3×3-Kronecker Pauli Matrices

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Abstract : The properties of what we call inverse-symmetric matrices have helped us for constructing a basis of $\mathbb{C}^{3\times 3}$ which satisfy four properties of the Kronecker generalized Pauli matrices. The Pauli group of this basis has been defined. In using some properties of the Kronecker commutation matrices, bases of $\mathbb{C}^{5\times 5}$ and $\mathbb{C}^{6\times 6}$ which share the same properties have also been constructed.

Keywords: Kronecker product, Pauli matrices, Kronecker commutation matrices, Kronecker generalized Pauli matrices.

1 Introduction

This paper tries to solve a problem posed in [1], of searching a set $\mathcal{K}_3 = \{\tau_1, \tau_2, \cdots, \tau_9\}$ of nine 3×3-matrices satisfy the following relation with the 3⊗3-Kronecker commutation matrix (KCM),

$$K_{3\otimes 3} = \frac{1}{3}\sum_{i=1}^{9} \tau_i \otimes \tau_i \qquad (1)$$

The usefulness of the Kronecker permutation matrices, particularly the Kronecker commutation matrices (KCMs) in mathematical physics can be seen in [2], [3], [4], [5]. In these papers, the 2⊗2- KCM is written in terms of the Pauli matrices, which are 2×2 matrices, by the following way

$$K_{2\otimes 2} = \frac{1}{2}\sum_{i=0}^{3} \sigma_i \otimes \sigma_i$$

The generalization of this formula in terms of generalized Gell-Mann matrices, which are a generalization of the Pauli matrices, is the topic of [6]. But there are other generalization of the Pauli matrices in other sense than the generalized Gell-Mann matrices, among others the Kibler matrices [7], the Kronecker generalized Pauli matrices, see for example [8]. These last are obtained by Kronecker product of the Pauli matrices. The more general relation giving the $2^k \otimes 2^k$-KCM in terms of the Kronecker generalized Pauli matrices may be seen in [1]. That makes us search for other generalization of the Pauli matrices in this sense, like the generalization of the Gell-Mann matrices to what we call rectangle Gell-Mann matrices in [9]. That is, we search for set of 3×3 matrices which have got some properties of the Kronecker generalized Pauli matrices, which are $2^k \times 2^k$ matrices. We will call these matrices 3×3-Kronecker Pauli matrices (KPMs).

These properties of the Kronecker generalized Pauli matrices or $N \times N$-KPMs $(\Sigma_\mathrm{i})_{0 \le i \le N^2-1}$, with $N = 2^k$ and $\Sigma_i = \sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_k}$, are

i. $(\Sigma_\mathrm{i})_{0 \le i \le N^2-1}$ is a basis of $\mathbb{C}^{N \times N}$

ii.
$$K_{N \otimes N} = \frac{1}{N} \sum_{i=0}^{N^2-1} \Sigma_i \otimes \Sigma_\mathrm{i}$$

iii. $\Sigma_i^+ = \Sigma_i$ (hermiticity)

iv. $\Sigma_i^2 = I_n$ (Square root of unit)

v. $Tr\left(\Sigma_j^+ \Sigma_\mathrm{k}\right) = N\delta_{jk}$ (Orthogonality)

vi. $Tr\left(\Sigma_j\right) = 0$ (Tracelessness)

However, there is no 3×3 matrix, formed by zeros in the diagonal which satisfies both the relations iii. and iv. [1]. Thus, at a first time, for the 3×3-KPMs we do not demand tracelessness.

We will call 3×3-KPMs a set of 3×3 matrices which satisfy, not only the formula (1), but the five properties above, tracelessness vi. moved apart.

Thus, in this paper we will talk at first about KCMs. In the next section, we will talk about what we call inverse-symmetric matrices. These matrices have got interesting properties for constructing the 3×3-KPMs. After, we will give the set of 3×3-KPMs, which are inverse-symmetric matrices. Finally, some way to the generalization will be discussed.

We know that the set of $2^k \times 2^k$-KPMs, up to multiplicative phases: 1, -1, i and $-i$, is the Pauli group for the usual matrix product. Thus, we will try to define the Pauli group of the 3×3-KPMs.

Some calculations such as the expression of 3⊗3-KCMs request calculations with software. We have used SCILAB for those calculations.

2 Kronecker Commutation matrices

The Kronecker product of matrices is not commutative, but there is a permutation matrix which, in multiplying to the product, commutes the product. We call such matrix Kronecker commutation matrix.

Definition 1 The permutation matrix $K_{n\otimes p} \in \mathbb{C}^{np\times np}$, such that for any matrices $a \in \mathbb{C}^{n\times 1}$, $b \in \mathbb{C}^{p\times 1}$

$$K_{n\otimes p}(a\otimes b) = b\otimes a$$

is called $n\otimes p$-Kronecker commutation matrix, $n\otimes p$-KCM.

$$K_{2\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_{3\otimes 3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For constructing $n\otimes p$ –KCM, we can use the following rule [13].

Rule 2 Let us start in putting 1 at first row, first column, then let us pass into the second column in going down at the rate of n rows and put 1 at this place, then pass into the third column in going down at the rate of n rows and put 1, and so on until there is only for us n-1 rows for going down (then we have obtained as number of 1 : p). Then pass into the next column which is the (p + 1)-th column, put 1 at the second row of this column and repeat the process until we have only n-2 rows for going down (then we have obtained as number of 1 : 2p). After that pass into the next column which is the (2p + 2)-th column, put 1 at the third row of this column and repeat the process until we have only n-3 rows for going down (then we have obtained as number of 1 : 3p). Continuing in this way we will have that the element at n×p-th row, n×p-th column is 1.

Proposition 3 Suppose

$$K_{n\otimes m} = \sum_{i,j=1}^{s} A_i \otimes B_j$$

and

$$K_{p\otimes q} = \sum_{k,l=1}^{r} C_k \otimes D_l$$

with the A_i's are elements of $\mathbb{C}^{m\times n}$, the B_j's are elements of $\mathbb{C}^{n\times m}$, the C_k's are elements of $\mathbb{C}^{q\times p}$and the D_l's are elements of $\mathbb{C}^{p\times q}$. Then,

$$K_{np\otimes mq} = \sum_{i,j=1}^{s} \sum_{k,l=1}^{r} A_i \otimes C_k \otimes B_j \otimes D_l$$

Proof

Let (a_α), (c_β), (b_γ) and (d_δ) be, respectively, bases of $\mathbb{C}^{n\times 1}$, $\mathbb{C}^{p\times 1}$, $\mathbb{C}^{m\times 1}$ and $\mathbb{C}^{q\times 1}$. Then, $\left(a_\alpha \otimes c_\beta \otimes b_\gamma \otimes d_\delta\right)$ is a basis of $\mathbb{C}^{npmq\times 1}$. It is enough to prove that

$$\sum_{i,j=1}^{s} \sum_{k,l=1}^{r} A_i \otimes C_k \otimes B_j \otimes D_l \left(a_\alpha \otimes c_\beta \otimes b_\gamma \otimes d_\delta\right) = b_\gamma \otimes d_\delta \otimes a_\alpha \otimes c_\beta$$

We use the proposition 11. From

$$\sum_{i,j=1}^{s} A_i \otimes B_j \left(a_\alpha \otimes b_\gamma\right) = b_\gamma \otimes a_\alpha$$

we have

$$\sum_{i,j=1}^{s} \sum_{k,l=1}^{r} A_i a_\alpha \otimes C_k c_\beta \otimes B_j b_\gamma \otimes D_l\, d_\delta = \sum_{k,l=1}^{r} b_\gamma \otimes C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta$$

$$= b_\gamma \otimes \sum_{k,l=1}^{r} C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta$$

Moreover

$$\sum_{k,l=1}^{r} C_k c_\beta \otimes a_\alpha \otimes D_l d_\delta = d_\delta \otimes a_\alpha \otimes c_\beta$$

and that ends the proof.

3 Inverse-symmetric matrices

In this section, we introduce what we call inverse-symmetric matrices. We think that this term will be useful for the continuation.

Definition 4 Let us call inverse-symmetric matrix an invertible complex matrix $A = \left(A_j^i\right)$ such that $A_i^j = \frac{1}{A_j^i}$ if $A_j^i \neq 0$.

Unlike an antisymmetric matrix, for the non-zero element of the matrix, its symmetric with respect to the diagonal is its inverse.

Example 5 The 2×2 unit matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are inverse-symmetric matrices.

Proposition 6 Let $A = \left(A_j^i\right)$ and $B = \left(B_j^i\right)$ be inverse-symmetric matrices.

Then, $A \otimes B$ is an inverse-symmetric matrix.

Proof. $A \otimes B$ is an invertible matrix. If $(A \otimes B)_{jl}^{ik} \neq 0$ if, only if $A_j^i \neq 0$ and $B_l^k \neq 0$. Its symmetric with respect to the diagonal is $(A \otimes B)_{ik}^{jl} = A_i^j B_k^l = \frac{1}{A_j^i} \frac{1}{B_l^k} = \frac{1}{(A \otimes B)_{jl}^{ik}}$.

Proposition 7 For any $n \times n$ inverse-symmetric matrix A, with only n non zero elements, $A^2 = I_n$.

Proof. Let $A = \left(A_j^i\right)_{1 \leq i,j \leq n}$ and then $A^2 = \left(\sum_{i=1}^n A_k^i A_j^k\right)_{1 \leq i,j \leq n}$. In order that A is invertible, there must be only a non zero element in each row and in each column. Let A_m^i be the non zero element in the row i and A_j^p the non zero element in the column j. $(A^2)_j^i = A_m^i A_j^m + A_p^i A_j^p$.

If $i \neq j$, $A_i^m = \frac{1}{A_m^i} \neq 0$ and $A_j^p \neq 0$, thus $A_j^m = 0$ and $A_p^j = 0$. Hence, $(A^2)_j^i = 0$.

If $i = j$, $(A^2)_j^i = \sum_{i=1}^n A_k^i A_j^k = A_m^i A_j^m = 1$.

Therefore, let us take some inverse-symmetric matrices formed by only three non zero elements for the nine 3×3 matrices we would like to search for, in order that iv. is satisfied.

4 Kronecker generalization of the Pauli matrices

The Kronecker generalized Pauli matrices are the matrices $\left(\sigma_i \otimes \sigma_j\right)_{0\le i,j\le 3}$[10], [11], $\left(\sigma_i \otimes \sigma_j \otimes \sigma_k\right)_{0\le i,j,k\le 3}$, $\left(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}\right)_{0\le i_1,i_2,\cdots,i_n\le 3}$ [8] obtained by Kronecker product of the Pauli matrices and the 2×2 unit matrix. According to the propositions above, they are inverse-symmetric matrices and share many of the properties of the Pauli matrices: basis of $\mathbb{C}^{2^n\times 2^n}$, ii., iii., iv., v. and vi. in the introduction, for [8].

Denote the set of $\left(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}\right)_{0\le i_1,i_2,\cdots,i_n\le 3}$ by $\mathcal{K}_{2^n}$.

$$\mathcal{K}_{2^n} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}^{\otimes n} = \left\{\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n} / 0 \le i_1, i_2, \cdots, i_n \le 3\right\}$$

The set

$$\mathcal{P}_{2^n} = \mathcal{K}_{2^n} \otimes \{1, -1, i, -i\}$$

is a group called the Pauli group of $\left(\sigma_{i_1} \otimes \sigma_{i_2} \otimes \cdots \otimes \sigma_{i_n}\right)_{0\le i_1,i_2,\cdots,i_n\le 3}$ [8].

We can check easily, in using the relation $\sigma_j \sigma_k = \delta_{jk}\sigma_0 + i\sum_{i=1}^{3} \varepsilon_{jkl}\,\sigma_l$, for $j,k \in \{1,2,3\}$ that $\mathcal{P}_{2^n}$ is equal to the set of the products of two elements of $\mathcal{K}_{2^n}$, up to multiplicative phases, which are elements of $\{1, -1, i, -i\}$.

$$\mathcal{P}_{2^n} = \mathcal{K}_{2^n}\mathcal{K}_{2^n} \otimes \{1, -1, i, -i\} \qquad (2)$$

where δ_{jk} is the Kronecker symbol, ε_{jkl} is totally antisymmetric, $\varepsilon_{123} = +1$.

It is normal to think that there should be nine 3×3 matrices which share many of the properties of these Kronecker generalized Pauli matrices or $2^n\times 2^n$-KPMs.

5 3×3-Kronecker-Pauli Matrices

Now, we are going to construct the nine 3×3 matrices which satisfy the five properties cited in the introduction, tracelessness vi. moved apart. As we have said above these matrices should be among the inverse-symmetric matrices formed by only three non zero elements. In order that the hermiticity iii. to be satisfied, let us take the 3×3 inverse-symmetric matrices formed by the cubic roots of unit, 1, $j = e^{\frac{2i\pi}{3}}$, $j^2 = e^{\frac{4i\pi}{3}}$. Our choice of the cubic roots of unit has been inspired by [7], [12].

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j \\ 0 & j^2 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & j^2 \\ 0 & j & 0 \end{pmatrix}$$

$$\tau_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 0 & j \\ 0 & 1 & 0 \\ j^2 & 0 & 0 \end{pmatrix}, \quad \tau_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 0 & 1 & 0 \\ j & 0 & 0 \end{pmatrix}$$

$$\tau_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_8 = \begin{pmatrix} 0 & j & 0 \\ j^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_9 = \begin{pmatrix} 0 & j^2 & 0 \\ j & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The set $\mathcal{K}_3$ of them is a basis of $\mathbb{C}^{3\times 3}$. We can check easily that these matrices satisfy the two other properties, orthogonality v. and ii., for n = 3.

In contrast with the $2^n\times 2^n$-KPMs the 3×3-KPMs are not traceless, but according to the orthogonality v. and hermiticity iii. any product of two different 3×3-KPMs is traceless. Thus $\mathcal{K}_3$ up to multiplicative phases can not be a group. However, according to the relation (2), for defining the Pauli group $\mathcal{P}_3$ of the 3×3-KPMs, we suggest to take the set of the products of two elements of the 3×3-KPMs up to multiplicative phases. We have got the following relations between these products

$$\tau_7\tau_3 = j^2\tau_9\tau_2 = j\tau_8\tau_1 = \tau_6\tau_7 = j\tau_5\tau_8 = j^2\tau_4\tau_9 = j\tau_3\tau_6 = j^2\tau_2\tau_4 = \tau_1\tau_5$$
$$\tau_7\tau_6 = j^2\tau_8\tau_5 = j\tau_9\tau_4 = \tau_6\tau_3 = j\tau_4\tau_2 = j^2\tau_5\tau_1 = j\tau_2\tau_9 = j^2\tau_1\tau_8 = \tau_3\tau_7$$
$$\tau_8\tau_3 = j^2\tau_4\tau_8 = j\tau_7\tau_2 = \tau_9\tau_1 = j\tau_5\tau_7 = j^2\tau_3\tau_4 = j\tau_1\tau_6 = j^2\tau_2\tau_5 = \tau_6\tau_9$$
$$\tau_9\tau_3 = j^2\tau_8\tau_2 = j\tau_7\tau_1 = \tau_5\tau_9 = j\tau_4\tau_7 = j^2\tau_6\tau_8 = j\tau_1\tau_4 = j^2\tau_2\tau_6 = \tau_3\tau_5$$
$$\tau_8\tau_6 = j^2\tau_9\tau_5 = j\tau_7\tau_4 = \tau_2\tau_8 = j\tau_4\tau_1 = j^2\tau_5\tau_3 = j\tau_1\tau_7 = j^2\tau_3\tau_9 = \tau_6\tau_2$$
$$\tau_9\tau_6 = j^2\tau_7\tau_5 = j\tau_8\tau_4 = \tau_6\tau_1 = j\tau_4\tau_3 = j^2\tau_5\tau_2 = j\tau_3\tau_8 = j^2\tau_2\tau_7 = \tau_1\tau_9$$

$$\tau_9\tau_7 = j^2\tau_3\tau_1 = j\tau_6\tau_5 = \tau_8\tau_9 = j\tau_5\tau_4 = j^2\tau_2\tau_3 = j\tau_4\tau_6 = j^2\tau_1\tau_2 = \tau_7\tau_8$$
$$\tau_9\tau_8 = j^2\tau_6\tau_4 = j\tau_3\tau_2 = \tau_8\tau_7 = j\tau_2\tau_1 = j^2\tau_5\tau_6 = j\tau_1\tau_3 = j^2\tau_4\tau_5 = \tau_7\tau_9$$

Therefore, we can check easily that

$$\mathcal{P}_3 = \mathcal{K}_3\mathcal{K}_3 \otimes \{1,\ j,\ j^2\}$$

is a group. We call this group the *Pauli group of 3×3-KPMs*. In using the equalities above

$$\mathcal{P}_3 = \tau_k\mathcal{K}_3 \otimes \{1,\ j,\ j^2\}$$

for $k \in \{1,\ 2,\ \cdots,\ 9\}$.

For example, $\tau_1\mathcal{K}_3$ is the set of the following nine 3×3 matrices :

$$\tau_1\tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_1\tau_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad \tau_1\tau_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}$$

$$\tau_1\tau_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_1\tau_5 = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_1\tau_6 = \begin{pmatrix} 0 & 0 & j^2 \\ j & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\tau_1\tau_7 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau_1\tau_8 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, \quad \tau_1\tau_9 = \begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix}$$

They are traceless, except the $\tau_1\tau_1 = I_3$.

The $\tau_k\mathcal{K}_3$'s are differents from the set of Kibler matrices in [7].

6 Roads to generalization

We are going talk two roads to generalization, the first one is by Kronecker product, which does not include the case of prime number, thus the second one is the case of prime number, different from 2.

6.1 Kronecker generalization

In this subsection, we give two examples of 6×6-KPMs, obtained by Kronecker product. The first one is $(\tau_j \otimes \sigma_k)_{1\le j\le 9, 0\le k\le 3}$ and the second one is $(\sigma_j \otimes \tau_k)_{0\le j\le 3, 1\le k\le 9}$. That is, in using the propositions above, they satisfy the six properties of the Kronecker generalized Pauli matrices, with $\mathrm{n} = 6$.

6.2 (prime number)×(prime number)-KPMs

The case of 3×3-KPMs suggests us how to construct a 5×5-KPMs. For starting, let us take 5×5 ones matrices, all elements are equals to $+1$. Decompose this matrix as a sum of five inverse-symmetric matrices the only five non zero elements are equals to $+1$,

$$\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix}+$$
$$\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

For each of these five inverse-symmetric matrices replace the $+1$s by the five fifth roots of unit : 1, u, u^2, u^3, u^4. But we arrange them in order that the orthogonality v. is satisfied and they are inverse-symmetric matrices. Then, we have got the following twenty five inverse-symmetric matrices, which are 5×5-KPMs. That is, they share also the five properties of the Kronecker generalized Pauli matrices, with $\mathrm{n} = 5$.

$$Q_1=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\end{pmatrix},\quad Q_2=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^4\\0&0&0&u^3&0\\0&0&u^2&0&0\\0&u&0&0&0\end{pmatrix},\quad Q_3=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^2\\0&0&0&u^4&0\\0&0&u&0&0\\0&u^3&0&0&0\end{pmatrix},$$
$$Q_4=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u^3\\0&0&0&u&0\\0&0&u^4&0&0\\0&u^2&0&0&0\end{pmatrix},Q_5=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&u\\0&0&0&u^2&0\\0&0&u^3&0&0\\0&u^4&0&0&0\end{pmatrix},$$

$$Q_6=\begin{pmatrix}0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\end{pmatrix},\quad Q_7=\begin{pmatrix}0&0&u&0&0\\0&1&0&0&0\\u^4&0&0&0&0\\0&0&0&0&u^2\\0&0&0&u^3&0\end{pmatrix},\quad Q_8=\begin{pmatrix}0&0&u^3&0&0\\0&1&0&0&0\\u^2&0&0&0&0\\0&0&0&0&u\\0&0&0&u^4&0\end{pmatrix},$$

$$Q_9=\begin{pmatrix}0&0&u^2&0&0\\0&1&0&0&0\\u^3&0&0&0&0\\0&0&0&0&u^4\\0&0&0&u&0\end{pmatrix}, Q_{10}=\begin{pmatrix}0&0&u^4&0&0\\0&1&0&0&0\\u&0&0&0&0\\0&0&0&0&u^3\\0&0&0&u^2&0\end{pmatrix},$$

$$Q_{11}=\begin{pmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\end{pmatrix}, Q_{12}=\begin{pmatrix}0&0&0&0&u\\0&0&0&u^2&0\\0&0&1&0&0\\0&u^3&0&0&0\\u^4&0&0&0&0\end{pmatrix}, Q_{13}=\begin{pmatrix}0&0&0&0&u^3\\0&0&0&u&0\\0&0&1&0&0\\0&u^4&0&0&0\\u^2&0&0&0&0\end{pmatrix},$$

$$Q_{14}=\begin{pmatrix}0&0&0&0&u^2\\0&0&0&u^4&0\\0&0&1&0&0\\0&u&0&0&0\\u^3&0&0&0&0\end{pmatrix}, Q_{15}=\begin{pmatrix}0&0&0&0&u^4\\0&0&0&u^3&0\\0&0&1&0&0\\0&u^2&0&0&0\\u&0&0&0&0\end{pmatrix},$$

$$Q_{16}=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\\0&0&0&1&0\\0&0&1&0&0\end{pmatrix}, Q_{17}=\begin{pmatrix}0&u&0&0&0\\u^4&0&0&0&0\\0&0&0&0&u^2\\0&0&0&1&0\\0&0&u^3&0&0\end{pmatrix}, Q_{18}=\begin{pmatrix}0&u^3&0&0&0\\u^2&0&0&0&0\\0&0&0&0&u\\0&0&0&1&0\\0&0&u^4&0&0\end{pmatrix},$$

$$Q_{19}=\begin{pmatrix}0&u^2&0&0&0\\u^3&0&0&0&0\\0&0&0&0&u^4\\0&0&0&1&0\\0&0&u&0&0\end{pmatrix}, Q_{20}=\begin{pmatrix}0&u^4&0&0&0\\u&0&0&0&0\\0&0&0&0&u^3\\0&0&0&1&0\\0&0&u^2&0&0\end{pmatrix}$$

$$Q_{21}=\begin{pmatrix}0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{22}=\begin{pmatrix}0&0&0&u&0\\0&0&u^2&0&0\\0&u^3&0&0&0\\u^4&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{23}=\begin{pmatrix}0&0&0&u^2&0\\0&0&u^4&0&0\\0&u&0&0&0\\u^3&0&0&0&0\\0&0&0&0&1\end{pmatrix},$$

$$Q_{24}=\begin{pmatrix}0&0&0&u^3&0\\0&0&u&0&0\\0&u^4&0&0&0\\u^2&0&0&0&0\\0&0&0&0&1\end{pmatrix}, Q_{25}=\begin{pmatrix}0&0&0&u^4&0\\0&0&u^3&0&0\\0&u^2&0&0&0\\u&0&0&0&0\\0&0&0&0&1\end{pmatrix}$$

For checking the formula ii., we can use the rule 2 for constructing a KCM, in particular the $5\otimes5$-KCM $K_{5\otimes5}$. The decomposition of 5×5 ones matrices as a sum of five inverse-symmetric matrices, the only five non zero elements are equals to $+1$, is not unique, thus we can construct other 5×5-KPMs than above.

Conclusion

For concluding, we think having given solution to the problem posed of searching 3×3 matrices sharing five properties of the Kronecker generalized Pauli matrices, tracelessness vi. moved apart. We call these matrices 3×3-Kronecker Pauli matrices. For the definition of the Pauli group, we would prefer to call Pauli group of the Kronecker generalized Pauli matrices the group of the set of the products of two elements, up to multiplicative phases, in order that it can be extended to the case of 3×3-KPMs.

The 3×3-KPMs we have obtained suggest us how to construct 5×5-KPMs.

We have introduced what we call inverse-symmetric matrices. Their properties and those of KCMs have made more obvious the construction of the 3×3-KPMs and some ways to generalization.

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A Kronecker Product

Definition 8 For any matrices $A = \left(A_j^i\right)_{1\le i\le n, 1\le j\le p} \in \mathbb{C}^{n\times p}$, $B = \left(B_j^i\right)_{1\le i\le m, 1\le j\le q} \in \mathbb{C}^{m\times q}$, the Kronecker product of the matrix A by the matrix B is the matrix

$$A\otimes B = \begin{pmatrix} A_1^1 B & A_2^1 B & \cdots & A_p^1 B \\ A_1^2 B & A_2^2 B & \cdots & A_p^2 B \\ \vdots & \vdots & \cdots & \vdots \\ A_1^n B & A_2^n B & \cdots & A_p^n B \end{pmatrix}$$

Properties 9

- $\otimes$ is associative.
- $\otimes$ is distributive with respect to the addition.
- For any matrices A, B, C, and D

$$(A\otimes B)(C\otimes D) = AC\otimes BD$$

- For any invertible matrices A and B

$$(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$$

- For any matrices A and B

$$(A \otimes B)^{+} = A^{+} \otimes B^{+}$$

Proposition 10 Let $(A_i)_{1 \leq i \leq np}$, and $(B_j)_{1 \leq j \leq mq}$ respectively be some bases of $\mathbb{C}^{n \times p}$ and $\mathbb{C}^{m \times q}$. Then, $(A_i \otimes B_j)_{1 \leq i \leq np, 1 \leq j \leq mq}$ is a basis of $\mathbb{C}^{nm \times pq}$.

Proposition 11 Suppose

$$\sum_{j=1}^{m} M_j \otimes N_j = \sum_{i=1}^{n} A_i \otimes B_i$$

with the M_j's, A_i's are elements of $\mathbb{C}^{p \times q}$ and the N_j's, B_i's are elements of $\mathbb{C}^{r \times s}$.

Then

$$\sum_{j=1}^{m} M_j \otimes K \otimes N_j = \sum_{i=1}^{n} A_i \otimes K \otimes B_i$$

for any matrix K [1].