n!}\right)$$

and find its values at natural points except $s = 1$.

Theorem: If we define the sequence of polynomials

$$P_1(x) = x, \ P_{n+1}(x) = \int_0^x \frac{P_n(t)(1-t)}{t} dt$$

Then

$$\forall x \in [0;1]: \quad \sum_{n=1}^{\infty} \frac{n^{n-k} x^n e^{-nx}}{n!} = P_k(x)$$

Proof by induction: For $k = 1$ it is true. Let

$$\sum_{n=1}^{\infty} \frac{n^{n-k} x^n e^{-nx}}{n!} = P_k(x)$$

Then

$$\sum_{n=1}^{\infty} \frac{n^{n-k-1} x^n e^{-nx}}{n!} = \int\limits_0^{xe^{-x}} \frac{P_k(-W_0(-t))}{t} dt = \int\limits_0^x \frac{P_k(t)(1-t)}{t} dt = P_{k+1}(x) \quad \square$$

So $\forall k \in \mathbb{N}_{k\geqslant 2}, \forall x \in [0;1]$:

$$\sum_{n=1}^{\infty} \frac{e^{-nx}}{n^k}\left(1+(nx)+\frac{(nx)^2}{2!}+...+\frac{(nx)^n}{n!}\right)=\zeta(k)-\int_0^x P_{k-1}(t)dt$$

But we can continue this sequence of polynomials at least to all real variable $\geqslant 1$ (but these functions of course won't be polynomials anymore).

$$\sum_{n=1}^{\infty} \frac{n^{n-y}x^n e^{-nx}}{n!} = \frac{1}{\Gamma(y-1)}\sum_{n=1}^{\infty} \frac{n^{n-1}x^n e^{-nx}}{n!}\int\limits_0^{\infty} t^{y-2}e^{-nt}dt =$$

$$= -\frac{1}{\Gamma(y-1)}\int\limits_0^{\infty} t^{y-2}W_0(-xe^{-x-t})dt$$

But $x \in [0;1]$, so we can change the variable $t = xz - x - \ln z$ to obtain:

$$P_y(x) = \frac{1}{\Gamma(y-1)}\int\limits_0^1 (xz-x-\ln z)^{y-2}(x-x^2z)dz = \frac{x}{\Gamma(y)}\int\limits_0^1 (xz-x-\ln z)^{y-1}dz$$

(at least for all $x \in [0;1]$, $y \in [1;+\infty)$). But now we are interested only in case when $y \in \mathbb{N}$ to derive the closed-form formula for the values of this Dirichlet series at natural points.

$$P_y(x) = \frac{x}{(y-1)!}\int\limits_0^1 \sum_{m=0}^{y-1}\binom{y-1}{m}(x(t-1))^m(-\ln t)^{y-1-m}dt =$$

$$= \frac{x}{(y-1)!}\sum_{m=0}^{y-1}\binom{y-1}{m}x^m(-1)^{y-1}\left.\frac{d^{y-1-m}}{ds^{y-1-m}}\frac{\Gamma(s+1)\Gamma(m+1)}{\Gamma(s+m+2)}\right|_{s=0}$$

The n-th derivative of Beta-function $\frac{d^n}{ds^n}B(s,m+1)$ might be obtained by using the fact that

$$\frac{1}{(s+1)(s+2)...(s+m)} = \sum_{j=1}^{m}\frac{1}{s+j}\lim_{x\to -j}\frac{(x+j)}{(x+1)(x+2)...(x+m)} =$$

$$= \sum_{j=1}^{m}\frac{1}{s+j}\frac{(-1)^{j-1}}{(j-1)!(m+1-j)!}$$

So

$$P_y(x) = \sum_{m=1}^{y}\frac{x^m}{m!}\sum_{j=1}^{m}\frac{(-1)^{m+j}}{j^{y-m}}\binom{m}{j}$$

And so we obtain the formula for values of this function at natural points

$$M(k)=\zeta(k)-\sum_{m=1}^{k-1}\sum_{j=1}^{m}\frac{(-1)^{m+j}}{(m+1)!j^{k-1-m}}\binom{m}{j}$$

or using the rearranging formula for double finite sums

$$\sum_{m=1}^{n-1}\sum_{k=1}^{m}f(k,m)=\sum_{m=1}^{n-1}\sum_{k=1}^{m}f(k,n-m+k-1)$$

to avoid binomial coefficients:

$$M(k)=\zeta(k)+\sum_{m=1}^{k-1}\frac{(-1)^{k-m}}{(k-m-1)!}\sum_{j=1}^{m}\frac{j^{j-m}}{(k-m+j)j!}$$

Now let's investigate the behavior of this Dirichlet series at $k=1$. It is well known that

$$\lim_{n\to\infty}e^{-n}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right)=\frac{1}{2}$$

so we can try to find the closed-form expression of the sum

$$\sum_{n=1}^{\infty}\frac{e^{-n}}{n}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right)-\frac{1}{2n}=M$$

By differentiating the definition of Lambert function, one can show that it satisfies the differential equation

$$W'(z)=\frac{W(z)}{z(1+W(z))}$$

So if we differentiate the Taylor's series of W and then substitute $-xe^{-x}$ we will obtain the series for $x(1-x)^{-1}$, which converges for all x in $[0,1)$. But to integrate this formula from 0 to 1 we need the uniform convergence on the whole segment $[0,1]$. So let's use the expansion of the function $f(x)=(1-x)^{-\frac{1}{2}}$ into Taylor series to have

$$\frac{x}{1-x}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2-2xe^{1-x}}}=\sum_{n=1}^{\infty}\frac{n^nx^ne^{-nx}}{n!}-\frac{\binom{2n}{n}x^ne^{n-nx}}{2^{2n}\sqrt{2}}$$

There's no limit for $x\to 1^+$, but to apply Abel's theorem we need only the limit for $x\to 1^-$, which is $\frac{1}{\sqrt{2}}-\frac{2}{3}$. Now we can integrate this formula:

$$\int_0^1\frac{x}{1-x}+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2-2xe^{1-x}}}dx=$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-n}\left(1 + n + \frac{n^2}{2} + ... + \frac{n^n}{n!}\right)\right) - \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} dx =$$

$$= -M + \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} dx = -M + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2n\binom{2n}{n}} - \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} dx$$

Γ-function duplication theorem implies that

$$\frac{1}{n\binom{2n}{n}} = \frac{1}{4^n} \int_0^1 \frac{x^{n-1}}{\sqrt{1-x}} dx$$

Then changing of variable $x = te^{1-t}$ we obtain

$$\frac{1}{n\binom{2n}{n}} = \frac{e^n}{4^n} \int_0^1 t^{n-1} e^{-nt} \frac{1-t}{\sqrt{1-te^{1-t}}} dt$$

and so the following identity holds

$$\int_0^1 \frac{x}{1-x} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-2xe^{1-x}}} dx =$$

$$= -M + \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}\sqrt{2}} \int_0^1 x^n e^{n-nx} \left(\frac{1-x}{x\sqrt{2-2xe^{1-x}}} - 1\right) dx$$

Now the series have uniform convergence and so we can change the order of summation and integration to have

$$\int_0^1 \frac{x}{1-x} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-2xe^{1-x}}} dx =$$

$$= -M + \int_0^1 \left(\frac{1}{\sqrt{2-2xe^{1-x}}} - \frac{1}{\sqrt{2}}\right)\left(\frac{1-x}{x\sqrt{2-2xe^{1-x}}} - 1\right) dx$$

That's why

$$M = \int_0^1 \frac{1-x}{2x\sqrt{1-xe^{1-x}}}\left(\frac{1}{\sqrt{1-xe^{1-x}}} - 1\right) - \frac{x}{1-x} dx$$

Using the substitution $t = xe^{1-x}$ and reverse we can find the antiderivative of this function:

$$\int \frac{1-x}{2x\sqrt{1-xe^{1-x}}}\left(\frac{1}{\sqrt{1-xe^{1-x}}} - 1\right) - \frac{x}{1-x} dx =$$

$$= x + \ln(1-x) + \frac{1}{2}\ln(xe^{1-x}) - \frac{1}{2}\ln(1-xe^{1-x}) - \frac{1}{2}\ln(1-\sqrt{1-xe^{1-x}}) +$$

$$+\frac{1}{2}\ln(1+\sqrt{1-xe^{1-x}}) + C$$

counting the limits of this function for $x \to 0$ and $x \to 1^-$ we finally have:

$$\boxed{\sum_{n=1}^{\infty} \frac{e^{-n}}{n}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right) - \frac{1}{2n} = 1 - \frac{1}{2}\ln 2}$$

5 Conclusion

At first, we gave some new infinite series, which have closed-form expressions in terms of Trigonometric Integral functions, using the formal approach to the derivation of Stirling's formula. Then we derived some results for new Dirichlet series. Defining

$$M(s) = \sum_{n=1}^{\infty} \frac{e^{-n}}{n^s}\left(1+n+\frac{n^2}{2}+...+\frac{n^n}{n!}\right)$$

we obtained less obvious results than

$$\lim_{s\to\infty} M(s) = \frac{2}{e}$$

We gave the formulas for $M(2)$, $M(3)$ and general closed-form formula for $M(k)$ for all natural k except 1 and integral formula for real $k > 1$. We also derived that

$$\lim_{s\to 1} M(s) - \frac{\zeta(s)}{2} = 1 - \frac{1}{2}\ln 2$$

or

$$\lim_{s\to 1} M(s) - \frac{1}{2s-2} = 1 + \frac{\gamma}{2} - \frac{1}{2}\ln 2$$

An interesting problem is to find analytic continuation of $M(s)$ and find its values at negative points.

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