International J.Math. Combin. Vol.1(2016), 57-64

On Net-Regular Signed Graphs

Nutan G.Nayak

Department of Mathematics and Statistics

S. S. Dempo College of Commerce and Economics, Goa, India

E-mail: nayaknutan@yahoo.com

Abstract: A signed graph is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is the underlying graph of Σ and $\sigma : E \to \{+1, -1\}$, called signing function from the edge set $E(G)$ of G into the set $\{+1, -1\}$. It is said to be homogeneous if its edges are all positive or negative otherwise it is heterogeneous. Signed graph is balanced if all of its cycles are balanced otherwise unbalanced. It is said to be net-regular of degree k if all its vertices have same net-degree k i.e. $k = d^{\pm}_{\Sigma}(v) = d^{+}_{\Sigma}(v) - d^{-}_{\Sigma}(v)$, where $d^{+}_{\Sigma}(v)(d^{-}_{\Sigma}(v))$ is the number of positive(negative) edges incident with a vertex v. In this paper, we obtained the characterization of net-regular signed graphs and also established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

Key Words: Smarandachely k-signed graph, net-regular signed graph,co-regular signed graphs, signed complete graphs.

AMS(2010): 05C22, 05C50.

§1. Introduction

We consider graph G is a simple undirected graph without loops and multiple edges with n vertices and m edges. A Smarandachely k-signed graph is defined as an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is an underlying graph of Σ and $\sigma : E \to \{\overline{e_1}, \overline{e_2}, \overline{e_3}, \cdots, \overline{e_k}\}$ is a function, where $\overline{e_i} \in \{+, -\}$. A Smarandachely 2-signed graph is known as signed graph. It is said to be homogeneous if its edges are all positive or negative otherwise it is heterogeneous. We denote positive and negative homogeneous signed graphs as $+G$ and $-G$ respectively.

The adjacency matrix of a signed graph is the square matrix $A(\Sigma) = (a_{ij})$ where (i, j) entry is $+1$ if $\sigma(v_iv_j) = +1$ and -1 if $\sigma(v_iv_j) = -1$, 0 otherwise. The characteristic polynomial of the signed graph Σ is defined as $\Phi(\Sigma : \lambda) = \det(\lambda I - A(\Sigma))$, where I is an identity matrix of order n. The roots of the characteristic equation $\Phi(\Sigma : \lambda) = 0$, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$ are called the eigenvalues of signed graph Σ. If the distinct eigenvalues of $A(\Sigma)$ are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and their multiplicities are $m_1, m_2, \ldots, m_n$, then the spectrum of Σ is $Sp(\Sigma) = \{\lambda_1^{(m_1)}, \lambda_2^{(m_2)}, \cdots, \lambda_n^{(m_n)}\}$.

Two signed graphs are cospectral if they have the same spectrum. The spectral criterion

[1]Received May 19, 2015, Accepted February 15, 2016.

for balance in signed graph is given by B.D.Acharya as follows:

Theorem 1.1([1]) *A signed graph is balanced if and only if it is cospectral with the underlying graph. i.e.* $Sp(\Sigma) = Sp(G)$.

The sign of a cycle in a signed graph is the product of the signs of its edges. Thus a cycle is positive if and only if it contains an even number of negative edges. A signed graph is said to be balanced (or cycle balanced) if all of its cycles are positive otherwise unbalanced. The negation of a signed graph $\Sigma = (G, \sigma)$, denoted by $\eta(\Sigma) = (G, \sigma)$ is the same graph with all signs reversed. The adjacency matrices are related by $A(-\Sigma) = -A(\Sigma)$.

Theorem 1.2([12]) *Two signed graphs* $\Sigma_1 = (G, \sigma_1)$ *and* $\Sigma_2 = (G, \sigma_2)$ *on the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In signed graph Σ, the degree of a vertex v is defined as $sdeg(v) = d(v) = d^+_\Sigma(v) + d^-_\Sigma(v)$, where $d^+_\Sigma(v)(d^-_\Sigma(v))$ is the number of positive(negative) edges incident with v. The net degree of a vertex v of a signed graph Σ is $d^\pm_\Sigma(v) = d^+_\Sigma(v) - d^-_\Sigma(v)$. It is said to be net-regular of degree k if all its vertices have same net-degree equal to k. Hence net-regularity of a signed graph can be either positive, negative or zero. We denote net-regular signed graphs as Σ^k_n. We know [13] that if Σ is a k net-regular signed graph, then k is an eigenvalue of Σ with j as an eigenvector with all 1's.

K.S.Hameed and K.A.Germina [6] defined co-regularity pair of signed graphs as follows:

Definition 1.3([6]) *A signed graph* $\Sigma = (G, \sigma)$ *is said to be co-regular, if the underlying graph* G *is regular for some positive integer* r *and* Σ *is net-regular with net-degree* k *for some integer* k *and the co-regularity pair is an ordered pair of* (r, k).

The following results give the spectra of signed paths and signed cycles respectively.

Lemma 1.4([3]) *The signed paths* $P^{(r)}_n$, *where* r *is the number of negative edges and* $0 \le r \le n-1$, *have the eigenvalues(independent of* r*) given by*

$$\lambda_j = 2\cos\frac{\pi j}{n+1}, j = 1, 2, \cdots, n.$$

Lemma 1.5([9]) *The eigenvalues* λ_j *of signed cycles* $C^{(r)}_n$ *and* $0 \le r \le n$ *are given by*

$$\lambda_j = 2\cos\frac{(2j - [r])\pi}{n}, j = 1, 2, \cdots, n$$

where r *is the number of negative edges and* $[r] = 0$ *if* r *is even,* $[r] = 1$ *if* r *is odd.*

Spectra of graphs is well documented in [2] and signed graphs is discussed in [3, 4, 5, 9]. For standard terminology and notations in graph theory we follow D.B.West [10] and for signed graphs T. Zaslavsky [14].

The main aim of this paper is to characterize net-regular signed graphs and also to prove that there exists a net-regular signed graph on every regular graph but the converse does not hold good. Further, we construct a family of connected net-regular signed graphs whose underlying graphs are not regular. We established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

§2. Main Results

Spectral properties of regular graphs are well known in graph theory.

Theorem 2.1([2]) *If G is an r regular graph, then its maximum adjacency eigenvalue is equal to r and $r=\frac{2m}{n}$.*

Here we generalize Theorem 2.1 to signed graphs as graph is considered as one case in signed graph theory. We denote total number of positive and negative edges of Σ as m^+ and m^- respectively. The following lemma gives the structural characterization of signed graph Σ so that Σ is net-regular.

Lemma 2.2 *If $\Sigma=(G,\sigma)$ is a connected net-regular signed graph with net degree k then $k=\frac{2M}{n}$, where $M=(m^+-m^-)$, m^+ is the total number of positive edges and m^- is the total number of negative edges in Σ.*

Proof Let $\Sigma=(G,\sigma)$ be a net-regular signed graph with net degree k. Then by definition, $d^{\pm}_{\Sigma}(v)=d^{+}_{\Sigma}(v)-d^{-}_{\Sigma}(v)$. Hence,

$$\sum_{i=1}^{n} d^{\pm}_{\Sigma}(v)=\sum_{i=1}^{n} d^{+}_{\Sigma}(v)-\sum_{i=1}^{n} d^{-}_{\Sigma}(v).$$

Thus,

$$nk=\sum_{i=1}^{n} d^{+}_{\Sigma}(v)-\sum_{i=1}^{n} d^{-}_{\Sigma}(v).$$

Whence,

$$\begin{aligned} k &= \frac{1}{n}\left[\sum_{i=1}^{n} d^{+}_{\Sigma}(v)-\sum_{i=1}^{n} d^{-}_{\Sigma}(v)\right]=\frac{1}{n}\left[2m^+-2m^-\right] \\ &= \left[\frac{2(m^+-m^-)}{n}\right]=\frac{2M}{n}. \end{aligned}$$ □

Corollary 2.3 *If $\Sigma=(G,\sigma)$ is a signed graph with co-regularity pair (r,k) then $r\geq k$.*

Proof Let Σ be a k net-regular signed graph then by Lemma 2.2, $k=\frac{2M}{n}$, where $M=(m^+-m^-)$. Since G is its underlying graph with regularity r on n vertices then $r=\frac{2m}{n}$,where $m=m^++m^-$. It is clear that $\frac{2m}{n}\geq\frac{2(m^+-m^-)}{n}$. Hence $r\geq k$. □

Remark 2.4 By Corollary 2.3, if $\Sigma=(G,\sigma)$ is a signed graph with co-regularity pair (r,k) on

n vertices then $-r \leq k \leq r$.

Now the question arises whether all regular graphs can be net-regular and vice-versa. From Lemma 2.2, it is evident that at least two net-regular signed graphs exist on every regular graph when $m^+ = 0$ or $m^- = 0$. We feel the converse also holds good. But contrary to the intuition, the answer is negative. Next result proves that underlying graph of all net-regular signed graphs need not be regular.

Theorem 2.5 *Let Σ be a net-regular signed graph then its underlying graph is not necessarily a regular graph.*

Proof Let Σ be a net-regular signed graph with net degree k. Then by Lemma 2.2, $k = \left[\frac{2(m^+ - m^-)}{n}\right]$. By changing negative edges into positive edges we get $k = \frac{2m}{n}$ where $m = m^+ + m^-$. If $k = \frac{2m}{n}$ is a positive integer then underlying graph is of order $k = r$. If $k = \frac{2m}{n}$ is not a positive integer then $k \neq r$. Hence the underlying graph of a net-regular signed graph need not be a regular graph. □

Shahul Hameed et.al. [7] gave an example of a connected signed graph on $n = 5$ whose underlying is not a regular graph. Here we construct an infinite family of net-regular signed graphs whose underlying graphs are not regular.

Example 2.6 Here is an infinite family of net-regular signed graphs with the property that whose underlying graphs are not regular. Take two copies of C_n, join at one vertex and assign positive and negative signs so that degree of the vertex common to both cycles will have net degree 0 and also assign positive and negative signs to other edges in order to get net-degree 0. The resultant signed graph is a net-regular signed graph with net-degree 0 whose underlying graph is not regular. We denote it as $\Sigma^{(0)}_{(2n-1)}$ for each C_n and illustration is shown in Fig.1, 2 and 3. In chemistry, underlying graphs of these signed graphs are known as spiro compounds.

In the following figures, solid lines represent positive edges and dotted lines represent negative edges respectively.

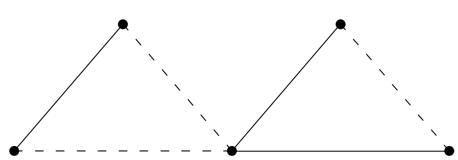

Fig.1 Net-regular signed graph Σ^0_5 for C_3

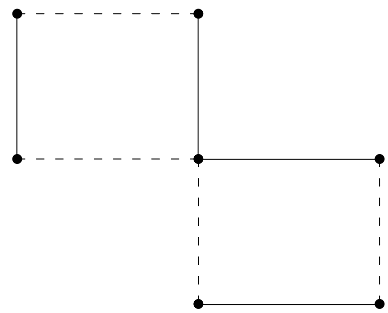

Fig.2 Net-regular signed graph Σ_7^0 for C_4

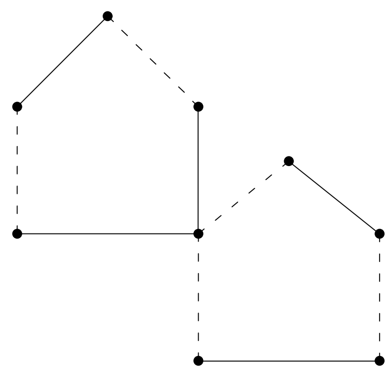

Fig.3 Net-regular signed graph Σ_9^0 for C_5

From Figures 1, 2 and 3, we can see that Σ_7^0 is a bipartite signed graph, but Σ_5^0 and Σ_9^0 are non-bipartite signed graphs. The spectrum of these net -regular signed graphs are

Sp(Σ_5^0)= $\{\pm 2.2361, \pm 1, 0\}$,

Sp(Σ_7^0)= $\{\pm 2.4495, \pm 1.4142, (0)^3\}$,

Sp(Σ_9^0)=$\{\pm 2.3028, \pm 1.6180, \pm 1.3028, \pm 0.6180, 0\}$.

Remark 2.7 Spectrum of this family of connected signed graphs $\Sigma_{(2n-1)}^{(0)}$ satisfy the pairing property i.e. spectrum is symmetric about the origin and also these are non-bipartite when cycle C_n is odd.

Heterogeneous signed complete graphs which are cycle isomorphic to the underlying graph $+K_n$ will have the spectrum $\{(n-1), (-1)^{(n-1)}\}$ and which are cycle isomorphic to $-K_n$ will have the spectrum $\{(1-n), (1)^{(n-1)}\}$. Here we established the spectrum for one class of heterogeneous unbalanced net-regular signed complete graphs.

Let C_n be a cycle on n vertices and $\overline{C}_n$ be its complement where $n \geq 4$. Define σ :

$E(K_n) \to \{1, -1\}$ by

$$\sigma(e) = \begin{cases} 1, & \text{if } \mathrm{e} \in \mathrm{C_n} \\ -1, & \text{if } \mathrm{e} \in \overline{\mathrm{C_n}} \end{cases}$$

Then $\Sigma = (K_n, \sigma)$ is an unbalanced net-regular signed complete graph and we denote it as K_n^{net} where $n \geq 4$.

The following spectrum for K_n^{net} is given by the author in [10].

Lemma 2.8([10]) *Let K_n^{net} be a heterogeneous unbalanced net-regular signed complete graph then*

$$Sp(K_n^{net}) = \begin{pmatrix} 5-n & 1+4\cos(\frac{2\pi j}{n}) \\ 1 & 1 \end{pmatrix} : j = 1, \ldots.., n-1,$$

where $(5-n)$ gives the net-regularity of K_n^{net}.

Lemma 2.9 $\omega^r + \omega^{n-r} = 2\cos\frac{2r\pi j}{n}$ *for $1 \leq j \leq n$ and $1 \leq r \leq n$, where ω is the n^{th} root of unity.*

Proof Let $1 \leq j \leq n$ and $1 \leq r \leq n$.

$$\begin{aligned} \omega^r + \omega^{n-r} &= e^{\frac{2r\pi ij}{n}} + e^{2r\pi ij} e^{\frac{-2r\pi ij}{n}} = e^{\frac{2r\pi ij}{n}} + e^{\frac{-2r\pi ij}{n}} \\ &= \cos\frac{2r\pi j}{n} + i\sin\frac{2r\pi j}{n} + \cos\frac{2r\pi j}{n} - i\sin\frac{2r\pi j}{n} \\ &= 2\cos\frac{2r\pi j}{n}. \end{aligned}$$

□

By using the properties of the permutation matrix [8] and from Lemma 2.9, we give a new spectrum for K_n^{net}.

Theorem 2.10 *Let K_n^{net} be a heterogeneous signed complete graph as defined above. If n is odd then*

$$Sp(K_n^{net}) = \{2\cos\frac{2\pi j}{n} - \sum_{r=2}^{\frac{n-1}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n\}$$

and if n is even then

$$Sp(K_n^{net}) = \{2\cos\frac{2\pi j}{n} - \cos\pi j - \sum_{r=2}^{\frac{n-2}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n\}.$$

Proof Label the vertices of a circulant graph as $0, 1, \cdots, (n-1)$. Then the adjacency matrix A is

$$A = \begin{pmatrix} 0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & 0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & 0 & \cdots & c_{n-3} \\ \vdots & & & \vdots & \\ c_1 & c_2 & c_3 & \cdots & 0 \end{pmatrix},$$

where $c_i = c_{n-i} = 0$ if vertices i and $n-i$ are not adjacent and $c_i = c_{n-i} = 1$ if vertices i and $n-i$ are adjacent.

Hence

$$\begin{aligned} A &= c_1P^1 + c_2P^2 + \cdots + c_{n-1}P^{n-1} \\ &= \sum_{r=1}^{n-1} c_rP^r, \end{aligned}$$

where P is a permutation matrix.

Let K_n^{net} be the heterogeneous signed complete graph and $A(K_n^{net})$ be its adjacency matrix. $A(K_n^{net})$ is a circulant matrix with first row $[0, 1, -1, -1, \cdots, -1, 1]$. Here $c_1 = 1, c_2 = -1, c_3 = -1, \cdots, c_{n-1} = 1$. Hence $A(K_n^{net})$ can be written as a linear combination of permutation matrix P. $A(K_n^{net}) = P^1 - P^2 - P^3 \cdots - P^{n-2} + P^{n-1}$.

Case 1. If n is odd then

$$A(K_n^{net}) = \left\{(P^1 + P^{n-1}) - (P^2 + P^{n-2}) - \cdots - (P^{\frac{n-1}{2}} + P^{\frac{n+1}{2}})\right\}$$

and $\omega \in Sp(P)$. Hence

$$\begin{aligned} Sp(K_n^{net}) &= \left\{(\omega^1 + \omega^{n-1}) - (\omega^2 + \omega^{n-2}) - \cdots - (\omega^{\frac{n-1}{2}} + \omega^{\frac{n+1}{2}})\right\} \\ &= \left\{2\cos\frac{2\pi j}{n} - \cdots - 2\cos\frac{2(\frac{n-1}{2})\pi j}{n}\right\} \\ Sp(K_n^{net}) &= \left\{2\cos\frac{2\pi j}{n} - \sum_{r=2}^{\frac{n-1}{2}} 2\cos\frac{2r\pi j}{n} : 1 \le j \le n\right\} \end{aligned}$$

Case 2. If n is even then

$$A(K_n^{net}) = \left\{(P^1 + P^{n-1}) - (P^2 + P^{n-2}) - \cdots - (P^{\frac{n-1}{2}} + P^{\frac{n+1}{2}}) - (P^{\frac{n}{2}})\right\}$$

and $\omega \in Sp(P)$. Hence

$$\begin{aligned} Sp(K_n^{net}) &= \left\{(\omega^1 + \omega^{n-1}) - (\omega^2 + \omega^{n-2}) - \cdots - (\omega^{\frac{n-1}{2}} + \omega^{\frac{n+1}{2}}) - (\omega^{\frac{n}{2}})\right\} \\ &= \left\{2\cos\frac{2\pi j}{n} - \cdots - 2\cos\frac{2(\frac{n-1}{2})\pi j}{n} - \cos\frac{2(\frac{n}{2})\pi j}{n} : 1 \le j \le n\right\}. \end{aligned}$$

So

$$Sp(K_n^{net}) = \left\{ 2\cos\frac{2\pi j}{n} - \cos\pi j - \sum_{r=2}^{\frac{n-2}{2}} 2\cos\frac{2r\pi j}{n} : 1 \leq j \leq n \right\}. \qquad \square$$

Acknowledgement The author thanks the University Grants Commission(India) for providing grants under minor research project No.47-902/14 during XII plan.

References

[1] B.D.Acharya, Spectral criterion for cycle balance in networks, *J. Graph Theory*, 4(1980), 1 - 11.

[2] D.M.Cvetkovic, M.Doob, H.Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[3] K.A.Germina, K.S.Hameed, On signed paths, signed cycles and their energies, *Applied Math Sci.*, 70(2010) 3455-3466.

[4] K.A.Germina, K.S.Hameed, T.Zaslavsky, On product and line graphs of signed graphs, their eigenvalues and energy, *Linear Algebra Appl.*, 435(2011) 2432-2450.

[5] M.K.Gill, B.D.Acharya, A recurrence formula for computing the characteristic polynomial of a sigraph, *J. Combin. Inform. Syst. Sci.*, 5(1)(1980) 68 - 72.

[6] K.S.Hameed, K.A.Germina, On composition of signed graphs, *Discussiones Mathematicae Graph Theory*, 32(2012) 507-516.

[7] K.S.Hameed, V.Paul, K.A.Germina, On co-regular signed graphs, *Australasian Journal of Combinatorics*, 62(2015) 8-17.

[8] R.A.Horn, C.R.Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[9] A.M.Mathai, T.Zaslavsky, On Adjacency matrices of simple signed cyclic connected graphs, *J. of Combinatorics, Information and System Sciences*, 37(2012) 369-382.

[10] N.G.Nayak, Equienergetic net-regular signed graphs, *International Journal of Contemporary Mathematical Sciences*, 9(2014) 685-693.

[11] D.B.West, *Introduction to Graph Theory*, Prentice-Hall of India Pvt. Ltd., 1996.

[12] T.Zaslavsky, Signed graphs, *Discrete Appl.Math.*, 4(1982) 47-74.

[13] T.Zaslavsky, Matrices in the theory of signed simple graphs, *Advances in Discrete Mathematics and Applications*, (Ramanujan Math. Soc. Lect. Notes Mysore, India), 13(2010) 207-229.

[14] T.Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, (Manuscript prepared with Marge Pratt), *Journal of Combinatorics*, DS, NO.8(2012), pp.1–340.