Observations on Zeta(3) from Piling Cubes

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Abstract

We look here at the geometry of zeta(3), $\zeta(3)$. By piling cubes a 3D shape is defined which has a volume $\zeta(3)$. This is a double integral form for $\zeta(3)$. Considering the centroid of this shape leads to an experimental estimate for $\zeta(3)$. Cutting the shape parallel to the x axis reproduces the dilogarithmic, $Li_2(2)$, relationship to $\zeta(3)$. Cutting the shape in the z axis reproduces the logarithmic version of Riemann's formula for $\zeta(3)$. Geometrical considerations also reproduce formula for the polylog of a half $Li_n(1/2)$ for n=2 and 3. These are illustrations of number geometry.

Background

Mikael Passare [1] gave a geometric proof of the famous Basel Problem first solved by Leonhard Euler in 1735

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

where

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \varsigma(2) = \text{zeta}(2).$$

He showed that by spreading and piling up the squares on the left of the z axis in Figure 1 limits to $e^{-x} + e^{-z} = 1$.

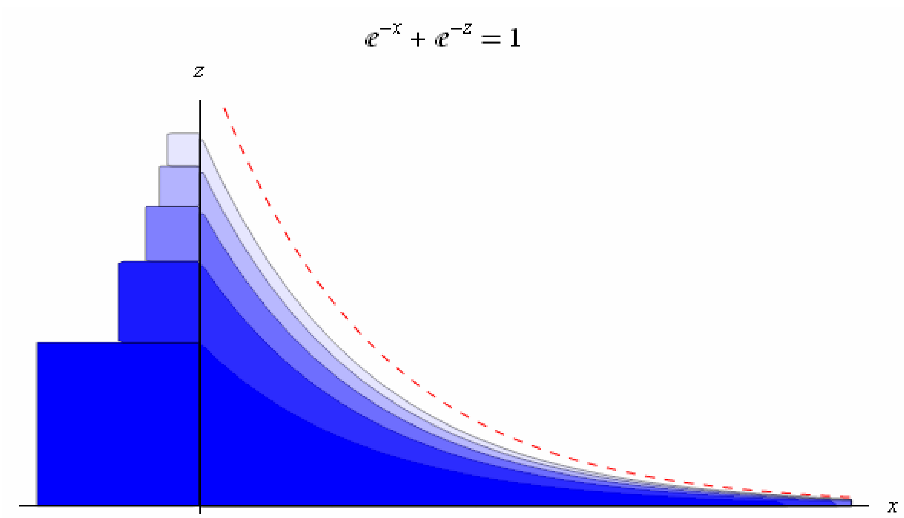

Figure 1. *The red dash curve is the limit of the piled spread out squares given by the formula* $e^{-x} + e^{-z} = 1$.

Passare showed,

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$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\int_0^{\infty}\frac{e^{-nx}}{n}dx=\int_0^{\infty}-\ln(1-e^{-x})dx=\varsigma(2)=\frac{\pi^2}{6}$$

That is, summing squares on the left of Figure 1, gives $\varsigma(2)=\frac{\pi^2}{6}$.

The red dashed line is the limit of the area of the vanishing small squares given by

$e^{-x}+e^{-z}=1$

$\Rightarrow e^{-z}=1-e^{-x}$

$\therefore \ln(e^{-z})=\ln(1-e^{-x})$

So the equation of this red-dashed line is,

$z=-\ln(1-e^{-x})$.

The area under this curve is,

$$\int_0^{\infty}-\ln(1-e^{-x})dx=\varsigma(2)=\frac{\pi^2}{6}$$

1 Extension to 3D

I try here to extend Passare's graphical approach to three dimensions by looking at the volume of piled cubes $\sum_{n=1}^{\infty}\frac{1}{n^3}$ as shown in Figure 2.

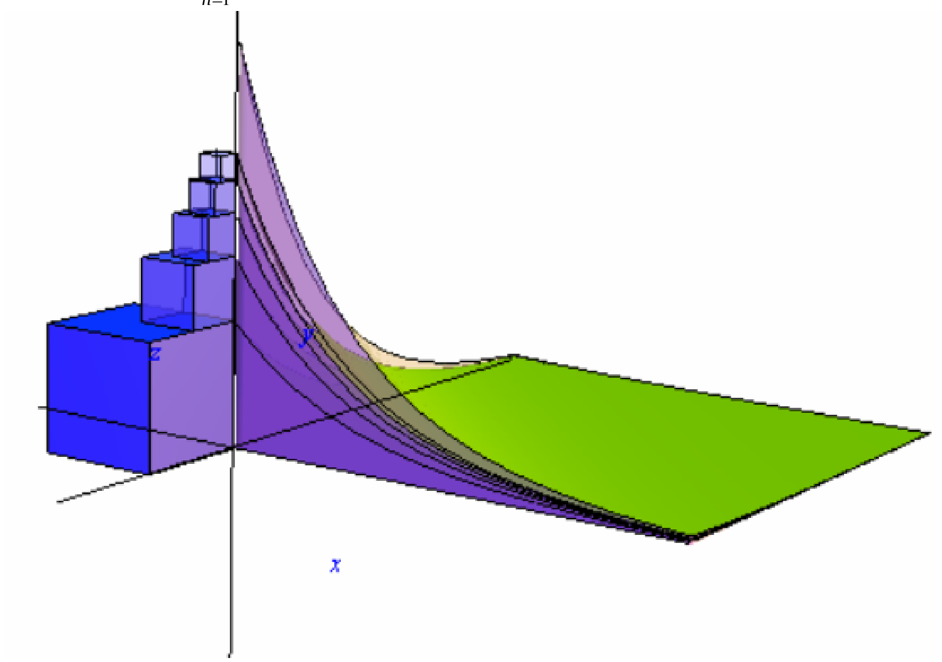


Figure 2. *Piled cubes spread out onto the surfaces shown. Similar shades have similar volumes.*

In two dimensions for squares we had,

$$\int_0^\infty -\ln(1-e^{-x})dx = \varsigma(2) = \frac{\pi^2}{6}.$$

Now what is the function that represents these cubes so that when the function is integrated over positive x and positive y it gives $\zeta(3)$?

It can not be just a volume of revolution of the original function $z = -\ln(1-e^{-x})$ over a quadrant since this will give a volume $\frac{\pi}{4}\int_0^\infty z^2 dx = \frac{\pi}{2}\zeta(3)$ with the shape of this surface given below, which is $\pi/2$ too big.

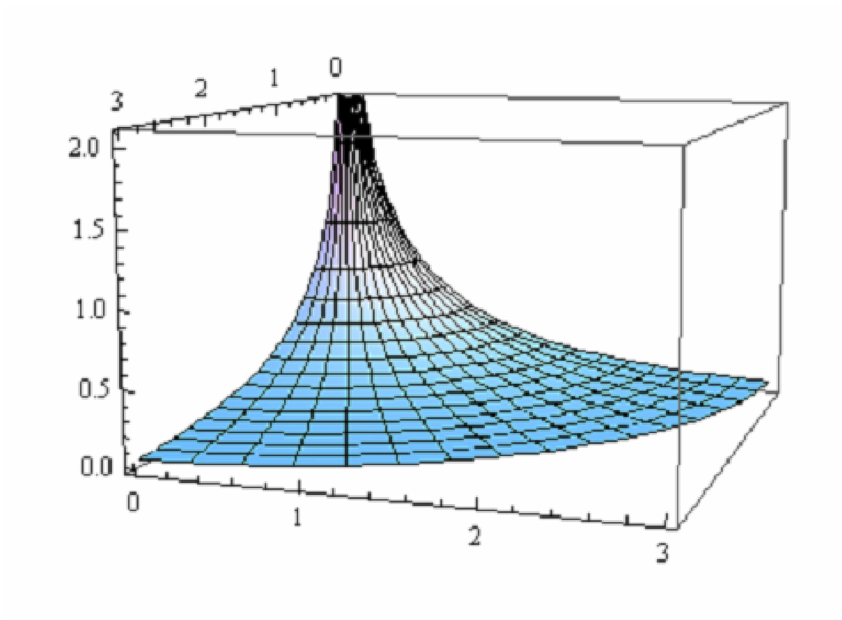


Figure 3. *Volume of revolution of* $\zeta(2)$ *area.*

2 Multi-integral form of the zeta function for n=3

Conjecture 1

The required surface is actually given by the function $z = -\ln(1-e^{-(x+y)})$ and therefore

$$\boxed{\int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y)})dxdy = \varsigma(3)}$$

Proof:

By Mathematica.

This surface is similar to the one above but just gradually dips as $x \Rightarrow y$ $(\theta \Rightarrow \pi/4)$ as we shall see below.

2(i) Multi-integral form of zeta function for n>3

Conjecture 2

By extending to more dimensions:

$$\int_0^\infty\int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y+z)})dxdydz = \varsigma(4) = \frac{\pi^4}{90}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y+z+a)})dxdydzda = \varsigma(5)$$

Proof:

By Mathematica

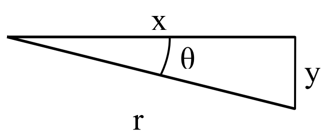
Or more generally:

$$\boxed{\int_0^\infty \text{...(n integrals)}\int_0^\infty\int_0^\infty\int_0^\infty -\ln(1-e^{-(x_1+x_2+...x_n)})dx_1dx_2...dx_n = \varsigma(n+1)}$$

We will now look at various observations about this surface. First, for ease of visualization, consider the polar form.

3 Zeta(3) surface in terms of r, θ;

We look at here defining the surface in terms of r,θ instead of x, y.



and the surface $\int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y)})dxdy$ transforms to

$$\int_0^{\pi/2}\int_0^\infty -\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})rdrd\theta = \varsigma(3)$$

We can consider the area under each radial cut of the surface.

4 The plane area under the $\varsigma(3)$ surface as a function of angle

At $\theta = 0$, that is, along the x axis the plane curve has area;

$$\int_0^\infty -\ln(1-e^{-r(\cos(0)+\sin(0))})dr = \varsigma(2)$$

At the $45°$ angle y=x, the area under the curve is;

$$\int_0^\infty -\ln(1-e^{-r(\cos(45)+\sin(45))})dr$$

$$\Rightarrow \int_0^\infty -\ln(1-e^{-\sqrt{2}r})dr = \frac{\varsigma(2)}{\sqrt{2}}$$

At any angle the area under the plane curve section is

$$\int_0^\infty -\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})dr \quad = \varsigma(2)/(\cos(\theta)+\sin(\theta))$$

So we can see the areas under a radial section of the surface are functions of $\zeta(2)$.

The plot below shows how this radial area normalized to $\varsigma(2)$ varies with θ. It has minimum area at $\theta = \pi/4$ and is symmetric about this angle.

It is the graph of $1/(\cos(\theta)+\sin(\theta))$

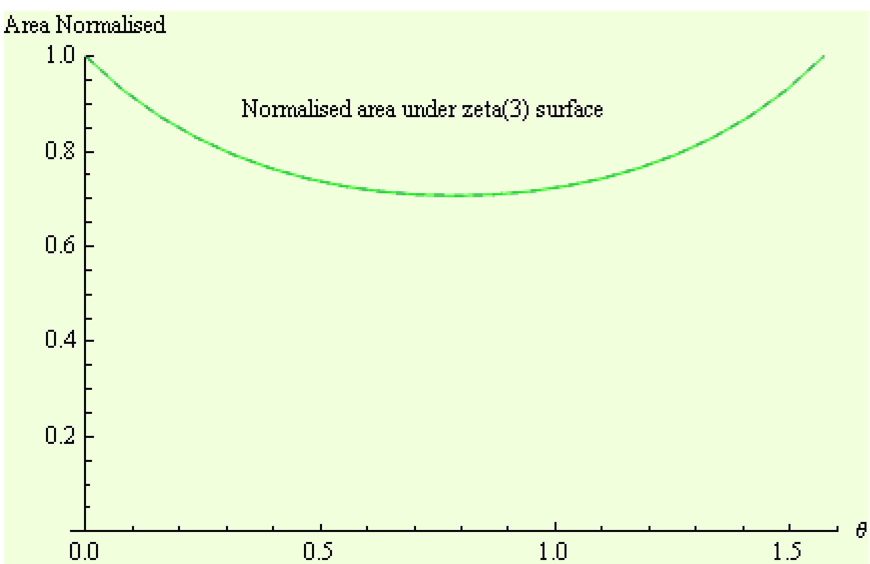


Figure 4. *The area normalised to* $\zeta(2)$ *under radial sections of the* $\zeta(3)$ *surface.*

We now try and show how the geometry of the stacked cubes relates to the surface.

4(i) Relating the hypotenuse of the stacked cubes to the surface plot in r,θ

So the $\varsigma(3)$ surface given by $-\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})$ for θ from 0 to $\pi/2$ and r from 0 to ∞. As stated this surface encloses a volume of $\varsigma(3)$ that is, $\int_0^{\pi/2}\int_0^{\infty}-\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})rdrd\theta=\varsigma(3)$. Three views of this surface are shown below for $r\le 1$.

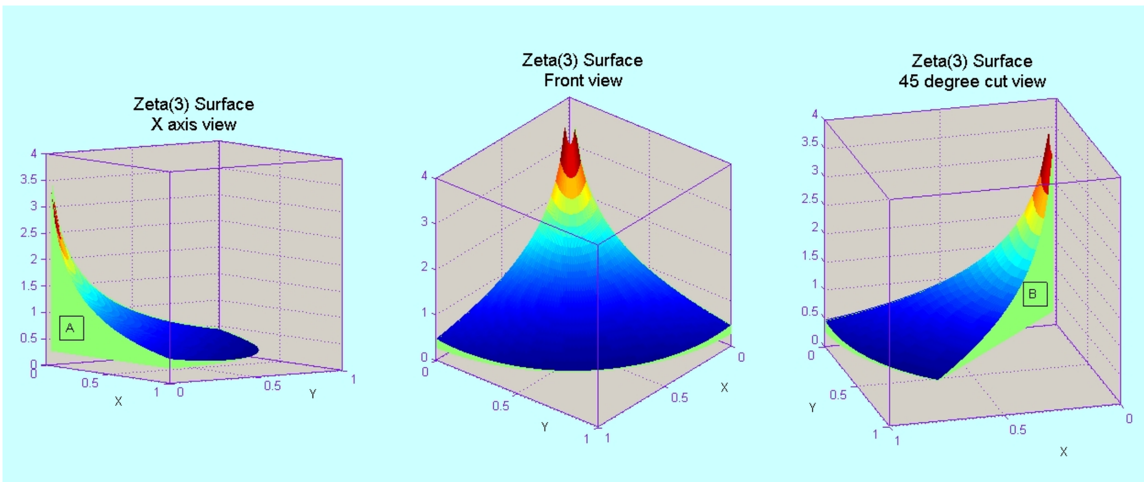


Figure 5. *Views of the zeta(3) surface for* $r\le 1$.

The left view shows the y=0 plane, that is $\theta=0°$. This plane area A is $\int_0^{\infty}-\ln(1-e^{-r(\cos(0)+\sin(0))})dr=\varsigma(2)$. As discussed previously it can be seen the surface dips to a minimum plane area B $\Rightarrow \int_0^{\infty}-\ln(1-e^{-\sqrt{2}r})dr=\dfrac{\varsigma(2)}{\sqrt{2}}$ for $\theta=45°$ as shown in the view on the right. The dipping is illustrated by the coloured contours which show the value of the height function, $\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})$ at constant radii.

The volume under the surface is $\varsigma(3)$ equals that of the stacked cubes $\sum_{n=1}^{\infty}\frac{1}{n^3}$. But is there a correspondence between the plane areas, A and B, and the plane section of the stacked cubes at a given θ? That is, between the plane section area of the surface and the hypotenuse of the stacked cubes? For example, the figure below shows a plan view of the $\varsigma(3)$ surface along with a z section of a cube of side a, where, a, is 1, 1/2 , 1/3, 1/4….$1/\infty$. For $\theta=45°$ the hypotenuse is $a\sqrt{2}$.

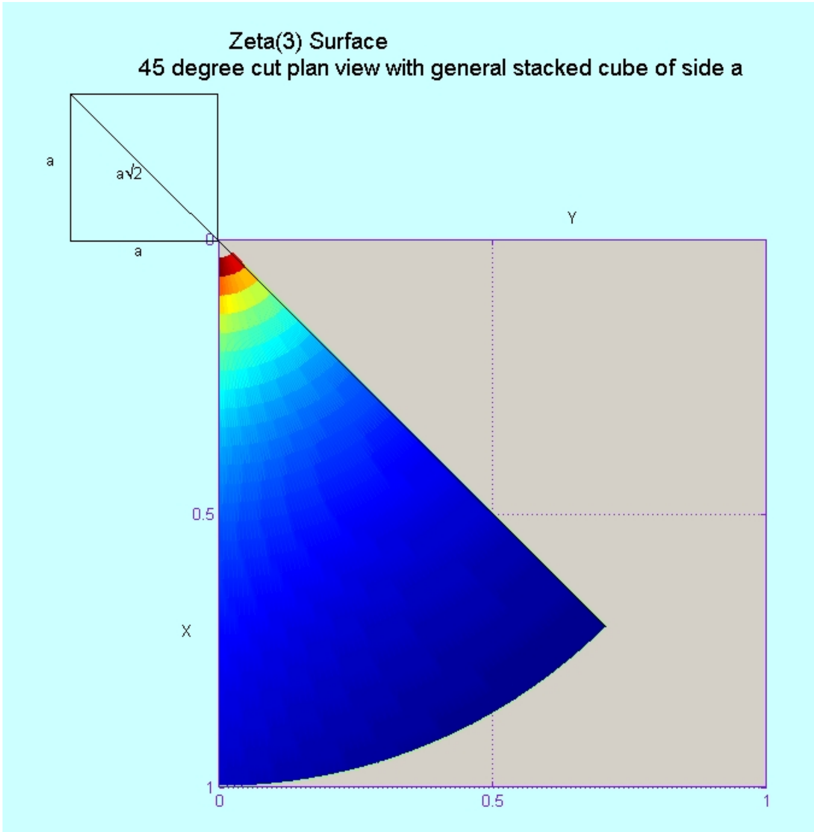


Figure 6. *Plan view of the zeta(3) surface along with a section of the stacked cube.*

The same figure but viewing the x=y plane is shown below. The cube sections now become squares. The first 3 stacked squares are shown on the right. The area B under the plane curve has previously been given as $\varsigma(2)/\sqrt{2}$. However stacking the squares of sections shown gives $\sqrt{2}.1.1+\sqrt{2}.\frac{1}{2}.\frac{1}{2}+\sqrt{2}.\frac{1}{3}.\frac{1}{3}+.... = \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{n^2} \;=\sqrt{2}\zeta(2)$ which is a factor 2 too large. So there is not a direct correspondence.

Generally, for any angle θ, the hypotenuse of the stacked squares grow as $[\sin(\theta)+\cos(\theta)]$ whereas the plane area reduces as $1/[\sin(\theta)+\cos(\theta)]$ and therefore the stacked squares are a factor $[\sin(\theta)+\cos(\theta)]^2$ larger than the plane area. There is only correspondence for $\theta = 0°$ and $\theta = 90°$ when both are $\varsigma(2)$.

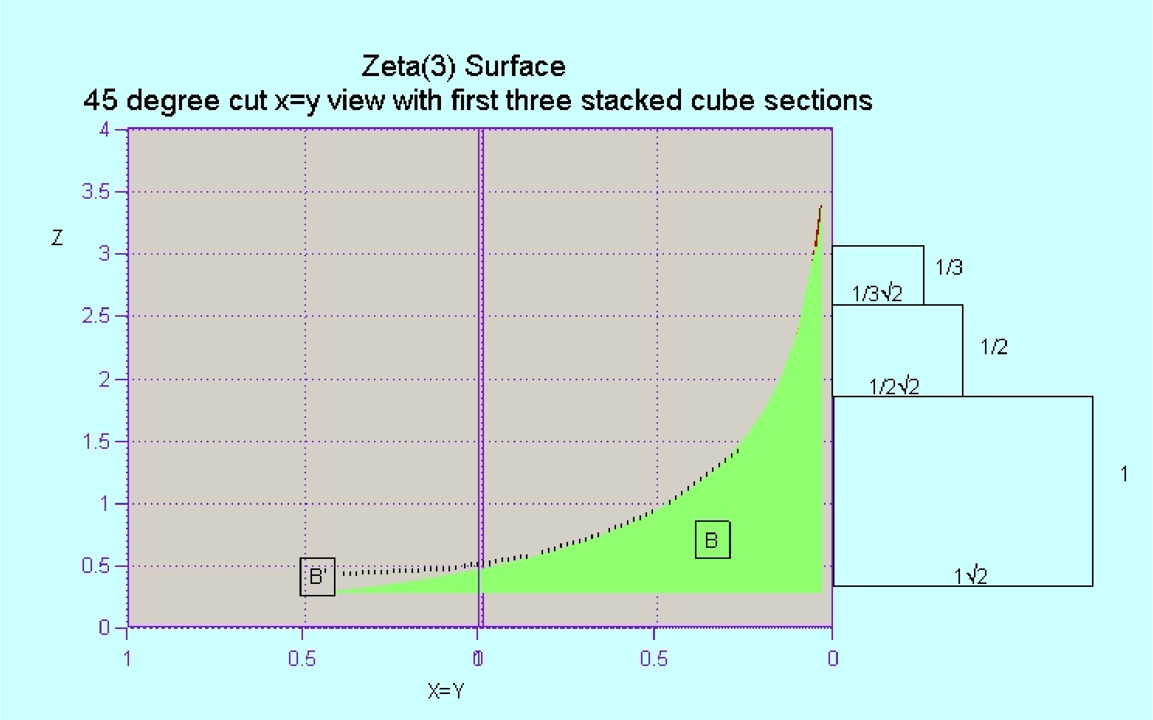


Figure 7. *The x=y section of the zeta(3) surface along with a section of the stacked cubes.*

So even though the volume under the surface of $-\ln(1-e^{-r(\cos(\theta)+\sin(\theta))})$ corresponds to the volume of the stacked cubes, which is $\zeta(3)$, there is no such correspondence between the areas under the radial sections of this surface and the corresponding summed areas of the rectangular sections of the cubes.

In order to make the plane area as large as the stacked rectangles, illustrated by B, we would require the height function to be $-\ln(1-e^{-r/\sqrt{2})})$ rather than $-\ln(1-e^{-\sqrt{2}r)})$ which gives the required plane area $\int_0^\infty -\ln(1-e^{-r/\sqrt{2})})dr = \sqrt{2}\varsigma(2)$, or generally for any θ,

$\int_0^\infty -\ln(1-e^{-r/(\cos(\theta)+\sin(\theta))})dr$ which is $[\sin(\theta)+\cos(\theta)]\varsigma(2)$.

Using this height function, however, now gives the volume under this surface as $\int_0^{\pi/2}\int_0^\infty -\ln(1-e^{-r/(\cos(\theta)+\sin(\theta))})rdrd\theta$ which is $(\pi/2+1)\varsigma(3)$ (from Mathematica) a factor $(\pi/2+1)$ too large for the stacked cubes. To see what this new surface looks like three views are shown below.

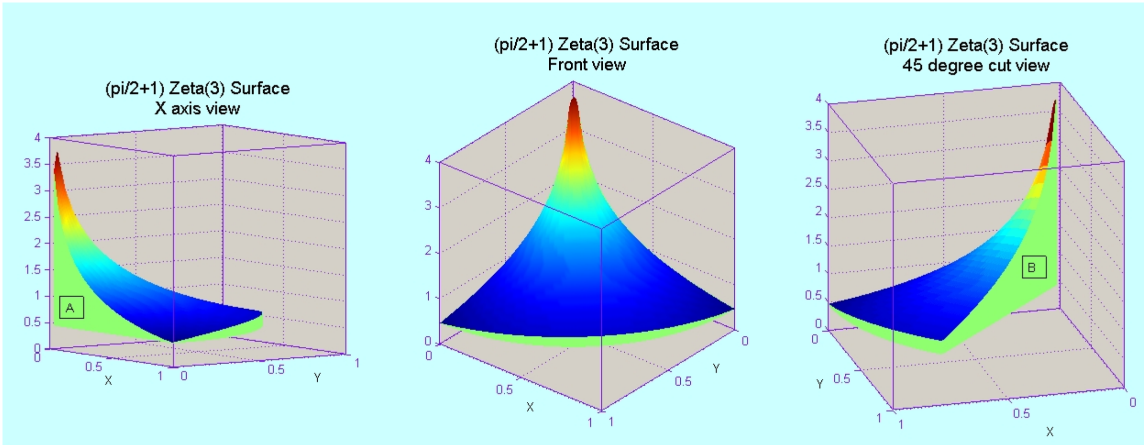


Figure 8. *The surface with a radial height function* $-\ln(1-e^{-r/(\sin\theta+\cos\theta)})$.

For a given radius this surface now peaks at $\theta = 45°$ whereas previously the $\varsigma(3)$ surface dipped. The area of section A is still $\varsigma(2)$ but of course the area of B is now $\sqrt{2}\varsigma(2)$ the same as the stacked square section.

So the formulations can either give the correct surface for piled cubes or the correct plane sections of the surface corresponding to the stacked rectangles of the cube sections.

4(ii) Comparison of shapes

For comparison the surface of revolution for the plane area A of $\varsigma(2)$ is

$\int_0^{\pi/2}\int_0^{\infty} -\ln(1-e^{-r}) r\,dr\,d\theta$ which is $\pi/2 \cdot \varsigma(3)$. This is shown below.

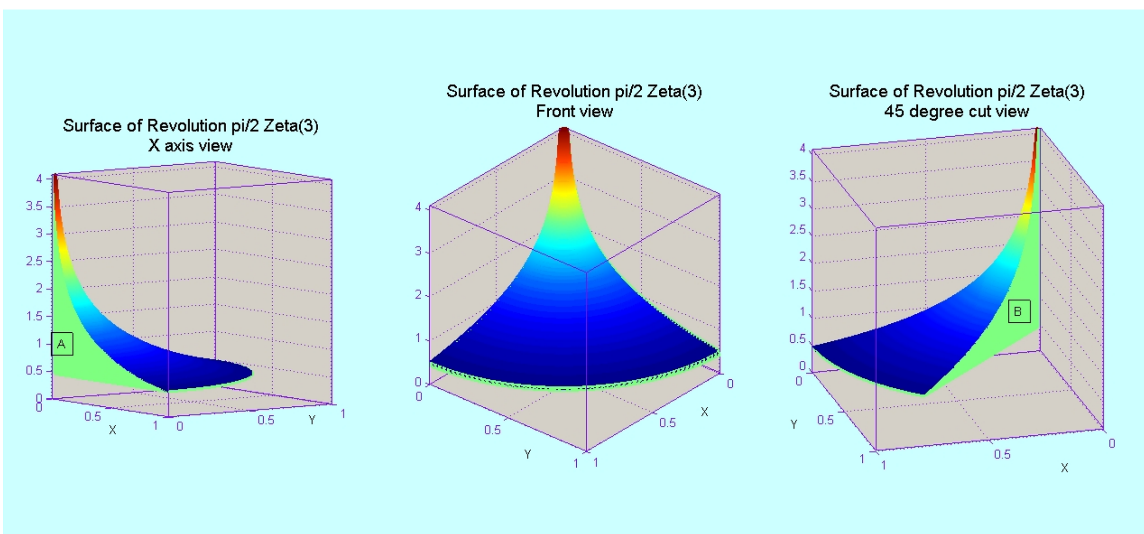


Figure 9. *Surface of revolution of plane curve with area* $\zeta(2)$.

There is now no dipping or peaking of the radial sections with angle so that the area under each radial section, and in particular B shown on the right, is still $\zeta(2)$.

The variations in the plot shapes can be seen by combining them as below. This shows the functions ($Log \equiv \ln$) to produce the shapes described above and the volume from integrating those functions. For clarity two views are shown for angles up to $45°$ and r$\le$2. The volumes are integrals over the limits r 0 to ∞.

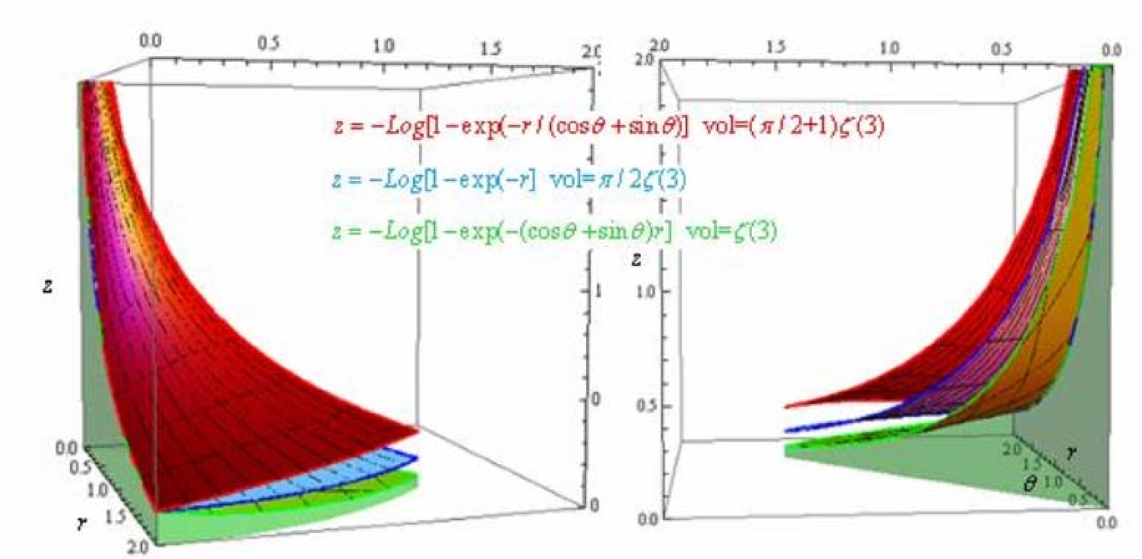


Figure 10. *The volumes below the surfaces give functions of* $\zeta(3)$. *The surfaces are cut-off for r*$\le$*2. Log is ln.*

If we subtract the blue volume from the red volume we get another volume of $\zeta(3)$ given by,

$$\zeta(3)=\int_0^{\pi/2}\int_0^{\infty} r[-\ln(1-e^{-r/(\cos\theta+\sin\theta)})+\ln(1-e^{-r})]drd\theta$$

Two views of this volume for r$\le\sqrt{2}$ are shown below. The view on the right is cut-off at $\theta = 45°$ for clarity.

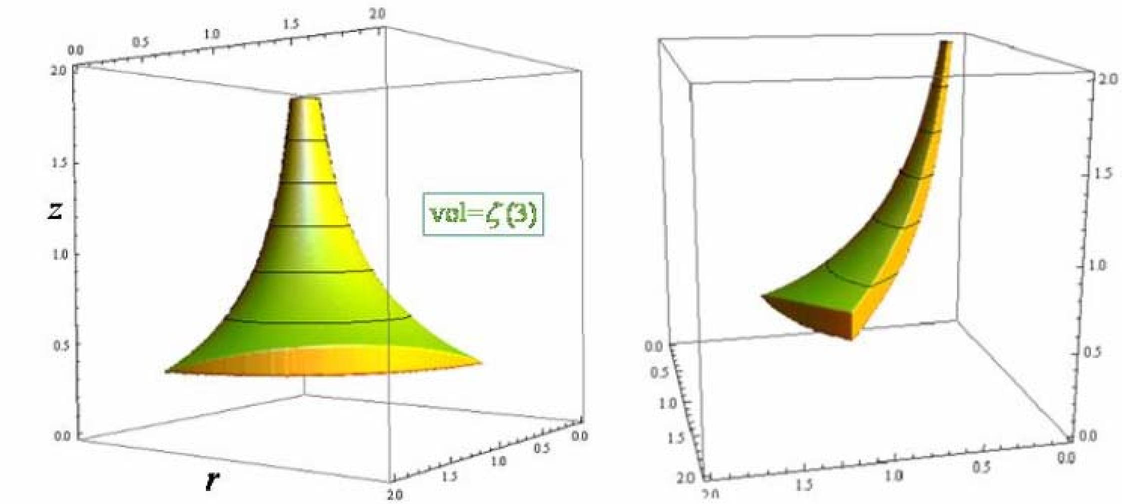


Figure 11. *Part of an alternative shape with volume* $\varsigma(3)$.

5 The mechanics of $\varsigma(3)$

Here we will relate $\varsigma(3)$ to the centroid of the graph $y=-\ln(1-\exp(-x))$. The centroid is the same here as the center of mass since the plane figure has uniform density. The centroid is the first moment of area (the integrand is multiplied by x^1).

5(i) $\varsigma(3)$ derived from the centroid of a plane figure

The x centroid, $\bar{x}$, of the plane figure A of the function, y, used in the definition of $\zeta(2)$ $y=-\ln(1-\exp(-x))$ is plotted below,

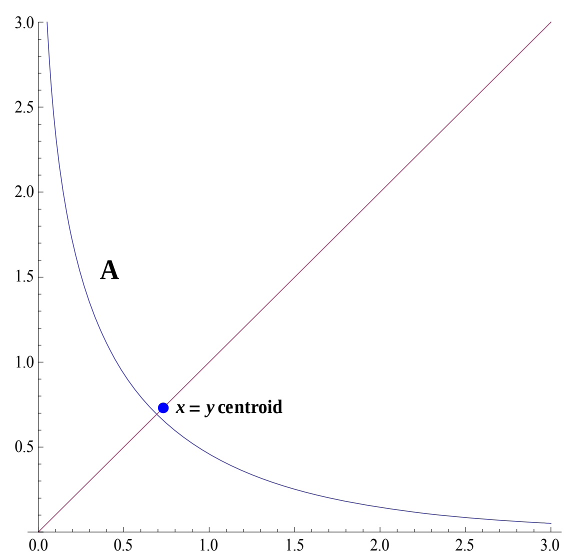


Figure 12. *Centroid of* $y=-\ln(1-\exp(-x))$.

It is computed from the standard formula,

$$\bar{x} = \frac{\int_0^\infty -\ln(1-\exp(-x)xdx}{\int_0^\infty -\ln(1-\exp(-x)dx}$$. The integrals give $\bar{x} = \frac{\varsigma(3)}{\varsigma(2)} \approx 0.73$ so therefore $\varsigma(3) = \bar{x}\varsigma(2)$

This opens up a way to approximate $\varsigma(3)$ by experimentally finding the centroid of the plane figure A. Note: because of the symmetry of the figure A, the x and y centroids are equal and both lie on the line $y = x$ as shown, which is a distance of $\sqrt{2}\bar{x}$ along this line from the origin.

5(ii) Experimental determination of $\varsigma(3)$

The function of the plane figure A which is $y = -\ln(1-\exp(-x))$ was plotted using Matlab and printed onto thick paper making sure there was equal scaling on the axes. The figure A was carefully cut out using scissors which due to the thinness of the paper limited the practical range of the x and y axes from about 0 to 4. The cut-out plane figure is shown below.

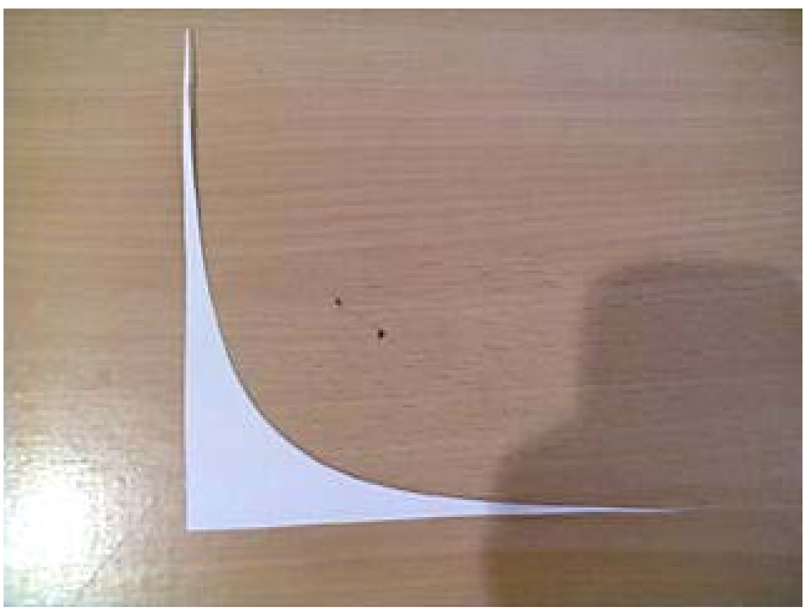


Figure 13. *Cut-out of plane figure A with x and y from 0 to about 4.*

The measurement of the centroid was achieved by pinning figure A to cardboard and using a plumbline to draw a line on A as shown below. Three measurements produced three lines drawn on A, each an estimate of the centroid, $\bar{x}$.

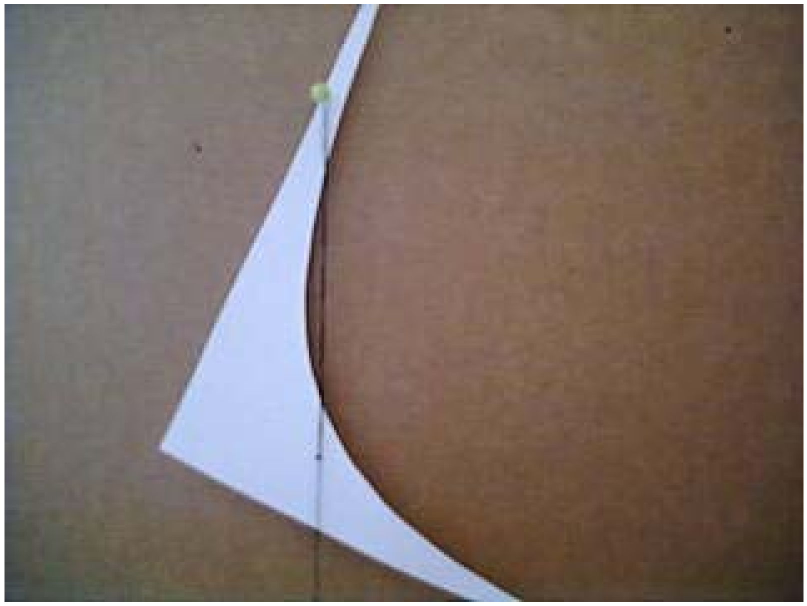


Figure 14. *Measuring the centroid.*

The measured lines on A were transcribed onto an identical printed figure of A except not cut-out. This was done by overlaying the two plots. By symmetry the centroid lies on the line y = x. The intersection of the x component of the measured lines on the figure with y = x gives three estimates for the centroid as shown below.

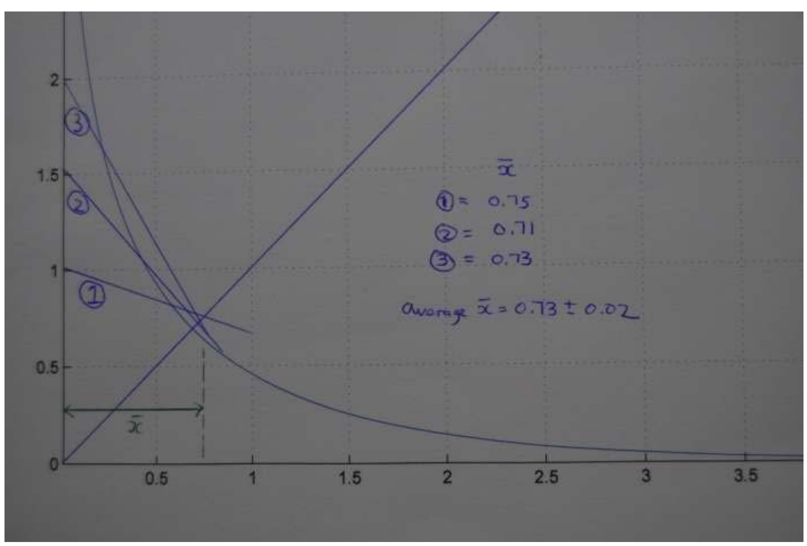


Figure 15. *Three measured estimates of the centroid.*

The results give a measured centroid $\bar{x} = 0.73 \pm 0.02$ and since $\varsigma(3) = \bar{x}\varsigma(2)$ the measured estimate of $\varsigma(3) \approx 1.20 \pm 0.03$ as compared to its value to 3 decimal places $\varsigma(3) = 1.202$.

We can also look at the second moment of area (the integrand is multiplied by x^2) of this figure which is also called the moment of inertia.

5(iii) $\varsigma(3)$ derived from the moment of inertia of the plane figure

The moment of inertia of the above plane figure A about the y axis is given by;

$$I_y = \int_0^\infty -\ln(1-\exp(-x)x^2 dx = 2\varsigma(4)$$

Using the parallel axis theorem the moment of inertia of A about its x centroid is;
$I_{cent} = I_y - M\bar{x}^2$, where using the centroid result above and for simplicity taking mass density M as 1 gives,

$$I_{cent} = 2\varsigma(4) - \left(\varsigma(3)/\varsigma(2)\right)^2$$

$$\Rightarrow \varsigma(3) = \varsigma(2)\sqrt{2\varsigma(4) - I_{cent}}$$

Again, $\varsigma(3)$ can be estimated from an experimental estimate of the moment of inertia of the plane figure A about its x centroid. This has not been experimentally investigated here.

6 Looking at cuts parallel to the x axis: the polylog

We can look at this $\varsigma(3)$ surface from a different point of view, that is, finding the area under the surface for plane sections parallel to the y axis as shown below.

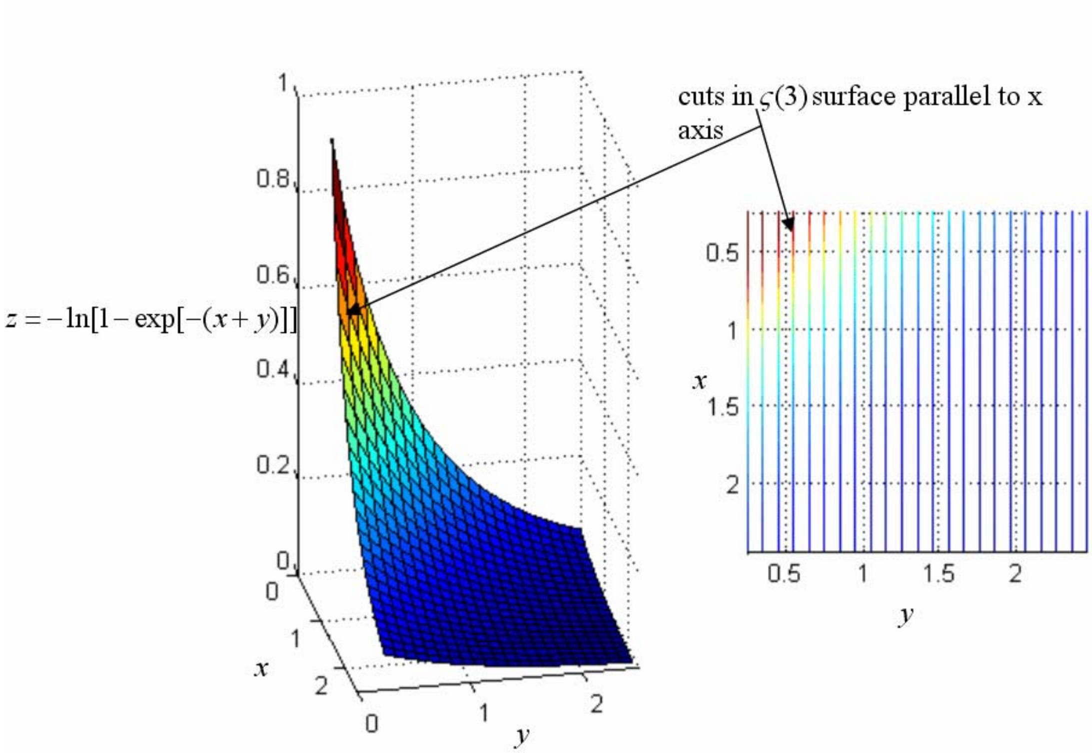


Figure 16. *The area under the $\zeta(3)$ surface for plane sections parallel to the x axis.*

So on the y=0 plane, that is, along the x-axis, the area under the curve as given by Mathematica is of course,

$$\int_0^\infty -\ln(1-e^{-(x+0)})dx = \varsigma(2).$$

As we move along the y-axis, at y=1 the area under the plane curve is now just the translation of the previous curve but starting at x=1 instead of x=0 which is from Mathematica,

$$\int_0^\infty -\ln(1-e^{-(x+1)})dx = Li_2(e^{-1}).$$

This is the polylogarithm of $(2, e^{-1})$ which for n=2 is called the dilogarithm (or dilog) of e^{-1}.

Similarly at the y=2 plane,

$$\int_0^\infty -\ln(1-e^{-(x+2)})dx = Li_2(e^{-2})$$

this area is the dilog of e^{-2}.

For a general planar y=n cut where y is a real number,

$$\int_0^\infty -\ln(1-e^{-(x+n)})dx = Li_2(e^{-n})$$

the area under this planar curve is the dilog of e^{-n}.
So equating the volume under the surface, which is, $\varsigma(3)$ to the summation of the parallel cuts as $dy \Rightarrow 0$, the integral of the dilog, gives another known form for the zeta function,

$$\boxed{\int_0^\infty Li_2(e^{-y})dy = \varsigma(3)}$$

Or more generally,

$$\int_0^\infty Li_{n-1}(e^{-y})dy = \varsigma(n)$$

Both proved by Mathematica.

7 Cutting the $\zeta(3)$ surface on the z axis

We can also look at the $\zeta(3)$ surface in terms of constant z cuts. You can see below that cuts in the $\zeta(3)$ surface along the z axis form triangles.

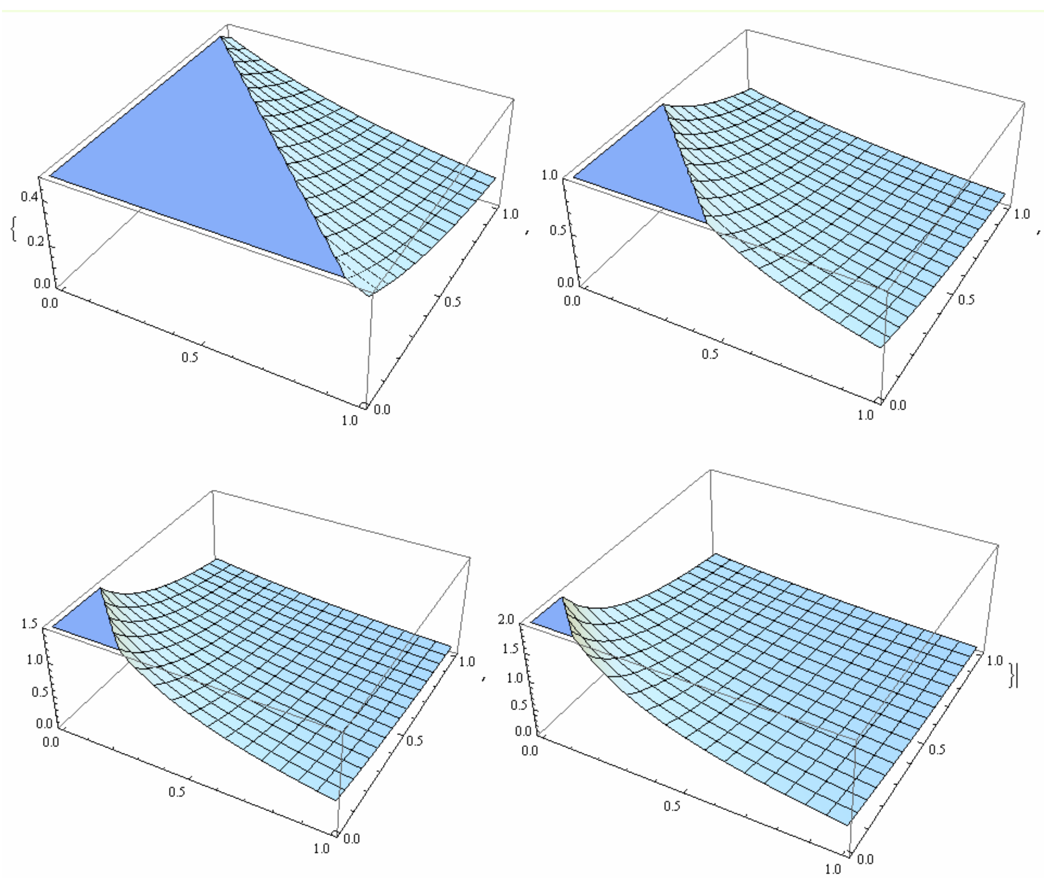


Figure 17. *Z sections of the* $\zeta(3)$ *surface are triangles.*

For a given x the height $z = f(x)$ is $-\ln(1-\exp(-x))$ which is the side-length of the triangle. The area of the triangle is $z^2/2$ which in terms of x is $[\ln(1-\exp(-x))]^2/2$, we have used here $y = x$ since the function is symmetric about that line.

The total volume under the surface is therefore $v = \int_0^\infty [\ln(1-\exp(-x))]^2/2dx$ which is of course $\zeta(3)$, that is,

$$\boxed{\zeta(3) = \int_0^\infty [\ln(1-\exp(-x))]^2/2dx}$$

which is the logarithmic version of Riemann's [2] formula for s=3 as shown in the Appendix.

Next we will make some observations on the limits of the volume of the $\varsigma(3)$ surface from two known theorems

8 The limits on the volume of the piled cubes.

8(i) Using volume of revolution

Here we denote $z = -\ln(1-e^{-at})$,
and we will use Riemann's integral,

$$\zeta(s) = \frac{a}{\Gamma(s)}\int_0^\infty z^{s-1}dt.......[1]$$

A solid of revolution has volume

$$v = \pi\int_0^\infty z^2dx$$

or over a quadrant

$$v = \frac{\pi}{4}\int_0^\infty z^2dx$$

So we can compute the limits on the integral of the volume v from the solid of revolution where

$$v = \int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y)})dxdy.$$

As we saw previously from the shape of this surface, the maximum sectional area is at when y=0 and this produces the largest volume of revolution over the quadrant as

$$\Rightarrow \frac{\pi}{4}\int_0^{\infty}[-\ln(1-e^{-x})]^2 dx$$

and using [1] $\Rightarrow \frac{\pi}{4}\Gamma(3)\varsigma(3) = \frac{\pi}{4}.2.\varsigma(3) = \frac{\pi}{2}\varsigma(3)$

And we have seen before that the minimum radial sectional area of the $\varsigma(3)$ surface occurs along the line y=x that is $\theta = 45°$ and therefore $e^{-(x+y)} = e^{-(x+x)} = e^{-2x}$. Rotating this minimum area gives a minimum volume of revolution,

$$\Rightarrow \frac{\pi}{4}\int_0^{\infty}[-\ln(1-e^{-2x})]^2 dx$$

and using [1] $= \frac{\pi}{4.2}\Gamma(3)\varsigma(3) = \frac{\pi}{4.2}.2.\varsigma(3) = \frac{\pi}{4}\varsigma(3)$

So the limits of the volume of the $\varsigma(3)$ surface $v = \int_0^{\infty}\int_0^{\infty} -\ln(1-e^{-(x+y)})dxdy$ is

$$\therefore \frac{\pi}{4}\varsigma(3) < v < \frac{\pi}{2}\varsigma(3)$$

As we know the actual volume is given by Mathematica as $v = \frac{\pi}{\pi}\varsigma(3) = \varsigma(3)$

8(ii) Using centroids (and Pappus's theorem of volumes)

Here we use the second theorem of the ancient Greek mathematician Pappus of Alexandria (c.300AD) which states that the volume *V* of a solid of revolution generated by rotating a plane figure *F* about an external axis is equal to the product of the area *A* of *F* and the distance *d* traveled by its geometric centroid. That is, *V* = *Ad* **.**

As (i) above the maximum volume is a solid of revolution of the y=0 plane figure.

As seen before the x centroid of this area is

$$\bar{x} = \frac{\int_0^{\infty} -\ln(1-\exp(-x)xdx}{\int_0^{\infty} -\ln(1-\exp(-x)dx}$$

This is

$$\bar{x} = \frac{\varsigma(3)}{\varsigma(2)} \approx 0.73$$

And the y centroid is

$$\bar{y} = \frac{\int_0^\infty [-\ln(1-\exp(-x)]^2 dx}{\int_0^\infty -\ln(1-\exp(-x)dx}$$ which is identical to $\bar{x}$.

Over the quadrant the distance travelled is;

$$d = \frac{\pi}{2}\bar{x}$$

The area under the curve at y=0 is,

$$A = \int_0^\infty -\ln(1-\exp(-x)dx = \varsigma(2)$$

Therefore the volume of revolution of the plane curve

$$V = Ad = \varsigma(2)\frac{\pi}{2}\frac{\varsigma(3)}{\varsigma(2)}$$

which gives, as before, the maximum volume of revolution as $V = \frac{\pi}{2}\varsigma(3) = V_{\max}$.

Again as in section (i) above, the minimum volume of revolution occurs when y=x ($\theta = 45$) $e^{-r(\cos(\vartheta)+\sin(\theta))} = e^{-r\sqrt{2}}$

The x centroid of this area is

$$\bar{x} = \frac{\int_0^\infty -\ln(1-\exp(-\sqrt{2}x)xdx}{\int_0^\infty -\ln(1-\exp(-\sqrt{2}x)dx}$$

This is

$$\bar{x} = \frac{\varsigma(3)}{2} / \frac{\varsigma(2)}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{\varsigma(3)}{\varsigma(2)} \approx 0.52$$

Over the quadrant the distance travelled is;

$$d = \frac{\pi}{2}\bar{x}$$

The area under the curve,

$$A = \int_0^\infty -\ln(1-\exp(-\sqrt{2}x)dx = \frac{\varsigma(2)}{\sqrt{2}}$$

Therefore the volume of revolution of the plane curve

$$V = Ad = \frac{\varsigma(2)}{\sqrt{2}}\frac{\pi}{2}\frac{\varsigma(3)}{\varsigma(2)\sqrt{2}}$$

$$\therefore V = \frac{\pi}{4}\varsigma(3) = V_{\min}$$

So again as in section (i) but by using Pappus's theorem, the limits of the volume of $\varsigma(3)$ surface $v = \int_0^\infty\int_0^\infty -\ln(1-e^{-(x+y)})dxdy$ is

$$\boxed{\therefore \frac{\pi}{4}\varsigma(3) < v < \frac{\pi}{2}\varsigma(3)}.$$

9 Geometry of the polylog of a half: $\mathbf{Li_n(1/2)}$

9(i) The dilog of a half: polylog for n=2, $Li_2(1/2)$

Here we will look at a geometrical derivation for the dilog of ½ $= Li_2(1/2)$.
Using the same plane curve A as before, the area under the blue curve $y = -\ln(1-e^{-x})$ below is of course $A = \int_0^\infty -\ln(1-e^{-x})dx = \varsigma(2) = \frac{\pi^2}{6}$.

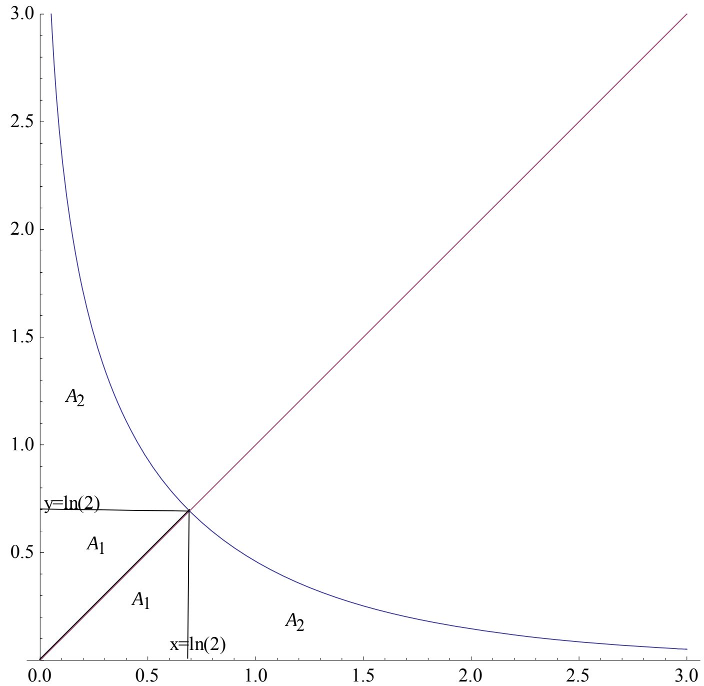


Figure 18. *Geometry of the* $\zeta(2)$ *area.*

It can be seen that the area is symmetric about y=x. The symmetry point on the blue curve is the intersection of y=x with the blue curve $y=-\ln(1-e^{-x})$. This x value is computed from $y=x=-\ln(1-e^{-x})\Rightarrow e^{-x}=1-e^{-x}\Rightarrow 2e^{-x}=1$
$\Rightarrow e^{-x}=1/2\Rightarrow x=-\ln(1/2)\Rightarrow x=\ln(2)$.

Form this into a square of side length $\ln(2)$ and area $\left[\ln(2)\right]^2$ as shown in the figure with half this area $A_1=\frac{\left[\ln(2)\right]^2}{2}$.

It can be seen from the figure that the total area under the curve A is a combination of A_1 and A_2, and therefore,

$$A=2(A_1+A_2)\Rightarrow A_2=A/2-A_1=\pi^2/12-\frac{\left[\ln(2)\right]^2}{2}$$

Now $A_2=-\int_{Log(2)}^{\infty}\ln(1-e^{-x})dx$,

and as shown earlier Mathematica gives;

$$\int_0^{\infty}-\ln(1-e^{-(x+n)})dx=\int_n^{\infty}-\ln(1-e^{-x})dx=Li_2(e^{-n})$$

Therefore

$$\int_0^{\infty}-\ln(1-e^{-(x+\ln(2))})dx=\int_{\ln(2)}^{\infty}-\ln(1-e^{-x})dx=A_2=Li_2(e^{-\ln(2)})=Li_2(1/2),$$

substituting for A_2 gives the geometrically derived known formula for the polylog of 0.5;

$$\boxed{Li_2(1/2)=\frac{\pi^2}{12}-\frac{\left[\ln(2)\right]^2}{2}}$$

9(ii) The trilog of a half: polylog for n=3, $Li_3(1/2)$

We look at the same plane curve A as before. This time we are going to revolve it a quarter turn about the z axis to produce a volume of revolution of total volume V_{tot}.

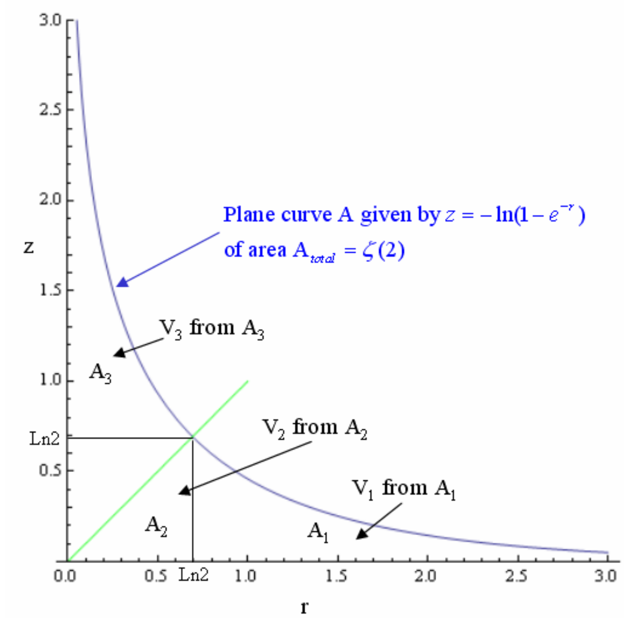


Figure 19. *Splitting the* $\zeta(2)$ *areas up to corresponding volumes of revolution.*

Consider splitting this area up into A_1, A_2, A_3 as shown where A_2 is the triangular region bounded by r=ln[2] and z=r.

The volume of revolution about the z axis of the plane curve A is given by $\frac{\pi}{2}\int_0^\infty rf(r)dr$ where $f(r)=-\ln[1-Exp[-r]$. This gives the total volume of revolution as

$V_{tot}=\frac{\pi}{2}\int_0^\infty -r\ln[1-Exp[-r]dr=\frac{\pi}{2}\zeta(3)$ or implicitly in terms of θ

$\int_0^{\pi/2}\int_0^\infty -\ln[1-Exp[-r]]rdrd\theta=\frac{\pi}{2}\zeta(3)$. This is the volume under the surface shown below.

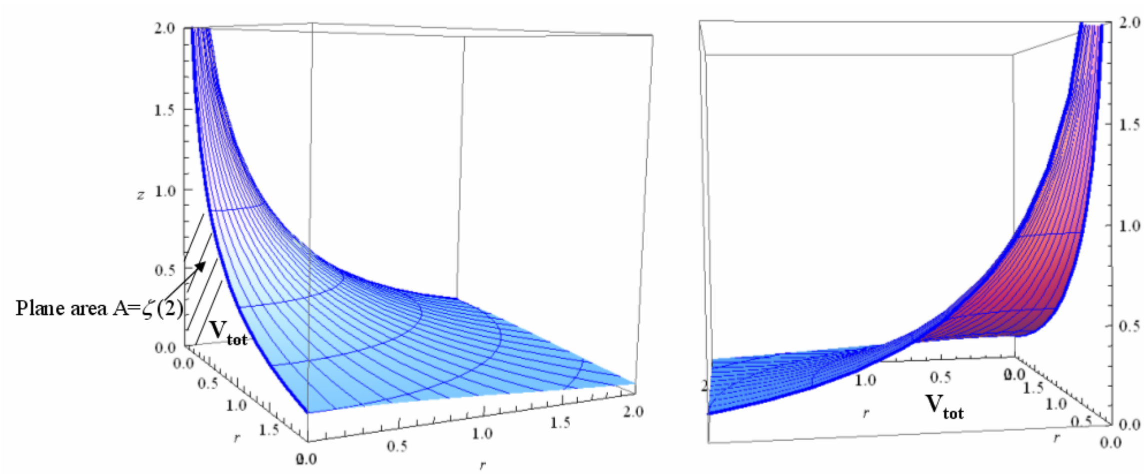


Figure 20. T*he volume of revolution of the plane* $\zeta(2)$ *area.*

The sub-volumes of revolution V_1, V_2, V_3 from the sub-areas A_1, A_2, A_3 are shown below.

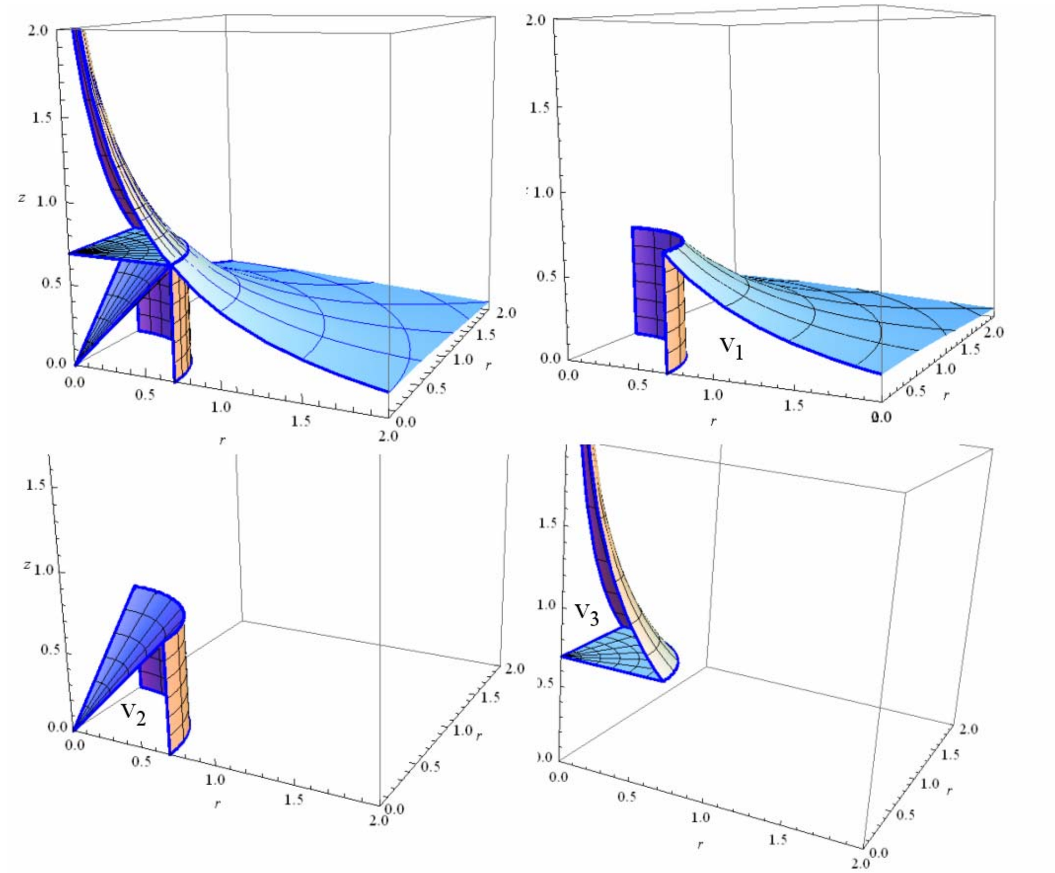


Figure 21. *Volumes of revolution corresponding to the* $\zeta(2)$ *sub-areas.*

Now V_1 has limits ln[2] to ∞ and is given from the general integral expression obtained from Mathematica;

$\int_0^{\pi/2}\int_n^{\infty} -\ln[1-Exp[-r]]rdrd\theta = \frac{\pi}{2}[nLi_2(1/e^n)+Li_3(1/e^n)]$. By substituting ln[2] for n gives,

$$V_1 = \int_0^{\pi/2}\int_{Log[2]}^{\infty} -\ln[1-Exp[-r]]rdrd\theta$$

$$\Rightarrow V_1 = \frac{\pi}{2}[\ln[2]Li_2(1/2)+Li_3(1/2)]$$

since $e^n = e^{\ln[2]} = 2$.

V_2 is the volume of revolution of the triangular area A_2 obtained from the intersection of $z = r$ with the function $z = -\ln[1 - Exp[-r]]$ which occurs at $r = z = \ln[2]$. Therefore $V_2 = \int_0^{\pi/2}\int_0^{\ln[2]} rrdrd\theta = \frac{\pi}{2}[\frac{\ln[2]^3}{3}]$.

To find V_3 we note that Mathematica gives the volume under the curve from r=0 to ln[2] as, $\int_0^{\pi/2}\int_0^{\ln[2]} -\ln[1-Exp[-r]]rdrd\theta = \frac{\pi}{2}[\frac{\ln[2]^3}{3} + \frac{\zeta(3)}{8}]$ which by inspection of the volumes is $V_3 + 2V_2$. Therefore $V_3 = \frac{\pi}{2}[\frac{\ln[2]^3}{3} + \frac{\zeta(3)}{8}] - 2\frac{\pi}{2}[\frac{\ln[2]^3}{3}] \; = \frac{\pi}{2}[\frac{\zeta(3)}{8} - \frac{\ln[2]^3}{3}]$

Equating the volumes gives $V_{tot} = V_1 + 2V_2 + V_3 = \frac{\pi}{2}\zeta(3)$

Therefore $\frac{\pi}{2}\zeta(3) = \frac{\pi}{2}[\ln[2]Li_2(1/2) + Li_3(1/2) + 2\frac{\ln[2]^3}{3} + \frac{\zeta(3)}{8} - \frac{\ln[2]^3}{3}]$

Rearranging for $Li_3(1/2)$ gives,

$$Li_3(1/2) = \frac{7\zeta(3)}{8} - \ln[2]Li_2(1/2) - \frac{\ln[2]^3}{3}$$

and substituting for $Li_2(1/2) = \frac{\pi^2}{12} - \frac{[\ln(2)]^2}{2}$

$Li_3(1/2) = \frac{7\zeta(3)}{8} - \ln[2](\frac{\pi^2}{12} - \frac{\ln[2]^2}{2}) - \frac{\ln[2]^3}{3}$. Rearranging gives the known formula derived geometrically,

$$\boxed{\therefore Li_3(1/2) = \frac{\ln[2]^3}{6} - \frac{\pi^2}{12}\ln[2] + \frac{7\zeta(3)}{8}}$$

Going back to the plane figure we can now see how the symmetrically split $\zeta(2)$ area is transformed into $\zeta(3)$ areas and volumes.

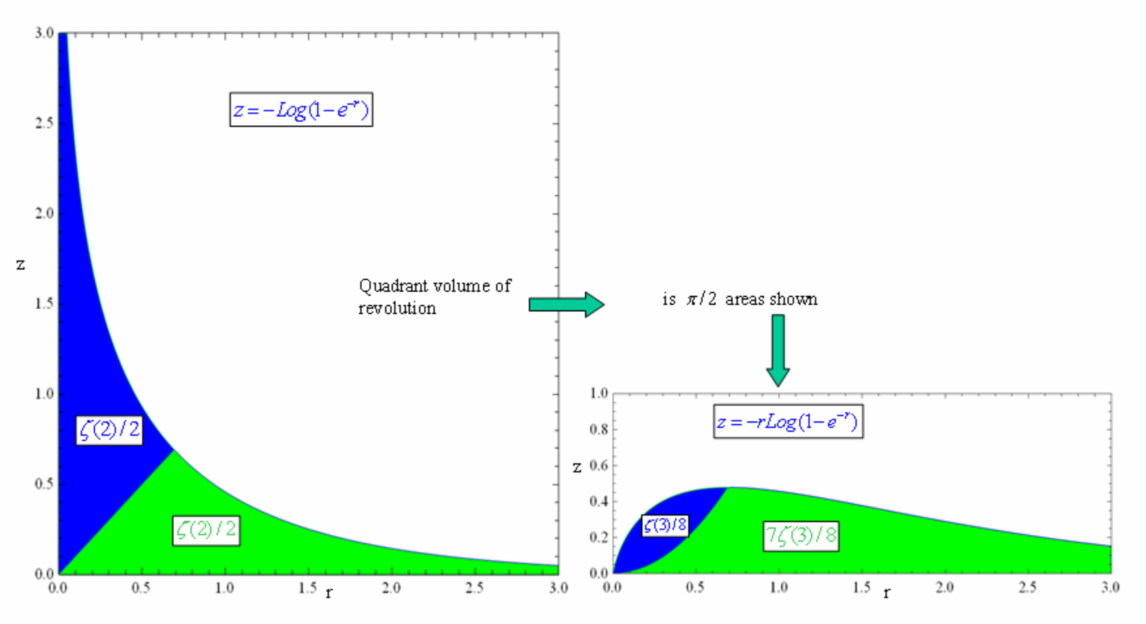


Figure 22. *The plane symmetric areas on the left are transformed by a volume of revolution over a quadrant into the areas and volumes shown on the right. Log is ln.*

The volume of transformation over a quadrant is $\pi/2$ r times the function. We can consider this as an area transformation by variable r followed by the volume transformation by a further factor $\pi/2$. The figure shows how the symmetrically split coloured $\zeta(2)$ areas are transformed. The splitting line $z=r$ in the left plot becomes $z=rr=r^2$ in the right plot. As shown previously the blue $\zeta(2)/2$ area becomes a quadrant of revolution with volume $\pi/2\ \zeta(3)/8$ and the green $\zeta(2)/2$ area is transformed to a volume $\pi/2\ \ 7\zeta(3)/8$.

Appendix

(i) Polylog

A useful function here is the polylog,

$$Li_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}$$

This relates to the Reimann zeta function, ς

$$\varsigma(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$\therefore Li_s(1)=\varsigma(s)$$

Integral representation of dilogarithm (polylog for s=2)

$$Li_2(z) = -\int_0^z \frac{\ln(1-t)}{t}dt$$

$$Li_2(1) = -\int_0^1 \frac{\ln(1-t)}{t}dt = \varsigma(2) = \frac{\pi^2}{6}$$

In terms of the exp function,

$$t = e^{-x},$$

$$\frac{dt}{dx} = -e^{-x} = -t,$$

$$\therefore \frac{dt}{t} = -dx,$$

$$Limits,$$

$$t = 1 \Rightarrow x = 0,$$

$$t = 0 \Rightarrow x = \infty,$$

$$\therefore Li_2(1) = \int_1^0 \frac{\ln(1-t)}{t}dt = -\int_0^\infty \ln(1-e^{-x})dx = \varsigma(2) = \frac{\pi^2}{6}.$$

(ii) Zeta Function

Riemann defined the zeta function as,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}dx \qquad \Re(s) > 1$$

This is put into a logarithmic format used here by a change of variable;

$$x = -\ln\left(1-e^{-t}\right)$$

$$e^x = \frac{1}{1-e^{-t}}$$

$$e^x - 1 = \frac{e^{-t}}{1-e^{-t}}$$

$$dx = \frac{-e^{-t}}{1-e^{-t}}dt$$

$$\frac{x^{s-1}}{e^x-1}dx = \frac{\left[-\ln(1-e^{-t})\right]^{s-1}}{\frac{e^{-t}}{1-e^{-t}}}\frac{-e^{-t}}{1-e^{-t}}dt = -\left[-\ln(1-e^{-t})\right]^{s-1}dt$$

The limits become

$x = 0 \quad t = \infty$

$x = \infty \quad t = 0$

$$= -\int_{\infty}^{0} \left[-\ln(1 - e^{-t}) \right]^{s-1} dt$$

$$\int_{0}^{\infty} \frac{x^{s-1}}{e^{x} - 1} dx = \int_{0}^{\infty} \left[-\ln(1 - e^{-t}) \right]^{s-1} dt$$

$$\Rightarrow \int_{0}^{\infty} \left[-\ln(1 - e^{-t}) \right]^{s-1} dt = \int_{0}^{\infty} \frac{x^{s-1}}{e^{x} - 1} dx = \Gamma(s)\zeta(s)$$

That is

$$\boxed{\Rightarrow \zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \left[-\ln(1 - e^{-t}) \right]^{s-1} dt}$$

Or more generally we will be using a scaling of t by factor a;

Again following the previous method, by change of variable this becomes:

$$x = -\ln\left(1 - e^{-at}\right)$$

$$e^{x} = \frac{1}{1 - e^{-at}}$$

$$e^{x} - 1 = \frac{e^{-at}}{1 - e^{-at}}$$

$$dx = \frac{-ae^{-at}}{1 - e^{-at}} dt$$

$$\frac{x^{s-1}}{e^{x} - 1} dx = \frac{\left[-\ln(1 - e^{-at}) \right]^{s-1}}{\frac{e^{-at}}{1 - e^{-at}}} \frac{-ae^{-at}}{1 - e^{-at}} dt = -a\left[-\ln(1 - e^{-at}) \right]^{s-1} dt$$

The limits become

$x = 0 \quad t = \infty$

$x = \infty \quad t = 0$

$$= -a\int_{\infty}^{0} \left[-\ln(1 - e^{-t}) \right]^{s-1} dt$$

$$\int_{0}^{\infty} \frac{x^{s-1}}{e^{x} - 1} dx = a\int_{0}^{\infty} \left[-\ln(1 - e^{-at}) \right]^{s-1} dt$$

$$\Rightarrow \int_0^\infty \left[-\ln(1-e^{-at})\right]^{s-1} dt = \frac{1}{a}\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \frac{\Gamma(s)\zeta(s)}{a}$$

That is,

$$\boxed{\Rightarrow \zeta(s) = \frac{a}{\Gamma(s)}\int_0^\infty \left[-\ln(1-e^{-at})\right]^{s-1} dt}$$

For a=1 and s=2,

$$\int_0^\infty \ln(1-e^{-x})dx = \varsigma(2) = \frac{\pi^2}{6}$$

Reference

1) Mikael Passare: How to compute $\sum \frac{1}{n^2}$ by plotting triangles arXiv:math/0701039v1 : 2007

2) Bernhard Riemann. On the number of prime numbers less than a given quantity. University of Berlin Nov 1859.