General Fourier Analysis and Gravitational Waves.

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Abstract

In a recent paper [1] of this author, we generalized quantum field theory to any curved spacetime. The key idea of the construction was to define a "universal" Fourier transform; we expand more on this theory here and use it to give an intrinsic definition of gravitational waves.

1 Introduction.

When being an undergraduate student in quantum mechanics, I learned about the Heisenberg commutation relations by means of the Fourier transform: that is, a multiplication operator in momentum space coincides with a derivative operator in position space and hence the famous commutation relations were born. Therefore, it seemed only natural that if you are asking what the correct generalization of the Heisenberg commutation relations on a general curved spacetime are that one should lift the Fourier theory to general manifolds. History has not chosen such a path however and has turned its head towards Dirac quantization based upon the Poisson Bracket. As explained in [2], this viewpoint may not be without difficulties when it comes down to the covariance of the theory; the latter being the main achievement of the novel construction in [1]. We will start here by treating those issues on a more detailed level than was the case in that paper which was somehow logical given that Fourier theory only constituted a small part of the new ideas to be imported; this will further elucidate in what sense our construction departs from standard thoughts. We are briefly interested in a first application of our Fourier theory in general relativity and proceed to setup an *intrinsic* definition of a gravitational wave - this is an unaccomplished feature indeed so far as the standard attempts to define such notion are all background dependent and hence in violation of the equivalence principle.

The key idea is to use physical coordinate systems relative to a point in defining the Fourier transform, such entities are given by the geodesic coordinates. Once it is clear that one has to think in a relational way and abondon the absolute point of view of Minkowski spacetime, which one recovers by means of

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translational invariance, the construction is almost obvious. To understand the ideas on that level is the purpose of section two where an interesting example is presented showing that our momenta do automatically discretize when one takes into account the geodesics wrapping around the circle with arbitrary periodicity. Section three extends the Fourier tranform to arbitrary tensors and in particular the metric tensor; we introduce some novel terminology reflecting deep physical ideas regarding the Fourier decomposition at a point and finish the discussion.

2 The Fourier waves.

Consider a spacetime $(\mathcal{M}, g)$ and start by defining the relative Fourier waves as it was done in [1]; for this, let x denote a base point, k^a a Lorentz vector at x defined with respect to $e_a(x)$ and y any point in $\mathcal{M}$. Let $\gamma(t)$ be a curve from x to y and denote by $k^\mu(t)$ the parallel transport of $k^\mu(x) = k^a e^\mu_a(x)$ along γ, then we can define a potential $\phi_\gamma(x, k^a, y)$ where

$$\phi_\gamma : T^\star\mathcal{M} \times \mathcal{M} \to U(1)$$

by means of the differential equation

$$\frac{d}{dt}\phi_\gamma(x, k^a, \gamma(t)) = -i\dot{\gamma}^\mu(t)k_\mu(t)\phi_\gamma(x, k^a, \gamma(t))$$

and $\phi_\gamma(x, k^a, x) = 1$. Then, one can show that in Minkowski spacetime, the potential is independent from the choice of γ and is given by the following group representation

$$\phi(x, k^a, y) = e^{ik.(x-y)}$$

where the formula is respect to global inertial coordinates defined by the vierbein $e_a(x)$. Minkowski is special in many ways: (a) every two events are connected by a unique geodesic (b) the ϕ are path independent and define a group representation. Motivated by some physical principle, we decided in [1] that (a) the paths γ had to be geodesics and (b) that we should sum over all distinct geodesics between x and y. This inspires one to consider the following mapping

$$\tilde{\phi} : T^\star\mathcal{M} \times T^\star\mathcal{M} \to U(t) : (x, k^a, w^a) \to \tilde{\phi}(x, k^a, w^a)$$

where $\tilde{\phi}(x, k^a, w^a)$ is defined as before by means of integrating the potential over the unique geodesic emanating from x with tangent vector w^a and parameter length one. One has that that

$$\phi(x, k^a, y) = \sum_{w:\exp_x(w)=y} \tilde{\phi}(x, k^a, w^a)$$

and although $\tilde{\phi}$ is more fundamental, we will sometimes swith between $\tilde{\phi}$ and ϕ by assuming that they are the same meaning that every two points in spacetime can be connected by a unique geodesic (this last assumption will be abbreviated to GS standing for "geodesic simplicity"). In a general spacetime,

$$\tilde{\phi}(x, k^a, w^b) = e^{-ik^a w_a} = e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,\exp_x(w))}$$

where we assume in the last equality GS to hold and

$$\sigma(x,y) = \frac{1}{2}\epsilon L^2$$

is Synge's function where $\epsilon = 1$ if x and y are connected by a spacelike geodesic and -1 if they are connected by a timelike geodesic and L denotes the geodesic length. Covariant derivatives of $\sigma(x,y)$ with respect to x will be denoted by unprimed indices μ, ν whereas their counterparts with respect to y are denoted with primed indices. It is clear that as usual the standard Fourier identities hold between the two tangent spaces at x, that is

$$\int_{T^\star \mathcal{M}_x} \frac{dk^a}{(2\pi)^4} \overline{e^{-ik^a w_a}} e^{-ik^a v_a} = \delta^4(w^a - v^a)$$

and

$$\int_{T^\star \mathcal{M}_x} \frac{dw^a}{(2\pi)^4} \overline{e^{-ik_a w^a}} e^{-il_a w^a} = \delta^4(k^a - l^a)$$

being the inverse Fourier transform. Under the hypothesis of GS, the first integral reduces to

$$\int_{T^\star \mathcal{M}_x} \frac{dk^a}{(2\pi)^4} \overline{e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,y)}} e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,z)} = \frac{\delta^4(y,z)}{\sqrt{-g(y)}\Delta(x,y)}$$

and the second one under the additional assumption of geodesic completeness (GC) becomes

$$\int_{\mathcal{M}} \frac{d^4 y}{(2\pi)^4} \sqrt{-g(y)}\Delta(x,y) \overline{e^{ik^a e^\mu_a(x)\sigma_{,\mu}(x,y)}} e^{il^a e^\mu_a(x)\sigma_{,\mu}(x,y)} = \delta^4(k^a - l^a).$$

Here,

$$\Delta(x,y) = \frac{|det(\sigma_{,\mu\nu'}(x,y))|}{\sqrt{-g(x)}\sqrt{-g(y)}}$$

is the absolute value of the Van Vleck-Morette determinant. Still working under the GS assumption, one recognizes the presence of a global coordinate system given by $\sigma_{,\mu}(x,y)$ which transforms as a covector under coordinate transformations at x; contracting with $e^{a\mu}(x)$, one obtains local Lorenz coordinates $\sigma^a(x,y)$ and momentum operators $-i\frac{\partial}{\partial \sigma^b(x,y)}$ which transform as a local Lorentz covector such that

$$-i\frac{\partial}{\partial \sigma^b(x,y)}\phi(x,k^a,y) = k_b \phi(x,k^a,y)$$

meaning our generalized exponentials are eigenfunctions of the relative momentum operators. Also,

$$-\eta^{ab}\frac{\partial}{\partial \sigma^a(x,y)}\frac{\partial}{\partial \sigma^b(x,y)}\phi(x,k^a,y) = k^2\phi(x,k^a,y)$$

meaning that the above operator is to be preferred over the generalized d'Alembertian. In Minkowski spacetime, something special happens as $\sigma^b(x,y) = x^b - y^b$ and one can substitute $-i\frac{\partial}{\partial \sigma^b(x,y)}$ by $-i\frac{\partial}{\partial x^b}$ or $i\frac{\partial}{\partial y^b}$. In other words, the x, y coordinates factorize and one can identify all pictures in this way and obtain one

Heisenberg pair only. As was explained in [3], this point of view is only meaningful for the free theory and in the interacting case some genuinely novel ideas are required which are explained in that paper. We also made it clear in [1] that the first order equation defined by deriving along the geodesic is the best point of view on $\phi(x, k^a, y)$ in comparison to the eigenvalue formulations of the Heisenberg operators constructed above. The former point of view reveals an objective process of what is going on while the latter does not and it is precisely this process line of thought we want to generalize further [2].

2.1 Some interesting example.

Let us study the example of the timelike cylinder $\mathbb{R} \times S^1$ with coordinates (t, θ) where θ has to be taken modulo $L > 0$ and see if only the discretized modes $k^1 = \frac{2\pi n}{L}$ for some $n \in \mathbb{Z}$ play a part in the propagator

$$W((0,0),(t,\theta)) = \int \frac{dk^1}{4\pi\sqrt{(k^1)^2 + m^2}} e^{i(\sqrt{(k^1)^2+m^2}t - k^1\theta)} \left[\sum_{n\in\mathbb{Z}} e^{ik^1 Ln} \right]$$

where we have chosen, without limitiation of generality, x to have coordinates $(0, 0)$. The factor between square brackets comes from the winding of geodesics around the circle and we will show now that it effectively agrees with

$$\frac{2\pi}{L} \sum_{n\in\mathbb{Z}} \delta\left(k^1 - \frac{2\pi n}{L}\right).$$

To prove this, note that the Wightman function is periodic in L and therefore Fourier decomposable by means of $\frac{1}{\sqrt{L}} e^{i\frac{2\pi m}{L}\theta}$. Now, we will compute

$$\sum_{p\in\mathbb{Z}} e^{-i\frac{2\pi p}{L}\theta} \frac{1}{L} \int_0^L e^{i\frac{2\pi p}{L}x} W((0,0),(t,x))dx$$

and the reader sees that

$$\frac{1}{L}\int_0^L e^{i\frac{2\pi p}{L}x} W((0,0),(t,x))dx = \frac{1}{L}\int \frac{dk^1}{2\sqrt{(k^1)^2+m^2}} e^{i\sqrt{(k^1)^2+m^2}t} \delta\left(k^1 - \frac{2\pi p}{L}\right)$$
$$= \frac{1}{2L\sqrt{\left(\frac{2\pi p}{L}\right)^2 + m^2}} e^{i\sqrt{\left(\frac{2\pi p}{L}\right)^2+m^2}t}$$

which is what we needed to prove. The reader notices that the periodic character of space settles itself in our definition by taking all possible geodesic paths into account which is a very reassuring feature indeed. This example obviously generalizes to higher dimensional cylinders over the spatial d-dimensional torus $\mathbb{T}^d$.

3 An intrinsic definition of gravitational waves.

What we are going to do here is to write out the relative Fourier decomposition at x of the spacetime metric at y with regard to derivatives with respect to y

of our basic function $\sigma_{\mu}(x,y)$. In what follows, we assume spacetime to be GS and the Van Vleck-Morette determinant to be nonzero meaning that $\sigma^{\nu'}_{,\mu}$ can be seen as a nonsingular mapping from $T^{\star}\mathcal{M}_x$ to $T^{\star}\mathcal{M}_y$. Hence,

$$g_{\alpha'\beta'}(y) = h^{\mu\nu}(x;y)\sigma_{,\mu\alpha'}(x,y)\sigma_{,\nu\beta'}(x,y)$$

and we shall Fourier decompose $h_{ab}(x;y) = e_{a\mu}(x)e_{b\nu}(x)h^{\mu\nu}(x;y)$ which is a symmetric tensor of the same signature. For general tensors, this is easy to define

$$V^{a_1\ldots a_r}_{b_1\ldots b_s}(x;y) = \int_{T^{\star}\mathcal{M}_x} \frac{d^4k}{(2\pi)^4} V^{a_1\ldots a_r}_{b_1\ldots b_s}(x;k)e^{ik^b\sigma_b(x,y)}$$

and $V^{a_1\ldots a_r}_{b_1\ldots b_s}(x;k)$ behaves as a local Lorentz tensor at x meaning that

$$V'^{a_1\ldots a_r}_{b_1\ldots b_s}(x;k') = \prod_{i,j} \Lambda^{a_i}_{c_i}(\Lambda^{-1})^{d_j}_{b_j} V^{c_1\ldots c_r}_{d_1\ldots d_s}(x;k).$$

For flat Minkowski in global inertial coordinates, we have that $\sigma_{\mu\alpha'} = -\eta_{\mu\alpha'}$ and therefore $h^{\mu\nu} = \eta^{\mu\nu}$ meaning that the Fourier transform $h_{ab}(x;k)$ is given by $(2\pi)^4\eta_{ab}\delta^4(k)$ which means there are no propagating waves. Before we come to a further discussion, let us remark one can develop the following terminology; spacetime is *future propagating* from x if and only if the support of $h_{ab}(x;k)$ is within the future lightcone at x. Spacetime is *past propagating* from x if and only if the support of $h_{ab}(x;k)$ is within the past lightcone at x; we say that it is *causally propagating* at x if and only if the support of $h_{ab}(x;k)$ is within the lightcone at x. If any spacetime would exist such that at some x, the support intersects the spacelike vectors, then we would call such spacetime *acausal* at x. One would suspect (a refinement of) such classification to play the equivalent role of various positive energy conditions in general relativity.

Coming back to our gravitational waves, we would somehow want to further specialize them to (future oriented) null vectors $k^2 = 0$; however, I am not sure at all if this is desirable. Also, one might want to add some Lorentz covariant conditions upon the polarization tensors $h_{ab}(x;k)$ as this is done in flat spacetime in some covariant gauge. It must be clear that one will have to go a long way before properly understanding the definitions and concepts elaborated upon in this paper. Albeit all definitions are clean and simple, the explicit calculation of Synge's function is a notoriously difficult task as geodesics can only be calculated numerically in general. However, there is good hope that strong and deep analytic results may be possible in this framework as it is very natural and gives rise to novel insights. As the principal aim of this paper was to widen the scope of our novel inventions beyond quantum field theory and not to investigate in depth the new concepts it gives rise too, we finish our preliminary discussion here.

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References

[1] J. Noldus, General Covariance: a new Paradigm for Relativistic Quantum Theory, Vixra

[2] J. Noldus, On the Foundations of Physics, Vixra.

[3] J. Noldus, Generally Covariant Quantum Theory: Examples, Vixra.

[4] Steven Weinberg, The quantum theory of fields, foundations, volume one, Cambridge university press.

[5] S. Hawking and G.Ellis, The large scale structure of spacetime, Cambridge University Press.

[6] E. Poisson, The motion of point particles in curved spacetime, Arxiv;gr-qc/0306052