Relative probability, an extended probability theory for the physical world.

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Abstract

As is well known, the standard Kolmogorov definition of probability may not be adequate for describing the physical world. In this work, we weaken the Kolmogorovian axioms and depart slightly from its measure theoretical context enabling one to describe systems with an infinite number of degrees of freedom. The resulting framework is more broad and physical than Kolmogorov's and constitutes the basis for the understanding of Quantum Theory as envisioned by this author.

1 Introduction.

The real physical world might consist out of an infinite number of degrees of freedom which makes it impossibe to construct a Lebesgue measure. Nevertheless, nature has no problems in behaving in a *local* or *relative* probabilistic fashion as we see every day in the laboratory. In order to understand the issue with the standard theory, consider a pot filled with an infinite number of point-like balls and ask someone to take a ball out of it; then the probability for a ball to be taken out is exactly zero. Nevertheless *some* ball is taken and we can make meaningful statements about the relative frequency between two balls to be chosen; the latter should be one after an infinite number of trials. Indeed, in the real physical world, it might be meaningless to ask the question concerning the probability that something happens but it could be more opportune to answer in a way revealing how many times more or less something occurs than something else. After all, this is the real basis of probability theory given that the latter says nothing about the number of events occuring. In quantum mechanics, this implies that the wave function should not be normalized, it might even have an infinite norm and still a consistent interpretation would exist. So, we are questioning the fact here if it is meaningful to define a probability function on a measure space to start with, better would be to specify a relational quantity behaving in an appropriate way. This is axiomatized in the following section.

As a matter of philosophy, we all like to believe that what is happening to us is in a sense unavoidable, close to the border of being deterministic; I believe

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this attitude to be wrong. In my experience, sample space is that large that everything which is happening is almost a pure coincidence by itself; the conspiracy being hidden in the relative amplitudes. Often, we tend to forget this as for example a medical doctor proclaims that you have only 3 months to live, he actually means given that the sun will rise 90 times you won't live for the 91 rise. The first condition is so obviously satisfied that we can take it for an absolute statement unless the sun explodes of course during these 90 days.

2 Definition.

Let $(\mathcal{M}, \Sigma)$ denote a sigma-algebra on some set $\mathcal{M}$, then consider

$$\lambda : \mathcal{D} \subset (\mathcal{M} \times \mathcal{M}, \Sigma \times \Sigma) \to \mathbb{R}_+ \cup \{\infty\}$$

where by definition $0.\infty = a$ where a is any number in $\mathbb{R}_+ \cup \{\infty\}$ and $b.\infty = \infty$ for every $b > 0$ so that the multiplication is still associative and commutative and we define the "inverse" of 0 to be ∞. Then, the inverse still satisfies the property that $(x^{-1})^{-1} = x$ and $(xy)^{-1} = y^{-1}x^{-1}$; however, we have not exactly a group structure but everything we say applies for any field. A symmetric[1] subset $\mathcal{D} \subset (\mathcal{M} \times \mathcal{M}, \Sigma \times \Sigma)$ is a symmetric subset of the sigma-algebra $\Sigma \times \Sigma$ on the same underlying space $\mathcal{M} \times \mathcal{M}$. λ defines a relative probability function if and only if

$$\begin{aligned} \lambda(A, B) &= \lambda(B, A)^{-1} \\ \lambda(\sqcup_n A_n, B) &= \sum_n \lambda(A_n, B) \\ \lambda(A, B) &\equiv \lambda(A, C)\lambda(C, B) \end{aligned}$$

where in the last sentence the equivalence means that some value of the right hand side must equal the left hand side. The union $\sqcup$ is the disjoint union meaning that the intersection of the A_n is empty. The implication of this point of view is nontrivial; as is well known, it is impossible to define a Lebesgue integral on $(\mathbb{R}^\infty, \mathcal{B})$ where $\mathcal{B}$ is the Borel sigma-algebra, but it is very well possible to define a relative Lebesgue measure and therefore integral by considering those Borel sets A which have a finite relative volume with respect to B. Here, the relative measure can be defined in a weak and strong sense; the former is given with respect to an increasing sequence of subspaces $\Gamma_n = \mathbb{R}^n$ with $\Gamma_n \subset \Gamma_{n+1}$ and the inductive limit of Γ_n is $\mathbb{R}^\infty$ by means of

$$\lambda(A, B) = \lim_{n \to \infty} \frac{\mu_n(\Gamma_n \cap A)}{\mu_n(\Gamma_n \cap B)}$$

where the limit is supposed to exist. The strong definition requires the above limit to exist and to be independent of any sequence chosen. Such strong relative measures are translation invariant and in the weak case, the relative measure is invariant with respect to any finite dimensional Euclidean group associated to the Γ_n. To define the relative integral is easy; denote by $\mathcal{B}(A)$ the set of all C such that $(C, A) \in \mathcal{D}$, then $\mathcal{B}(A)$ is not necessarily a Borel-Sigma algebra but it has all its salient features since it is a subset of Σ and the relative measure just

[1] Symmetric means symmetric with respect to the (x, y) interchange in $\mathcal{M} \times \mathcal{M}$.

reduces to an ordinary one. This suggests one to simply take over the definition of the standard Lebesgue integral with respect to $\mathcal{B}(A)$ to the infinite dimensional case. This calls for a point of attention though which is that decimating an infinite dimensional cube in every direction produces an $\aleph_1$ number of cubes. Clearly, this is not what we are doing and also the domain of integration is relative to A so that only a subcover with an $\aleph_0$ number of elements with finite relative volume are used. Hence, it appears that every theorem for Lebesgue theory generalizes in a way to relative measures; this work can henceforth also serve as a basis to rigorously define relative path integrals and it might be that one can detach oneself from limiting procedures.

3 Afterword.

It might be that stressing the relative character of amplitudes as we did is perhaps not the good thing to do, but only future work on this notion may give some more insight into this. It is certainly not my intention to do so as my aim was to stress the relative character of the probability interpretation and some of its salient features, *not* to develop a new type of Lebesgue theory which is rather something else.