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General one-sided Clifford Fourier transform, and convolution products in the spatial and frequency domains☆,☆☆

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Abstract

In this paper we use the general steerable one-sided Clifford Fourier transform (CFT), and relate the classical convolution of Clifford algebra-valued signals over $\mathbb{R}^{p,q}$ with the (equally steerable) Mustard convolution. A Mustard convolution can be expressed in the spectral domain as the point wise product of the CFTs of the factor functions. In full generality do we express the classical convolution of Clifford algebra signals in terms of a linear combination of Mustard convolutions, and vice versa the Mustard convolution of Clifford algebra signals in terms of a linear combination of classical convolutions.

Keywords: Convolution, Mustard convolution, Clifford Fourier transform, Clifford algebra signals, spatial domain, frequency domain

1. Introduction

The steerable one-sided Clifford Fourier transformation (CFT) was introduced in [20]. It generalizes related transforms, like the classical complex Fourier transform, the one-sided single kernel quaternion Fourier transform [10], and the Clifford Fourier transforms with pseudoscalar kernels [9, 14] to higher dimensions. These CFTs essentially replace the imaginary unit $i \in \mathbb{C}$ by a general multivector square root of -1, which usually populate continuous Clifford algebra submanifolds [17, 21]. The classical complex Fourier transform needs only one fully commuting kernel factor, due to the commutativity of complex numbers. To have a non-commutative kernel factor under the transform integral on one side of the signal function is meaningful due to the inherent non-commutativity in Clifford algebras. An extensive discussion of the historical development and the application relevance of the CFTs can be found in [5] and [26].

☆Soli Deo Gloria.

☆☆Dedicated to the unknown Syrian refugee, sent back to Syria from Turkey today [1]. The use of this paper is subject to the *Creative Peace License* [16].

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A key strength of the classical complex Fourier transform is its easy and fast application to filtering problems. The convolution of a signal with its filter function becomes in the spectral domain a point wise product of the respective Fourier transformations. This is generally not the case for the convolution of two Clifford algebra-valued signals (Clifford signals) over $\mathbb{R}^{p,q}$, due to Clifford algebra non-commutativity. Yet it is possible to define from the point wise product of the CFTs of two Clifford signals a new type of steerable convolution, called Mustard convolution [30, 7]. This Mustard convolution can be expressed in terms of sums of classical convolutions and vice versa. For the left-sided QFT this has recently been carried out in [8], for the two-sided QFT in [23], for the space-time Fourier transform in [24] and for the two-sided CFT in [25]. Here we extend this approach in full generality to the steerable one-sided CFT for signal functions which map non-degenerate quadratic form vector spaces to Clifford algebras in all dimensions.

This paper is organized as follows. Section 2 introduces Clifford algebra, multivector signal functions, and the continuous manifolds of multivector square roots of -1. Then, Section 3 gives some background on the steerable one-sided CFT. Finally, Section 4 defines the classical convolution of two Clifford signal functions, as well as the steerable Mustard convolution. The rest of the section is devoted to representing the classical convolution in terms of a sum of Mustard convolutions (Theorem 4.3) and dually to expressing the Mustard convolution in terms of a sum of classical convolutions (Theorem 4.4). Furthermore direct single convolution product identities between classical and Mustard convolutions are established (Theorem 4.6), together with the theoretical equivalence (for general Clifford signal convolution product factor functions) of expressing the classical convolution in terms of the Mustard convolution and the reverse (Equation (4.14)).

2. Clifford's geometric algebra

2.1. Multivector algebra

Definition 2.1 (Clifford's geometric algebra [11, 28, 13, 18])**.** Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_n\}$, with $n = p + q$, $e_k^2 = Q(e_k)1 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p + 1, \ldots, n$, be an *orthonormal base* of the non-degenerate inner product vector space $(\mathbb{R}^{p,q}, Q)$, Q the non-degenerate quadratic form, with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k, l = 1, \ldots n, \tag{2.1}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of *associativity* generate the 2^n-dimensional Clifford geometric algebra $Cl(p, q) = Cl(\mathbb{R}^{p,q}) = Cl_{p,q} = \mathcal{G}_{p,q} = \mathbb{R}_{p,q}$ over $\mathbb{R}$. For Euclidean vector spaces ($n = p$) we use $\mathbb{R}^n = \mathbb{R}^{n,0}$ and $Cl(n) = Cl(n, 0)$. The set $\{e_A : A \subseteq \{1, \ldots, n\}\}$ with $e_A = e_{h_1} e_{h_2} \ldots e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$, $e_\emptyset = 1$, the unity in the Clifford algebra, forms a graded (blade) basis of $Cl(p, q)$. The grades k range from 0 for scalars, 1 for vectors, 2 for bivectors, s for s-vectors, up to n for pseudoscalars. The quadratic space $(\mathbb{R}^{p,q}, Q)$ is embedded into $C\ell_{p,q}$ as a subspace, which is identified with the subspace of 1-vectors. All linear combinations of basis elements of grade k, $0 \leq k \leq n$, form the subspace $C\ell_{p,q}^k \subset C\ell_{p,q}$ of k-vectors. The general elements of $Cl(p,q)$ are real linear combinations of basis blades e_A, called Clifford numbers, multivectors or hypercomplex numbers.

In general $\langle A \rangle_k$ denotes the grade k part of $A \in Cl(p,q)$. Following [13], the parts of grade 0 and $k+s$, respectively, of the geometric product of a k-vector $A_k \in Cl(p,q)$ with an s-vector $B_s \in Cl(p,q)$

$$A_k * B_s := \langle A_k B_s \rangle_0, \qquad A_k \wedge B_s := \langle A_k B_s \rangle_{k+s}, \tag{2.2}$$

are called *scalar product* and *outer product*, respectively. They are bilinear products mapping a pair of multivectors to a resulting product multivector in the same algebra. The outer product is also associative, the scalar product not.

Every k-vector B that can be written as the outer product $B = \boldsymbol{b}_1 \wedge \boldsymbol{b}_2 \wedge \ldots \wedge \boldsymbol{b}_k$ of k vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbb{R}^{p,q}$ is called a *simple* k-vector or *blade*.

Multivectors $M \in Cl(p,q)$ have k-vector parts $(0 \leq k \leq n)$: scalar part $Sc(M) = \langle M \rangle = \langle M \rangle_0 = M_0 \in \mathbb{R}$, vector part $\langle M \rangle_1 \in \mathbb{R}^{p,q}$, bi-vector part $\langle M \rangle_2 \in \bigwedge^2 \mathbb{R}^{p,q}$, ..., and pseudoscalar part $\langle M \rangle_n \in \bigwedge^n \mathbb{R}^{p,q}$

$$M = \sum_A M_A \boldsymbol{e}_A = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \ldots + \langle M \rangle_n \,. \tag{2.3}$$

The *principal reverse* of $M \in Cl(p,q)$ defined as

$$\widetilde{M} = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle \overline{M} \rangle_k, \tag{2.4}$$

often replaces complex conjugation and quaternion conjugation. Taking the *reverse* is equivalent to reversing the order of products of basis vectors in the basis blades e_A. The operation $\overline{M}$ means to change in the basis decomposition of M the sign of every vector of negative square $\overline{e_A} = \varepsilon_{h_1} e_{h_1} \varepsilon_{h_2} e_{h_2} \ldots \varepsilon_{h_k} e_{h_k}$, $1 \leq h_1 < \ldots < h_k \leq n$. Reversion, $\overline{M}$, and principal reversion are all involutions. In $Cl(n)$ the principal reverse and the reverse are identical.

For $M, N \in Cl(p,q)$ we get $M * \widetilde{N} = \sum_A M_A N_A$. Two multivectors $M, N \in Cl(p,q)$ are *orthogonal* if and only if $M * \widetilde{N} = 0$. The modulus $|M|$ of a multivector $M \in Cl(p,q)$ is defined as

$$|M|^2 = M * \widetilde{M} = \sum_A M_A^2. \tag{2.5}$$

2.2. Multivector signal functions

A multivector valued function $h : \mathbb{R}^{p,q} \to Cl(p',q')$, has $2^{n'}$ blade components, $n' = p' + q'$ $(h_A : \mathbb{R}^{p,q} \to \mathbb{R})$

$$h(\boldsymbol{x}) = \sum_A h_A(\boldsymbol{x}) \boldsymbol{e}_A. \tag{2.6}$$

We define the *inner product* of two functions $h, m : \mathbb{R}^{p,q} \to Cl(p',q')$ by

$$(h,m) = \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x})\widetilde{m(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A\widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^{p,q}} h_A(\boldsymbol{x})m_B(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.7}$$

with the *symmetric scalar part*

$$\langle h,m\rangle = \int_{\mathbb{R}^{p,q}} h(\boldsymbol{x}) * \widetilde{m(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A} \int_{\mathbb{R}^{p,q}} h_A(\boldsymbol{x})m_A(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.8}$$

and the $L^2(\mathbb{R}^{p,q}; Cl(p',q'))$-*norm*

$$\|h\|^2 = \langle (h,h)\rangle = \int_{\mathbb{R}^{p,q}} |h(\boldsymbol{x})|^2 d^n\boldsymbol{x} = \sum_{A} \int_{\mathbb{R}^{p,q}} h_A^2(\boldsymbol{x})\, d^n\boldsymbol{x}, \tag{2.9}$$

$$L^2(\mathbb{R}^{p,q}; Cl(p',q')) = \{h : \mathbb{R}^{p,q} \to Cl(p',q') \mid \|h\| < \infty\}. \tag{2.10}$$

Notation 2.2 (Argument reflection)**.** *For a function $h : \mathbb{R}^{p,q} \to Cl(p',q')$ we set*[1]

$$h^1(\boldsymbol{x}) := h(-\boldsymbol{x}). \tag{2.11}$$

Note that we obviously have

$$(h^1)^1(\boldsymbol{x}) = h^1(-\boldsymbol{x}) = h(\boldsymbol{x}). \tag{2.12}$$

2.3. Square roots of -1 in Clifford algebras

Every Clifford algebra $Cl(p,q)$, $s_8 = (p-q) \bmod 8$, is isomorphic to one of the following (square) matrix algebras[2] $\mathcal{M}(2d,\mathbb{R})$, $\mathcal{M}(d,\mathbb{H})$, $\mathcal{M}(2d,\mathbb{R}^2)$, $\mathcal{M}(d,\mathbb{H}^2)$ or $\mathcal{M}(2d,\mathbb{C})$. The first argument of $\mathcal{M}$ is the dimension, the second the associated ring[3] $\mathbb{R}$ for $s_8 = 0,2$, $\mathbb{R}^2$ for $s_8 = 1$, $\mathbb{C}$ for $s_8 = 3,7$, $\mathbb{H}$ for $s_8 = 4,6$, and $\mathbb{H}^2$ for $s_8 = 5$. For even n: $d = 2^{(n-2)/2}$, for odd n: $d = 2^{(n-3)/2}$.

It has been shown [17, 21] that $Sc(f) = 0$ for every square root of $-\mathbf{1}$ in every matrix algebra $\mathcal{A}$ isomorphic to $Cl(p,q)$. One can distinguish *ordinary* square roots of $-\mathbf{1}$, and *exceptional* ones. All square roots of $-\mathbf{1}$ in $Cl(p,q)$ can be computed using the package CLIFFORD for Maple [3, 4, 19, 29].

In all cases the *ordinary* square roots f of $-\mathbf{1}$ constitute a *unique conjugacy class* of dimension $\dim(\mathcal{A})/2$, which has *as many connected components as the group* $\mathrm{G}(\mathcal{A})$ of invertible elements in $\mathcal{A}$. Furthermore, for ordinary square roots of $-\mathbf{1}$ we always have $\mathrm{Spec}(f) = 0$ (zero pseudoscalar part) if the associated ring is $\mathbb{R}^2$, $\mathbb{H}^2$, or $\mathbb{C}$. The exceptional square roots of $-\mathbf{1}$ *only* exist if $\mathcal{A} \cong \mathcal{M}(2d,\mathbb{C})$.

[1] We are aware that this notation could be confused with an ordinary taking to the power of 1, but as will be seen in the current context no danger of confusion is likely to arise.

[2] Compare chapter 16 on *matrix representations and periodicity of 8*, as well as Table 1 on p. 217 of [28].

[3] Associated ring means, that the matrix elements are from the respective ring $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{H}^2$.

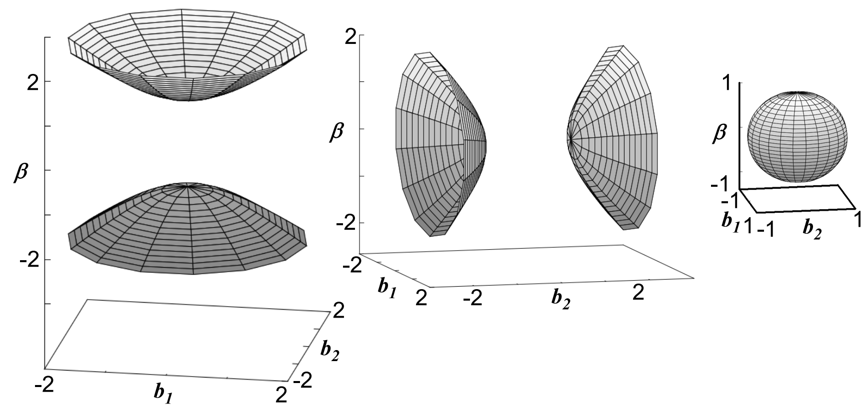

Figure 1: Manifolds of square roots f of -1 in $Cl(2,0)$ (left), $Cl(1,1)$ (center), and $Cl(0,2) \cong \mathbb{H}$ (right). The square roots are $f = \alpha + b_1 e_1 + b_2 e_2 + \beta e_{12}$, with $\alpha, b_1, b_2, \beta \in \mathbb{R}$, $\alpha = 0$, and $\beta^2 = b_1^2 e_2^2 + b_2^2 e_1^2 + e_1^2 e_2^2$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{R})$, the centralizer (set of all elements in $Cl(p,q)$ commuting with f) and the conjugacy class of a square root f of $-\mathbf{1}$ both have $\mathbb{R}$-dimension $2d^2$ with *two connected components*. For the simplest case $d = 1$ we have the algebra $Cl(2,0)$ isomorphic to $\mathcal{M}(2, \mathbb{R})$, see the left side of Fig. 1.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{R}^2) = \mathcal{M}(2d, \mathbb{R}) \times \mathcal{M}(2d, \mathbb{R})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots of $-\mathbf{1}$ in $\mathcal{M}(2d, \mathbb{R})$. They constitute a unique conjugacy class with *four connected components*, each of dimension $4d^2$. Regarding the four connected components, the group of inner automorphisms $\mathrm{Inn}(\mathcal{A})$ induces the permutations of the Klein group, whereas the quotient group $\mathrm{Aut}(\mathcal{A})/\mathrm{Inn}(\mathcal{A})$ is isomorphic to the group of isometries of a Euclidean square in 2D. The simplest example with $d = 1$ is $Cl(2,1)$ isomorphic to $M(2, \mathbb{R}^2) = \mathcal{M}(2, \mathbb{R}) \times \mathcal{M}(2, \mathbb{R})$.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H})$, the submanifold of the square roots f of $-\mathbf{1}$ is a *single connected conjugacy class* of $\mathbb{R}$-dimension $2d^2$ equal to the $\mathbb{R}$-dimension of the centralizer of every f. The easiest example for $d = 1$ is $\mathbb{H}$, isomorphic to $Cl(0,2)$, see the right side of Fig. 1.

For $\mathcal{A} = \mathcal{M}(d, \mathbb{H}^2) = \mathcal{M}(d, \mathbb{H}) \times \mathcal{M}(d, \mathbb{H})$, the square roots of $(-\mathbf{1}, -\mathbf{1})$ are pairs of two square roots (f, f') of $-\mathbf{1}$ in $\mathcal{M}(d, \mathbb{H})$ and constitute a *unique connected conjugacy class* of $\mathbb{R}$-dimension $4d^2$. The group $\mathrm{Aut}(\mathcal{A})$ has two connected components: the neutral component $\mathrm{Inn}(\mathcal{A})$ connected to the identity and the second component containing the swap automorphism $(f, f') \mapsto (f', f)$. The simplest case for $d = 1$ is $\mathbb{H}^2$ isomorphic to $Cl(0,3)$.

For $\mathcal{A} = \mathcal{M}(2d, \mathbb{C})$, the square roots of $-\mathbf{1}$ are in *bijection to the idempotents* [2]. First, the *ordinary* square roots of $-\mathbf{1}$ (with $k = 0$) constitute a conjugacy class of $\mathbb{R}$-dimension $4d^2$ of a *single connected component* which is invariant under $\mathrm{Aut}(\mathcal{A})$. Second, there are $2d$ *conjugacy classes* of *exceptional* square roots of $-\mathbf{1}$, each composed of a *single connected component*, characterized by the equality $\mathrm{Spec}(f) = k/d$ (the pseudoscalar coefficient) with $\pm k \in \{1, 2, \ldots, d\}$, and their $\mathbb{R}$-dimensions are $4(d^2 - k^2)$. The group $\mathrm{Aut}(\mathcal{A})$ includes conjugation of the pseudoscalar $\omega \mapsto -\omega$ which maps the conjugacy class associated with k to the class associated with $-k$. The simplest case for $d = 1$ is the Pauli matrix algebra isomorphic to the geometric algebra $Cl(3,0)$ of 3D Euclidean space $\mathbb{R}^3$, and to complex biquaternions [31].

2.4. The multivector split with respect to a square root of -1

With respect to any square root $f \in Cl(p,q)$ of -1, $f^2 = -1$, every multivector $A \in Cl(p,q)$ can be split into *commuting* and *anticommuting* parts [21].

Lemma 2.3. *Every multivector $A \in Cl(p,q)$ has, with respect to a square root $f \in Cl(p,q)$ of -1, i.e., $f^{-1} = -f$, the unique decomposition*

$$A_{+f} = \frac{1}{2}(A + f^{-1}Af), \qquad A_{-f} = \frac{1}{2}(A - f^{-1}Af)$$
$$A = A_{+f} + A_{-f}, \qquad A_{+f}\,f = fA_{+f}, \qquad A_{-f}\,f = -fA_{-f}, \tag{2.13}$$

$A_{+f} \in \mathrm{centralizer}(f, Cl_{p,q})$.

3. General steerable one-sided Clifford Fourier transforms

The *general steerable one-sided Clifford Fourier transform* (CFT) [20], can be understood as a generalization of previously known one-sided CFTs [14], to a general Clifford algebra setting. Most known CFTs (prior to [20]) used in their kernels specific square roots of -1, like bivectors, pseudoscalars, unit pure quaternions, or sets of coorthogonal blades (commuting or anticommuting blades) [6]. All those restrictions on the square roots of -1 used in a CFT do not apply in our definition below. Note further, that the definition we are about to introduce is even more general than Definition 3.1 given in [20], because we generalize to multivector signal functions in $L^1(\mathbb{R}^{p,q}; Cl(p',q'))$ and not only in $L^1(\mathbb{R}^{p,q}; Cl(p,q))$.

Definition 3.1 (Steerable CFT with respect to one square root of -1)**.** Let $i \in Cl(p',q')$, $i^2 = -1$, be any square root of -1. The general Clifford Fourier transform (CFT) of $f \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$, with respect to i is

$$\mathcal{F}^i\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{p,q}} f(\boldsymbol{x})\, e^{-iu(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{x}, \tag{3.1}$$

where $d^n\boldsymbol{x} = dx_1 \dots dx_n$, $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^{p,q}$, and $u : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$.

Since square roots of -1 in $Cl(p',q')$ populate *continuous submanifolds* in $Cl(p',q')$, the CFT of Definition 3.1 is generically *steerable* within these manifolds, see (3.3). In Definition 3.1, the square roots $i \in Cl(p',q')$ of -1 may be from any component of any conjugacy class. The choice of the Clifford's geometric product between multivector signal function f and the multivector kernel $e^{-iu(\boldsymbol{x},\boldsymbol{\omega})}$, in the integrand of (3.1) is very important. Because only this choice allowed, e.g. in [9], to define and apply a holistic vector field convolution, without loss of information.

Note that two-sided CFTs can be decomposed to pairs of one-sided CFTs [22].

Remark 3.2. *In order to avoid clutter we often drop the upper index* i *as in* $\mathcal{F}\{h\} = \mathcal{F}^i\{h\}$*, but in principle the one-sided CFT always depends on the particular choice* i *of the multivector square root of* -1*. Since square roots of* -1 *in* $Cl(p',q')$ *populate continuous submanifolds in* $Cl(p',q')$*, the CFT of Definition 3.1 is generically steerable within these submanifolds. In Definition 3.1, the square root* $i \in Cl(p',q')$ *of* -1*, may be from any conjugacy class and component, respectively.*

Within the same conjugacy class of square roots of -1 the CFTs of Definition 3.1 are related by the following equation, and therefore steerable. Let $i, i' \in Cl(p',q')$ be any two square roots of -1 in the same conjugacy class, i.e. $i' = a^{-1}ia$, $a \in Cl(p',q')$, a being invertible. As a consequence of this relationship we also have

$$e^{-i'u} = a^{-1}e^{-iu}a, \ \ \forall u \in \mathbb{R}. \tag{3.2}$$

This in turn leads to the following *steerability relationship* of all CFTs with square roots of -1 from the same conjugacy class:

$$\mathcal{F}^{i'}\{h\}(\boldsymbol{\omega}) = \mathcal{F}^i\{ha^{-1}\}(\boldsymbol{\omega})a, \tag{3.3}$$

where ha^{-1} means to multiply the signal function h by the constant multivector $a^{-1} \in Cl(p',q')$.

For establishing an *inversion* formula and other properties of the CFT in Definition 3.1, certain *assumptions* about the phase function $u(\boldsymbol{x},\boldsymbol{\omega})$ need to be made. In principle these assumptions could be made based on the desired properties of the resulting CFT. One possibility is, e.g., to assume

$$u(\boldsymbol{x},\boldsymbol{\omega}) = \boldsymbol{x} * \widetilde{\boldsymbol{\omega}} = \sum_{l=1}^{n} x^l\omega^l = \sum_{l=1}^{n} x_l\omega_l, \tag{3.4}$$

which will be assumed in the rest of this paper.

We then get the following *inversion* theorem[4].

Theorem 3.3 (Inversion of one-sided CFT)**.** *For* $\mathcal{F}^i\{h\} \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$ *we have*

$$h(\boldsymbol{x}) = \mathcal{F}^i_{-1}\{\mathcal{F}^i\{h\}\}(\boldsymbol{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{p,q}} \mathcal{F}^i\{h\}(\boldsymbol{\omega})\, e^{iu(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega}, \tag{3.5}$$

where $d^n\boldsymbol{\omega} = d\omega_1 \dots d\omega_n$*,* $\boldsymbol{x},\boldsymbol{\omega} \in \mathbb{R}^{p,q}$*.*

[4]Note, that we show the inversion symbol -1 as lower index in $\mathcal{F}^i_{-1}$, in order to avoid a possible confusion by using two upper indice. The inversion could also be written with the help of the CFT itself as $\mathcal{F}^i_{-1} = \frac{1}{(2\pi)^n}\mathcal{F}^{-i}$.

The proof of theorem 3.3 is strictly analogous to the proof of equation (4.8) on page 231 of [20], and therefore left as an exercise to the reader.

We further note the following useful relationship using the argument reflection of Notation 2.2

$$\mathcal{F}^{-i}\{h\} = \mathcal{F}^{i}\{h^{1}\} = \mathcal{F}\{h^{1}\}. \tag{3.6}$$

The main properties of the CFT of Definition 3.1 have been studied for the special case of multivector signal functions $f \in L^1(\mathbb{R}^{p,q}; Cl(p,q))$ in detail in [20], and can easily be generalized to the more general case of $f \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$.

4. Convolution and steerable Mustard convolution

We define the *convolution* of two Clifford (algebra) signals $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$ as

$$(a \star b)(\boldsymbol{x}) = \int_{\mathbb{R}^2} a(\boldsymbol{y}) b(\boldsymbol{x}-\boldsymbol{y}) d^2\boldsymbol{y}, \tag{4.1}$$

provided that the integral exists.

Note that the real continuous Clifford geometric algebra wavelet transform can be written as a convolution of the multivector signal function with the daughter wavelet (a rotated, dilated and translated mother wavelet), essentially evaluated at the center of the daughter wavelet, see [15].

The *Mustard* convolution [30, 7] of two Clifford signals $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p',q'))$ is defined as

$$(a \star_M b)(\boldsymbol{x}) = (\mathcal{F}^{i})^{-1}(\mathcal{F}^{i}\{a\}\mathcal{F}^{i}\{b\})(\boldsymbol{x}), \tag{4.2}$$

provided that the integral exists.

Remark 4.1. *The Mustard convolution has the conceptual and computational advantage to simply yield, independent of the particular Clifford algebra $Cl(p',q')$ involved and of the particular multivector square root of -1 in the CFT kernel, as spectrum in the CFT Fourier domain the point wise product of the CFTs of the two signals, just as for the classical complex Fourier transform. On the other hand, by its very definition, the Mustard convolution itself depends on the choice of i, i.e. of the multivector square root of -1, used in the Definition 3.1 of the CFT. The Mustard convolution* (4.2) *is therefore a steerable operator, dependent on the choice of i.*

In the following two Subsections we will express the convolution (4.1) in terms of the Mustard convolution (4.2), and vice versa, and study the mutual relations of these expressions.

4.1. Expressing the convolution in terms of the Mustard convolution

In this Subsection we assume the use of the one-sided CFT with a general multivector square roots of -1, $i \in Cl(p',q')$. The definition of the classical convolution (4.1) is independent of the application of a CFT. The Mustard convolution of (4.2) depends on the definition of the CFT and in particular on the choice of the multivector square root i of -1.

In our approach we generalize equation (4.17) on page 233 of [20], which expresses the convolution of two Clifford signal functions in the Clifford Fourier domain with the help of the CFT of Definition 3.1. We generalize this equation to the case of multivector signal functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q')$, and to the CFT of Definition 3.1. Nevertheless the proof works perfectly analogous to the one given in [20], we therefore leave this as an exercise to the reader.

Theorem 4.2 (CFT of convolution)**.** *We assume that the function u is linear with respect to its first argument. The CFT of the convolution* (4.1) *of two multivector signals $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q')$ can then be expressed as*

$$\begin{aligned} \mathcal{F}^i\{a \star b\} &= \mathcal{F}^{-i}\{a\}\mathcal{F}^i\{b_{-i}\} + \mathcal{F}^i\{a\}\mathcal{F}^i\{b_{+i}\} \\ &= \mathcal{F}^i\{a^1\}\mathcal{F}^i\{b_{-i}\} + \mathcal{F}^i\{a\}\mathcal{F}^i\{b_{+i}\}. \end{aligned} \tag{4.3}$$

We can now easily express the convolution of two multivector signals $\mathcal{F}^i\{a \star b\}(\boldsymbol{\omega})$ in terms of only two Mustard convolutions (4.2), by applying the inverse CFT.

Theorem 4.3 (Convolution in terms of Mustard convolution)**.** *Assuming a general multivector square root i of -1, the convolution* (4.1) *of two Clifford functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$ can be expressed in terms of two Mustard convolutions* (4.2) *as*

$$a \star b = a^1 \star_M b_{-i} + a \star_M b_{+i}. \tag{4.4}$$

4.2. Expressing the Mustard convolution in terms of the convolution

Now we will first simply write out the Mustard convolution (4.2) and simplify it until only standard convolutions (4.1) remain.

We begin by writing the Mustard convolution (4.2) of two multivector functions $a, b \in L^2(\mathbb{R}^{p,q}; Cl(p',q'))$

$$
\begin{aligned}
& a \star_M b(\boldsymbol{x}) \\
& = \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \mathcal{F}\{a\}(\boldsymbol{\omega}) \mathcal{F}\{b\}(\boldsymbol{\omega}) e^{iu(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega} \\
& = \tfrac{1}{(2\pi)^n} \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) e^{-iu(\boldsymbol{y},\boldsymbol{\omega})} d^n\boldsymbol{y} \int_{\mathbb{R}^{p,q}} b(\boldsymbol{z}) e^{-iu(\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{z} e^{iu(\boldsymbol{x},\boldsymbol{\omega})} d^n\boldsymbol{\omega} \\
& = \tfrac{1}{(2\pi)^n} \iiint a(\boldsymbol{y}) e^{-iu(\boldsymbol{y},\boldsymbol{\omega})} [b_{+i}(\boldsymbol{z}) + b_{-i}(\boldsymbol{z})] e^{iu(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
& = \tfrac{1}{(2\pi)^n} \iiint a(\boldsymbol{y}) b_{+i}(\boldsymbol{z}) e^{-iu(\boldsymbol{y},\boldsymbol{\omega})} e^{iu(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
& \quad + \tfrac{1}{(2\pi)^n} \iiint a(\boldsymbol{y}) b_{-i}(\boldsymbol{z}) e^{iu(\boldsymbol{y},\boldsymbol{\omega})} e^{iu(\boldsymbol{x}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
& = \tfrac{1}{(2\pi)^n} \iiint a(\boldsymbol{y}) b_{+i}(\boldsymbol{z}) e^{iu(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
& \quad + \tfrac{1}{(2\pi)^n} \iiint a(\boldsymbol{y}) b_{-i}(\boldsymbol{z}) e^{iu(\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{y} d^n\boldsymbol{z} d^n\boldsymbol{\omega} \\
& = \iint a(\boldsymbol{y}) b_{+i}(\boldsymbol{z}) \delta(\boldsymbol{x}-\boldsymbol{y}-\boldsymbol{z}) d^n\boldsymbol{y} d^n\boldsymbol{z} \\
& \quad + \iint a(\boldsymbol{y}) b_{-i}(\boldsymbol{z}) \delta(\boldsymbol{x}+\boldsymbol{y}-\boldsymbol{z}) d^n\boldsymbol{y} d^n\boldsymbol{z} \\
& = \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b_{+i}(\boldsymbol{x}-\boldsymbol{y}) d^n\boldsymbol{y} + \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b_{-i}(\boldsymbol{x}+\boldsymbol{y}) d^n\boldsymbol{y} \\
& = \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b_{+i}(\boldsymbol{x}-\boldsymbol{y}) d^n\boldsymbol{y} + \int_{\mathbb{R}^{p,q}} a(\boldsymbol{y}) b_{-i}(-(-\boldsymbol{x}-\boldsymbol{y})) d^n\boldsymbol{y} \\
& = a \star b_{+i}(\boldsymbol{x}) + a \star b^1_{-i}(-\boldsymbol{x}) \\
& = a \star b_{+i}(\boldsymbol{x}) + a^1 \star b_{-i}(\boldsymbol{x}).
\end{aligned}
\tag{4.5}
$$

We have abbreviated $\int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}}$ to $\iint$, and $\int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}} \int_{\mathbb{R}^{p,q}}$ to $\iiint$. For the third equality we applied the split of Lemma 2.3 to $b(\boldsymbol{x})$ and used the linearity of u with respect to its first argument. For the fourth equality we used the linearity of Clifford's geometric product, the linearity of the triple integral, and we used the commutation and anti-commutation properties of $b_{\pm i}(\boldsymbol{x})$ with the multivector square root $i \in Cl(p',q')$, which produces the sign change $e^{-iu(\boldsymbol{y},\boldsymbol{\omega})} \to e^{+iu(\boldsymbol{y},\boldsymbol{\omega})}$ in the case of anti-commutation. For the fifth equality we again applied the linearity of u with respect to its first argument. The integrations $\frac{1}{(2\pi)^n} \int e^{iu(\boldsymbol{x}\pm\boldsymbol{y}-\boldsymbol{z},\boldsymbol{\omega})} d^n\boldsymbol{\omega}$ produce the n-dimensional Dirac delta functions $\delta(\boldsymbol{x} \pm \boldsymbol{y} - \boldsymbol{z})$, giving the sixth equality.

We illustrate the last identity of (4.5), $a \star b^1_{-i}(-\boldsymbol{x}) = a^1 \star b_{-i}(\boldsymbol{x})$, in the onedimensional case $\mathbb{R}^{p,q} = \mathbb{R}$, the generalization to $\mathbb{R}^{p,q}$ is then straightforward

$$
\begin{aligned}
a \star b^1(-x) &= \int_{\mathbb{R}} a(y)b(-(-x-y))dy = \int_{-\infty}^{+\infty} a(y)b(x+y)dy \\
&= \int_{+\infty}^{-\infty} a(-g)b(x-g)(-1)dg = \int_{-\infty}^{+\infty} a(-g)b(x-g)dg \\
&= \int_{\mathbb{R}} a^1(g)b(x-g)dg = a^1 \star b(x).
\end{aligned}
\tag{4.6}
$$

where we have substituted $g = -y$, $dg = -dy$, including substitution of the integration boundaries for the third equality. The interchange of the integration boundaries eliminates the overall minus sign in the fourth equality of (4.6).

Note that in (4.5), $a \star b^1_{-i}(-\boldsymbol{x})$, means to first apply the convolution to the pair of functions a and b^1_{-i}, and only then to evaluate the result of the convolution integral with the argument $(-\boldsymbol{x})$. So in general $a \star b^1_{-i}(-\boldsymbol{x}) \neq a \star b_{-i}(+\boldsymbol{x})$.

We finally obtain the desired decomposition of the Mustard convolution (4.2) in terms of the classical convolution.

Theorem 4.4 (Mustard convolution in terms of standard convolution)**.** *The Mustard convolution* (4.2) *of two multivector signal functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$ can be expressed in terms of two standard convolutions* (4.1) *as*

$$
a \star_M b(\boldsymbol{x}) = a^1 \star b_{-i}(\boldsymbol{x}) + a \star b_{+i}(\boldsymbol{x}).
\tag{4.7}
$$

Remark 4.5 (Theorem duality)**.** *Comparing Theorems 4.3 and 4.4 we notice an interesting duality: interchanging convolution and Mustard convolution in either theorem yields the other, independent over which vector space $\mathbb{R}^{p,q}$ the multivector signals are defined, independent from the signal value Clifford algebra $Cl(p', q')$, and independent from the particular choice of multivector square root of -1, $i \in Cl(p', q')$. The last form of independence also means, that the observed duality is stable with respect to steering the CFT and the Mustard convolution by changing $i \in Cl(p', q')$. Note further, that a corresponding duality will be valid for the left-sided version of the CFT in Definition 3.1, by placing the kernel factor on the left side and going analogously through all arguments up to Theorem 4.4.*

Yet, it is an interesting non-trivial question, whether a similar duality may hold for other forms of the CFT, e.g. with more than one kernel factor, see e.g. [22, 25].

4.3. Single convolution product identities for classical and Mustard convolutions

Let us now apply Theorem 4.3 to the three functions $a, b_{\pm i}$, observing that

$$
(b_{+i})_{-i} = (b_{-i})_{+i} = 0, \quad (b_{+i})_{+i} = b_{+i}, \quad (b_{-i})_{-i} = b_{-i}.
\tag{4.8}
$$

Then we obtain

$$
a \star b_{+i} = a^1 \star_M (b_{+i})_{-i} + a \star_M (b_{+i})_{+i} = 0 + a \star_M b_{+i} = a \star_M b_{+i},
\tag{4.9}
$$

and similarly,

$$a \star b_{-i} = a^1 \star_M b_{-i} \quad \Longleftrightarrow \quad a^1 \star b_{-i} = a \star_M b_{-i}, \tag{4.10}$$

since double reflection of the argument returns the function itself (2.12). Note, that the very same identities are easily obtained by analogously applying Theorem 4.4 to $a, b_{\pm i}$. We therefore summarize them in the following theorem.

Theorem 4.6 (Partial identities between convolutions and Mustard convolutions)**.** *For pairs of functions (a, b_{-i}) and (a, b_{+i}) with $a, b_{\pm i} \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$, where the second factor either commutes or anti-commutes with the multivector square root of -1, $i \in Cl(p', q')$ of the Definition 3.1, the following convolution product identites between convolution* (4.1) *and Mustard convolution* (4.2) *hold*

$$\begin{aligned} a^1 \star b_{-i} = a \star_M b_{-i} \quad &\Longleftrightarrow \quad a \star b_{-i} = a^1 \star_M b_{-i}, \\ a \star b_{+i} &= a \star_M b_{+i}. \end{aligned} \tag{4.11}$$

Theorem 4.6 can therefore either be derived from Theorem 4.3 or from Theorem 4.4. Moreover, Theorem 4.6 can also be established independently by *direct* computation. Then adding two convolution terms would give the Mustard convolution

$$a^1 \star b_{-i} + a \star b_{+i} \overset{\text{Th. 4.6}}{=} a \star_M b_{-i} + a \star_M b_{+i} = a \star_M b. \tag{4.12}$$

And conversely adding two Mustard convolution terms would give the convolution

$$a^1 \star_M b_{-i} + a \star_M b_{+i} \overset{\text{Th. 4.6}}{=} a \star b_{-i} + a \star b_{+i} = a \star b. \tag{4.13}$$

This establishes the following important *threefold theorem equivalence*

$$\text{Theorem 4.3} \quad \Longleftrightarrow \quad \text{Theorem 4.6} \quad \Longleftrightarrow \quad \text{Theorem 4.4}. \tag{4.14}$$

Remark 4.7. *Note that the need to always decompose the right convolution product factor function $b = b_{-i} + b_{+i}$ is manifestly due to the kernel in Definition 3.1 being placed on the right side. Using a corresponding left side kernel CFT, would lead to analogous results with decomposing the left convolution product factor $a = a_{-i} + a_{+i}$.*

Furthermore, we can ask under what conditions we get a *full direct single convolution product identity* of the two convolution products $a \star b = a \star_M b$? This identity holds under any of the following conditions:

1. For all functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$, with $b_{-i} \equiv 0$. This condition depends on the choice of i.
2. For central multivector square roots $i \in Cl(p', q')$ of -1 and *all* functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$. An important practical example is $i = e_1 e_2 e_3 \in Cl(3, 0)$ [9].
3. For all functions $a, b \in L^1(\mathbb{R}^{p,q}; Cl(p', q'))$, with reflection symmetry $a^1 = a$. This condition does not depend on the choice of i, and poses no restriction on b.

5. Conclusion

In this paper we have briefly reviewed non-degenerate Clifford algebras, their continuous manifolds of multivector square roots of -1, Clifford algebra decomposition with respect to a pair of square roots of -1, and the general steerable one-sided Clifford Fourier transform. We defined the notions of (classical non-steerable) convolution of two Clifford algebra valued functions over $\mathbb{R}^{p,q}$, and the steerable Mustard convolution (with its CFT as the point wise product of the CFTs of the factor functions).

The main results are: A decomposition of the classical convolution of Clifford algebra signals in terms of two Mustard convolutions. Next, we showed how in a dual way to fully generally express the Mustard convolution of two Clifford algebra signals in terms of two classical convolutions. Finally, we studied direct single convolution product identities between classical and Mustard convolutions, and showed how even for general Clifford signal factor functions the dual convolution product decompositions are theoretically fully equivalent, including equivalence to pairs of single convolution product identities.

In view of the many potential applications of the CFT [5], including already its lower-dimensional realizations as QFT [23, introduction], and space-time FT [24, introduction], we expect our new results to be of great interest in physics, pure and applied mathematics, and engineering, e.g., for filter design and feature extraction in multi-dimensional signal and (color) image processing. Finally, the CFT and all convolutions described above can be implemented for simulations and real data applications in the recently released Clifford Multivector Toolbox (for MATLAB) [32, 33].

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