Algebraic Quantum Thermodynamics

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Abstract Density matrices can be used to describe ensembles of particles in thermodynamic equilibrium. We assume that the density matrices are a more natural description of quantum objects than state vectors. This suggests that we generalize density matrices from the usual operators on a Hilbert space to be elements of an algebra. Using density matrix renormalization to change the temperature of the ensembles, we show how the choice of algebra determines the symmetry and particle content of these generalized density matrices. The symmetries are of the form SU(N)xSU(M)x...U(1). We propose that the Standard Model of elementary particles should include a dark matter SU(2) doublet.

1 Introduction

There are various reasons for considering density matrices to be a better description of nature than state vectors. Density matrices model measurement without requiring instantaneous alteration of distant entangled states and they avoid the many-worlds interpretation. [1] In addition, as we explore in this paper, mixed density matrices have more degrees of freedom than state vectors and these extra degrees of freedom allow us to determine symmetry, particle content and conserved charges from the assumption of the algebra that contains the density matrices.

In section 2 we review classical thermodynamics in mixed density matrix form, including the process of "density matrix renormalization" that reduces the temperature of a mixed density matrix. Section 3 reviews quantum thrermodynamics in mixed density matrices. Quantum thermodynamics (in finite dimension) assumes complex square matrices; to extend this to algebraic quantum thermodynamics (AQT) we need

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to generalize the algebra in which the mixed density matrices appear. One easy way of defining an algebra is from a fixed group. These "complex group algebras" are reviewed in section 4 with the example of the complex algebra over the permutation group of 2 objects, i.e. $\mathbb{C}[S_2]$. Section 5 discusses the case of general group algebras and shows that their symmetries are of the form SU(N)×SU(M)×...×U(1). Section 6 reviews the block diagonal form of $\mathbb{C}[S_3]$, the complex algebra over the group of permutations on three objects and shows that for the general case, the particles in a complex group algebra will be singlets or K-plets of each SU(K) symmetry of the complex group algebra.

The symmetry of the mixed density matrices of any complex group algebra is easily determined by the character table for the finite group. This is shown in section 7 with the example of $\mathbb{C}[S_3]$. Linear superposition of states with different electric charges are forbidden in elementary particle theory by "superselection rules". These are discussed in section 8. Conserved charges and the projection operators for different fermions are the subject of section 9. The process of density matrix renormalization induces a renormalization flow in the particles. This is discussed in section 10, with illustrations. The density matrix renormalization symmetry of the Standard Model is not quite that of the quantum field theory symmetry SU(3)×SU(2)×U(1). This is discussed in section 11. In section 12 we discuss $\mathbb{C}[S_4]$. This turns out to be the smallest complex group algebra that contains the symmetry of the Standard Model. There is only one particle beyond those of the standard model. This is our dark matter candidate, an SU(2) doublet. Finally, section 13 is a brief discussion.

Section titles: 1) Classical Thermodynamics, 2) Quantum Thermodynamics, 3) Algebra 4)

2 Classical Thermodynamics

Suppose an ensemble of classical particles is at temperature T and that the particles can be in either of two states with energies $\pm E$. Then the probability that a particle is found in one or the other of the states is proportional to the Boltzmann factor:

$$p_{\pm}(T) \propto \exp(\mp E/k_B T). \tag{1}$$

Squaring the Boltzmann factors gives factors for a temperature reduced by half:

$$p_{\pm}(T/2) \propto \exp(\mp E/k_B(T/2)) = (\exp(\mp E/k_B T))^2. \tag{2}$$

We can put $p_+(T)$ and $p_-(T)$ on the diagonal of a 2x2 matrix. Since $p_+(T) + p_-(T) = 1$, the result has trace 1 and is reminiscent of a mixed density matrix:

$$\rho(T) = \begin{pmatrix} p_+(T) & 0 \\ 0 & p_-(T) \end{pmatrix}. \tag{3}$$

To reduce the temperature of this matrix by half, we square and divide by the trace:

$$\rho(T/2) = \rho^2(T)/\operatorname{tr}(\rho^2(T)). \tag{4}$$

This is the same relation as with mixed density matrices and with the algebraic density matrices we explore in this paper. The difference is that the above is a diagonal matrix. We will see that the algebraic density matrices are block diagonal, and naturally fit between the classical and quantum cases.

3 Quantum Thermodynamics

Suppose that an ensemble of spin-1/2 particles is at temperature T and that there is an energy difference of $\pm E$ for states with spin parallel or antiparallel to the z axis. The density matrix representing this ensemble is the same as the classical density matrix given by equation (3). We can write this in terms of the Pauli spin matrices by replacing $p_\pm(T)$ with the Boltzmann factors divied by the trace:

$$\rho_z(T) = \exp(-E\sigma_z/k_BT) \; / \; \mathrm{tr}(\exp(-E\sigma_z/k_BT)). \tag{5}$$

This immediately generalizes to the case with spin parallel or antiparallel to a direction given by a unit vector $\mathbf{u} = (u_x, u_y, u_z)$; one replaces σ_z with $\sigma_u = u_x\sigma_x + u_y\sigma_y + u_z\sigma_z$.

As the temperature in equation (5) approaches ∞, the matrix becomes the unit matrix divided by its trace. This infinite temperature limit is shared with the classical and algebraic cases. For $T \gg k_B/E$, the quantum density matrix is approximated by:

$$\begin{aligned}
\rho_u(T) &\approx \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \tfrac{E}{k_BT}(u_x\sigma_x + u_y\sigma_y + u_z\sigma_z), \\
&= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \tfrac{E}{k_BT}\mathbf{u}\cdot\overrightarrow{\sigma}.
\end{aligned} \tag{6}$$

The small difference from the high temperature limit is parameterized by the Pauli spin matrices.

Applying density matrix renormalization to the above matrix gives a sequence of density matrices with lower and lower temperatures. As the temperature approaches zero, these matrices converge to the pure density matrix corresponding to spin-1/2 in the $\mathbf{u}$ direction:

$$\rho_u(T) \longrightarrow \rho_u(T=0) = \frac{1}{2}(1 + u_x\sigma_x + u_y\sigma_y + u_z\sigma_z). \tag{7}$$

One obtains the spinor for spin in the $\mathbf{u}$ direction by taking a nonzero column from the above matrix and normalizing it. Allowing $\mathbf{u}$ to range over all possible directions we obtain a set of spin-1/2 particles.

The case for $K \times K$ matrices is similar to the above 2×2 case. Instead of characterizing high temperature matrices with the Pauli spin matrices we use the infinitesimal generators of SU(K). These are Hermitian and have trace 0 so adding them to the unit matrix divided by K gives matrices that are Hermitian with trace 1. For example with $K = 3$:

$$\rho_u(T) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{E}{k_BT}\mathbf{u}\cdot\overrightarrow{\lambda}, \tag{8}$$

where **u** is a 7×1 real vector of length 1 and $\overrightarrow{\lambda}$ is a vector of the seven generators of SU(3) symmetry such as the Gell-Mann matrices. The pure density matrices obtained this way carry the symmetry of the fundamental representation of SU(K). That is, they are SU(2) doublets, or SU(3) triplets, etc.

Note that the conversion from a mixed density matrix to a pure density matrix by cooling involves a loss of degrees of freedom. The SU(N) fundamental representation mixed density matrices have the same number of degrees of freedom as the generators, that is, there are $N^2 - 1$ real degrees of freedom. But pure SU(N) density matrices have the same degrees of freedom as the corresponding state vectors; $2N$ for the N complex numbers in a N-vector minus one for the arbitrary complex phase and one more for normalization to give $2N - 1 - 1 = 2N - 2$. Thus the loss in real degrees of freedom is $(N^2 - 1) - (2N - 2) = (N - 1)^2$. For the classical case there was no loss in degrees of freedom; the algebraic cases will depend on the algebra and be between 0 and $(N - 1)^2$.

4 $\mathbb{C}[S_2]$, Group Algebras

To generalize quantum thermodynamics, we need to reconsider the complex K×K matrices where the density matrices are found. For normalization we need to be able to multiply by a scalar which suggests we should assume that our objects form a vector space over the complex numbers. In this sense, the K×K matrices are a vector space with K^2 basis elements. A natural choice of basis elements for the matrices are the matrices that have a single element equal to 1 and all other elements zero.

A vector space gives the ability to add two objects which is required for linear superposition. Necessary for the renormalization process is the ability to multiply two objects. For both the classical and quantum case the high temperature limit was given by the unit matrix divided by its trace so we will here assume that a unit matrix or "multiplicative identity" is included in our objects. Finally, we will assume associativity; the result is that we will be considering what the mathematicians call an "associative unital algebra over the complex numbers," or sometimes an "algebra over the complex numbers."

A common way of obtaining one of these algebras is to use a finite group as the basis for a vector space. Multiplication is then defined by distributing over addition and using the finite group to reduce products of basis vectors to a single basis vector. This is called the "group algebra", see section (3-17) of Hammermesh.[2] This may sound complicated; perhaps it is best shown by example.

The simplest nontrivial finite group is S_2, the group of permutations on two objects. There are only two permutations, we either swap the two elements: "(12)", or we leave them alone: "( )". With this notation, the multiplication table for the finite group is:

$$
\begin{array}{c|cc}
 & (\,) & (12) \\
\hline
(\,) & (\,) & (12) \\
(12) & (12) & (\,)
\end{array}
\tag{9}
$$

The complex algebra over this group is called $\mathbb{C}[S_2]$. As a vector space, the natural basis for $\mathbb{C}[S_2]$ is ( ), (12). Elements of the algebra $\mathbb{C}[S_2]$ are sums of complex mul-

tiples of the basis elements. Given complex numbers a_I and a_S, a general element of the algebra is:

$$a = a_I(\;) + a_S(12). \qquad (10)$$

Addition is performed as in any vector space. Given a and b defined as above,

$$a + b = [a_I + b_I](\;) + [a_S + b_S](12). \qquad (11)$$

Multiplication is defined by distributing multiplication over addition and then simplying the products by using the group multiplication table for S_2:

$$
\begin{aligned}
ab &= [a_I(\;) + a_S(12)]\,[b_I(\;) + b_S(12)], \\
&= a_I b_I(\;)(\;) + a_I b_S(\;)(12) + a_S b_I(12)(\;) + a_S b_S(12)(12), \\
&= a_I b_I(\;) + a_I b_S(12) + a_S b_I(12) + a_S b_S(\;), \\
&= [a_I b_I + a_S b_S](\;) + [a_I b_S + a_S b_I](12).
\end{aligned}
\qquad (12)
$$

For example, $[(\;) + 2(12)]\,[3(\;) + 5(12)] = [13(\;) + 11(12)]$.

5 Block Diagonal Forms and SU(N)×SU(M)...

Given a complex finite group algebra, it is an elementary fact of group representation theory that it is always possible to faithfully represent it with complex matrices. Such a representation is determined by specifying how to represent the basis elements of the finite group. Given g in the finite group, we write $R(g)$ for the matrix representation. Given a complex matrix S and its inverse S^{-1} we can transform the matrix representation $R(g)$ by $R(g) \rightarrow S R(g) S^{-1}$. This transforms a faithful representation into another faithful representation.

Suppose that the faithful representation uses $K \times K$ matrices. If it happens that the finite group has K^2 elements then there are the same number of basis elements in the algebra and its representation and since the density matrix symmetry of the representation is SU(K) this must also be the symmetry of density matrices taken from the algebra.

More interesting are the cases where the matrix representation is not the same as a matrix algebra. In such a case we can transform to a block diagonal representation of the algebra, see section (3-13) of [2]. If the sizes of the blocks are N, M, ... P, then the symmetry of the density matrices of the algebra will be SU(N)×SU(M)×...SU(P) where, as noted below in section 6, we interpret "SU(1)" as U(1).

For the case of $\mathbb{C}[S_2]$ the size of the finite group S_2 is 2. We will end up with a block diagonal form with two 1×1 blocks. The representation for the group identity $(\;)$ has to be the unit matrix. So to find a representation of $\mathbb{C}[S_2]$, we need only choose a matrix for (12). Thinking of permutation, one obvious choice is:

$$(12) \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (13)$$

which swaps the entries in a spinor, but the representation will not be block diagonal. The above matrix is the Pauli spin matrix σ_x so a rotation exists that takes it to a

diagonal matrix, σ_z. This gives a block diagonal matrix:

$$(12) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

And the block diagonal representation is:

$$[a_I(\ ) + a_S(12)] \rightarrow \begin{pmatrix} a_I + a_S & 0 \\ 0 & a_I - a_S \end{pmatrix}. \tag{15}$$

The representation gives multiplication identical to $\mathbb{C}[S_2]$, for example translating $[(\ ) + 2(12)]$ and $[3(\ ) + 5(12)]$ into matrix form, the product of their representations is

$$\begin{pmatrix} 1+2 & 0 \\ 0 & 1-2 \end{pmatrix} \begin{pmatrix} 3+5 & 0 \\ 0 & 3-5 \end{pmatrix} = \begin{pmatrix} 24 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 13+11 & 0 \\ 0 & 13-11 \end{pmatrix} \tag{16}$$

which is the same result as above.

Suppose that ρ is a mixed density matrix in $\mathbb{C}[S_2]$:

$$\rho = \rho_I(\ ) + \rho_S(12). \tag{17}$$

The requirement that ρ be Hermitian and have trace 1 gives that ρ_I and ρ_S are real. Positivity requires that $|\rho_S| \le 1$. These are exactly the classical density matrix in equation (3) so we have that $\mathbb{C}[S_2]$ gives classical thermodynamics. In the cold limit, there are two states that for spin-1/2 would be called spin-up and spin-down. But in the $\mathbb{C}[S_2]$ language they are:

$$\begin{aligned} \rho_+ &= 0.5[(\ ) + (12)], \\ \rho_- &= 0.5[(\ ) - (12)] \end{aligned} \tag{18}$$

A physicist might treat this system as an example of a high temperature SU(2) doublet symmetry that is broken in the cold limit to give a spin-up and spin-down state.

6 $\mathbb{C}[S_3]$ and Fundamental Representations

The group S_3 has six elements corresponding to the permutations on three elements. The elements come in three classes, the identity, 3-cycles, and swaps:

$$\begin{aligned} &(\ ) \\ &(123), (132) \\ &(12), (13), (23). \end{aligned} \tag{19}$$

The group multiplication table is:

	( )	(12)	(13)	(23)	(123)	(132)
( )	( )	(12)	(13)	(23)	(123)	(132)
(12)	(12)	( )	(132)	(123)	(23)	(13)
(13)	(13)	(123)	( )	(132)	(12)	(23)
(23)	(23)	(132)	(123)	( )	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	( )
(132)	(132)	(23)	(12)	(13)	( )	(123).

$$\tag{20}$$

Since S_3 has six elements, $\mathbb{C}[S_3]$ is six dimensional. Suppose that we are able to put $\mathbb{C}[S_3]$ into block diagonal form. If the sizes of the blocks are $n_1, n_2, ...$, then the dimensionality of the block diagonal representation is $n_1^2 + n_2^2 + ... = 6$. There are only two solutions to the problem of writing 6 as a sum of squares:

$$\begin{aligned} 6 &= 2^2 + 1^2 + 1^2, \\ &= 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2. \end{aligned} \tag{21}$$

If we use $6 = 1 + 1 + 1 + 1 + 1 + 1$, we will have $\mathbb{C}[S_3]$ written in terms of diagonal matrices. This is impossible because diagonal matrices commute but $\mathbb{C}[S_3]$ does not: in particular $(12)(13) \neq (13)(12)$. Therefore the block diagonal form for $\mathbb{C}[S_3]$ has to be the first case: $6 = 2^2 + 1^2 + 1^2$.

The block diagonal form for $\mathbb{C}[S_3]$ gives an arbitrary member as:

$$\begin{pmatrix} a & & & \\ & b & & \\ & & c_{11} & c_{12} \\ & & c_{21} & c_{22} \end{pmatrix} \tag{22}$$

where a, b and c_{jk} are six complex numbers and the blank matrix entries are always zero. From the matrix form we see that $\mathbb{C}[S_3]$ density matrices have symmetry SU(2)xU(1)xU(1).

For SU(N), the mixed density matrix giving the high temperature limit was $1/n$. When we square and normalize this matrix, (that is, when we apply density matrix renormalization to it), we get back the same matrix. The corresponding matrix for the block diagonal form for $\mathbb{C}[S_3]$ is $1/4$.

Let's consider matrices in $\mathbb{C}[S_3]$ that are near the high temperature limit. As before, we will use the Pauli spin matrices and the unit vector $\mathbf{u} = (u_x, u_y, u_z)$ to parameterize the SU(2) portion. We also need to parameterize the three blocks; we will use ϵ_a, ϵ_b and ϵ_c, chosen randomly:

$$\frac{1}{4} \begin{pmatrix} 1 + \epsilon_a & & & \\ & 1 + \epsilon_b & & \\ & & 1 + \epsilon_c u_z & \epsilon_c(u_x - iu_y) \\ & & \epsilon_c(u_x + iu_y) & 1 - \epsilon_c u_z \end{pmatrix} \tag{23}$$

In the above, note that ϵ_c preserves the trace while ϵ_a and ϵ_b do not. This is why the 1×1 blocks are U(1) symmetry instead of "SU(1)". Consequently, while ϵ_a, ϵ_b and ϵ_c all need to be small real numbers, the U(1) values can be negative.

For small ϵ_j, squaring the above matrix approximately doubles each ϵ_j. But since they are chosen randomly, the probability that any of the ϵ_j are equal is zero. Consequently, as the above matrix is repeatedly cooled, one of the blocks will be growing exponentially compared to the others. If ϵ_a and ϵ_b are both negative those blocks will exponentially shrink on squaring and the 2×2 block will grow. Then the low temperature limit will be:

$$\rho(T = 0) = \frac{1}{2} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 + u_z & u_x - iu_y \\ & & u_x + iu_y & 1 - u_z \end{pmatrix}. \tag{24}$$

On the other hand, if ϵ_a or ϵ_b is positive, then whichever is larger will exponentially grow fastest and the cold limit will be:

$$\rho(T = 0) \; = \; \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}. \tag{25}$$

depending on whether ϵ_a or ϵ_b is larger.

The above argument is general and applies to any algebra which can be put into block diagonal form with blocks of size N, M, ... P. Then the cold density matrices will have symmetry SU(N)×SU(M)×...SU(P) and the particles will have the fundamental or singlet representation for each of the SU(K) symmetries. In addition, at least for random samples cooled from near the high temperature limit, the particle can take the fundamental representation for only one of the SU(K); it must be a singlet for the others. This last restriction, "only one fundamental representation" does not apply to particles taken from a restricted subset of the initial conditions. If some of the ϵ are equal, one could end up with a particle that was fundamental in more than one SU(K).

7 $\mathbb{C}[S_3]$ Characters and Particles

A central concept in the theory of representations of a finite group are the "characters" or traces of the representation of a group element, see sections (3-18) to (4-3) of [2]. For elements which are in the same class, it is easily shown that the character is the same. Since S_3 has three classes, namely $\{(\ )\}$, $\{(12),(13),(23)\}$ and $\{(123),(132)\}$ a representation of S_3 has its characters defined by three different characters.

Each of the three block diagonals in the block diagonal version of $\mathbb{C}[S_3]$ is in itself a representation of S_3. And if two of the diagonal blocks were identical (for all the group elements) we could eliminate one of them and so the three diagonal blocks are three different representations of S_3.

The two 1×1 blocks are easy to determine. They are commuting representations of S_3 so all six elements of S_3 must be represented by a complex number. Since $(12),(23),(13)$ all square to $(\ ) \equiv 1$, they must be represented by ± 1. And for 1×1 matrices, the trace is the same as the only entry and therefore all three of $(12),(13),(23)$ have the same 1×1 representation. Since $(123) = (12)(23)$ we then have the representation of (123) must be $(\pm 1)^2 = 1$. With (12) represented by either $+1$ or -1 there are two possible 1×1 representations. This solves the two 1×1 blocks, a and b:

$$\begin{array}{c|cccccc} & (\ ) & (12) & (13) & (23) & (123) & (132) \\ \hline a & 1 & 1 & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & -1 & -1 & -1 \end{array}. \tag{26}$$

The 2×2 block requires a little algegbra.

Since $(123)^3 = (\ )$, the 2×2 block will represent (123) with a cubed root of the unit 2×2 matrix. A general 2×2 matrix can be written with four complex coefficients

$\alpha, \beta, \gamma, \delta$ giving (123) as:

$$(123) \equiv \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \sigma_x + \gamma \sigma_y + \delta \sigma_z. \qquad (27)$$

Taking advantage of the fact that the Pauli spin matrices anticommute and square to unity, we get:

$$(123)^2 = (132) \equiv \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2\alpha(\beta\sigma_x + \gamma\sigma_y + \delta\sigma_z). \qquad (28)$$

Multiplying again by (123) gives:

$$(123)^3 = (\ ) \equiv 1 = \alpha^3 + 3\alpha(\beta^2 + \gamma^2 + \delta^2) + (3\alpha^2 + \beta^2 + \gamma^2 + \delta^2)(\beta\sigma_x + \gamma\sigma_y + \delta\sigma_z). \qquad (29)$$

solving this gives:

$$\begin{aligned} \alpha^3 &= -1/8, \\ 3\alpha^2 &= \beta^2 + \gamma^2 + \delta^2. \end{aligned} \qquad (30)$$

A convenient solution is $\alpha = -1/2$, $\beta = \gamma = 0$, $\delta = i\sqrt{3}/2$. This gives:

$$(123) \equiv -1/2 + i\sigma_z \sqrt{3}/2 = \begin{pmatrix} \omega & 0 \\ 0 & \omega^* \end{pmatrix}, \quad (132) \equiv \begin{pmatrix} \omega^* & 0 \\ 0 & \omega \end{pmatrix} \qquad (31)$$

where $\omega = -1/2 + i\sqrt{3}/2$ is the complex cubed root of one. The remaining three elements are the swaps $(12),(13),(23)$. They need to square to one. By equation 28 the α component will need to be zero and $\beta^2 + \gamma^2 + \delta^2 = 1$. This defines them as generalized Pauli spin-1/2 matrices, i.e. $(12) \equiv \sigma_u = \mathbf{u} \cdot \vec{\sigma} = u_x\sigma_x + u_y\sigma_y + u_z\sigma_z$ where $\mathbf{u}$ is a unit vector so that σ_u is the spin matrix for spin in the $\mathbf{u}$ direction. The products of two such objects is given by a dot product and a cross product:

$$\sigma_u \sigma_v = \mathbf{u} \cdot \mathbf{v} + i\mathbf{u} \times \mathbf{v}. \qquad (32)$$

The products of swaps are equal to (123) or (132) which are equivalent to $-1/2 \pm i\sqrt{3}\sigma_z/2$ so equating the scalar and vector parts give that the swaps are equivalent to spin-1/2 in the x, y plane with angles between them of $2\pi/3$. We can use

$$\begin{aligned} (12) &\equiv \sigma_x, \\ (23) &\equiv -\sigma_x/2 + \sqrt{3}\sigma_y/2, \\ (13) &\equiv -\sigma_x/2 - \sqrt{3}\sigma_y/2, \end{aligned} \qquad (33)$$

This completes the representation of $\mathbb{C}[S_3]$ in block diagonal form.

The characters for the three representations given above are

$$\begin{array}{c|cccc} \chi(g) & (\ ) & (123),(132) & (12),(13),(23) \\ \hline a,\ 1\times 1 & 1 & 1 & 1 \\ b,\ 1\times 1 & 1 & 1 & -1 \\ c,\ 2\times 2 & 2 & -1 & 0 \end{array} \qquad (34)$$

Of course our representations of $\mathbb{C}[S_3]$ are also representations of S_3; we simply restrict the mapping to the basis elements of $\mathbb{C}[S_3]$ given by the elements of S_3. And the above character table happens to be a complete character table for S_3. That is,

it has one line for each possible irreducible representation of S_3. This is a general property of the representations of complex algebras.

Thus we can read off the density matrix renormalization structure from the character table of a finite group. The characters of the diagonal blocks of $\mathbb{C}[G]$ are exactly the character table for all possible irreps of G. The column under the identity gives the symmetry of that component. In the above case the numbers under the identity ( ) are $1, 1, 2$ so the symmetry is U(1)×U(1)×SU(2) and there are three particles. The first two are singlets and the last is an SU(2) doublet.

8 Linear Superposition and Superselection Sectors

Linear superposition is important in the state vector representation of quantum mechanics. To see how this extends to the pure density matrices taken from a group algebra let's define a restricted type of linear superposition. Let α be a real number between 0 and 1 and define linear superposition between two states $|u\rangle$ and $|v\rangle$ as:

$$|\alpha\rangle = \alpha|u\rangle + (1-\alpha)|v\rangle, \tag{35}$$

perhaps followed by a renormalization to get $\langle\alpha|\alpha\rangle = 1$. Since there is an arbitrary complex phase available in the definition of $|u\rangle$ and $|v\rangle$, we get an equivalent to the more general linear superposition usually defined with complex coefficients. Topologically, $|\alpha\rangle$ gives a continuous path in quantum states as α is taken over $[0, 1]$. The path begins at $|u\rangle$ and ends at $|v\rangle$. We can define a similar path in density matrices by:

$$\rho_\alpha = \alpha|u\rangle\langle u| + (1-\alpha)|v\rangle\langle u| = \alpha\rho_u + (1-\alpha)\rho_v, \tag{36}$$

but the path goes through mixed density matrix states that have temperatures greater than zero and so cannot be represented by state vectors.

If $|u\rangle$ and $|v\rangle$ are both in one of the block diagonals we have a situation identical to standard quantum mechanics and the relationship between density matrices and state vectors. But if $|u\rangle$ and $|v\rangle$ are taken from two different fundamental representations, there is no possible path between them among the cold density matrix states and hence linear superpositions of the state vectors are forbidden. We propose that this limitation on linear superposition is the origin of the "superselection sectors" seen in the standard model of elementary particles. [3,4] This paper proposes that superselection sectors arise from the fact that it is not possible to create linear superpositions between states that have different charges, until one considers high temperature states in mixed density matrices.

9 $\mathbb{C}[S_3]$ Conserved Charges

Suppose that a projection operator $P = P^2$ of $\mathbb{C}[S_3]$ commutes with everything in the algebra. Such an element is preserved by the same transformations that define the classes. Thus P must have the same projection on each element of a class and we can write:

$$P = \alpha_1(\,) + \alpha_2[(12) + (13) + (23)] + \alpha_3[(123) + (132)] \tag{37}$$

for three complex constants $\alpha_1, \alpha_2, \alpha_3$. In the block diagonal matrix representation, the projection operators P_a, P_b, P_c for the three blocks are:

$$
\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0\ 0 & \\ & & 0\ 0 & \end{pmatrix}, \quad
\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0\ 0 & \\ & & 0\ 0 & \end{pmatrix}, \quad
\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1\ 0 & \\ & & 0\ 1 & \end{pmatrix}. \tag{38}
$$

These three projection operators can be translated into $\mathbb{C}[S_3]$ by using the details of the representation we derived above, but since they depend on the characters we can derive them from the character table alone. Since superselection sectors are distinguished by conserved charges, this means we can derive the conserved charges from the character table of the finite group.

The projection operator for a singlet describes a single particle but the projection operator for an SU(K) K-plet projects out K particles. So the projection operator for an SU(K) block needs to have a trace that is K times the projection operator for a single SU(K) particle. So we modify the character table by multiplying an SU(K) row by K. For the S_3 character table only the SU(2) row is multiplied (by 2) and we have:

P_χ	( )	{(123)}	{(12)}
$a, 1 \times 1$	1	1	1
$b, 1 \times 1$	1	1	−1
$c, 2 \times 2$	4	−2	0

$$\tag{39}$$

where from here on we use "{(12)}" to indicate the class of elements equivalent to (12), etc. We can now read the projection operators off of the table by dividing by the size of the finite group, in this case, 6:

$$
\begin{aligned}
P_a &= [(\,) + (123) + (132) + (12) + (13) + (23)]/6, \\
P_b &= [(\,) + (123) + (132) - (12) - (13) - (23)]/6, \\
P_c &= [4(\,) - 2(123) - 2(132)]/6.
\end{aligned}
\tag{40}
$$

Note that the sum of the projection operators is the identity as required.

The projection operators defined by equation (37) use only three of the six degrees of freedom present in $\mathbb{C}[S_3]$. In equation (40) we use those three degrees of freedom to give projection operators for the three blocks. The remaining three degrees of freedom correspond to the generators of the SU(2) symmetry, $\{\sigma_x, \sigma_y, \sigma_z\}$. This is a general property of complex group algebras. If there is a K×K block it will use one degree of freedom for its projection operator and $K^2 - 1$ more for its SU(K) symmetry generators to give a total of K^2 degrees of freedom - the same as the number of degrees of freedom for that block. And the U(1) blocks will each use one degree of freedom for their projection operators. Note that each of these projection operators or symmetry generators are linearly independent in the vector sense. Thus all the degrees of freedom of the group algebra are accounted for in the projection operators and SU(K) symmetry generators.
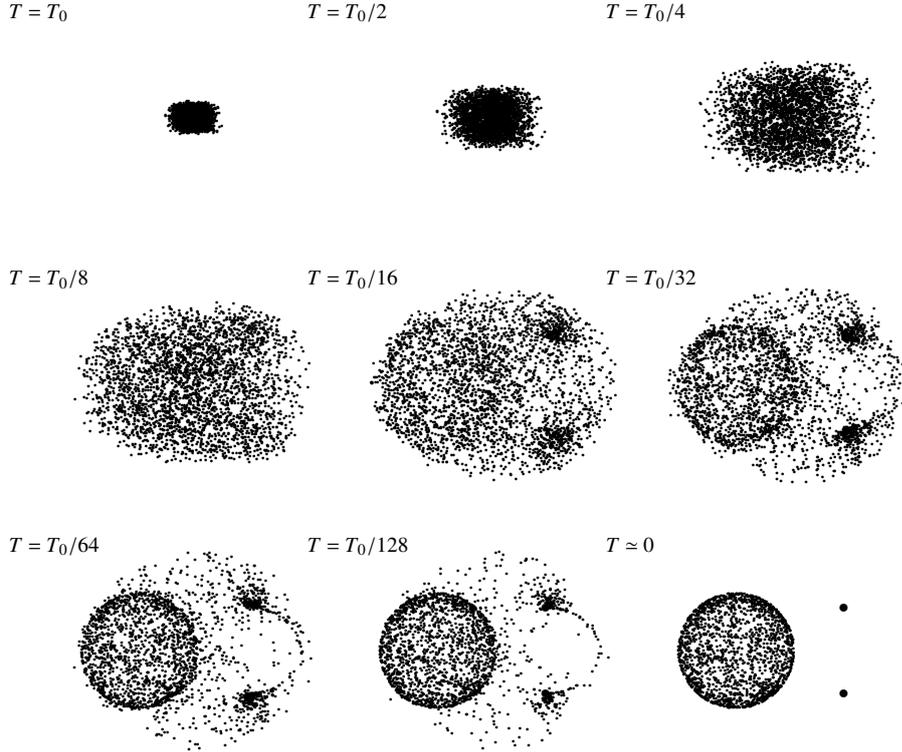
Fig. 1 Density matrix renormalization flow of $\mathbb{C}[S_3]$ with 3,000 states. The symmetry and representation is $SU(2) \times U(1) \times U(1)$ with one SU(2) doublet and two singlets. The two axes of the SU(2) Bloch sphere are given by $(123) - (132)$ and $(12) - (23)$. The vertical separation between the two U(1) singlets is given by the commutative element $(12) + (13) + (23)$ and these two states are horizontally separated from the SU(2) doublet by another commutative element $(123) + (132)$. These drawings use a horizontal axis of $3.0[(123) - (132)] + 2.8[(123) + 132)]$ and vertical axis $3.0[(12) - (23)] + 1.5[(12) + (23) + (13)]$.

10 $\mathbb{C}[S_3]$ **Renormalization Flow**

The block diagonal representation of $\mathbb{C}[S_3]$ allows us to graphically display the SU(2)×U(1)×U(1) symmetry of the density matrices and see how they renormalize. The elements of $\mathbb{C}[S_3]$ that correspond to the Pauli spin matrices of the SU(2) particle can be deduced from equations (33 or we can look for three more vectors that are linearly independent of the three given in equation (40) such as $[(123) - (132)]$, $[(12) - (13)]$ and $[(12) + (13) - 2(12)]$. One view of the renormalization flow is shown in figure 1.

11 Standard Model Symmetry

The Standard Model is written in quantum field theory (QFT). The QFT formulation of quantum mechanics allows the number of particles to change. Density matrix renormalization is a process that applies to density matrices and these are suitable only for situations where the number of each particle type does not change. So in using density matrix renormalization to understand the Standard Model we have to restrict ourselves to situations where the particle number does not change. The simplest such situations are the propagators for elementary fermions.

Consider a mixed density matrix model of an electron at low temperature. As the temperature increases the pure state becomes mixed and the 2 degrees of freedom of the pure state become the 3 degrees of freedom of the mixed state. From a field theory point of view what is happening is that the electron is emitting and absorbing photons from the temperature bath.

Once the temperature becomes hot enough to include $W^{\pm}$ particles, the bath is able to exchange charge with the system and the electron is able to explore neutrino degrees of freedom. Instead of an electron density matrix we now have a lepton density matrix. The pure states will have twice as many degrees of freedom now. From the point of view of algebraic density matrix theory, the density matrix now becomes block diagonal with an electron block and a neutrino block.

The simplest model of a two block system is $\mathbb{C}[S_2]$. There are two cold states $0.5[(\,) \pm (12)]$ as given in equation (18). The quantum numbers match the weak hypercharge and weak isospin quantum numbers of the left handed lepton SU(2) doublets when we use

$$
\begin{aligned}
T_3 &= (12), \\
Y_W &= -2(\,), \\
Q = T_3 + Y_W/2 &= (12) - (\,).
\end{aligned}
\tag{41}
$$

With the right handed states included, the quantum numbers are:

$$
\begin{array}{l|ccc}
 & Q & T_3 & -Y_W/2 \\
 & (\,)-(12) & (12) & -(\,) \\
\hline
\nu_{eL} = 0.5[(\,)+(12)] & 0 & +1/2 & -1/2 \\
e_L = 0.5[(\,)-(12)] & -1 & -1/2 & -1/2 \\
\nu_{eR} = 0 & 0 & 0 & 0 \\
e_R = (\,) & -1 & 0 & -1
\end{array}
\tag{42}
$$

While the right handed states do not correspond to $\mathbb{C}[S_2]$ pure states, they are both idempotents of $\mathbb{C}[S_2]$ and in addition, e_R is the identity, which is a fixed point of the renormalization process.

There is another problem, this one with the left handed quarks. These particles are an SU(2) doublet and also SU(3) triplets. As was shown at the end of section 6 this is impossible for randomly chosen initial conditions as only one of the SU(K) can be a K-plet, the rest have to be singlets.

From the point of view of quantum mechanics, one obtains these left or right handed states as a limit process of boosting the electron or neutrino either parallel or antiparallel to its spin direction. But density matrix renormalization from a random

initial state should give random leptons. They will have random spin orientations and random boosts and none are likely to be lucky enough to receive an infinite boost in their spin direction.

So what symmetry can we expect from an algebra that models the Standard Model fermions? Instead of looking at the left and right handed states, we can only assume the internal symmetries that are shared between these states. Such a model should include the external symmetries, and the external symmetry needs to be coupled to the internal so that when the particle is boosted along its spin axis, it approaches the QFT symmetries of the Stanadard Model, but for a finite model, only the shared internal symmetries can be included.

The up and down quark will each be SU(3) triplets but the up and down quarks differ in that one is a triplet while the other is an anti-triplet. So we will need two different SU(3) symmetries for them, one for the up quarks and one for the down quarks. That their symmetries be different is required by the results of section 6. Since weak isospin is broken at the low temperature limit, the electron and neutrino will be separate U(1) singlets

There is currently no known fractionally charged boson that can convert between quarks and leptons. Assuming that there is no such boson, we will need different algebras for the quarks and leptons. The quark algebra will need to have two triplets. Every algebra has the trivial representation where all group elements are represented by 1 and this representation corresponds to a density matrix U(1) singlet. So in addition to the quarks, a quark algebra will need extra states. For the leptons we already have an algebra, $\mathbb{C}[S_2]$ and this algebra has the required two U(1) singlets and no other particles.

It's more interesting to assume that all the Standard Model fermions mix at some high temperature and we will explore this possibility for the remainder of this paper. Such a "grand unification algebra" will have symmetry of (at least) SU(3)×SU(3) ×U(1)×U(1) with particles that are multiplets or singlets of each SU(K) according to:

$$
\begin{array}{c|cccc}
 & SU(3) & SU(3) & U(1) & U(1) \\
\hline
\nu & 1 & 1 & 1 & 1 \\
e & 1 & 1 & 1 & 1 \\
u & 3 & 1 & 1 & 1 \\
d & 1 & 3 & 1 & 1 \\
\end{array}
\tag{43}
$$

Counting degrees of freedom for the SU(K) generators and the block projection operators, the up and down quarks will use 9 degrees of freedom each while the leptons will use 1 each. This is a total of 20 degrees of freedom.

If there are any additional degrees of freedom we will suppose that they correspond to a dark matter particle X^0. So the character table for a prospective Standard Model (internal) symmetry will need to have at least four lines and those four lines will need to have the chracter of the identity equal to 1, 1, 3 and 3, with possible additional line(s) for dark matter. A complete model will need a way of explaining the three generations but since the higher generation fermions decay into the first generation they cannot show up as the result of density matrix renormalization.

The high temperature limit state for the Standard Model fermions deserves a name. We will use[1] "sitronen", a name that means "lemon" in several Scandinavian languages. The renormalization flow from a neighborhood of the sitronen forms a lemon tree, very pretty. See figures below. It is a neutral, spin-1/2 fermion.

12 $\mathbb{C}[S_4]$, the Standard Model, and Dark Matter

On looking for a finite group that has sufficient symmetry to contain the Standard Model quarks and leptons the obvious candidate is $\mathbb{C}[S_4]$. The finite group S_4 has 24 elements which is 4 more than the 20 needed for the quarks and leptons. They fall into 5 conjugacy classes so there is one extra class for dark matter which will be an SU(2) doublet:

$$
\begin{aligned}
&\{\,(\,)\,\} \\
&\{\,(12)(34),\ (13)(24),\ (14)(23)\,\} \\
&\{\,(123),\ (132),\ (124),\ (142),\ (134),\ (143),\ (234),\ (243)\,\} \qquad (44)\\
&\{\,(12),\ (13),\ (14),\ (23),\ (24),\ (34)\,\} \\
&\{\,(1234),\ (1243),\ (1324),\ (1342),\ (1423),\ (1432)\,\}.
\end{aligned}
$$

The (12) and (1234) classes consist of odd permutations. The others are even. The character table:

	( )	(12)(34)	(123)	(12)	(1234)
L^1	1	1	1	1	1
L^2	1	1	1	−1	−1
X^0	2	2	−1	0	0
Q^1	3	−1	0	1	−1
Q^2	3	−1	0	−1	1

$$(45)$$

where we use L and Q to specify the leptons and quarks and X^0 is the dark matter doublet. From the above we see that the odd permutations (12) and (1234) define the difference between the weak isospin doublets (ν, e) and (u, d), and that the dark matter particle is distinct from the visible matter in that it has no odd component.

The projection operators for the five particles are:

$$
\begin{aligned}
P_{L1} &= [\ (\,)\ +\{(12)(34)\}\ +\{(123)\}\ +\{(12)\}\ +\{(1234)\}\ ]/24, \\
P_{L2} &= [\ (\,)\ +\{(12)(34)\}\ +\{(123)\}\ -\{(12)\}\ -\{(1234)\}\ ]/24, \\
P_{X0} &= [\ 4(\,)\ +4\{(12)(34)\}\ -2\{(123)\}\qquad\qquad\qquad\ ]/24, \qquad (46)\\
P_{Q1} &= [\ 9(\,)\ -3\{(12)(34)\}\qquad\qquad\ +3\{(12)\}\ -3\{(1234)\}\ ]/24, \\
P_{Q2} &= [\ 9(\,)\ -3\{(12)(34)\}\qquad\qquad\ -3\{(12)\}\ +3\{(1234)\}\ ]/24,
\end{aligned}
$$

where by an abuse of notation, we abbreviate: $\{(12)(34)\} = (12)(34) + (13)(24) + (14)(23)$, etc. From a thermodynamic point of view, these projection operators give the high temperature limit of particles that are restricted to not change their particle type. For the leptons this is the same as the cold temperature limit but for the others it is the mixed density matrix limit appropriate to that particle.

---

[1] Neologism due to Forrest LeDuc.

The generators of the SU(N) symmetries are arbitrary in that they depend on how we pack the algebra into block diagonal form. For a particular choice, the SU(2) generators for X are:

$$
\begin{aligned}
\sigma_x &= [\,2(12)-(13)-(14)-(23)-(24)+2(34)-(1234)-(1243)+2(1324) \\
&\quad -(1342)+2(1423)-(1432)\,]/12, \\
\sigma_y &= [\,(13)-(14)-(23)+(24)+(1234)-(1243)-(1342)+(1432)\,]/\sqrt{48}, \\
\sigma_z &= [\,-(123)+(132)+(124)-(142)-(134)+(143)+(234)-(243)\,]i/\sqrt{48}
\end{aligned}
\tag{47}
$$

Note that the above are orthogonal to each other and orthogonal to the projection operators. The SU(3) generators for Q^1 and Q^2 are:

$$
\begin{aligned}
\lambda_1^\pm &= [\,\mp 2(14)\pm 2(23)+(123)+(132)-(124)-(142)-(134)-(143) \\
&\quad +(234)+(243)]/8, \\
\lambda_2^\pm &= [\,(123)-(132)+(124)-(142)-(134)+(143)-(234)+(243) \\
&\quad \pm 2(1243)\mp 2(1342)]i/8, \\
\lambda_3^\pm &= [\,\pm(12)\mp(13)\mp(24)\pm(34)\pm(1234)\mp(1324)\mp(1423)\pm(1432) \\
&\quad +2(12)(34)-2(13)(24)]/8, \\[6pt]
\lambda_4^\pm &= [\,\pm 2(13)\mp 2(24)+(123)+(132)-(124)-(142)+(134)+(143) \\
&\quad -(234)-(243)]/8, \\
\lambda_5^\pm &= [\,-(123)+(132)-(124)+(142)-(134)+(143)-(234)+(243) \\
&\quad ;\mp 2(1234)\pm 2(1432)]i/8, \\
\lambda_6^\pm &= [\,\pm 2(12)\mp 2(34)+(123)+(132)+(124)+(142)-(134)-(143) \\
&\quad -(234)-(243)]/8, \\[6pt]
\lambda_7^\pm &= [\,(123)-(132)-(124)+(142)-(134)+(143)+(234)-(243) \\
&\quad \mp 2(1324)\pm 2(1423)]i/8, \\
\lambda_8^\pm &= [\,\pm(12)\pm(13)\mp 2(14)\mp 2(23)\pm(24)\pm(34)\mp(1234)\pm 2(1243)\mp(1324) \\
&\quad \pm 2(1342)\mp(1423)\mp(1432)+2(12)(34)+2(13)(24)-4(14)(23)]/\sqrt{192}.
\end{aligned}
\tag{48}
$$

where the upper signs are for Q^1. The λ^+ and λ^- generators differ in having their odd permutation terms negated.

If we know the charges of the particles we can use the projection operators to figure out the charges of the elements of the algebra. As noted above, these charges are carried by the classes, for example, the charges of (123) are the same as the charges of (124). As shown in section 9 for the projection operators, we use the character table to obtain a set of equations for the particle charges from the charges of the group elements:

$$
\begin{aligned}
q(L^1) &= [\;\; q(()) +3q((12)(34)) +8q((123)) +6q((12)) +6q((1234))\;]/24, \\
q(L^2) &= [\;\; q(()) +3q((12)(34)) +8q((123)) -6q((12)) -6q((1234))\;]/24, \\
q(X^0) &= [\,2q(()) +6q((12)(34)) -8q((123)) \qquad\qquad\qquad\quad\;\;]/24, \\
q(Q^1) &= [\,3q(()) -3q((12)(34)) \qquad\qquad +6q((12)) -6q((1234))\;]/24, \\
q(Q^2) &= [\,3q(()) -3q((12)(34)) \qquad\qquad -6q((12)) +6q((1234))\;]/24.
\end{aligned}
\tag{49}
$$

These solve to give:

$$
\begin{aligned}
q(()) &= q(L^1) + q(L^2) + 2q(X^0) + 3q(Q^1) + 3q(Q^2), \\
q((12)(34)) &= q(L^1) + q(L^2) + 2q(X^0) - q(Q^1) - q(Q^2), \\
q((123)) &= q(L^1) + q(L^2) - q(X^0) \qquad\qquad\qquad\;\;, \\
q((12)) &= q(L^1) - q(L^2) \qquad\quad + q(Q^1) - q(Q^2), \\
q((1234)) &= q(L^1) - q(L^2) \qquad\quad - q(Q^1) + q(Q^2).
\end{aligned}
\tag{50}
$$

If we take L^1, L^2 to be the leptons and Q^1, Q^2 to be the quarks, and assume that the assignment is for the particles rather than some mixture of particles and antiparticles, there are four ways of assigning them. We can put L^1 as the electron or neutrino and Q^1 as the up or down quark. Using the equations (50) immediately above, the four cases give charges for the S_4 elements as follows:

	( )	(12)(34)	(123)	(12) or (1234)
Q	0	$-4/3$	-1	0 or ± 2
$Y_W/2$	0	$-4/3$	-1	0
T_3	0	0	0	0 or ± 2

$$\tag{51}$$

where the T_3 and $Y_W/2$ values are for the left hand states. Depending on the assignment, the Q and T_3 charges on one of (12) or (1234) is zero and the charges on the other are either $+2$ or -2.

The charges are somewhat simpler if we sum over the classes. This amounts to multiplying by the class sizes 1,3,8,6, and 6:

	( )	$\{(12)(34)\}$	$\{(123)\}$	$\{(12)\}$ or $\{(1234)\}$
Q	0	-4	-8	0 or ± 12
$Y_W/2$	0	-4	-8	0
T_3	0	0	0	0 or ± 12

$$\tag{52}$$

Other than the identity, S_4 has four classes: $\{(12)(34)\}$, $\{(123)\}$, $\{(12)\}$ and $\{(1234)\}$. Their electric charges are small integer multiples of four times that of the electron. This paper concentrates on the symmetry of the Standard Model fermions but it appears that the method might help explain their coupling to the gauge bosons.

We will use the assignment $L^1 = e$, $L^2 = \nu$, $Q^1 = u$ and $Q^2 = d$. This gives electric charge as:

$$
Q = -4[\, \{(12)(34)\} + 2\{(123)\} + 3\{(1234)\} \,]
\tag{53}
$$

See figure 2 for a plot of the Standard Model fermions by electric charge.

13 Discussion

This paper is an attempt at organizing the Standard Model fermions from physical rather than mathematical principles. The assumption here is that symmetry is something that arises from systems of equations rather than symmetry. In that sense, attributing the Standard Model to an algebra is not the end of the road. Instead, the algebra needs to arise from a simple set of equations of motion. At best, the algebra discussed here might give hints on those equations of motion.
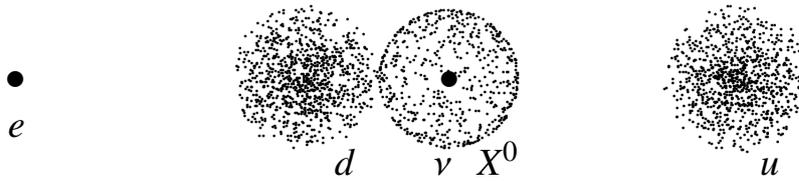
Fig. 2 3000 random Standard Model fermions at zero temperature. The leptons are the small dots. The X^0 is the SU(2) Bloch sphere centered on the neutrino. The u and d quarks are SU(3) fuzzy spheres. The horizontal axis is $Q + 0.16\sigma_x + 0.2\lambda_1^+ + 0.2\lambda_1^+$, vertical axis is $0.16\sigma_y + 0.2\lambda_2^+ + 0.2\lambda_2^-$. Equation (53) gives Q while the σs and $\lambda^\pm$s are defined below equation (46).

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