Ramanujan Theta Functions: The Route to Chaos

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Abstract

Motivated by the call for a more flexible, easy-to-tune means of generating chaos, the present work elaborates upon the development of the Ramanujan Theta Function as a source of chaos. By setting the variables of this function to sinusoids of competing frequencies the chaotic output signal is generated, with the ratio of the input frequencies serving as control parameter. The characteristics of such chaos generated are studied using the iterative map and bifurcation plots, and the presence of chaos is ascertained using Lyapunov Exponents and Kolmogorov Entropy. Following this, the route from order to chaos of the proposed system are studied using three techniques – phase portrait, Fourier Spectra and wavelet analysis. In the phase portraits, it is seen that for non-chaotic regimes, phase portraits are orderly with definite number of loops, whereas for chaotic regimes, trajectories are spread all over the phase space, suggesting ergodicity and giving the phase portrait a rich, ornamental look. The Fourier spectra highlighted the discrete frequency components in non-chaotic regimes, with well formed sidebands, whereas in chaotic regimes, a lot of new frequency components are seen, giving the spectral profile a 'grassy' appearance. Finally, a hyperbolic wavelet, termed the Solitary Wavelet seen to possess vanishing higher order moments with a negative logarithmic slope, is used as the basis to perform wavelet analysis. The results reveal that the rhythmic periodicity observed in large scale values for non-chaotic regimes is significantly absent for chaotic regimes, with variations in the trends of new pulses emerging alongside the main pulse train. It is seen that the wavelet analyses combine the best features of phase portrait analyses (ergodicity detected by peak sporadicity and pulse variance), and Fourier Spectral analyses (new frequency component generation seen by observing dominance at various scales, and new peaks at lower scales corresponding to new pulses), while revealing additional features such as fractal nature, not seen in the other two analysis tools. In summary, the present article ushers in a novel perspective pertaining to signal oriented chaos, calling for a change in the way bifurcation plots and iterative maps are perceived, as also the means to generate and control such chaos. It is hoped that the wavelet analysis, highlighting both spectral and temporal aspects of the signal, emerges as a reliable and assertive qualitative means to identify, detect and to an extent, characterize the nature of chaos, either stand-alone, or in conjunction with tools such as phase portraits. Progress in such an area will eventually drive chaos analyses away from Lyapunov Exponents, which are most useful in system based chaos where initial conditions are well known, and are at best computed with approximations from output chaotic signals, using methods such as the Rosenstein Algorithm.

1. Introduction

The advancement and development of simulation and visualization technologies in recent decades have enabled a better understanding into the evolution mechanics of various natural and man-made systems. A key offshoot of such understanding, especially in systems governed by nonlinear laws, is the rise and growth of chaos theory. From describing a narrow set of phenomena related to meteorology and celestial mechanics, chaos theory has grown to encompass a wide range of fields and applications, with some works going so far as to pin chaos as the underlying aspect of quantum mechanics [1,2].

In the engineering and signal processing domains, chaos, usually targeted towards developing secure communications, informatics or nonlinear control applications, is usually generated using circuits and systems physically realizing a set of coupled nonlinear partial differential equations, such as the Lorenz system. Such systems, the Chua diode being an example, typically use system based chaos, such as RLC circuits and piecewise linear transfer functions to generate the required nonlinearity and hence drive the system towards chaos [3].

However, the present day demands for chaos induced security and control, coupled with flexibility, miniaturization and ultrafast operating speeds call for a novel and radical approach to generate chaos – signal based chaos generation.

It is in this spirit that the present work explores one of the well known nonlinear special functions, namely the Ramanujan Theta Function (RTF), as a possible means to generate signal based chaos [4]. This is achieved by setting the two variables of the RTF as sinusoidal signals with competing frequencies. With the ratio of input frequencies as the control parameter, the iterative map of such a formation is derived and the bifurcation plot is plotted, revealing patterns of order and chaos. The presence of chaos, as well as its dependence on the control parameter is ascertained and characterized using the Lyapunov Exponent.

Having ascertained the presence of chaos in the RTF, the primary objective of the work is focused upon, namely the study and characterization of the route to chaos. This is done using three powerful analysis tools – the Fourier spectrum of the RTF signal, phase portraits, and Wavelet analysis. The Solitary wavelet, seen to possess higher order moments vanishing with a negative logarithmic slope, is used as the basis wavelet function. The patterns of order and chaos seen herein are in contrast with conventional system based chaos, and among the analysis tools used, the wavelet analysis is seen to provide crucial insights into the nature of the chaotic output signal. The formulations, analysis, and characterization discussed in the present work purports to a radically innovative approach in chaos generation, while also calls for changes in the perception of chaos from a purely system oriented view to a more inclusive perspective.

2. Ramanujan Theta Function - Iterative Maps and Bifurcation Analysis

We start with the general form of the Ramanujan Theta Function (RTF), given as follows:

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}$$

where *a* and *b* are variables with the specific condition that $|ab|<=1$. Since the present work pertains to a signal based chaos, the variables *a* and *b* are represented as sinusoidal signals with frequencies f1 and f2=rf1 respectively, where r represents the ratio of the frequencies r=f2/f1. Consequently, the output *f(a,b)* is represented as a time varying signal X(t). Thus, the output signal is given by

$$X(t) = \sum_{n=-\infty}^{\infty} sin(2\pi f_1 t)^{n(n+1)/2} sin(2\pi r f_1 t)^{n(n-1)/2}$$

From this relation it is seen that the output signal results from a mixing (multiplication) operation of two nonlinearly wave-shaped (exponents of sinusoids) inputs, both mathematical terms giving rise to new frequencies other than f1 and rf1.

To understand the nonlinear evolution and dynamics of the system, an iterative map has to be formed. In order to do this, the derivative of X(t) is found as X'(t). By discretizing X(t) as well as its derivative, the latter is expressed as the difference equation between successive samples of X(n) as X(i+1)-X(i). By rearranging, X(i+1) is obtained as a function of its previous sample X(i) and the derivative of X(i) as follows, depicting the dependence of the current sample on previous samples, and for this reason termed the `Iterative Map' [1].

$$T1(i) = [\pi f_1 n(n+1) sin(2\pi f_1 i)^{(n(n+1)/2)-1} cos(2\pi f_1 i) sin(2\pi r f_1 i)^{n(n-1)/2}]$$

$$T2(i) = [\pi r f_1 n(n-1) sin(2\pi r f_1 i)^{(n(n-1)/2)-1} cos(2\pi r f_1 i) sin(2\pi f_1 i)^{n(n+1)/2}]$$

$$X(i+1) = X(i) + \sum_{n=-\infty}^{\infty} [T1(i) + T2(i)]$$

As seen from the iterative map, the evolution of X depends intricately on r. Thus, a mapping of X as a function of r, termed the `Bifurcation Plot' is the ideal tool to study the evolution and dynamics of the system represented by the RTF, and this is plotted as follows for values of r from 0 to 4.

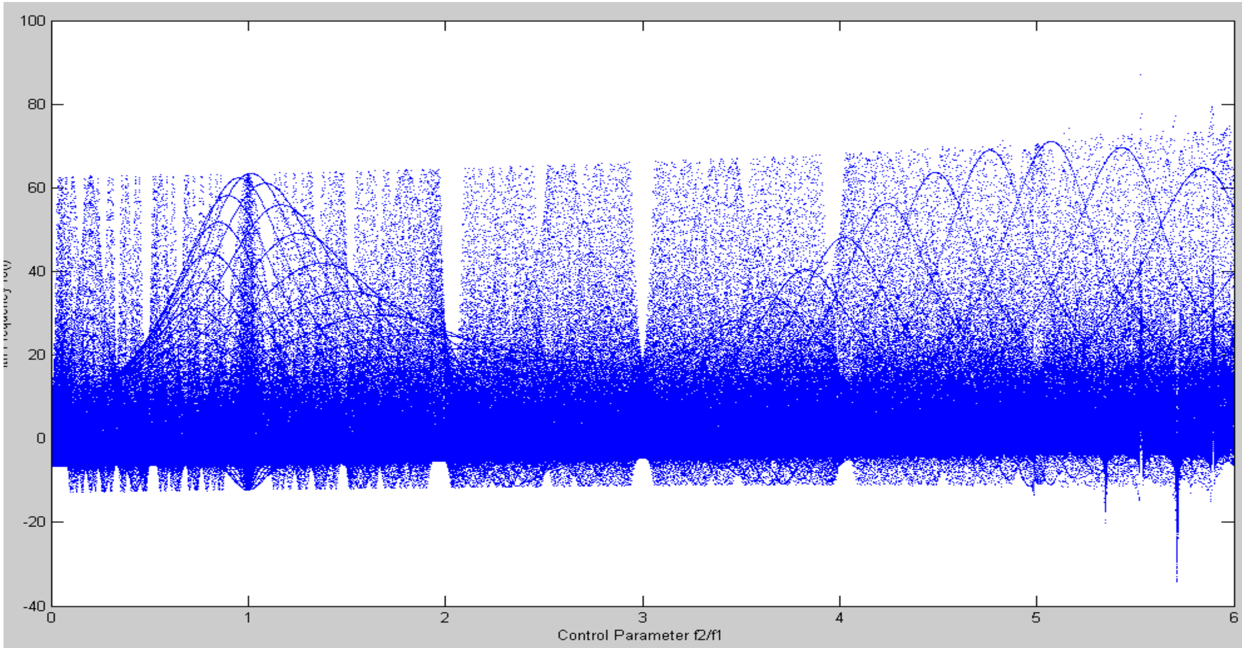

Figure 1 Bifurcation Plot of the RTF

It is clear from the bifurcation plot that the system represented by the RTF indeed shows an evolutionary behavior dependent on r, with certain regions, viewed as `sparse' and `dense' corresponding to non-chaotic and chaotic regimes of operation respectively. Specifically it is seen that the ‘grassy’ nature of the bifurcation plot trends disappear while approaching integer values of r, such as 1,2,3,4…

It is seen that the proposed system radically differs from bifurcation plots seen in conventional chaotic systems, such as the logistic map, since a conventional period doubling, tripling etc route is not followed here. This, by itself calls for a more detailed and inclusive perspective of the concept of an iterative map, as well as the very term “bifurcation” map.

The nature of chaos in X is assertively established by calculating the largest Lyapunov Exponent (LLE), quantifying the system’s sensitive dependence on initial conditions. The Rosenstein’s algorithm is used to compute the Lyapunov Exponents from the time series, where the sensitive dependence is characterized by the divergence samples between nearest trajectories [1]. It is seen that the LLE corresponding to the non integer values of r such as 2.0322 is obtained as 3.36, whereas negative LLE value is obtained for r=1. These values and trends ascertain the presence of chaos and validate the ratio dependent trends observed in the bifurcation plot.

The worthiness of the generated signal as a potential information carrier can be precisely quantified by the Kolmogorov Entropy K, a statistical measure of the uncertainty of the signal. The values of K are computed and tabulated for selected values of r, along with their corresponding LLE Values as follows:

Ratio r	LLE	K (nats/sym)
1.0	-0.45	0.67
1.1	2.85	2.92
1.2	1.08	2.23
1.3	2.76	2.54
1.4	1.57	2.18
1.5	2.66	2.77
1.6	2.91	3.01
1.7	-0.02	0.93
1.8	0.87	2.28
1.9	2.95	3.23

Figure 2 LLE and K values for RTF Chaos

3. The Route to Chaos

With the presence of chaos ascertained, it is now important to understand how the system transits from order to chaos and vice versa, while also characterizing various aspects of the RTF system when in order and when in chaotic regimes.

In order to achieve this, three analysis tools are used, as elaborated below.

A. Phase Portraits

In order to examine the system dynamics for specific values of r, the phase portrait, a plot of the time derivative of a signal X' as a function of the signal X illustrating the phase space dynamics, qualitatively serving as a tool to assess various chaotic parameters such as sensitivity and ergodicity, is used [1]. It is seen from the bifurcation analysis that non-integral r values such as 2.0322, pi (3.14) and the golden ratio (phi) correspond to dense patches indicative of chaos, and to validate this inference, three phase portraits of X corresponding to the three non integral values are plotted alongside six integer values of r ranging from 1 to 6.

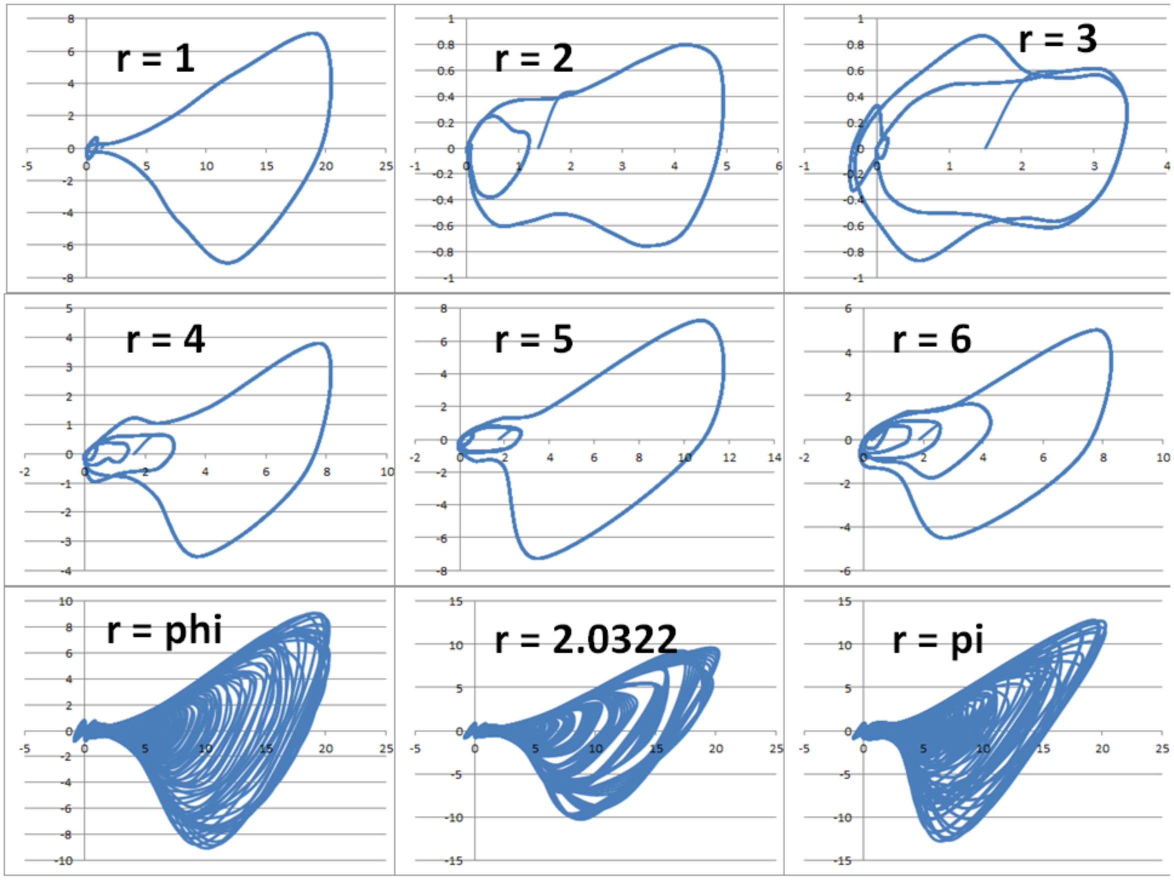

Figure 3 Phase Portraits of the RTF

A clear distinction is observed between the six phase portraits corresponding to the integer r values, and the three non-integer r value phase portraits. In particular, one observes a rich, ornamental appearance for chaotic cases, with the evolution trajectories well spread around the phase space, whereas for non chaotic cases, phase portrait trajectories are limited to finite number of loops. This spreading is indicative of the ergodicity of the system. Among the chaotic cases, the r value of 2.0322 shows less ergodicity than irrational, well established ratios phi and pi.

Thus, it is seen that the phase portrait, while offering an insight into the evolution of the chaotic system for a particular r value, is also used as a first qualitative tool for identifying chaotic and non-chaotic regimes of operation.

B. Fourier Spectra

The second analysis tool used is the Absolute Fourier Spectra, computed using the Fast Fourier Transform function. These are plotted for the nine cases, as follows:

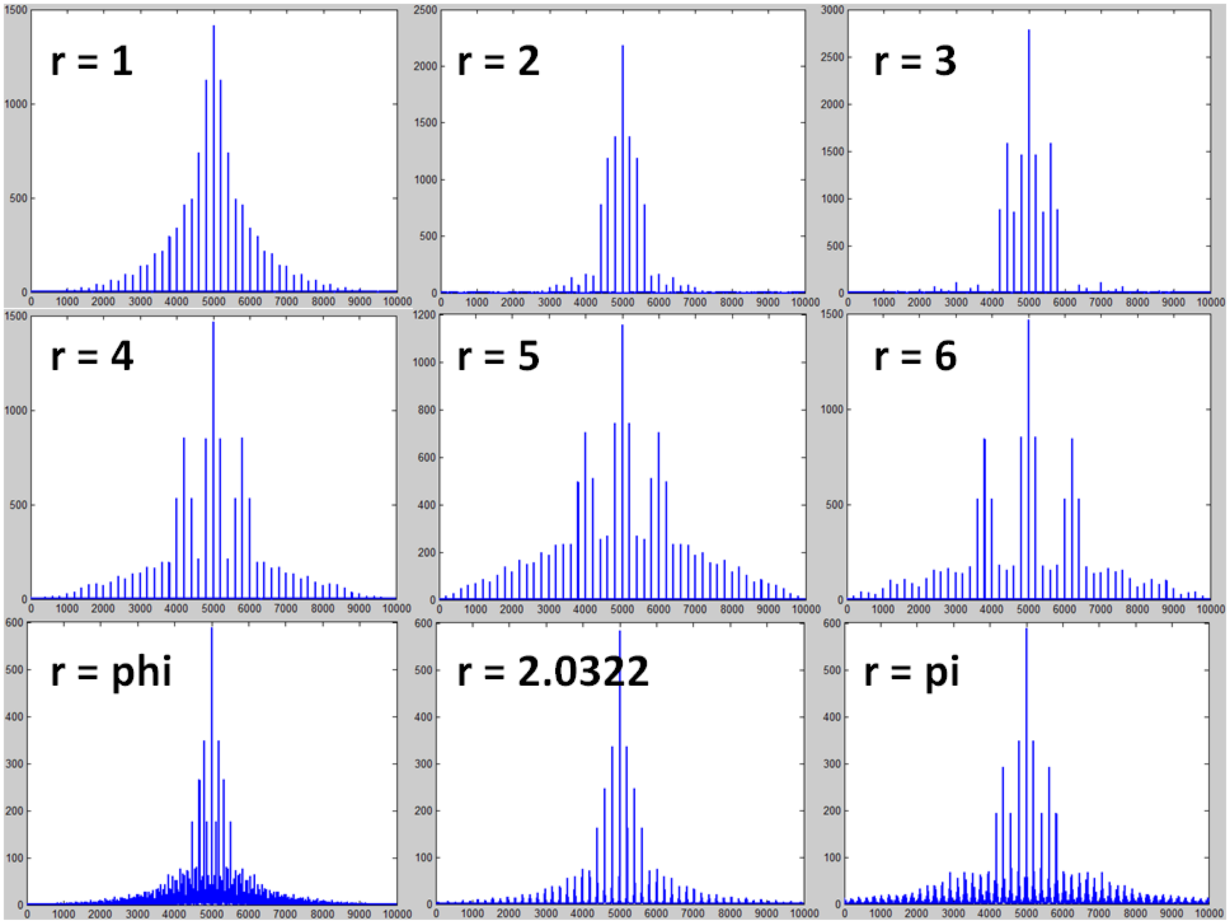

Figure 4 Fourier Spectra of the RTF

From the spectra, the following can be inferred:

1. The RTF has an inherent nonlinearity, which performs wave-shaping on the input sinusoidal signals. This is seen in the r=1 case, where $a=b$, and the spectrum shows an exponentially decaying profile with a significant number of harmonics.
2. As r increases in integer values, one observes the evolution and distinct segregation of sidebands, with the sidebands themselves developing side frequencies on either side, resembling a fractal structure.
3. In all integer r cases, it is seen that the spectra shows discrete components, with particular frequencies, without and leakage or noise in between frequency components.
4. In all three non-integer r values, one observes a stark contrast to the integer cases. Specifically, frequency components are no longer discrete, with a lot of ‘grass’ seen in between frequencies. This suggests that the aperiodicity driving chaos in the system generates a whole array of new frequencies, often expressed as a combination of several harmonics upto large orders.
5. Among the non-integer cases, the presence of ‘grass’ is least dominant in the r=2.0322 case, which, in accordance with the earlier phase portrait observations, has lower chaoticity than pi and phi values of r.

In summary, it is seen that the Fourier Spectra provides valuable information on the evolution and creation of frequency components in the output signal for various chaotic and non-chaotic cases.

C. Solitary Wavelet Analysis

The efficient understanding of the rich dynamics of most real-time signals and waveforms requires a thorough analysis at multiple levels of temporal and spectral resolution. One of the most significant applications of the wavelet based analysis – a comprehensive extension to the frequency-localized Fourier Transform by adding time localization, is in detecting, identifying, characterizing and predicting trends and features in real-time signals, with the key clues in such feature detections being discontinuities and bursts in the time series data, which usually tend to be extremely compact and localized in time [5]. It has been a constant challenge to formulate various mathematical functions as bases for wavelets, which are able to capture such bursts with the least possible level of decomposition, reconstruction and filtering. The ability of a wavelet to capture such bursts effectively translates analytically to the wavelet function, also called the 'Mother Wavelet' having the most possible number of zero higher order central moments. In recent works, a new kind of wavelet, the 'Solitary Wavelet' has been proposed based on the hyperbolic secant function, which is known to possess an extremely smooth and compact structure. It is seen that the higher order moments of this wavelet is closer to zero and vanishes at a rate logarithmically faster than existing wavelets mentioned above. The primary application targeted for this wavelet is the effective detection of bursts in signals.

The primary inspiration for the concept of solitary wavelet arises from the hyperbolic secant (sech) function, popular as a solution to various nonlinear differential equations, and the waveform of the sech function is known to be compact and extremely smooth. Based on these properties, the solitary wavelet is formulated according to the following procedure:

The first step is to define the Scaling Function, also called the 'Father Wavelet' ϕ in continuous time, based on the hyperbolic secant as $\phi(t) = \operatorname{sech}(t)$.

The Solitary Father Wavelet ϕ thus defined is used as the basis to form the Solitary 'Mother Wavelet' ψ, such that the following criteria are satisfied:

1. ψ(t) belongs to a subspace of the space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the space of absolutely and square integrable measurable functions.
2. ϕ(t) and ψ(t) are orthogonal to each other.
3. ψ(t) has zero mean, i.e. the following holds: $\int_{-\infty}^{\infty} \psi(t) dt = 0$
4. ψ(t) has unity square norm, as per the following equation: $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$

It is preferable, but not a mandatory criterion to ensure that ψ(t) possesses a higher number M vanishing moments. In other words, for all m<M, $\int_{-\infty}^{\infty} t^m \psi(t) dt = 0$

The Solitary Mother Wavelet ψ is used to define the solitary daughter wavelets $\psi_{(a,b)}(t)$ in the following fashion with a > 0 denoting the 'scale' and b ϵ ℝ denoting the 'shift' : $\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$

The Father, Mother and Daughter Solitary Wavelets are expressed as discrete signals ϕ(n), ψ(n) and

$\psi_{(a,b)}(n)$ centered around zero. The Father and Mother Wavelet Signals are plotted as follows:

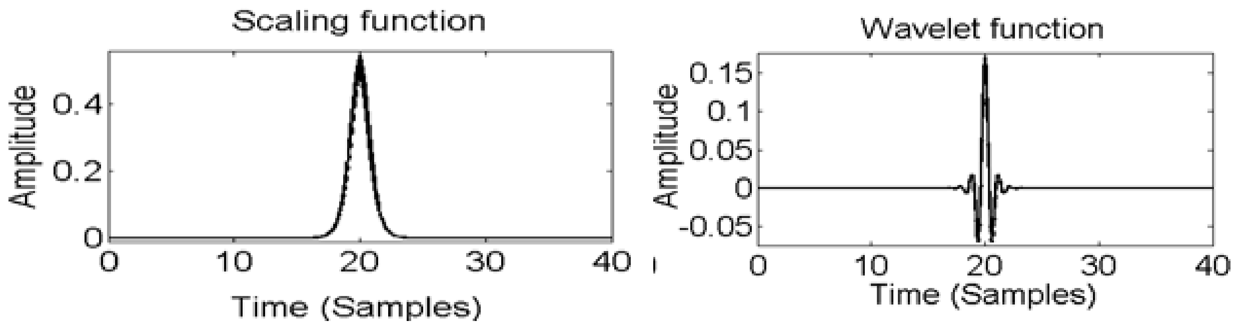

Figure 5 The Solitary Wavelet

In order to investigate and characterize the performance of the solitary wavelet, the moments upto the tenth order of the solitary mother wavelet (SOL) are computed and compared with the corresponding moments of six established wavelets, namely Daubechies 4 (DB4), Biorthogonal 4.4 (BIOR4.4), Reverse Biorthogonal 4.4 (RBIO4.4), Symlet 4 (SYM4), Coiflet 4 (COIF4) and the Discrete Meyer Wavelet (DMEY) [5]. The moments of the various wavelets from the third order onwards are plotted on a logarithmic scale as follows:

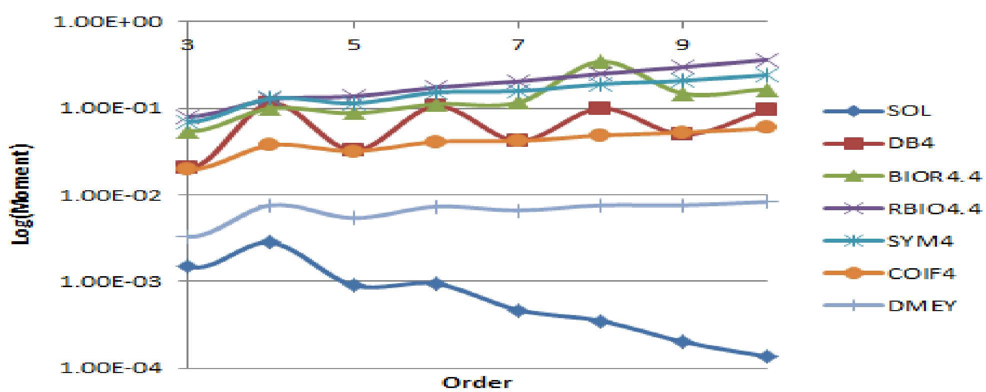

Figure 6 Higher Order Moments of various wavelets

It is clearly seen that while all the other wavelet moments such as the Daubechies and Meyer show an increasing trend, the solitary wavelet moments show a decreasing trend with a negative logarithmic slope. This indicates that the moments of the solitary wavelet rapidly decay and vanish toward zero. This gives the solitary wavelet the exclusive advantages of smoothness, compactness and effective detection of bursts.

The time waveforms, for the six non-chaotic and the three chaotic cases, as plotted as follows:

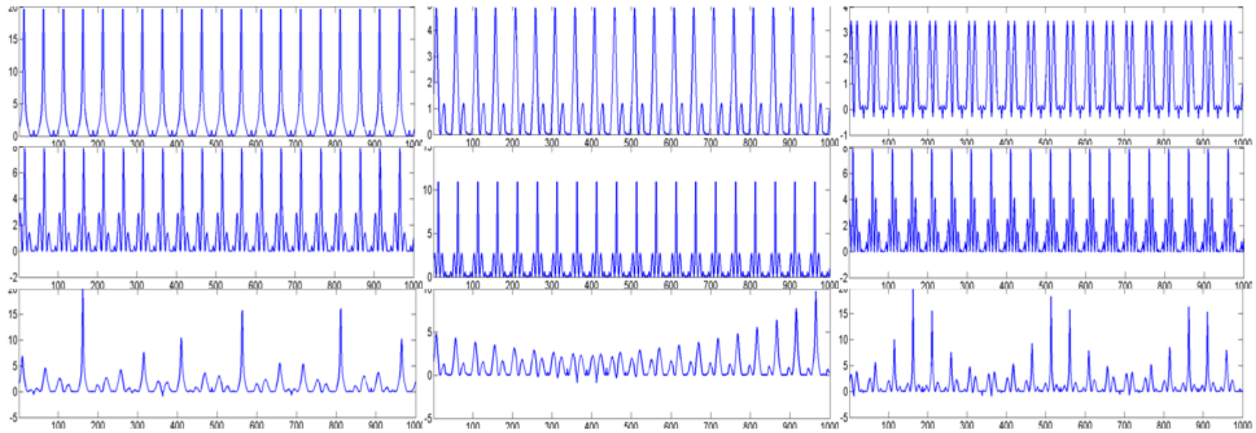

Figure 7 Time Series Waveforms (Top: r=1,2,3; Middle: r=4,5,6; Bottom: r=phi,2.0322,pi)

The corresponding wavelet analyses, plotted as a colormap of coefficients for various shifts and scales, are plotted alongside:

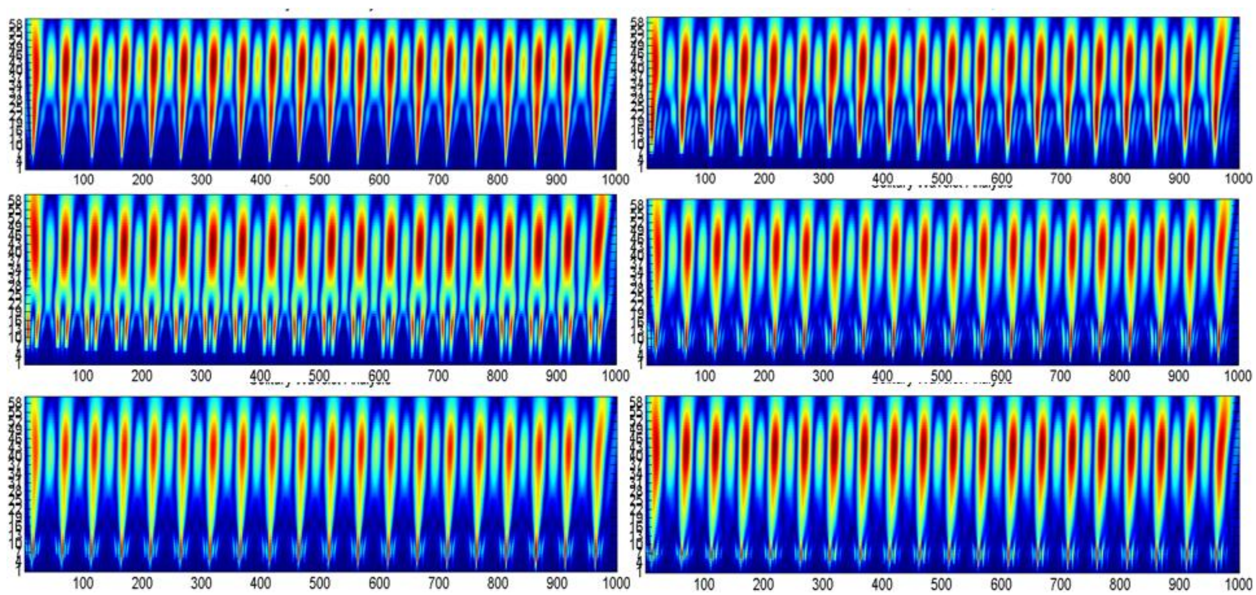

Figure 8 Solitary Wavelet Analysis (Top: r=1,2; Middle: r=3,4; Bottom: r=5,6)

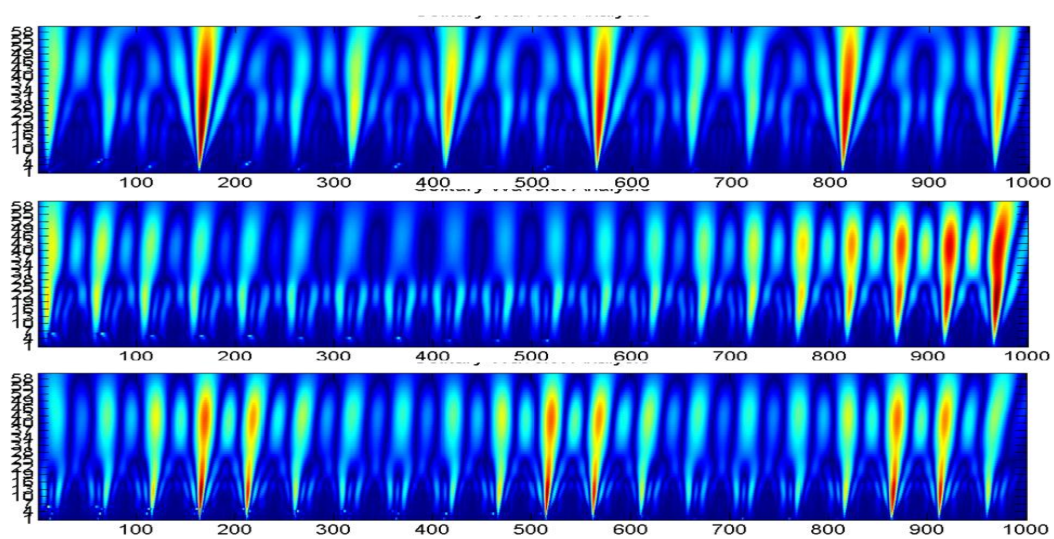

Figure 9 Solitary Wavelet Analysis (Top: r=phi; Middle: r=2.0322; Bottom: r=pi)

The following can be inferred from the analysis results:

1. In all integer r values, a remarkable periodicity, seen by the rhythmic alternation of red and blue, is observed in the coefficient trends pertaining to scales between 35 and 55. The three chaotic cases show a clear absence of such rhythmic periodicity, with peaks in the said coefficient ranges occurring in sporadic bursts, or as a pattern similar to a modulating envelope.
2. At lower scales (between 4 and 14), one observes that as r increases in integer values, new peaks, corresponding to emergence of smaller pulses and bifurcations of the fundamental pulse train in Fig. 7 are observed. The number of peaks is well in accordance with the number of new pulses formed.
3. From Fig. 7, one observes that in the chaotic cases, the number of smaller pulses constantly keep varying, and this change is reflected in the number of peaks in wavelet analysis of Fig. 9.
4. An interesting observation in Fig. 9 in all three chaotic cases is the phenomenon where a large scale peak is followed by bunching of smaller peaks at lower scales, which are followed by bunching of even lower scales and so on, giving a fractal appearance. Such an appearance has been previously detected in wavelet analysis of share revenue and foreign exchange markets.
5. The above mentioned sporadicity at larger scales, as well as variations in pulse trends, as seen in wavelet analyses of chaotic signals assertively emerge as indicative factors of the new frequency components creating a grassy nature in the Fourier Spectral profile.

In summary, it is seen that the wavelet analyses combine the best features of phase portrait analyses (ergodicity detected by peak sporadicity and pulse variance), and Fourier Spectral analyses (new frequency component generation seen by observing dominance at various scales, and new peaks at lower scales corresponding to new pulses), while revealing additional features such as fractal nature, not seen in the other two analysis tools.

4. Conclusion

Motivated by the call for a more flexible, easy-to-tune means of generating chaos, the present work elaborates upon the development of the Ramanujan Theta Function as a source of chaos. By setting the variables of this function to sinusoids of competing frequencies the chaotic output signal is generated, with the ratio of the input frequencies serving as control parameter. The characteristics of such chaos generated are studied using the iterative map and bifurcation plots, and the presence of chaos is ascertained using Lyapunov Exponents and Kolmogorov Entropy. Following this, the route from order to chaos of the proposed system are studied using three techniques – phase portrait, Fourier Spectra and wavelet analysis. In the phase portraits, it is seen that for non-chaotic regimes, phase portraits are orderly with definite number of loops, whereas for chaotic regimes, trajectories are spread all over the phase space, suggesting ergodicity and giving the phase portrait a rich, ornamental look. The Fourier spectra highlighted the discrete frequency components in non-chaotic regimes, with well formed sidebands, whereas in chaotic regimes, a lot of new frequency components are seen, giving the spectral profile a 'grassy' appearance. Finally, a hyperbolic wavelet, termed the Solitary Wavelet seen to possess vanishing higher order moments with a negative logarithmic slope, is used as the basis to perform wavelet analysis. The results reveal that the rhythmic periodicity observed in large scale values for non-chaotic regimes is significantly absent for chaotic regimes, with variations in the trends of new pulses emerging alongside the main pulse train. It is seen that the wavelet analyses combine the best features of phase portrait analyses (ergodicity detected by peak sporadicity and pulse variance), and Fourier Spectral analyses (new frequency component generation seen by observing dominance at various scales, and new peaks at lower scales corresponding to new pulses), while revealing additional features such as fractal nature, not seen in the other two analysis tools. In summary, the present article ushers in a novel perspective pertaining to signal oriented chaos, calling for a change in the way bifurcation plots and iterative maps are perceived, as also the means to generate and control such chaos. It is hoped that the wavelet analysis, highlighting both spectral and temporal aspects of the signal, emerges as a reliable and assertive qualitative means to identify, detect and to an extent, characterize the nature of chaos, either stand-alone, or in conjunction with tools such as phase portraits. Progress in such an area will eventually drive chaos analyses away from Lyapunov Exponents, which are most useful in system based chaos where initial conditions are well known, and are at best computed with approximations from output chaotic signals, using methods such as the Rosenstein Algorithm.

References

[1] Strogatz, Steven H. Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering. Westview press, 2014.

[2] Interpreting Quantum Mechanics using Chaos Theory, viXra:1510.0438 submitted on 2015-10-27 21:14:32

[3] Bilotta, Eleonora, and Pietro Pantano. A gallery of Chua attractors. Singapore: World Scientific, 2008.

[4] Chan, Heng Huat, Zhi-Guo Liu, and Say Tiong Ng. "Circular summation of theta functions in Ramanujan's Lost Notebook." Journal of mathematical analysis and applications 316, no. 2 (2006): 628-641.

[5] Daubechies, Ingrid. Ten lectures on wavelets. Vol. 61. Philadelphia: Society for industrial and applied mathematics, 1992.