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N^*C^*- Smarandache Curve of Bertrand Curves Pair According to Frenet Frame

Süleyman Şenyurt , Abdussamet Çalışkan and Ünzile Çelik

(Faculty of Arts and Sciences, Department of Mathematics, Ordu University, Ordu, Turkey)

E-mail: senyurtsuleyman@hotmail.com, abdussamet65@gmail.com, unzile.celik@hotmail.com

Abstract: In this paper, let (α, α^*) be Bertrand curve pair, when the unit Darboux vector of the α^* curve are taken as the position vectors, the curvature and the torsion of Smarandache curve are calculated. These values are expressed depending upon the α curve. Besides, we illustrate example of our main results.

Key Words: Bertrand curves pair, Smarandache curves, Frenet invariants, Darboux vector.

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§1. Introduction

It is well known that many studies related to the differential geometry of curves have been made. Especially, by establishing relations between the Frenet Frames in mutual points of two curves several theories have been obtained. The best known of the Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topics of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve, called Bertrand mate or Bertrand curve Partner. If $\alpha^* = \alpha + \lambda N$, $\lambda = const.$, then (α, α^*) are called Bertrand curves pair. If α and α^* Bertrand curves pair, then $\langle T, T^* \rangle = \cos\theta = constant$, [9], [10]. The definition of n-dimensional Bertrand curves in Lorentzian space is given by comparing a well-known Bertrand pair of curves in n- dimensional Euclidean space. It shown that the distance between corresponding of Bertrand pair of curves and the angle between the tangent vector fields of these points are constant. Moreover Schell and Mannheim theorems are given in the Lorentzian space, [7]. The Bertrand curves are the Inclined curve pairs. On the other hand, it gave the notion of Bertrand Representation and found that the Bertrand Representation is spherical, [8]. Some characterizations for general helices in space forms were given, [11].

A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [14]. Special Smarandache curves have been studied by some authors. Melih Turgut and Süha Yılmaz studied a special case of such curves and called it Smarandache TB_2 curves in the space $\mathbb{E}_1^4$ ([14]). Ahmad T.Ali

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studied some special Smarandache curves in the Euclidean space. He studied Frenet-Serret invariants of a special case, [1]. Şenyurt and Çalışkan investigated special Smarandache curves in terms of Sabban frame of spherical indicatrix curves and they gave some characterization of Smarandache curves, [4]Özcan Bektaş and Salim Yüce studied some special Smarandache curves according to Darboux Frame in $\mathbb{E}^3$, [3]. Kemal Taşköprü and Murat Tosun studied special Smarandache curves according to Sabban frame on S^2 ([2]). They defined NC-Smarandache curve, then they calculated the curvature and torsion of NB and TNB- Smarandache curves together with NC-Smarandache curve, [12]. It studied that the special Smarandache curve in terms of Sabban frame of Fixed Pole curve and they gave some characterization of Smarandache curves, [12]. When the unit Darboux vector of the partner curve of Mannheim curve were taken as the position vectors, the curvature and the torsion of Smarandache curve were calculated. These values were expressed depending upon the Mannheim curve, [6].

In this paper, special Smarandache curve belonging to α curve such as N^*C^* drawn by Frenet frame are defined and some related results are given.

§2. Preliminaries

The Euclidean 3-space $\mathbb{E}^3$ be inner product given by

$$\langle\ ,\ \rangle = x_1^2 + x_2^3 + x_3^2$$

where $(x_1, x_2, x_3) \in \mathbb{E}^3$. Let $\alpha : I \to \mathbb{E}^3$ be a unit speed curve denote by $\{T, N, B\}$ the moving Frenet frame . For an arbitrary curve $\alpha \in \mathbb{E}^3$, with first and second curvature, κ and τ respectively, the Frenet formulae is given by [9], [10].

$$\begin{cases} T' = \kappa N \\ N' = -\kappa T + \tau B \\ B' = -\tau N. \end{cases} \tag{2.1}$$

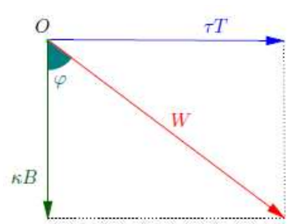

Figure 1 Darboux vector

For any unit speed curve $\alpha : I \to \mathbb{E}^3$, the vector W is called Darboux vector defined by

$$W = \tau T + \kappa B. \tag{2.2}$$

If we consider the normalization of the Darboux, we have

$$\sin\varphi = \frac{\tau}{\|W\|}, \quad \cos\varphi = \frac{\kappa}{\|W\|} \tag{2.3}$$

and

$$C = \sin\varphi T + \cos\varphi B, \tag{2.4}$$

where $\angle(W, B) = \varphi$.

Definition 2.1([9]) *Let $\alpha : I \to \mathbb{E}^3$ and $\alpha^* : I \to \mathbb{E}^3$ be the C^2- class differentiable unit speed two curves and let $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ be the Frenet frames of the curves α and α^*, respectively. If the principal normal vector N of the curve α is linearly dependent on the principal vector N^* of the curve α^*, then the pair (α, α^*) is said to be Bertrand curves pair.*

The relations between the Frenet frames $\{T(s), N(s), B(s)\}$ and $\{T^*(s), N^*(s), B^*(s)\}$ are as follows:

$$\begin{cases} T^* = \cos\theta T + \sin\theta B \\ N^* = N \\ B^* = -\sin\theta T + \cos\theta B. \end{cases} \tag{2.5}$$

where $\angle(T, T^*) = \theta$

Theorem 2.2([9], [10]) *The distance between corresponding points of the Bertrand curves pair in $\mathbb{E}^3$ is constant.*

Theorem 2.3([10]) *Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{cases} \kappa^* = \dfrac{\lambda\kappa - \sin^2\theta}{\lambda(1-\lambda\kappa)}\ , \lambda = constant \\ \\ \tau^* = \dfrac{\sin^2\theta}{\lambda^2\tau}. \end{cases} \tag{2.6}$$

Theorem 2.4([9]) *Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$. For the curvatures and the torsions of the Bertrand curves pair (α, α^*) we have*

$$\begin{cases} \kappa^* \frac{ds^*}{ds} = \kappa\cos\theta - \tau\sin\theta, \\ \\ \tau^* \frac{ds^*}{ds} = \kappa\sin\theta + \tau\cos\theta. \end{cases} \tag{2.7}$$

By using equation (2.2), we can write Darboux vector belonging to Bertrand mate α^*.

$$W^* = \tau^* T^* + \kappa^* B^*. \tag{2.8}$$

If we consider the normalization of the Darboux vector, we have

$$C^* = \sin\varphi^* T^* + \cos\varphi^* B^*. \tag{2.9}$$

From the equation (2.3) and (2.7), we can write

$$\begin{aligned} \sin\varphi^* &= \frac{\tau^*}{\|W^*\|} = \frac{\kappa\sin\theta + \tau\cos\theta}{\|W\|} = \sin(\varphi+\theta), \\ \cos\varphi^* &= \frac{\kappa^*}{\|W^*\|} = \frac{\kappa\cos\theta - \tau\sin\theta}{\|W\|} = \cos(\varphi+\theta), \end{aligned} \tag{2.10}$$

where $\|W^*\| = \sqrt{\kappa^{*2} + \tau^{*2}} = \|W\|$ and $\angle(W^*, B^*) = \varphi^*$. By the using (2.5) and (2.10), the final version of the equation (2.9) is as follows:

$$C^* = \sin\varphi T + \cos\varphi B. \tag{2.11}$$

§3. N^*C^*- Smarandache Curve of Bertrand Curves Pair According to Frenet Frame

Let (α, α^*) be a Bertrand curves pair in $\mathbb{E}^3$ and $\{T^*, N^*, B^*\}$ be the Frenet frame of the curve α^* at $\alpha^*(s)$. In this case, N^*C^* - Smarandache curve can be defined by

$$\psi(s) = \frac{1}{\sqrt{2}}(N^* + C^*). \tag{3.1}$$

Solving the above equation by substitution of N^* and C^* from (2.5) and (2.11), we obtain

$$\psi(s) = \frac{\sin\varphi T + N + \cos\varphi B}{\sqrt{2}}\cdot \tag{3.2}$$

The derivative of this equation with respect to s is as follows,

$$\psi' = T_\psi \frac{ds_\psi}{ds} = \frac{(-\kappa + \varphi'\cos\varphi)T + (\tau - \varphi'\sin\varphi)B}{\sqrt{2}} \tag{3.3}$$

and by substitution, we get

$$T_\psi = \frac{(-\kappa + \varphi'\cos\varphi)T + (\tau - \varphi'\sin\varphi)B}{\sqrt{\|W\|^2 - 2\varphi'\|W\| + \varphi'^2}}, \tag{3.4}$$

where

$$\frac{ds_\psi}{ds} = \sqrt{\frac{\|W\|^2 - 2\varphi'\|W\| + \varphi'^2}{2}}\cdot \tag{3.5}$$

In order to determine the first curvature and the principal normal of the curve $\psi(s)$, we formalize

$$T'_\psi(s) = \frac{\sqrt{2}\big[(\omega_1\cos\theta+\omega_3\sin\theta)T+\omega_2 N+(-\omega_1\sin\theta+\omega_3\cos\theta)B\big]}{\big[\|W\|^2-2\varphi'\|W\|+\varphi'^2\big]^2}, \tag{3.6}$$

where

$$\begin{cases}\omega_1=\big(-\kappa\cos\theta+\tau\sin\theta+\varphi'\cos(\varphi+\theta)\big)'\big(\|W\|^2-2\varphi'\|W\|+\varphi'^2\big)-\big(-\kappa\cos\theta\\ \qquad+\tau\sin\theta+\varphi'\cos(\varphi+\theta)\big)\big(\|W\|\|W\|'-\varphi''\|W\|-\varphi'\|W\|'+\varphi'\varphi''\big)\\ \omega_2=\big(-\|W\|^2+\varphi'\|W\|\big)\big(\|W\|^2-2\varphi'\|W\|+\varphi'^2\big)\\ \omega_3=\big(\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)'\big(\|W\|^2-2\varphi'\|W\|+\varphi'^2\big)-\big(\kappa\sin\theta\\ \qquad+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)\big(\|W\|\|W\|'-\varphi''\|W\|-\varphi'\|W\|'+\varphi'\varphi''\big)\end{cases}$$

The first curvature is

$$\kappa_\psi=\|T'_\psi\|,\quad \kappa_\psi=\frac{\sqrt{2({\omega_1}^2+{\omega_2}^2+{\omega_3}^2)}}{\big[\|W\|^2-2\varphi'\|W\|+\varphi'^2\big]^2}.$$

The principal normal vector field and the binormal vector field are respectively given by

$$N_\psi=\frac{\big[(\omega_1\cos\theta+\omega_3\sin\theta)T+\omega_2 N+(-\omega_1\sin\theta+\omega_3\cos\theta)B\big]}{\sqrt{{\omega_1}^2+{\omega_2}^2+{\omega_3}^2}}, \tag{3.7}$$

$$B_\psi=\frac{\begin{array}{l}\omega_2\big[-2\kappa\sin\theta\cos\theta+\tau(\sin^2\theta-\cos^2\theta)+\varphi'\sin\varphi\big]T\\ +\omega_1\big[\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big]N+\omega_2\big[2\tau\sin\theta\cos\theta\\ +\kappa(\sin^2\theta-\cos^2\theta)+\varphi'\cos\varphi\big]B\end{array}}{\sqrt{\big(\|W\|^2-2\varphi'\|W\|+\varphi'^2\big)({\omega_1}^2+{\omega_2}^2+{\omega_3}^2)}} \tag{3.8}$$

The torsion is then given by

$$\tau_\psi=\frac{det(\psi',\psi'',\psi''')}{\|\psi'\wedge\psi''\|^2}\ ,$$

$$\tau_\psi=\frac{\sqrt{2}(\vartheta\eta+\varrho\lambda+\mu\rho)}{\vartheta^2+\varrho^2+\mu^2}$$

where

$$\begin{cases}\eta=\big(\varphi'\cos(\varphi+\theta)-\kappa\cos\theta+\tau\sin\theta\big)''+(\kappa\cos\theta-\tau\sin\theta)\|W\|^2\\ \qquad-(\kappa\cos\theta-\tau\sin\theta)\varphi'\|W\|\\ \lambda=(\kappa\cos\theta-\tau\sin\theta)\big(\varphi'\cos(\varphi+\theta)-\kappa\cos\theta+\tau\sin\theta\big)'+(-\|W\|^2\\ \qquad+\varphi'\|W\|)'-(\kappa\sin\theta+\tau\cos\theta)\big(\kappa\sin\theta+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)'\\ \rho=(-\kappa\sin\theta-\tau\cos\theta)\|W\|^2+(\kappa\sin\theta+\tau\cos\theta)\varphi'\|W\|+\big(\kappa\sin\theta\\ \qquad+\tau\cos\theta-\varphi'\sin(\varphi+\theta)\big)'',\end{cases}$$

$$\begin{cases} \vartheta = -\big(-\|W\|^2 + \varphi'\|W\|\big)\big(\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big) \\ \varrho = -\Big[\big(\varphi'\cos(\varphi+\theta) - \kappa\cos\theta + \tau\sin\theta\big)\big(\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)' \\ \qquad +\big(\varphi'\cos(\varphi+\theta) - \kappa\cos\theta + \tau\sin\theta\big)'\big(\kappa\sin\theta + \tau\cos\theta - \varphi'\sin(\varphi+\theta)\big)\Big] \\ \mu = \big(\varphi'\cos(\varphi+\theta) - \kappa\cos\theta + \tau\sin\theta\big)\big(-\|W\|^2 + \varphi'\|W\|\big). \end{cases}$$

Example 3.1 Let us consider the unit speed α curve and α^* curve:

$$\alpha(s) = \frac{1}{\sqrt{2}}\big(-\cos s, -\sin s, s\big) \ \ and \ \ \alpha^*(s) = \frac{1}{\sqrt{2}}(\cos s, \sin s, s)\cdot$$

The Frenet invariants of the curve, $\alpha^*(s)$ are given as following:

$$\begin{cases} T^*(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1), N^*(s) = \big(-\cos s, -\sin s, 0\big) \\ B^*(s) = \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1), C^*(s) = (0,0,1) \\ \kappa^*(s) = \frac{1}{\sqrt{2}}, \tau^*(s) = \frac{1}{\sqrt{2}}\cdot \end{cases}$$

In terms of definitions, we obtain special Smarandache curve, see Figure 1.

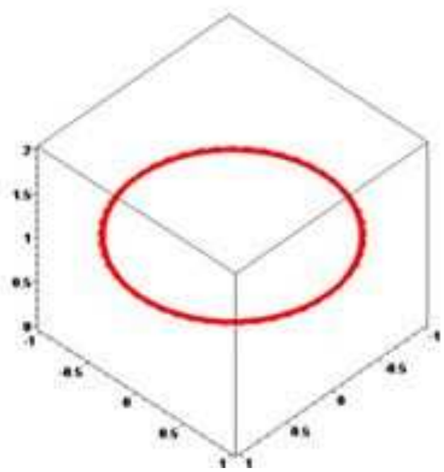

Figure 2 N^*C^*-Smarandache Curve

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