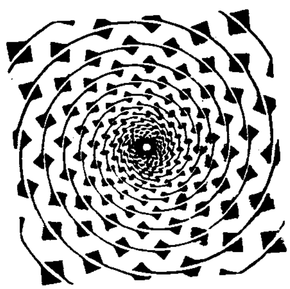
C. Dumitrescu V. Seleacu

The Smarandache Function



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Introduction

The function named in the title of this book is originated from the exiled Romanian mathematician Florentin Smarandache, who has significant contributions not only in mathematics, but also in literature. He is the father of *The Paradoxist Literary Movement* and is the author of many stories, novels, dramas, poems.

The Smarandache function, say S, is a numerical function defined such that for every positive integer n, its image $S(n)$ is the smallest positive integer whose factorial is divisible by n.

The results already obtained on this function contain some surprises. Such a surprise is the fact that to expresse $S(p^\alpha)$ the exponent α is *written* in a (generalised) numerical scale, say $[p]$, and is *"read"* in another (usual) scale, say (p) (eq. 1.21). More details on this subject may be found in section 1.2.

Another surprise is that *"the complement until the identity"* (equation 1.34) of $S(p^\alpha)$ may be expressed in a dual manner with the exponent of the prime p in the expression of $n!$, given by Legendre's formula (eq. 1.15 and eq. 1.36).

Finally, we mention that the Smarandache function may be generalised in various ways, one of these generalisations, the Smarandache function attached to a strong divisibility sequence (eq. 2.59), and particulary to Fibonacci sequence, has a dual property with the strong divisibility (theorem 2.4.7).

Of course, all these pleasant surprises are *"attractors"* for us, the mathematicians, that we are in a permanent search for new wonderful results.

But *"the attraction"* itself on the initial concept, started by Florentin Smarandache, permitted to obtain the interesting results mentioned above. Indeed, many mathematicians are already inquired about this subject and have obtained these and other results, permitting the publication of the present book. Among these we mention here Ch. Asbacher, I. Balacenoiu, P. Erdös, H. Ibstedt, P. Gronas, T. Tomita.

We mentione also two of the most interesting problems, still unsolved:

1) Find a formula expressing $S(n)$ by means of n itself and not using the decomposition of the number into primes.

2) Solve the equation $S(n) = S(n+1)$.

The (positive) answer to first of these problems will permet to have more important information on the distribution of the prime numbers.

Let the future permit to reach the knowledge until these, and other, exciting results.

THE AUTHORS

Chapter 1

The Smarandache Function

1.1 Generalised Numerical Scale

It is said that every positive integer r, strictly greater than 1, determine a numerical scale. That is, given r, every positive integer n may be written under the form:

$$n = c_m r^m + c_{m-1} r^{m-1} + \ldots + c_1 r + c_0 \tag{1.1}$$

where m and c_i are non-negative integers and $0 \leq c_i \leq r-1$, $c_m \neq 0$.

We can attach a symbol to each number from the sequence $0, 1, 2, \ldots, r-1$. These are the *digits* of the scale, and the equality (1.1) may be written as:

$$n_{(r)} = \overline{\gamma_m \gamma_{m-1} \cdots \gamma_1 \gamma_0} \tag{1.2}$$

where γ_i is the digit symbolising the number c_i.

In this manner every integer may be *uniquely* written in a numerical scale (r) and if we note $a_i = r^i$, one observe that the sequence $(a_i)_{i\in N}$ satisfies the recurrence relation

$$a_{i+1} = ra_i \tag{1.3}$$

and (1.1) becomes

$$n = c_m a_m + c_{m-1}a_{m-1} + \dots + c_1 a_1 + c_0 a_0 \tag{1.4}$$

The equality (1.4) may be generalised in the following way. Let $(b_i)_{i\in N}$ be an arbitrary increasing sequence. Then the non-negative integer n may be uniquely written under the form:

$$n = c_h b_h + c_{h-1}b_{h-1} + \dots + c_1 b_1 + c_0 b_0 \tag{1.5}$$

But the conditions satisfied by the digits in this case are not so simple as those from (1.3), satisfied for the scale determined by the sequence $(a_i)_{i\in N}$.

For instance Fibonacci sequence, determined by the conditions:

$$F_1 = F_2 = 1, \quad F_{i+2} = F_{i+1} + F_i \tag{1.6}$$

may be considered as a *generalised numerical scale*, in the sense described above.

From the inequality

$$2F_i > F_{i+1}$$

it results the advantage that the corresponding digits are only 0 and 1, as for the standard scale determined by $r = 2$.

So, using the generalised scale determined by Fibonacci sequence for representing the numbers in the memory of computers we may utilise only two states of the circuits (as when the scale (2) is used) but we need a few memory working with Fibonacci scale, because the digits are less in this case.

Another generalised scale, which we shall use in the following, is the scale determined by the sequence

$$a_i(p) = \frac{p^i - 1}{p - 1} \tag{1.7}$$

where $p > 1$ is a prime number.

Let us denote this scale by $[p]$. So we have:

$$[p]: \quad 1,\ a_2(p),\ a_3(p),\ \ldots.,\ a_i(p), \ldots \tag{1.8}$$

and the corresponding recurrence relation is:

$$a_{i+1}(p) = pa_i(p) + 1 \tag{1.9}$$

This is a relatively simple recurrence, but it is different from the classical recurrence relation (1.3).

Of course, every positive integer may be written as:

$$n_{[p]} = c_m a_m(p) + c_{m-1} a_{m-1}(p) + \ldots + c_1 a_1(p) \tag{1.10}$$

so it may be written in the scale $[p]$.

To determine the conditions satisfied by the digits c_i in this case we prove the following lemme:

1.1.1 Lemme. Let n be an arbitrary positive integer. Then for every integer $p > 1$ the number n may be written uniquely as:

$$n = t_1 a_{n_1}(p) + t_2 a_{n_2}(p) + \ldots + t_l a_{n_l}(p) \tag{1.11}$$

with $n_1 > n_2 > \ldots > n_l > 0$ and

$$1 \le t_j \le p - 1 \ \text{ for } j = 1, 2, \ldots, l-1, \ \ 1 \le t_l \le p \tag{1.12}$$

Proof. From the recurrence relation satisfied by the sequence $(a_i(p))_{n\in N^*}$ it results:

$$a_1(p) = 1,\ a_2(p) = 1 + p,\ a_3(p) = 1 + p + p^2 ...$$

So, because

$$[a_i(p),\ a_{i+1}(p)) \cap [a_{i+1}(p),\ a_{i+2}(p)) = \emptyset$$

it results

$$N^* = \bigcup_{i\in N^*} \{[a_i(p),\ a_{i+1}(p)) \cap N^*\}$$

Then for every $n \in N^*$ it exists uniquely $n_1 \geq 1$ such that $n \in [a_{n_1}(p),\ a_{n_1+1}(p))$ and we have

$$n = \left[\frac{n}{a_{n_1}(p)}\right] a_{n_1}(p) + r_1$$

where $[x]$ denote the integer part of x.

If we note

$$t_1 = \left[\frac{n}{a_{n_1}(p)}\right]$$

it results

$$n = t_1 a_{n_1}(p) + r_1 \ \text{ with } r_1 < a_{n_1}(p)$$

If $r_1 = 0$, from the inequalities

$$a_{n_1}(p) \leq n \leq a_{n_1+1}(p) - 1 \tag{1.13}$$

it results $1 \leq t_1 \leq p$.

If $r_1 \neq 0$, it exists uniquely $n_2 \in N^*$ such that

$$r_1 \in [a_{n_2}(p),\ a_{n_2+1}(p))$$

and because $a_{n_1}(p) > r_1$it results $n_1 > n_2$.

Also, because

$$t_1 \leq \frac{a_{n_1+1}(p) - 1 - r_1}{a_{n_1}(p)} < p$$

from (1.13) it results $1 \leq t_1 \leq p - 1$. Now, it exists uniquely n_2 such that

$$r_1 = t_2 a_{n_2}(p) + r_2$$

and so one. After a finite number of steps we obtain:

$$r_{l-1} = t_l a_{n_l}(p) + r_l \text{ with } r_l = 0$$

and $n_l < n_{l-1}$, $1 \leq t_l \leq p$, so the lemme is proved.

Let us observe that in (1.11) unlike from (1.10) all the digits t_i are greater than zero. Consequently all the digits c_i from (1.10) are between *zero* and $p-1$, except the last non-nul digit, which can take also the value p.

If we note by (p) the standard scale determined by the prime number p:

$$(p): \quad 1, p, p^2, p^3, ..., p^i, ... \tag{1.14}$$

it results that the difference between the recurrence relations (1.3) and (1.9) induces essential differences for the calculus in the two scales (p) and $[p]$.

Indeed, as it is proved in [1] if

$$m_{[5]} = 442, \; n_{[5]} = 412 \text{ and } r_{[5]} = 44$$

then writing

$$\begin{array}{rr} m+n+r = & 442+ \\ & 412 \\ & \underline{\ \ 44\ } \\ & dcba \end{array}$$

to determine the digits a, b, c, d we start the addition from the second column (the column corresponding to $a_2(5)$). We have

$$4a_2(5) + a_2(5) + 4a_2(5) = 5a_2(5) + 4a_2(5)$$

Now, using a unit from the first column it results:

$$5a_2(5) + 4a_2(5) = a_3(5) + 4a_2(5)$$

so (for the moment) $b = 4$.

Continuing, we get:

$$4a_3(5) + 4a_3(5) + a_3(5) = 5a_3(5) + 4a_3(5)$$

and using a new unit from the first column it results:

$$4a_3(5) + 4a_3(5) + a_3(5) = a_4(5) + 4a_3(5)$$

so $c = 4$ and $d = 1$.

Finally, adding the remainder digits:

$$4a_1(5) + 2a_1(5) = 5a_1(5) + a_1(5) = 5a_1(5) + 1 = a_2(5)$$

it results that the value of b must be modified, and $a = 0$. So

$$m + n + r = 1450_{[5]}$$

1.2 A New Function in Number Theory

This function is *the Smarandache function* $S : N^* \longrightarrow N^*$ defined by the conditions:

(s_1) $S(n)!$ is divisible by n,
(s_2) $S(n)$ is the smallest positive integerwith the property (s_1)

Let $p > 0$ be a prime number. We start by the construction of the function

$$S_p : N^* \longrightarrow N^*$$

such that

(s_3) $S_p(a_i(p)) = p^i$
(s_4) If $n \in N^*$ is written under the form given by (1.11) then $S_p(n) = t_1 S_p(a_{n_1}(p)) + t_2 S_p(a_{n_2}(p)) + \ldots + t_l S_p(a_{n_l}(p))$

1.2.1 Lemme. For every $n \in N^*$ the exponent of the prime p in the decomposition into primes of $n!$ is greater or equal to n.

Proof. From the properties of the integer part we deduce:

$$\left[\frac{a_1 + a_2 + \ldots + a_n}{b}\right] \geq \left[\frac{a_1}{b}\right] + \left[\frac{a_2}{b}\right] + \ldots + \left[\frac{a_n}{b}\right]$$

for every $a_i, b \in N^*$.

A result does to Legendre assert that the exponent of the prime p in the decomposition into primes of $n!$ is:

$$e_p(n) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots \qquad (1.15)$$

Then if n has the decomposition (1.11) it results:

$$\left[\frac{t_1p^{n_1}+t_2p^{n_2}+\ldots+t_lp^{n_l}}{p}\right] \geq \left[\frac{t_1p^{n_1}}{p}\right]+\left[\frac{t_2p^{n_2}}{p}\right]+\ldots+\left[\frac{t_lp^{n_l}}{p}\right]=$$
$$= t_1p^{n_1-1}+t_2p^{n_2-1}+\ldots+t_lp^{n_l-1}$$

....

$$\left[\frac{t_1p^{n_1}+t_2p^{n_2}+\ldots+t_lp^{n_l}}{p}\right] \geq \left[\frac{t_1p^{n_1}}{p^{n_l}}\right]+\left[\frac{t_2p^{n_2}}{p^{n_l}}\right]+\ldots+\left[\frac{t_lp^{n_l}}{p^{n_l}}\right]=$$
$$= t_1p^{n_1-n_l}+t_2p^{n_2-n_l}+\ldots+t_lp^{0}$$

....

$$\left[\frac{t_1p^{n_1}+t_2p^{n_2}+\ldots+t_lp^{n_l}}{p^{n_1}}\right] \geq \left[\frac{t_1p^{n_1}}{p^{n_1}}\right]+\left[\frac{t_2p^{n_2}}{p^{n_1}}\right]+\ldots+\left[\frac{t_lp^{n_l}}{p^{n_1}}\right]=$$
$$= t_1p^{0}+\left[\frac{t_2p^{n_2}}{p^{n_1}}\right]+\ldots+\left[\frac{t_lp^{n_l}}{p^{n_1}}\right]$$

and so

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\ldots+\left[\frac{n}{p^{n_1}}\right] \geq t_1(p^{n_1-1}+p^{n_1-2}+\ldots+p^0)+\ldots$$
$$+t_l(p^{n_l-1}+p^{n_l-2}+\ldots+p^0=$$
$$= t_1a_{n_1}(p)+t_2a_{n_2}(p)+\ldots+t_la_{n_l}(p)=n$$

1.2.2 Theorem. The function S_p defined by the conditions (s_3) and (s_4) from above satisfies:

(1) $S_p(n)!$ is divisible by p^n
(2) $S_p(n)$ is the smallest positive integer with the property (1).

Proof. The property (1) results from the preceding lemme. To prove (2) let $n \in N^*$ and $p \geq 2$ an arbitrary prime. Considering n written as in (1.11) we note

$$z = t_1p^{n_1}+t_2p^{n_2}+\ldots t_lp^{n_l}$$

and we shall prove that the number z is the smallest positive integer with the property (1).

Indeed, if there exists $u \in N^*, u < z$ such that $u!$ is divisible by p^n,then

$$u < z \Longrightarrow u \leq z - 1 \Longrightarrow (z-1)! \text{ is divisible by } p^n$$

But

$$z - 1 = t_1 p^{n_1} + t_2 p^{n_2} + \ldots + t_l p^{n_l} - 1$$

and $n_1 > n_2 > \ldots > n_l \geq 1$.

Because $[k + \alpha] = k + [\alpha]$ for every integer k, it results:

$$\left[\frac{z-1}{p}\right] = t_1 p^{n_1 - 1} + t_2 p^{n_2 - 1} + \ldots + t_l p^{n_l - 1} - 1$$

Analogously we have for instance

$$\left[\frac{z-1}{p^{n_l}}\right] = t_1 p^{n_1 - n_l} + t_2 p^{n_2 - n_l} + \ldots + t_{l-1} p^{n_{l-1} - n_l} + t_l p^0 - 1$$

$$\left[\frac{z-1}{p^{n_l+1}}\right] = t_1 p^{n_1 - n_l - 1} + t_2 p^{n_2 - n_l - 1} + \ldots + t_{l-1} p^{n_{l-1} - n_l - 1} + \left[\frac{t_l p^{n_l} - 1}{p^{n_l + 1}}\right] = t_1 p^{n_1 - n_l - 1} + \ldots t_{l-1} p^{n_{l-1} - n_l - 1}$$

because $0 < t_l p^{n_l} - 1 \leq p \cdot p^{n_l} - 1 < p^{n_l + 1}$.

Also,

$$\left[\frac{z-1}{p^{n_{l-1}}}\right] = t_1 p^{n_1 - n_{l-1}} + \ldots t_{l-1} p^0 + \left[\frac{t_l p^{n_l} - 1}{p^{n_{l-1}}}\right] = t_1 p^{n_1 - n_{l-1}} + t_{l-1} p^0$$

The last equality of this kind is:

$$\left[\frac{z-1}{p^{n_1}}\right] = t_1 p^0 + \left[\frac{t_2 p^{n_2} + \ldots + t_l p^{n_l} - 1}{p^{n_1}}\right] = t_1 p^0$$

because

$$\begin{aligned} & 0 < t_2 p^{n_2} + \ldots + t_l p^{n_l} \leq (p-1)p^{n_2} + \ldots + (p-1)p^{n_l - 1} + \\ & p \cdot p^{n_l} - 1 \leq (p-1) \sum_{i=n_{l-1}}^{n_2} p^i + p^{n_l+1} - 1 \leq (p-1)\tfrac{p^{n_2+1}}{p-1} = \\ & = p^{n_2+1} - 1 < p^{n_1} - 1 < p^{n_1} \end{aligned}$$

Indeed, for the next power of p we have

$$\left[\frac{z-1}{p^{n_1+1}}\right] = \left[\frac{t_1 p^{n_1} + t_2 p^{n_2} + \ldots + t_l p^{n_l}}{p^{n_1+1}}\right] = 0$$

because

$$0 < t_1 p^{n_1} + t_2 p^{n_2} + \ldots + t_l p^{n_l} - 1 < p^{n_1+1} - 1 < p^{n_1+1}$$

From these equalities it results that the exponent of p in the dcomposition into primes of $(z-1)!$ is

$$\begin{aligned} & \left[\tfrac{z-1}{p}\right] + \left[\tfrac{z-1}{p^2}\right] + \ldots + \left[\tfrac{z-1}{p^{n_1}}\right] = t_1(p^{n_1-1} + p^{n_1-2} + \ldots + p^0) + \ldots \\ & + t_{l-1}(p^{n_{l-1}-1} + \ldots + p^0) + t_l(p^{n_l-1} + \ldots + p^0) - n_l = n - n_l < n \end{aligned}$$

and the theorem is proved.

Now we may construct the function $S : N^* \longrightarrow N^*$ having the properties (s_1) and (s_2) as follows:

(i) $S(1) = 1$
(ii) For every $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_s^{\alpha_s}$, with $\alpha_i \geq 1$,
and p_i primes, $p_i \neq p_j$ we define:

$$S(n) = \max S_{p_i}(\alpha_i) \tag{1.16}$$

1.2.3 Theorem. The function S defined by the conditions (i) and (ii) from above satisfies the properties (s_1) and (s_2).

Proof. Let us suppose $n \neq 1$. We shall note by $M(x)$ an arbitrary multiple of x and

$$S_{p_{i_0}}(\alpha_{i_0}) = \max S_{p_i}(\alpha_i) \tag{1.17}$$

Of course,

$$S_{p_{i_0}}(\alpha_{i_0})! = M(p_{i_0}^{\alpha_{i_0}})$$

and because $S_{p_i}(\alpha_i)! = M(p_i^{\alpha_i})$ for $i = \overline{1,s}$, it results:

$$S_{p_{i_0}}(\alpha_{i_0})! = M(p_i^{\alpha_i}) \text{ for } i = \overline{1,s}$$

Moreover, because $p_i \underset{d}{\wedge} p = 1$ it results:

$$S_{p_{i_0}}(\alpha_{i_0})! = M(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s})$$

and so (s_1) is proved.

To prove (s_2) let us observe that for every $u < S_{p_{i_0}}(\alpha_{i_0})$ we have $u! \neq M(p_{i_0}^{\alpha_{i_0}})$, because $S_{p_{i_0}}(\alpha_{i_0})$ is the smallest positive integer with the property $k! = M(p_{i_0}^{\alpha_{i_0}})$. So,

$$u! \neq M(\, p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_s^{\alpha_s}) = M(n)$$

and the property (s_2) is proved.

1.2.4 Proposition. For every prime p the function S_p is increasing and surjective, but not injective. The function S is *generaly increasing*, in the sense that:

$$(\forall)\; n \in N^* \; (\exists)\; k \in N^* \; S(k) \geq n$$

and it is surjective but not injective.

1.2.5 Consequences. 1) For every $\alpha \in N^*$ holds:

$$S_p(\alpha) = S(p^\alpha) \tag{1.18}$$

2) For every $n > 4$ we have:

$$n \text{ is a prime } \Longleftrightarrow S(n) = n$$

Indeed, if $n \geq 5$ is a prime then $S(n) = S_n(1) = n$.

Conversely, if $n > 4$ is not a prime but $S(n) = n$, let $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_s^{\alpha_s}$ with $s \geq 2$, $\alpha_i \in N^*$, for $i = \overline{1,s}$. Then if $S_{p_{i_0}}(\alpha_{i_0})$ is given by (1.17), from Legendre's formula (1.15) it results the contradiction:

$$S_{p_{i_0}}(\alpha_{i_0}) < \alpha_{i_0} p_{i_0} < n$$

Also, if $n = p^\alpha$, with $\alpha \geq 2$, it results:

$$S(n) = S_p(\alpha) \leq p \cdot \alpha < p^\alpha = n$$

and the theorem is proved.

1.2.6 Examples. 1) If $n = 2^{31} \cdot 3^{27} \cdot 7^{13}$ we have:

$$S(n) = \max\{S_2(31), S_3(27), S_7(13)\} \tag{1.19}$$

and to calculate $S_2(31)$ we consider the generalised numerical scale

$$[2]: \quad 1,\ 3,\ 7,\ 31,\ 63, ...$$

Then $31 = 1 \cdot a_5(2)$, so $S_2(31) = 1 \cdot 2^5 = 32$.

For the calculus of $S_3(27)$ we consider the scale

$$[3]: \quad 1,\ 4,\ 13,\ 40, ...$$

and we have $27 = 2 \cdot 13 + 1 = 2a_3(3) + a_1(3)$ so

$$S_3(27) = S_3(2a_3(3) + a_1(3)) = 2S_3(a_3(3)) + S_3(a_1(3)) =$$
$$= 2 \cdot 3^3 + 1 \cdot 3^1 = 57$$

Finally, to calculate $S_7(13)$ we consider the generalised scale

$$[7]: \quad 1,\ 8,\ 57,\ \ldots.$$

and it results

$$13 = a_2(7) + 5a_1(7), \text{ so } S_7(13) = 1 \cdot S_7(8) + 5 \cdot S_7(1) = \\ = 1 \cdot 7^2 + 5 \cdot 7 = 84$$

From (1.19) one deduces $S(n) = 84$. So 84 is the smallest positive integer whose factorial is divisible by $2^{31} \cdot 3^{27} \cdot 7^{13}$.

2) Which are the numbers with the factorial ending in 1000 zeros?

To answer this question we observe that for $n = 10^{1000}$ it results $S(n)! = M(10^{1000})$ and $S(n)$ is the smallest positive integer whose factorial endsin 1000 zeros.

We have $S(n) = S(2^{1000} \cdot 5^{1000}) = \max\{S_2(1000), S_5(1000)\} = S_5(1000)$.

Considering the generalised numerical scale

$$[5]: \quad 1,\ 6,\ 31,\ 156,\ 781, \ldots$$

it results:

$$S_5(1000) = S_5(a_5(5) + a_4(5) + 2a_3(5) + a_1(5)) = \\ = 5^5 + 5^4 + 2 \cdot 5^3 + 5 = 4005$$

The numbers $4006, 4007, 4008, 4009$ have also the required property, but the factorial of 4010 ends in 1001 zeros.

To calculate $S(p^\alpha)$ we need to writte the exponent α in the generalised scale $[p]$. For this we observe that:

$$a_m(p) \leq \alpha \iff (p^m - 1)/(p - 1) \leq \alpha \iff \\ p^m \leq (p-1)\alpha + 1 \iff m \leq \log_p((p-1)\alpha + 1)$$

and if

$$\alpha_{[p]} = k_v a_v(p) + k_{v-1} a_{v-1}(p) + \ldots + k_1 a_1(p) = \overline{k_v k_{v-1} \ldots k_1} \quad (1.20)$$

is the expression of the exponent α in the scale $[p]$, then v is the integer part of $\log_p((p-1)\alpha + 1)$ and *the digit* k_v is obtained by the equality

$$\alpha = k_v a_v(p) + r_{v-1}$$

Using the same procedure for r_{v-1} it results the next non-zero digit from (1.20)

1.3 Some Formulae for the Calculus of S(n)

From the property (s_4) satisfied by the function S_p, one deduce:

$$S(p^\alpha) = p(\alpha_{[p]})_{(p)} \quad (1.21)$$

that is the value of $S(p^\alpha)$ is obtained multiplying the prime p by the number obtained writing the exponent α in the generalised scale $[p]$ and "reading" it in the usual scale (p).

1.3.1 Example. To calculate $S(11^{1000})$ we consider first the generalised scale

$$[11]: \quad 1,\ 12,\ 133,\ 1464, \ldots$$

Using the considerations from the end of the preceding section we get:

$$1000 = 7a_3(11) + 5a_2(11) + 9a_1(11) = 759_{[11]}$$

so $S(11^{1000}) = 11(759)_{(11)} = 11(7\cdot 11^2+5\cdot 11+9) = 10021$. Consequently 10021 is the smallest positive integer whose factorial is divisible by 11^{1000}.

The equality (1.21) prove the importance of the scales (p) and $[p]$ for the calculus of $S(n)$.

Let now

$$\alpha_{(p)} = \sum_{i=0}^{u} c_i p^i, \quad \alpha_{[p]} = \sum_{j=1}^{v} k_j a_j(p) = \sum_{j=1}^{v} k_j \frac{p^j - 1}{p-1} \tag{1.22}$$

be the expression of the the exponent α in the two scales. It results:

$$(p-1)\alpha = \sum_{j=1}^{v} k_j p^j - \sum_{j=1}^{v} k_j$$

Then noting

$$\sigma_{(p)}(\alpha) = \sum_{i=0}^{u} c_i, \quad \sigma_{[p]}(\alpha) = \sum_{j=1}^{v} k_j \tag{1.23}$$

and taking into acount that $\sum\limits_{j=1}^{v} k_j p^j = p \sum\limits_{j=0}^{v-1} k_j p^j$ is exactly $p(\alpha_{[p]})_{(p)}$, one obtain

$$S(p^\alpha) = (p-1)\alpha + \sigma_{[p]}(\alpha) \tag{1.24}$$

Using the first equality from (1.23) we get:

$$p\alpha_{(p)} = \sum_{i=0}^{u} c_i(p^{i+1} - 1) + \sum_{i=0}^{u} c_i$$

or

$$\frac{p}{p-1}\alpha = \sum_{i=0}^{u} c_i a_{i+1}(p) + \frac{1}{p-1}\sigma_{(p)}(\alpha)$$

consequently

$$\alpha = \frac{p-1}{p}(\alpha_{(p)})_{[p]} + \frac{1}{p}\sigma_{(p)}(\alpha) \tag{1.25}$$

where $(\alpha_{(p)})_{[p]}$ denote the number obtained writing the exponent α in the scale (p) and reading it in the scale $[p]$.

Replacing this expression of α in (1.24) we get:

$$S(p^\alpha) = \frac{(p-1)^2}{p}(\alpha_{(p)})_{[p]} + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \tag{1.26}$$

One may obtain also a connection between $S(p^\alpha)$ and the exponent $e_p(\alpha)$ defined by Legendre's formula (1.15). It is said that $e_p(\alpha)$ may be expressed also as:

$$e_p(\alpha) = \frac{\alpha - \sigma_{(p)}(\alpha)}{p-1} \tag{1.27}$$

so using (1.25) one get:

$$e_p(\alpha) = (\alpha_{(p)})_{[p]} - \alpha \tag{1.28}$$

An other formula for $e_p(\alpha)$ may be obtained as follows: if α given by the first equality from (1.22) is:

$$\alpha_{(p)} = c_u p^u + c_{u-1}p^{u-1} + \dots + c_1 p + c_0 \tag{1.29}$$

then because

$$e_p(\alpha) = \left[\tfrac{\alpha}{p}\right] + \left[\tfrac{\alpha}{p^2}\right] + \dots + \left[\tfrac{\alpha}{p^u}\right] = (c_u p^{u-1} + c_{u-1}p^{u-2} + \dots + c_1) + \\ +(c_u p + c_{u-1}) + c_u$$

we get:

$$e_p(\alpha) = ((\alpha - c_0)_{(p)})_{[p]} = ((\left[\frac{\alpha}{p}\right])_{(p)})_{[p]} \tag{1.30}$$

where $\alpha_{(p)} = \overline{c_u c_{u-1} \ldots c_0}$ is the expression of α in the scale (p).

From (1.26) and (1.28) it results:

$$S(p^\alpha) = \frac{(p-1)^2}{p}(e_p(\alpha) + \alpha) + \frac{p-1}{p}\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \quad (1.31)$$

Using the equalities (1.21) and (1.26) one deduce a connection between the following two numbers:

$(\alpha_{(p)})_{[p]} =$ the number α written in the scale (p) and readed in the scale $[p]$
$(\alpha_{[p]})_{(p)} =$ the number α written in the scale $[p]$ and readed in the scale(p)

namely:

$$p^2(\alpha_{[p]})_{(p)} - (p-1)^2(\alpha_{(p)})_{[p]} = p\sigma_{[p]}(\alpha) + (p-1)\sigma_{(p)}(\alpha) \quad (1.32)$$

To obtain other expressions for $S(p^\alpha)$ let us observe that from Legendre's formula (1.15) it results:

$$S(p^\alpha) = p(\alpha - i_p(\alpha)) \text{ with } 0 \leq i_p(\alpha) \leq [\frac{\alpha - 1}{p}] \quad (1.33)$$

Then using for $S(p^\alpha)$ the notation $S_p(\alpha)$ one obtain:

$$\frac{1}{p}S_p(\alpha) + i_p(\alpha) = \alpha \quad (1.34)$$

and so, for each function S_p there exists a function $i_p$such that the linear combination (1.34), to obtain the identity, holds.

To obtain expressions of i_p let us observe that from (1.27) it results:

$$\alpha = (p-1)e_p(\alpha) + \sigma_{(p)}(\alpha)$$

and from (1.24) it results $\alpha = (S_p(\alpha) - \sigma_{[p]}(\alpha))/(p-1)$, so

$$(p-1)e_p(\alpha) + \sigma_{(p)}(\alpha) = \frac{S_p(\alpha) - \sigma_{[p]}(\alpha)}{p-1}$$

or

$$S(p^\alpha) = (p-1)^2 e_p(\alpha) + (p-1)\sigma_{(p)}(\alpha) + \sigma_{[p]}(\alpha) \qquad (1.35)$$

Let us return now to the function i_p and observe that from (1.24) and (1.34) it results:

$$i_p(\alpha) = \frac{\alpha - \sigma_{[p]}(\alpha)}{p} \qquad (1.36)$$

consequently we can say that there exists a duality between the expression of $e_p(\alpha)$ in (1.27) and the above expression of $i_p(\alpha)$.

One may obtain other connections between i_p and e_p.For instance from (1.27) and (1.36) it results:

$$i_p(\alpha) = \frac{(p-1)e_p(\alpha) + \sigma_{(p)}(\alpha) - \sigma_{[p]}(\alpha)}{p} \qquad (1.37)$$

Also, from

$$\alpha_{[p]} = \overline{k_v k_{v-1} \ldots k_1} = k_v(p^{v-1} + p^{v-2} + \ldots + 1) + k_{v-1}(p^{v-2} + p^{v-3} + \ldots + p + 1) + \ldots. + k_2(p+1) + k_1$$

one obtain

$$\alpha = (k_v p^{v-1} + k_{v-1}p^{v-2} + \ldots + k_2 p + k_1) + k_v(p^{v-2} + p^{v-3} + \ldots + 1) + k_{v-1}(p^{v-3} + p^{v-4} + \ldots + 1) + \ldots + k_3(p+1) + k_2 = (\alpha_{[p]})_{(p)} + \left[\frac{\alpha}{p}\right] - \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]$$

that because

$$\left[\frac{\alpha}{p}\right] = k_v(p^{v-2} + p^{v-3} + \ldots + p + 1) + \frac{k_v}{p} + k_{v-1}(p^{v-3} + \\ + p^{v-4} + \ldots + p + 1) + \frac{k_{v-1}}{p} + \ldots. + k_3(p+1) + \\ + \frac{k_3}{p} + k_2 + \frac{k_2}{p} + \frac{k_1}{p}$$

and $[n + x] = n + [x]$.

One obtain

$$\alpha = (\alpha_{[p]})_{(p)} + \left[\frac{\alpha}{p}\right] - \left[\frac{\sigma_{[p]}(\alpha)}{p}\right] \tag{1.38}$$

and we can writte:

$$S(p^\alpha) = p(\alpha - (\left[\frac{\alpha}{p}\right] - \left[\frac{\sigma_{[p]}(\alpha)}{p}\right])) \tag{1.39}$$

and from 1.36) and (1.39) we dededuce

$$i_p(\alpha) = \left[\frac{\alpha}{p}\right] - \left[\frac{\sigma_{[p]}(\alpha)}{p}\right] \tag{1.40}$$

This equality results also directly, from (1.36), taking into acount that

$$\frac{m-n}{p} \in N \Longrightarrow \frac{m-n}{p} = \left[\frac{m}{p}\right] - \left[\frac{n}{p}\right]$$

consequently

$$\frac{\alpha - \sigma_{[p]}(\alpha)}{p} = \left[\frac{\alpha}{p}\right] - \left[\frac{\sigma_{[p]}(\alpha)}{p}\right]$$

An other expression of $i_p(\alpha)$ is obtained from (1.21) and (1.36) or from (1.38) and (1.40). Namely

$$i_p(\alpha) = \alpha - (\alpha_{[p]})_{(p)} \tag{1.41}$$

From the definition of the function S it results:

$$S_p(e_p(\alpha)) = p\left[\frac{\alpha}{p}\right] = \alpha - \alpha_p$$

where α_p is the remainder of α modulo p, and also:

$$e_p(S_p(\alpha)) \geq \alpha,\ e_p(S_p(\alpha) - 1) < \alpha \qquad (1.42)$$

so

$$\frac{S_p(\alpha) - \sigma_{(p)}(S_p(\alpha))}{p-1} \geq \alpha,\ \ \frac{S_p(\alpha) - 1 - \sigma_{(p)}(S_p(\alpha) - 1)}{p-1} < \alpha$$

Using (1.24) it results that $S_p(\alpha)$ is the unique solution of the system:

$$\sigma_{(p)}(x) \leq \sigma_{[p]}(\alpha) \leq \sigma_{(p)}(x-1) + 1 \qquad (1.43)$$

At the end of this section we return to the function i_p, to find an asimpthotic behaviour for this "complement until the identity" of the function S_p.

From the conditions satisfied by this function in (1.33) it results for

$$\Delta(\alpha, p) = \left[\frac{\alpha - 1}{p}\right] - i_p(\alpha)$$

that $\Delta(\alpha, p) \geq 0$.

To find an expression for this function we observe that:

$$\left[\frac{\alpha-1}{p}\right] - i_p(\alpha) = \left[\frac{\alpha-1}{p}\right] - \left[\frac{\alpha}{p}\right] + \left[\frac{\sigma_{[p]}(\alpha)}{p}\right] \qquad (1.44)$$

and supposing that $\alpha \in [hp+1, hp+p-1]$ it results $\left[\frac{\alpha-1}{p}\right] = \left[\frac{\alpha}{p}\right]$, so:

$$\Delta(\alpha, p) = \left[\frac{\alpha - 1}{p}\right] - i_p(\alpha) = \left[\frac{\sigma_{[p]}(\alpha)}{p}\right] \tag{1.45}$$

Also, if $\alpha = hp$, it results

$$\left[\frac{\alpha - 1}{p}\right] = \left[\frac{hp - 1}{p}\right] = h - 1 \quad \text{and} \quad \left[\frac{\alpha}{p}\right] = h$$

so (1.44) becomes:

$$\Delta(\alpha, p) = \left[\frac{\sigma_{[p]}(\alpha)}{p}\right] - 1 \tag{1.46}$$

Analogously, if $\alpha = hp + p$, one obtains

$$\left[\frac{\alpha - 1}{p}\right] = \left[h + 1 - \frac{1}{p}\right] = h$$

and $\left[\frac{\alpha}{p}\right] = h + 1$, so (1.44) has the form (1.46).

It results that for every α for which $\Delta(\alpha, p)$ has the form (1.45) or (1.46), the value of $\Delta(\alpha, p)$ is maximum if $\sigma_{[p]}(\alpha)$ is maximum, so for $\alpha = \alpha_M$, where

$$\alpha_M = \overline{\underbrace{(p-1)(p-1)\ldots(p-1)}_{v \text{ terms}}\, p}_{[p]}$$

We have then

$$\begin{gathered}\alpha_m = (p-1)a_v(p) + (p-1)a_{v-1}(p) + \ldots + (p-1)a_2(p) + p = \\ (p-1)\left(\tfrac{p^v - 1}{p-1} + \tfrac{p^{v-1} - 1}{p-1} + \ldots + \tfrac{p^2 - 1}{p-1}\right) + p = \\ (p^v + p^{v-1} + \ldots + p^2 + p) - (v-1) = pa_v(p) - (v-1)\end{gathered}$$

It results that α_M is not divisible by p if and only if $v - 1$ is not divisible by p. In this case

$$\sigma_{[p]}(\alpha_M) = (v-1)(p-1) + p = pv - v + 1$$

and

$$\Delta(\alpha_M, p) = \left[\frac{\sigma_{[p]}(\alpha_M)}{p}\right] = \left[v - \frac{v-1}{p}\right] = v - \left[\frac{v-1}{p}\right]$$

So,

$$i_p(\alpha_M) \geq \left[\frac{\alpha_M - 1}{p}\right] - v$$

that is

$$i_p(\alpha_M) \in [\left[\frac{\alpha_M - 1}{p}\right] - v, \left[\frac{\alpha_M - 1}{p}\right]]$$

If $v - 1 \in (hp, hp + p)$ it results $[\frac{v-1}{p}] = h$, and

$$h(p-1) + 1 < \Delta(\alpha_M, p) < h(p-1) + p + 1$$

so

$$\lim_{\alpha_M \longrightarrow \infty} \Delta(\alpha_M, p) = \infty$$

We also observe that

$$\left[\frac{\alpha_M - 1}{p}\right] = a_v(p) - \left[\frac{v-1}{p}\right] =$$

$$= \frac{p^{v+1} - 1}{p-1} - \left[\frac{v-1}{p}\right] \in [\frac{p^{hp+1} - 1}{p-1} - h, \frac{p^{hp+p+1} - 1}{p-1} - h]$$

So, if $\alpha_M \longrightarrow \infty$ as p^x then $\Delta(\alpha_M, p) \longrightarrow \infty$ as x.
Also, from

$$\frac{i_p(\alpha_M)}{\left[\frac{\alpha_M - 1}{p}\right]} = \frac{a_v(p) - v}{a_v(p) - \left[\frac{v-2}{p}\right]} \longrightarrow 1$$

it results

$$\lim_{\alpha \longrightarrow \infty} \frac{i_p(\alpha)}{\left[\frac{\alpha-1}{p}\right]} = 1$$

1.4 Connections with Some Classical Numerical Functions

In this section we shall present some connections of Smarandache function with Euller's totient function, von Mangolt's function, Riemann's function and the function $\Pi(x)$ denoting the number of primes not greater than x.

1.4.1 Definition. The function of von Mangolt is:

$$\Lambda(n) = \begin{cases} \ln n & \text{if } n = p^m \\ 0 & \text{if } n \neq p^m \end{cases} \tag{1.47}$$

This function is not a multiplicative function, that is from $n \overset{d}{\vee} m = 1$ does not result $\Lambda(n \cdot m) = \Lambda(n) \cdot \Lambda(m)$. For instance, if $n = 3$ and $m = 5$ we have $\Lambda(n) = \ln 3, \Lambda(m) = \ln 5$ and $\Lambda(m \cdot n) = \Lambda(15) = 0$.

We remember the following results:

1.4.2 Theorem. The following equalities hold:

$$(i) \ \sum_{d/n} \Lambda(d) = \ln n$$

$$(ii) \ \Lambda(n) = \sum_{d/n} \mu(d) \ln \frac{n}{d}$$

where μ is *Mõbius function*, defined by:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^k & \text{if } n - p_1 \cdot p_2 \cdot \ldots \cdot p_k \end{cases} \tag{1.48}$$

1.4.3 Definition. The function $\Psi : R \longrightarrow R$ is defined by:

$$\Psi(x) = \sum_{p^m \leq x} \ln p \tag{1.49}$$

From the properties of this function we mention only the following two:

1.4.4 The function Ψ satisfies:

$$(i)\ \Psi(x) = \sum_{n \leq x} \Lambda(n)$$
$$(ii)\ \Psi(x) = \ln[1, 2, 3, ..., [x]]$$

where $[1, 2, 3, ..., [x]]$ denotes the lowest common multiple of $1, 2, 3, ..., [x]$.

It is said that on the set N^* of the positive integers one may consider two latticeal structures:

$$\mathcal{N}_o = (N^*, \wedge, \vee) \quad \text{and} \quad \mathcal{N}_d = (N^*, \underset{d}{\wedge}, \overset{d}{\vee}) \tag{1.50}$$

where

$\wedge = \min$, $\vee = \max$

$\underset{d}{\wedge} =$ the greatest common divisor

$\overset{d}{\vee} =$ the lowest common multiple

We shall note also $n \underset{d}{\wedge} m = (n, m)$ and $n \overset{d}{\vee} m = [n, m]$.

The order in the lattice $\mathcal{N}_o$ is noted by $\leq$ and the order from $\mathcal{N}_d$ is noted by $\underset{d}{\leq}$. It is said that:

$$n_1 \underset{d}{\leq} n_2 \Longleftrightarrow n_1 \text{ divides } n_2 \Longleftrightarrow n_1/n_2 \tag{1.51}$$

and we also observe that the Smarandache function is not a *monotonous* function:

$$n_1 \leq n_2 \ \text{ does not implique } \ S(n_1) \leq S(n_2)$$

But, taking into account that

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2) \tag{1.52}$$

we can consider the function S as a function defined on the lattice $\mathcal{N}_d$ with values in the lattice N_o:

$$S : \mathcal{N}_d \longrightarrow \mathcal{N}_o \tag{1.53}$$

In this way the Smarandache function becomes an *order preserving* function, in the sense that:

$$n_1 \underset{d}{\leq} n_2 \Longrightarrow S(n_1) \leq S(n_2) \tag{1.54}$$

It is said [31] that if $(V, \wedge, \vee)$ is a finite lattice, $V = \{x_1, x_2, ..., x_n\}$, with the induced order $\prec$, then for every function $f : V \longrightarrow R$, the corresponding generating function is defined by:

$$F(n) = \sum_{y \prec n} f(y) \tag{1.55}$$

Now we may return to von Mangolt's function. Let us observe that to every function:

$$f : N^* \longrightarrow N^* \tag{1.56}$$

one may attach two generating functions, namely the generating functions F^d and F^o determined by the lattices $\mathcal{N}_d$ and $\mathcal{N}_o$.

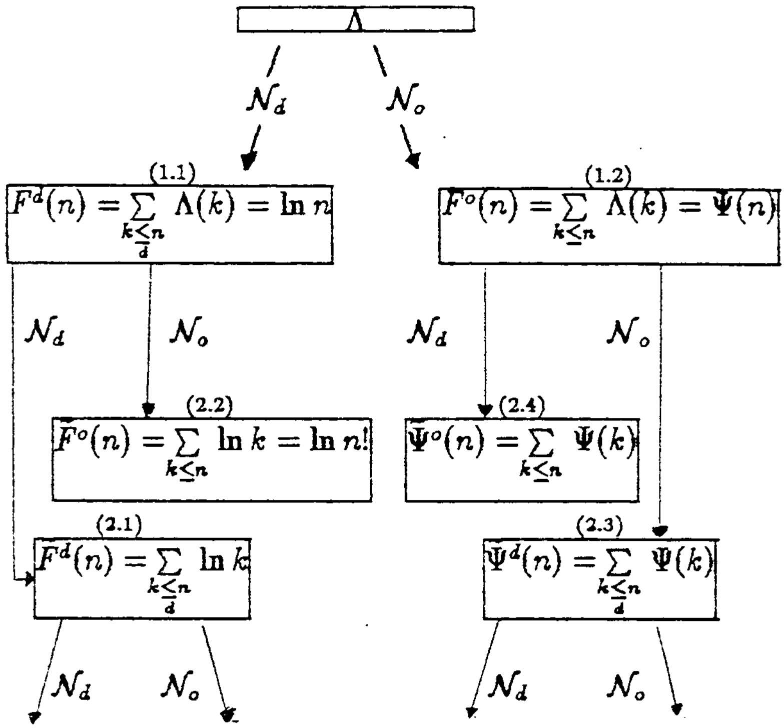
Then, by the theorem (1.4.2), for $f(x) = \Lambda(x)$ it results:

$$F^d(n) = \sum_{k \underset{d}{\leq} n} \Lambda(k) = \ln n \tag{1.57}$$

and also

$$F^o(n) = \sum_{k \leq n} \Lambda(k) = \Psi(n) = \ln[1, 2, ..., n]$$

Then it results the following diagram:



....

It results a strong connection between the definition of the Smarandache function S and the equalities (1.1) and (2.2) from this diagram.

Let f from (1.56) be the function of von Mangolt's. Then

$$[1, 2, ..., n] = e^{F(n)} = e^{f(1)} \cdot e^{f(2)} ... e^{f(n)} = e^{\Psi(n)}$$
$$n! = e^{\tilde{F}(n)} = e^{F^d(1)} \cdot e^{F^d(2)} ... e^{F^d(n)}$$

and so, using the definition of S, we are conducted to consider functions of the form:

$$\gamma(n) = \min\{m \;/\; n \underset{d}{\leq} [1, 2, ...m]\} \qquad (1.58)$$

We shall study this kind of functions in the section 2.2 of the following chapter.

Returning now to the idea of finding connections between the Smarandache function and some classical numerical functions, we present such a connection, with Euller's function φ. Let us remeber that if p is a prime number then:

$$\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} \qquad (1.59)$$

and for $\alpha \geq 2$ we have

$$p^{\alpha-1} = (p-1)a_{\alpha-1}(p) + 1 \quad \text{so} \quad \sigma_{[p]}(p^{\alpha-1}) = p$$

Using the equality (1.24) it results:

$$S_p(p^{\alpha-1}) = (p-1)p^{\alpha-1} + \sigma_{[p]}(p^{\alpha-1}) = \varphi(p^\alpha) + p \qquad (1.60)$$

1.4.5 Definition. Let C be the set of all complex numbers.Then the Dirichlet series attached to a function

$$f : N^* \longrightarrow C$$

is

$$D_f(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^z} \qquad (1.61)$$

For some $z = x + iy$ this series may be convergent or not.
The simplest Dirichlet series is:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

named *the function of Riemann* or *zeta function*. This function converges for $Re(z) > 1$.

It is said that the Diriclet series attached to Möbius function μ is:

$$D_\mu(z) = \frac{1}{\zeta(z)} \quad \text{for} \quad Re(z) > 1$$

and the Diriclet series attached to Euller's function φ is:

$$D_\varphi(z) = \frac{\zeta(z-1)}{\zeta(z)} \quad \text{for} \quad Re(z) > 2$$

We also have:

$$D_\tau(z) = \zeta^2(z) \quad \text{for} \quad Re(z) > 1$$

where $\tau(n)$ is *the number of divisors of* n, including 1and n.

More general,

$$D_{\sigma_k}(z) = \zeta(z) \cdot \zeta(z-k) \quad \text{for} \quad Re(z) > k+1$$

where $\sigma_k(n)$ is the sum of $k^{th}-$ powers of the divisors of n.

In the sequel we shall writte $\sigma(n)$ instead of $\sigma_1(n)$ and $\tau(n)$ instead of $\sigma_0(n)$. We also suppose that $z = x$, so z is a real number.

1.4.6 Theorem. If

$$n = \prod_{i=1}^{t_n} p_i^{\alpha_{i_n}}$$

is the decomposition of n into primes then the Smarandache function and Riemann's function are linked by the following equality:

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{n \geq 1} \prod_{i=1}^{t_n} \frac{S_{p_i}(p_i^{\alpha_{i_n}-1}) - p_i}{p_i^{x\alpha_{i_n}}} \tag{1.62}$$

Proof. We have seen that between the functions φ and ζ there exists a connection given by:

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{n\geq 1} \frac{\varphi(n)}{n^x} \tag{1.63}$$

Moreover,

$$\varphi(n) = \prod_{i=1}^{t_n} \varphi(p_i^{\alpha_i}) = \prod_{i=1}^{t_n} (S_{p_i}(p_i^{\alpha_{i_n}-1}) - p_i)$$

and replacing this expression of $\varphi(n)$ in (1.63) it results the equality (1.62).

The Dirichlet series corresponding to the function S is:

$$D_S = \sum_{n=1}^{\infty} \frac{S(n)}{n^x}$$

and noting by $D_{F_S^d}$ the Dirichlet series attached to the generating function F_S^d it results:

1.4.7 Theorem. For every $x > 2$ we have:

$$(i)\ \zeta(x) \leq D_S(x) \leq \zeta(x-1)$$
$$(ii)\ \zeta^2(x) \leq D_{F_S^d}(x) \leq \zeta(x)\cdot\zeta(x-1)$$

Proof. The inequalities (i) result from the fact that

$$1 \leq S(n) \leq n \ \text{ for every } \ n \in N^* \tag{1.64}$$

(ii) We have:

$$\zeta(x)\cdot D_S(x) = \left(\sum_{k=1}^{\infty} \frac{1}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = S91) + \frac{S(1)+S(2)}{2^x} +$$
$$+ \frac{S(1)+S(3)}{3^x} + \frac{S(1)+S(2)+S(4)}{4^x} + \ldots.. = D_{F_S^d}(x)$$

and the inequalities results using (i).

One observe that (ii) is equivalent with

$$D_\tau(x) \leq D_{F_S^d}(x) \leq D_\sigma(x)$$

This equality may be also deduced observing that from (1.64) it results:

$$\sum_{k \leq \frac{n}{d}} 1 \leq \sum_{k \leq \frac{n}{d}} S(k) \leq \sum_{k \leq \frac{n}{d}} k$$

and consequently:

$$\tau(n) \leq F_S^d(n) \leq \sigma(n) \tag{1.65}$$

In [19] has been proved for F_S^d even that:

$$\tau(n) \leq F_s^d(n) \leq n + 4$$

To prove other inequalities satisfied by the Dirichlet series D_S we remember first that if f and g are two unbounded functions defined on the set R of real numbers satisfying $g(x) > 0$, and if there exist the constants C_1, C_2 such that

$$/f(x)/ < C_1 g(x) \text{ for every } x > C_2$$

then the functions f and g are said to be of the same order of magnitude and one note

$$f(x) = O(g(x))$$

Particularly, is noted by $O(1)$ any function which is bounded for $x > C_2$.

The fact that it exists

$$\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = 0$$

is noted by

$$f(x) = o(g(x))$$

Particularly is noted by $o(1)$ any function tending to zero when x tends to infinity and evidently we have:

$$f(x) = o(g(x)) \Longrightarrow f(x) = O(g(x))$$

It is said that Rieman's function satisfies the properties given bellow:

1.4.8 Theorem. For all complex number z we have:

$$\begin{array}{l}(i)\ \zeta(z) + \frac{1}{z-1} + O(1)\\ (ii)\ \ln\zeta(z) = \ln\frac{1}{z-1} + O(z-1)\\ (iii)\ \zeta'(z) = -\frac{1}{(z-1)^2} + O(1)\end{array}$$

Using the theorems (1.4.7) and (1.4.8) now we obtain:

1.4.9 Theorem. The Dirichlet series D_S attached to the Smarandache function S and his derivative D'_S satisfy:

$$\begin{array}{l}(i)\ \frac{1}{x-1} + O(1) \leq D_S(x) \leq \frac{1}{x-2} + O(1)\\ (ii)\ -\frac{1}{(x-1)^2} + O(1) \leq D'_S(x) \leq -\frac{1}{(x-1)^2} + O(1)\end{array}$$

The number of primes not exceeding a given number x is usually denoted by $\Pi(x)$. In [39] is given a connection between the Smarandache function S and the function Π.

Starting from the fact that $S(n) \leq n$ for every n and that, for $n > 4$ we have $S(n) = n$ if and only if n is a prime, it is obtained the equality:

$$\Pi(x) = \sum_{k=2}^{[x]} \left[\frac{S(k)}{k}\right] - 1.$$

1.5 The Smarandache Function as Generating Function

It is said that Möbius inversion formula permet to obtain any numerical function f from his generating function F^d. Namely,

$$f(n) = \sum_{d/n} \mu(d) F^d\left(\frac{n}{d}\right) \tag{1.66}$$

if

$$F^d(n) = \sum_{d/n} f(d)$$

So, we can consider every numerical function f in two distinct positions: one is that in which we are interested to consider its generating function, and in the second we consider the function f itself as a generating function, for some numerical function g.

$$g(n) = \sum_{d/n} \mu(d) f\left(\frac{n}{d}\right) \longleftarrow \boxed{f} \longrightarrow F^d(n) = \sum_{d/n} f(d) \tag{1.67}$$

For instance if $f(n) = n$ is the identity map of N^* we get:

$$g(n) = \sum_{d/n} \mu(d) \frac{n}{d} = \varphi(n) \ ; \ \ F^d(n) = \sum_{d/n} d = \sigma(d) \tag{1.68}$$

In the case when f is the Smarandache function S, it is difficult to calculate for any positive integer n the value of $F_S^d(n)$. That because :

$$F_S^d(n) = \sum_{d/n} S(d) = \sum_{d/n} \max(S(\delta_i^{\beta_i}) \tag{1.69}$$

where δ_i are the prime factors of d.

However, there are two situations in which the explicite forme of $F_S^d(n)$ may be obtained easily. These are for $n = p^\alpha$ and for n a square free number.

In the first case we have

$$\begin{aligned} F_S^d(p^\alpha) &= \sum_{j=1}^{\alpha} S(p^j) = \sum_{j=1}^{\alpha} ((p-1)j + \sigma_{[p]}(j)) = \\ &= (p-1)\tfrac{\alpha(\alpha-1)}{2} + \sum_{j=1}^{\alpha} \sigma_{[p]}(j) \end{aligned} \tag{1.70}$$

Let consider $n = p_1 \cdot p_2 \ldots\ldots p_k$ a square free number, where $p_1 < p_2 < \ldots < p_k$ are the prime factors of n. It results:

$$S(n) = p_k \text{ and}$$
$$F_S^d(1) = S(1) = 1$$
$$F_S^d(p_1) = S(1) + S(p_1) = 1 + p_1$$
$$F_S^d(p_1 \cdot p_2) = S(1) + S(p_1) + S(p_2) + S(p_1 \cdot p_2) = 1 + p_1 + 2p_2$$
$$F_S^d(p_1 \cdot p_2 \cdot p_3) = 1 + p_1 + 2p_2 + 2^3 p_3 + F_S^d(p_1 \cdot p_2) + 2^3 p_3$$

and also:

$$F_S^d(n) = 1 + F_S^d(p_1 \cdot p_2 \ldots\ldots p_{k-1}) + 2^{k-1} p_k$$

Then

$$F_S^d(n) = 1 + \sum_{i=1}^{k} 2^{i-1} p_i \tag{1.71}$$

One observe that because $S(n) = p_k$, replacing the values of $F_S^d(t)$ given by (1.71) in

$$S(n) = \sum_{r.t=n} \mu(r) F_S^d(t) \tag{1.72}$$

apparently we get an expression of the prime number p_k by means of the preceding primes $p_1, p_2, \ldots p_{k-1}$. In reality (1.72) is an identity in which ,after the reduction of all similar terms, the prime numbers p_i has the coefficient equal to zero.

In [19] it is solved the equation

$$F_S^d(n) = n \tag{1.73}$$

under the hypothesis

$$S(1) = 0 \tag{1.74}$$

and it is found the following result:

1.5.1 Proposition. The equation (1.73) has as solutions only: all the prime numbers n and the composit numbers $n = 9, 16, 24$.

Proof. Because

$$F_S^d(n) = \sum_{d/n} S(d) \tag{1.75}$$

under the hypothesis (1.74) one observe that every prime is a solution of our equation. Let now suppose $n > 4$ be a composit number:

$$n = \prod_{i=1}^{k} p_i^{r_i}$$

where the primes p_i and the exponents r_i are ranged such that

(c_1) $p_1 r_1 \geq p_i r_i$ for every $i \in \{1, 2, ..., k\}$
(c_2) $p_i < p_{i+1}$ for $i \in \{2, 3, ..., k-1\}$ whenever $k \geq 3$

Let us suppose first $k = 1$ and $r_1 \geq 2$. From the inequality

$$S(p_1^{s_1}) \leq p_1 s_1$$

it results

$$p_1^{r_1} = n = F_S^d(n) = F_S^d(p_1^{r_1}) = \sum_{s_1=0}^{r_1} S(p_1^{s_1}) \leq \sum_{s_1=0}^{r_1} p_1 s_1 = \frac{p_1 r_1(r_1+1)}{2}$$

so

$$2p_1^{r_1-1} \leq r_1(r_1+1) \quad \text{if} \quad r_1 \geq 2 \tag{1.76}$$

This inequality is not verified for $p_1 \geq 5$ and $r_1 \geq 2$, so we must have $p_1 < 5$. That is $p_1 \in \{2, 3\}$.

By means of (1.76) we can find a supremum for r_1. This supremum depends on the value of p_1.

If $p_1 = 2$ it results for r_1 only the values $2, 3, 4$, and for $p_1 = 3$ it results $r_1 = 2$.

So, for $n = p_1^{r_1}$ there are at most four solutions of the equation (1.73), namely $n \in \{4, 8, 9, 16\}$. In each of these cases calculating the value of $F_S^d(n)$ we obtain:

$$F_S^d(4) = 6, \; F_S^d(8) = 10, \; F_S^d(9) = 9, \; F_S^d(16) = 16$$

Consequently the solutions are $n = 9$ and $n = 16$.

Let now suppose $k \geq 2$. Writing in the equation (1.73) the decomposition into primes of n we get:

$$\begin{aligned}
\prod_{i=1}^{k} p_i^{r_i} = F_S^d\Big(\prod_{i=1}^{k} p_i^{r_i}\Big) &= \sum_{d/n} S(d) = \sum_{s_1=0}^{r_1} \cdots\cdots \sum_{s_k=0}^{r_k} S\Big(\prod_{i=1}^{k} p_i^{s_i}\Big) = \\
&= \sum_{s_1=0}^{r_1} \cdots\cdots \sum_{s_k=0}^{r_k} \max\{S(p_1^{s_1}), S(p_2^{s_2}), \ldots, S(p_k^{s_k})\} \leq \\
&= \sum_{s_1=0}^{r_1} \cdots\cdots \sum_{s_k=0}^{r_k} \max\{p_1 s_1, p_2 s_2, \ldots p_k s_k\} < \\
&= \sum_{s_1=0}^{r_1} \cdots\cdots \sum_{s_k=0}^{r_k} \max\{p_1 r_1, p_2 r_2, \ldots p_k r_k\} = \\
&= \sum_{s_1=0}^{r_1} \cdots\cdots \sum_{s_k=0}^{r_k} p_1 r_1 \leq p_1 r_1 \prod_{i=1}^{k} (r_i + 1)
\end{aligned}$$

Consequently, the inequality:

$$\prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{p_1 r_1 (r_1+1)}{p_1^{r_1}} = \frac{r_1(r_1+1)}{p_1^{r_1-1}} \tag{1.77}$$

holds, and we are then conducted to study the functions:

$$f(x) = \frac{a^x}{x+1} \quad \text{and} \quad g(x) = \frac{x(x+1)}{b^{x-1}} \quad \text{for } x \geq 0$$

where $a, b \geq 2$.

The derivatives of these functions are:

$$f'(x) = \frac{a^x}{(x+1)^2}[(x+1)\ln a - 1] \quad \text{and}$$
$$g'(x) = \frac{(-\ln b)x^2 + (2-\ln b)x + 1}{b^{x-1}}$$

Because $(x+1)\ln a - 1 \geq (1+1)\ln 2 - 1 = 2\ln 2 - 1 > 0$ it results $f'(x) > 0$ for $x \geq 1$. In addition the maximum of this function is obtained for $x = \max\{1, \hat{x}\}$, where

$$\hat{x} = \frac{2 - \ln b + \sqrt{(\ln b)^2 + 4}}{2\ln b}$$

and we deduce $\sqrt{(\ln b)^2 + 4} < \ln b + 2$, for $b \geq 2$, so

$$\hat{x} < \frac{(2-\ln b) + (\ln b + 2)}{2\ln b} = \frac{2}{\ln b} \leq \frac{2}{\ln 2} < 3$$

We also have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and then $p_1^{r_1}/(r_1 + 1)$ increase from $p_1/2$ to infinity, when $r_1 \in N^*$. Moreover, because

$$\frac{6}{p_1} \geq \frac{12}{p_1^2} \quad \text{if } p_1 \geq 2$$

it results

$$\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}, \frac{12}{p_1^2}\} = \max\{2, \frac{6}{p_1}\} \leq 3$$

Using (1.77) we obtain:

$$\prod_{i=2}^{k} \frac{p_1}{2} \leq \prod_{i=2}^{k} \frac{p_i^{r_i}}{r_i+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1-1}} \leq 3, \qquad (1.78)$$

for $r_1 \in N^*$, and so

$$\prod_{i=2}^{k} \frac{p_i}{2} < 3$$

But we have also

$$\prod_{i=2}^{4} \frac{p_i}{2} \geq \frac{2}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} = \frac{15}{4} > 3$$

and then it results $k \leq 3$.

For $k = 2$, using (1.77) and (1.78) it results:

$$\frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \quad \text{and} \quad \frac{p_2}{2} < 3$$

so $p_2 < 6$.

If we suppose $r_2 \geq 3$, it results

$$p_1 \cdot p_2 \geq 2 \cdot 3 = 6 \ \text{ or } \quad p_2 > \frac{6}{p_1}$$

and then

$$\frac{p_2^3}{4} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \max\{2, \frac{6}{p_1}\} \leq \max\{2, p_2\} = p_2$$

so it results the contradiction $p_2^2 < 4$, and we have $p_2 \in \{2, 3, 5\}$, $r_2 \in \{1, 2\}$. Moreover, from

$$1 \leq \frac{p_2}{2} \leq \frac{p_2^{r_2}}{r_2+1} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{2^{r_1-1}}$$

it results $r_1 \leq 6$.

Then, for fixed values of p_2 and r_2, the inequalities

$$\frac{r_1(r_1+1)}{p_1^{r_1-1}} > \frac{p_2^{r_2}}{r_2+1}, \quad p_1 r_1 > p_2 r_2$$

give us iformations for finding an upper bound of r_1, for every value of p_1. It results $r_1 < 7$ and the conclusions are given in the table bellow.

	p_2	r_2	p_1	r_1	$n = p_1^{r_1} p_2^{r_2}$	$F_S^d(n)$
a)	2	1	3	$1 \leq r_1 \leq 3$	$2 \cdot 3^{r_1}$	$2 + 3r_1(r_1+1)$
b)	2	1	5	$1 \leq r_1 \leq 2$	$2 \cdot 5^{r_1}$	$2 + 5r_1(r_1+1)$
c)	2	1	$p_1 \geq 7$	1	$2 \cdot p_1$	$2 + 2p_1$
d)	2	2	3	2	36	34
e)	2	2	$p_1 \geq 5$	1	$4p_1$	$3p_1 + 6$
f)	3	1	2	$2 \leq r_1 \leq 5$	$3 \cdot 2^{r_1}$	$2r_1^2 - 2r_1 + 12$
g)	3	1	$p_1 \geq 5$	1	$3p_1$	$2p_1 + 3$
h)	3	1	2	3	40	30

Conclusions:

	If $F_S^d(n) = n$ *then*
a)	3 *divides* 2
b)	5 *divides* 2
c)	$0 = 2$
d)	$34 = 36$
e)	$p_1 = 6$
f)	$r_1 = 3$
g)	$p_1 = 3$
h)	$30 = 40$

It results that we must have

$$n = 3 \cdot 2^{r_1} \quad \text{or} \quad r_1 = 3$$

so $n = 3 \cdot 2^3 = 24$. That is for $k = 2$ the equation (1.73) has as solution only $n = 24$.

Finally, supposing $k = 3$, from

$$\frac{p_2}{2} \cdot \frac{p_2}{2} < 3$$

it results $p_2 \cdot p_3 < 12$, so $p_2 = 2$ and $p_3 \in \{3, 5\}$.

Using (1.78) from

$$\frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{3^{r_1-1}} \leq 2 \tag{1.79}$$

it results $p_2 = 3$.

Also, from (1.78) and (1.79) we obtain

$$\frac{2^{r_2}}{r_2+1} \cdot \frac{3^{r_3}}{r_3+1} < 2$$

and because the left hand side of this inequality is the product of two increasing functions on $[0, \infty)$, it results for r_2 and r_3 only the values $r_2 = r_3 = 1$.

With these values in (1.77) one obtain:

$$\frac{3}{2} < \frac{r_1(r_1+1)}{p_1^{r_1-1}} \leq \frac{r_1(r_1+1)}{5^{r_1-1}}$$

and so $r_1 = 1$. Consequently, the equation (1.73) is satisfied only for $n = 2 \cdot 3 \cdot p_1 = 6p_1$.

But

$$6p_1 = F_S^d(6p_1) = S(1) + S(2) + S(3) + S(6) +$$
$$+ \sum_{i=0}^{1} \sum_{j=0}^{1} S(2^i \cdot 3^j \cdot p_1) = 8 + \sum_{i=0}^{1} \sum_{j=0}^{1} \max\{S(2^i \cdot 3^j), p_1\} = 8 + 4p_1$$

because $S(2^i \cdot 3^j) \leq 3 < p_1$ for $i, j \in \{0, 1\}$, and so it results the contradiction $p_1 = 4$.

Then for $k = 3$ the equation has no solution and the theorem is proved.

1.5.2 Consequence. The solutions of the inequation

$$F_S^d(n) > n \tag{1.80}$$

result from the fact that this inequation implies (1.77). So,

$$F_S^d(n) > n \Longleftrightarrow n \in \{8, 12, 18, 20\} \text{ or } n = 2p, \text{ with } p \text{ a prime}$$

We deduce also that

$$F_S^d(n) \leq n + 4, \text{ for every } n \in N^*$$

Moreover, because we have the solutions of the inequation

$$F_S^d(n) \geq n$$

we may deduce the solutions of the inequation $F_S^d(n) < n$.

In [40] is studied the limit of the sequence

$$T(n) = 1 - \ln F_S^d(n) + \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{1}{F_S^d(p_i^k)}$$

which contains the generating function. It is proved that

$$\lim_{n \to \infty} T(n) = -\infty$$

In the sequel we focus the attention on the left side of (1.67), namely we shall regard the Smarandache function as a generating function of a certain numerical function s.

By definition we have

$$s(n) = \sum_{d/n} \mu(d) S\left(\frac{n}{d}\right)$$

If the decomposition into primes of the number n is

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} .....p_t^{\alpha_t}$$

it results

$$s(n) = \sum_{p_{i_1} p_{i_2} \cdots p_{i_r}} (-1)^r S(\frac{n}{p_{i_1} p_{i_2} \cdots p_{i_r}})$$

Let us consider that

$$S(n) = \max\{S(p_i^{\alpha_i}) = S(p_{i_0}^{\alpha_{i_0}}) \qquad (1.81)$$

We have the following cases:

(a_1) There exists $i_0 \in \{1, 2, ..., t\}$ such that:
$S(p_{i_0}^{\alpha_{i_0}-1}) \geq S(p_i^{\alpha_i})$ for $i \neq i_0$

The divisors d of n for which $\mu(d) \neq 0$ are of the form: $d = 1$ or $d = p_{i_1} \cdot p_{i_2} .... p_{i_r}$.

A divisor of the aecond kind may contains p_{i_0} or not. Using $(ref\ 1510)$, with the notation $C_t^k = \frac{t!}{k!(t-k)!}$, it results:

$$s(n) = S(p_{i_0}^{\alpha_{i_0}})(1 - C_{t-1}^1 + C_{t-1}^2 + ... + (-1)^{t-1} C_{t-1}^{t-1}) +$$
$$S(p_{i_0}^{\alpha_{i_0}-1})(-1 + C_{t-1}^1 - C_{t-1}^2 + ... + (-1)^t C_{t-1}^{t-1})$$

and so, we have:

$$s(n) = \begin{cases} 0 & \text{if } t \geq 2 \text{ or } S(p_{i_0}^{\alpha_{i_0}}) = S(p_{i_0}^{\alpha_{i_0}-1}) \\ p_{i_0} & otherwise \end{cases}$$

(a_2) There exists $j_0 \in \{1, 2, ..., t\}$ such that we have:
$S(p_{i_0}^{\alpha_{i_0}-1}) < S(p_{j_0}^{\alpha_{j_0}})$ and $S(p_{j_0}^{\alpha_{j_0}-1}) \geq S(p_i^{\alpha_i})$ for $i \notin \{i_0, j_0\}$

In this case, supposing in addition that

$$S(p_{j_0}^{\alpha_{j_0}}) = \max\{S(p_j^{\alpha_j}) \ / \ S(p_{i_0}^{\alpha_{i_0}-1}) < S(p_j^{\alpha_j})\}$$

one obtain:

$$s(n) = S(p_{i_0}^{\alpha_{i_0}})(1 - C_{t-1}^1 + C_{t-1}^2 - \dots + (-1)^{t-1}C_{t-1}^{t-1}) + \\ + S(p_{j_0}^{\alpha_{j_0}})(-1 + C_{t-2}^1 - C_{t-2}^2 + \dots + (-1)^{t-1}C_{t-2}^{t-2}) + \\ + S(p_{j_0}^{\alpha_{j_0}-1})(1 - C_{t-2}^1 + C_{t-2}^2 - \dots + (-1)^{t-2}C_{t-2}^{t-2})$$

and it results:

$$s(n) = \begin{cases} 0 & \text{if } t \geq 3 \text{ or } S(p_{j_0}^{\alpha_{j_0}-1}) = S(p_{j_0}^{\alpha_{j_0}}) \\ -p_{j_0} & \text{otherwise} \end{cases}$$

Consequently, to obtain $s(n)$ we construct, as above, a maximal sequence $i_1, i_2, \dots, i_k$, such that

$$S(n) = S(p_{i_1}^{\alpha_{i_1}}), S(p_{i_1}^{\alpha_{i_1}-1}) < S(p_{i_2}^{\alpha_{i_2}}), \dots, S(p_{i_{k-1}}^{\alpha_{i_{k-1}}-1}) < S(p_{i_k}^{\alpha_{i_k}}$$

and it results:

$$s(n) = \begin{cases} 0 & \text{if } t \geq k+1 \text{ or } S(p_{i_k}^{\alpha_{i_k}}) = S(p_{i_k}^{\alpha_{i_k}-1}) \\ (-1)^{k+1}p_{i_k} & \text{otherwise} \end{cases}$$

Now, because

$$S(p^\alpha) = S(p^{\alpha-1}) \Longleftrightarrow (p-1)\alpha + \sigma_{[p]}(\alpha) = (p-1)(\alpha-1) + \\ + \sigma_{[p]}(\alpha-1) \Longleftrightarrow \sigma_{[p]}(\alpha-1) - \sigma_{[p]}(\alpha) = p-1$$

and

$$S(p^\alpha) \neq S(p^{\alpha-1}) \Longleftrightarrow \sigma_{[p]}(\alpha-1) - \sigma_{[p]}(\alpha) = -1$$

it results

$$s(n) = \begin{cases} 0 & \begin{array}{l} \text{if } t \geq k+1 \text{ or} \\ \sigma_{[p_k]}(\alpha_k - 1) - \sigma_{[p_k]}(\alpha_k) = p_k - 1 \end{array} \\ (-1)^{k+1}p_k & \text{otherwise} \end{cases}$$

1.5.3 Consequence. It is said [31] that if $(V, \wedge, \vee)$ is a finit lattice with the induced order $\prec$, then considering a function $f : V \longrightarrow R$ as well as its generating function F, defined by the equality 1.55), and noting

$$g_{ij} = F(x_i \wedge x_j)$$

it results

$$\det(g_{ij}) = f(x_1) \cdot f(x_2)...f(x_n)$$

In [31] it is proved a generalisation of this result to an arbitrary partial ordered set, namely, defining the function g_{ij} by:

$$g_{ij} = \sum_{\substack{x \prec x_i \\ x \prec x_j}} f(x)$$

Using these results and noting $\Delta(r) = \det(S(i \underset{d}{\wedge} j))$, for $i, j = \overline{1, r}$, we get:

$$\Delta(r) = s(1) \cdot s(2)...s(r)$$

so, for sufficiently large r (in fact for $r \geq 8$) we have $\Delta(r) = 0$. Moreover, for every $n \in N^*$ there exists a sufficiently large $r \in N^*$ such that noting $\Delta(n, k) = \det S((n + i) \underset{d}{\wedge} (n + j))$, for $i, j = \overline{1, k}$, we have $\Delta(n, k) = 0$ for $k \geq r$. Indeed, this assertion is valid because

$$\Delta(n, k) = \prod_{i=1}^{k} s(n + i)$$

Ending this section we consider the Dirichlet series D_s attached to the function s to prove the following result:

1.5.4 Theorem. The Dirichlet series D_s of the function s, given by

$$D_s(x) = \sum_{n=1}^{\infty} \frac{s(n)}{n^x}$$

satisfies:

$$\begin{array}{l} (i)\ \ 1 \le D_s(x) \le D_\varphi(x) \ \text{ for } x > 2 \\ (ii)\ \ 1 \le D_s(x) \le \frac{x-1}{e^A(x-2)} \end{array}$$

for some positive constant A.

Proof. (i) Using the multiplication of Dirichlet series we obtain:

$$\frac{1}{\zeta(x)} D_s(x) = \left(\sum_{k=1}^{\infty} \frac{\mu(k)}{k^x}\right)\left(\sum_{k=1}^{\infty} \frac{S(k)}{k^x}\right) = \mu(1)S(1) + \frac{\mu(1)S(2)+\mu(2)S(1)}{2^x} +$$
$$+ \frac{\mu(1)S(3)+\mu(3)S(1)}{3^x} + \frac{\mu(1)S(4)+\mu(2)S(2)+\mu(4)S(1)}{4^x} + \dots\dots$$
$$= \sum_{k=1}^{\infty} \frac{s(k)}{k^x} = D_s(x)$$

and the afirmation results using the inequalities (i) from the theorem (1.4.7). The inequalities (ii) also results using the same theorem.

1.6 Numerical Series Containing the Function S

It is difficult to study the variation of the function S on the set N^* of all positive integers, because this function is not monotonous in the usual sense. Then the study of some numerical series involving this function may be an useful instrument to obtain new informations about the function.

In this section we add to the study begun by the Dirichlet series, the study of some new series, which shall give us information about the order of average of the Smarandache function.

1.6.1 Theorem. The series

$$\sum_{k=2}^{\infty} \frac{S(k)}{(k+1)!} \tag{1.82}$$

converges. If β, is its sum, then $\beta \in (e - \frac{3}{2}, \frac{1}{2})$.

Proof. Let us note

$$E_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

Then we shall prove the inequality

$$E_{n+1} - \frac{3}{2} < \sum_{k=2}^{n} \frac{S(k)}{(k+1)!} < \frac{1}{2} \tag{1.83}$$

Indeed, we have

$$\sum_{k=1}^{n} \frac{k}{(k+1)!} = \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) = \sum_{k=1}^{n} \frac{1}{k!} - \sum_{k=1}^{n} \frac{1}{(k+1)!} =$$
$$= \frac{1}{2} - \frac{1}{(n+1)!}$$

and from $S(k) \leq k$ it results:

$$\sum_{k=1}^{n} \frac{S(k)}{(k+1)!} \leq \sum_{k=1}^{n} \frac{k}{(k+1)!} = \frac{1}{2} - \frac{1}{(k-1)!} < \frac{1}{2}$$

On the other hend, for $k \geq 2$ we have $S(k) > 1$ and consequently

$$\sum_{k=1}^{n} \frac{S(k)}{(k+1)!} > \sum_{k=1}^{n} \frac{1}{(k+1)!} = \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n+1)!} = E_{n+1} - \frac{3}{2}$$

1.6.2 Proposition. The series

(i) $\displaystyle\sum_{k=r}^{\infty} \frac{S(k)}{(k-r)!}$, with $r \in N^*$ and (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{S(k)}{(k+r)!}$, with $r \in N$

converges.

Proof. We have

$$\sum_{k=r}^{n}\frac{S(k)}{(k-r)!}\leq\sum_{k=r}^{n}\frac{k}{(k-r)!}=\frac{r}{0!}+\frac{r+1}{1!}+\ldots+\frac{r+(n-r)}{(n-r)!}=$$
$$=r\left(\frac{1}{0!}+\frac{1}{1!}+\ldots+\frac{1}{(n-r)!}\right)+\left(\frac{1}{1!}+\frac{2}{2!}+\ldots+\frac{n-r}{(n-r)!}\right)=$$
$$rE_{n-r}+E_{n-r-1}$$

and it results:

$$\sum_{k=r}^{n}\frac{S(k)}{(k-r)!}<rE_{n-r}+E_{n-r-1}$$

so the series from (i) is convergent. Analogously one may prove the convergence of second series.

1.6.3 Remark. Because if $n \geq 3$ and $m = \frac{n!}{2}$ we have:

$$\frac{m}{S(m)!}=\frac{\frac{n!}{2}}{n!}=\frac{1}{2}$$

it results the divergence of the series:

$$\sum_{k=1}^{\infty}\frac{k}{S(k)!} \tag{1.84}$$

We may consider the series:

$$f_S(z)=\sum_{k=1}^{\infty}\frac{S(k)}{(k+1)!}z^k \tag{1.85}$$

For

$$a_k=\frac{S(k)}{(k+1)!}$$

it results $a_{k+1}/a_k \longrightarrow 0$. Indeed,

$$\frac{a_{k+1}}{a_k}=\frac{S(k+1)}{(k+2)S(k)}\leq\frac{k+1}{(k+2)S(k)}\leq\frac{1}{S(k)}$$

and so the series 1.85 converges, for all $z \in C$.

1.6.4 Propoisition. The function f_S from (1.85) satisfies: $/f_S(z)/ \leq \beta z$ on the unit disc $u(0,1) = \{ z \;/\; /z/ < 1 \}$, where β is the sum of the series (1.82).

Proof. A lemme does to Schwarz assert that if a function f is holomorphic on the unit disc $u(0,1)$ and satisfies $f(0) = 0$, $/f'(z)/ < 1$ on this disc, then $/f(z)/ \leq z$ on $u(0,1)$ and $/f(0)/ \leq 1$.

For f_S it results

$$/f_S(z) < \beta \quad \text{if} \quad /z/ < 1$$

so the function $(1/\beta)f_S$ satisfies the conditions of Schwarz's lemme.

The connection between the function S and the factorial justifies to consider *the complement of a number until the most appropriate factorial.*

So, let us consider the function:

$$b : N^* \longrightarrow N^*$$

defined by the condition that

$$b(n) = \frac{S(n)!}{n} \tag{1.86}$$

1.6.5 Proposition. The sequences $(b(n))_{n \in N^*}$ and $(b(n)/n^k)_{n \in N^*}$, for $k > 0$, are divergent.

Proof. Of course, $b(n!) = 1$, and if $(p_n)_{n \in N^*}$ is the sequence of all the primes, we have

$$b(p_n) = \frac{S(p_n)!}{p_n} = \frac{p_n!}{p_n} = (p_n - 1)!$$

Noting

$$x_n = \frac{b(n)}{n^k}$$

for fixed $k > 0$ it results:

$$x_n = \frac{S(n)!}{n^{k+1}}$$

and so

$$x_{n!} = \frac{S(n!)!}{(n!)^{k+1}} = \frac{n!}{(n!)^{k+1}} \longrightarrow 0$$
$$x_{p_n} = \frac{p_n!}{(p_n)^{k+1}} = \frac{(p_n-1)!}{p_n^k} > \frac{p_1 \cdot p_2 \cdots p_{n-1}}{p_n^k} > p_n$$

because it is said [33] that for fixed k and sufficiently large n we have

$$p_1 \cdot p_2 \dots p_{n-1} > p_n^{k+2}$$

1.6.6 Proposition. The sequence

$$T(n) = 1 + \sum_{i=2}^{n} \frac{1}{b(n)} - \ln b(n) \qquad (1.87)$$

has no limit.

Proof. Let us suppose that $\lim_{n \to \infty} T(n) = l < \infty$. From (1.84) it results

$$\sum_{n=2}^{\infty} \frac{1}{b(n)} = \infty$$

and then by the hypothesis, using (1.87) it results

$$\lim_{n \to \infty} \ln b(n) = \infty$$

If we suppose $\lim_{n \to \infty} T(n) = -\infty$, using the expression of $b(n)$ from (1.87) it also results $\lim_{n \to \infty} \ln b(n) = \infty$.

We can't have $\lim_{n \to \infty} T(n) = \infty$, because $T(n) < 0$ for infinitely many n.Indeed, from $i \leq S(i)!$, it results

$$\frac{i}{S(i)!} \leq 1 \text{ for } i \geq 2$$

so

$$T(p_n) = 1 + \tfrac{2}{S(2)!} + \tfrac{3}{S(3)!} + ... + \tfrac{p_n}{S(p_n)!} - \ln((p_n - 1)!) <$$
$$1 + (p_n - 1) - \ln((p_n - 1)!) = p_n - \ln((p_n - 1)!)$$

But for sufficiently large k we have $e^k < (k-1)!$, and consequently there exists $m \in N$ such that $p_n < \ln((p_n - 1)!)$ for $n \geq m$, and the proposition is proved.

Let us consider now the function

$$H_b(x) = \sum_{2 \leq n \leq x} b(n) \tag{1.88}$$

1.6.7 Proposition. The series

$$\sum_{n=2}^{\infty} H_b^{-1}(n) \tag{1.89}$$

converges.

Proof. The sequence $(b(2) + b(3) = ...b(n))_{n \geq 2}$ strictly increase to infinity and

$$\frac{S(2)!}{2} + \frac{S(3)!}{3} > \frac{S(2)!}{2}$$

$$\frac{S(2)!}{2} + \frac{S(3)!}{3} + \frac{S(4)!}{4} > \frac{S(3)!}{3}$$

$$\frac{S(2)!}{2} + \frac{S(3)!}{3} + \frac{S(4)!}{4} + \frac{S(5)!}{5} > \frac{S(5)!}{5}$$

$$\frac{S(2)!}{2} + \frac{S(3)!}{3} + \frac{S(4)!}{4} + \frac{S(5)!}{5} + \frac{S(6)!}{6} > \frac{S(5)!}{5}$$

$$\frac{S(2)!}{2}+\frac{S(3)!}{3}+\frac{S(4)!}{4}+\frac{S(5)!}{5}+\frac{S(6)!}{6}+\frac{S(7)!}{7}>\frac{S(7)!}{7}$$

....

so it results:

$$\begin{aligned}\sum_{n=2}^{\infty} H_b^{-1}(n) &= \frac{1}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}}+\dots+\frac{1}{\frac{S(2)!}{2}+\frac{S(3)!}{3}+\dots+\frac{S(n)!}{n}} < \\ &< \frac{2}{\frac{S(2)!}{2}}+\frac{1}{\frac{S(3)!}{3}}+\frac{2}{\frac{S(5)!}{5}}+\dots+\frac{p_{k+1}-p_k}{\frac{S(p_k)!}{p_k}}+\dots < \\ &< 1+\sum_{k=2}^{\infty}\frac{p_k(p_{k+1}-p_k)}{S(p_k)!} = 1+\frac{1}{2}+\frac{1}{12}+\sum_{k=4}^{\infty}\frac{p_k(p_{k+1}-p_k)}{p_k!}\end{aligned}$$

But $(p_n-1)! > p_1\cdot p_2\dots p_n$ for $n\geq 4$ and so

$$\sum_{n=2}^{\infty} H_b^{-1}(n) < \frac{19}{12}+\sum_{k=4}^{\infty} a_k$$

where

$$a_k=\frac{p_k(p_{k+1}-p_k)}{p_k!}=\frac{(p_{k+1}-p_k)}{(p_k-1)!}<\frac{p_{k+1}-p_k}{p_1\cdot p_2\dots p_k}<\frac{p_{k+1}}{p_1\cdot p_2\dots p_k}$$

Because for sufficiently large k we have $p_1\cdot p_2\dots p_k > p_{k+1}^3$, it results:

$$a_k<\frac{p_{k+1}}{p_{k+1}^3}=\frac{1}{p_{k+1}^2}$$

and then the convergence of the series (1.89) results from the convergence of the series

$$\sum_{k\geq k_0}\frac{1}{p_{k+1}^2}$$

We shall give now an elementary proof of the series

$$\sum_{k=2}^{\infty} \frac{1}{(S(k)^{\alpha})\sqrt{S(k)!}}, \text{ with } \alpha > 1 \tag{1.90}$$

and using this convergence we shall prove the convergence of the series

$$\sum_{k=2}^{\infty} \frac{1}{S(k)!} \tag{1.91}$$

1.6.8 Proposition. The series (1.90) converges, for all $\alpha > 1$.

Proof. We have succesively:

$$\sum_{k=2}^{\infty} \frac{1}{(S(k)^{\alpha})\sqrt{S(k)!}} = \frac{1}{2^{\alpha}\sqrt{2!}} + \frac{1}{3^{\alpha}\sqrt{3!}} + \frac{1}{4^{\alpha}\sqrt{4!}} + \frac{1}{5^{\alpha}\sqrt{5!}} +$$
$$+ \frac{1}{3^{\alpha}\sqrt{3!}} + \frac{1}{7^{\alpha}\sqrt{7!}} + \frac{1}{4^{\alpha}\sqrt{4!}} + \ldots = \sum_{t=2}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{t!}}$$

where m_t is the cardinal of the set

$$\begin{aligned} M_t = \{\, k \;/\; S(k) = t \,\} = \\ = \{\, k \;/\; k \text{ divides } t! \text{ and does not divide } (t-1)! \,\} \end{aligned} \tag{1.92}$$

It results that $M_t \subset \{\, k \;/\; k \text{ divides } t! \,\}$, so m_t is lowest than the number of divisors of $t!$. So we have

$$m_t < \tau(t!)$$

But it is said that $\tau(n) < 2\sqrt{n}$, for every positive integer n, consequently

$$\sum_{k=2}^{\infty} \frac{m_t}{(t^{\alpha})\sqrt{t!}} < \sum_{k=2}^{\infty} \frac{2\sqrt{t!}}{(t^{\alpha})\sqrt{t!}} = 2\sum_{k=2}^{\infty} \frac{1}{t^{\alpha}}$$

and the proposition is proved.

1.6.9 Consequence. From the convergence of the series (1.90) it results the convergence of the series (1.91). To prove this we shall use the following result:

1.6.10 Proposition. For $\alpha > 0$ let us note

$$t^* = [e^{2\alpha+1}]$$

Then the inequality $t^\alpha \sqrt{t!} < t!$ holds for every $t > t^*$.

Proof. We have

$$(t^\alpha)\sqrt{t!} < t! \Longleftrightarrow (t^{2\alpha})t! < (t!)^2 \Longleftrightarrow t^{2\alpha} < t!$$

On the other hand

$$t^{2\alpha} < (\tfrac{t}{e})^t \Longleftrightarrow (e\tfrac{t}{e})^{2\alpha} < (\tfrac{t}{e})^t \Longleftrightarrow e^{2\alpha}(\tfrac{t}{e})^{2\alpha} < (\tfrac{t}{e})^t \Longleftrightarrow$$
$$\Longleftrightarrow e^{2\alpha} < (\tfrac{t}{e})^{t-2\alpha}$$

But

$$t > e^{2\alpha+1} \Longrightarrow (\tfrac{t}{e})^{t-2\alpha} > (\tfrac{e^{2\alpha+1}}{e})^{t-2\alpha} =$$
$$= (e^{2\alpha})^{t-2\alpha} > (e^{2\alpha})^{e^{2\alpha+1}-2\alpha}$$

Now, for $x > 0$ we have $e^x > 1 + x$, and so, taking $x = 2\alpha + 1$ and $t > 2\alpha + 1$, it results

$$(\frac{t}{e})^{t-2\alpha} > e^{4\alpha} > e^{2\alpha}$$

Then for $t > t^*$ we get

$$e^{2\alpha} < (\frac{t}{e})^{t-2\alpha} \Longleftrightarrow t^{2\alpha} < (\frac{t}{e})^t < t!$$

It results $t^{2\alpha} < t!$ if $t > t^*$.

Using this result we may writte:

$$(t^\alpha)\sqrt{t!} < t! \Longleftrightarrow \frac{m_t}{(t^\alpha)\sqrt{t!}} > \frac{m_t}{t!} \text{ for } t > t^*$$

and from the proposition (1.89) it results the convergence of the series

$$\sum_{t=2}^{\infty} \frac{m_t}{t!}$$

and of course we have

$$\sum_{k=2}^{\infty} \frac{1}{S(k)!} = \sum_{k=2}^{\infty} \frac{m_t}{t!}$$

1.6.9 Theorem. Let $f : N^* \longrightarrow R$ be a function which satisfies the condition

$$f(t) \leq \frac{c}{t^{\alpha}(d(t!) - d((t-1)!))}$$

for $t \in N^*$ and the constants $\alpha > 1, c > 0$. Then the series

$$\sum_{k=1}^{\infty} f(S(k))$$

is convergent.

Proof. For M_t given by (1.92) we have $M_t = d(t!) - d((t-1)!)$ and

$$\sum_{k=1}^{\infty} f(S(k)) = \sum_{k=1}^{\infty} M_t f(t)$$

Then because $M_t \cdot f(t) \leq M_t \cdot \frac{c}{t^{\alpha} M_t} = \frac{c}{t^{\alpha}}$ it resultsthe convergence of the series.

1.6.10 Proposition. If $(x_n)_{n \in N^*}$ is any strict increasing sequence of positive integers, then the series

$$\sum_{n=1}^{\infty} \frac{x_{n+1} - x_n}{S(x_n)}$$

is divergent.

Proof. Let consider the function

$$f : [x_n, x_{n+1}] \longrightarrow R, \quad f(x) = \ln \ln x$$

From the theorem of Lagrange it results that there exists $c_n \in (x_n, x_{n+1})$ such that

$$\ln \ln x_{n+1} - \ln \ln x_n = \frac{1}{c_n \ln c_n}(x_{n+1} - x_n)$$

and because $x_n < c_n < x_{n+1}$, we have

$$\frac{x_{n+1} - x_n}{x_{n+1} \ln x_{n+1}} < \ln \ln x_{n+1} - \ln \ln x_n < \frac{x_{n+1} - x_n}{x_n \ln x_n} \tag{1.93}$$

for every $n \in N^*$.Then for $n > 1$

$$\frac{S(n)}{n} \leq 1 \Longrightarrow 0 < \frac{S(n)}{n \ln n} \leq \frac{1}{\ln n}$$

That is

$$\lim_{n \longrightarrow \infty} \frac{S(n)}{n \ln n} = 0$$

and hence for every $n \in N^*$ there exists $k > 0$ such that $\frac{S(n)}{n \ln n} < k$, or $n \ln n > \frac{S(n)}{k}$. Then

$$\frac{1}{x_n \ln x_n} < \frac{k}{S(x_n)} \tag{1.94}$$

Introducing (1.94) in (1.93) we obtain

$$\ln \ln x_{n+1} - \ln \ln x_n < k \frac{x_{n+1} - x_n}{S(x_n)}$$

for every $n > 1$. Summing it results

$$\sum_{n=1}^{m} \frac{x_{n+1} - x_n}{S(x_n)} > \frac{1}{k}(\ln \ln x_{m+1} - \ln \ln x_1)$$

and the divergence of the series results from the fact that $\ln \ln x_m$ tends to infinity.

Consequences. 1) For $x_n = n$ it results the divergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{S(n)}$$

2) If $x_n = p_n$ (the $n-$th prime), it results the divergence of the series

$$\sum_{n=1}^{\infty} \frac{p_{n+1} - p_n}{p_n}$$

3) If $(x_n)_{n \in N^*}$ is an arithmetical progression of positive integers then the series

$$\sum_{n=1}^{\infty} \frac{1}{S(x_n)}$$

is divergent.

1.6.11 Proposition. The series

$$\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)...S(n)}$$

is convergent to a number $s \in (1.71, 2.01)$.

Proof. From the definition of the Smarandache function it results the inequality

$$\frac{1}{S(n)} \geq \frac{1}{n}$$

and summing we get

$$\sum_{n=1}^{\infty} \frac{1}{S(1)S(2)...S(n)} \geq \sum_{n=1}^{\infty} \frac{1}{n!} = e - 2$$

On the other hand the product $S(1)S(2)...S(n)$ is greater than the product of primes from the set $\{1, 2, ...n\}$, because $S(i) = i$ if i is a prime. Therefore

$$\frac{1}{\prod_{i=1}^{n} S(i)} < \frac{1}{\prod_{i=1}^{k} p_i}$$

where p_k is the greatest prime number not exceeding n. Then

$$S = \sum_{n=1}^{\infty} \frac{1}{S(1)S(2)...S(n)} = \frac{1}{S(1)} + \frac{1}{S(1)S(2)} + ... +$$
$$+ \frac{1}{S(1)S(2)...S(k)} + ... < 1 + \frac{1}{2} + \frac{2}{2 \cdot 3} + \frac{2}{2 \cdot 3 \cdot 5} +$$
$$\frac{4}{2 \cdot 3 \cdot 5 \cdot 7} + ... + \frac{p_{k+1} - p_k}{p_1 p_2 ... p_k} + ...$$

and using the inequality $p_1 p_2 ... p_k > p_{k+1}^3$ for every $k \geq 5$ (see [33]) it results:

$$s < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{1}{p_6^2} + \frac{1}{p_7^2} + ... + \frac{1}{p_{k+1}^2} + ... \quad (1.95)$$

let us note $P = \frac{1}{p_6^2} + \frac{1}{p_7^2} + ...$ and observe that $P < \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} + ...$

It results

$$P < \frac{\pi^2}{6} - (1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{12^2}$$

because $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

Introducing in (1.95) we obtain:

$$s < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{15} + \frac{2}{105} + \frac{\pi^2}{6} - (1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{12^2}$$

Estimating with an approximation of order not more than 10^{-2} it results $s \in (1.71, 2.01)$.

1.6.12 Proposition. For every $\alpha \geq 1$, the series

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{S(1)S(2)...S(n)}$$

converges.

Proof. If $(p_k)_{k\in N^*}$ is the sequence of primes, we can writte:

$$\frac{2^{\alpha}}{S(2)} = \frac{2^{\alpha}}{2} = 2^{\alpha-1}$$
$$\frac{3^{\alpha}}{S(2)S(3)} \frac{3^{\alpha}}{p_1p_2}$$
$$\frac{4^{\alpha}}{S(2)S(3)S(4)} < \frac{4^{\alpha}}{p_1p_2} < \frac{p_3^{\alpha}}{p_1p_2}$$
$$\frac{5^{\alpha}}{S(2)S(3)S(4)S(5)} < \frac{5^{\alpha}}{p_1p_2p_3} < \frac{p_4^{\alpha}}{p_1p_2p_3}$$
$$\frac{6^{\alpha}}{S(2)S(3)S(4)S(5)S(6)} < \frac{6^{\alpha}}{p_1p_2p_3} < \frac{p_4^{\alpha}}{p_1p_2p_3}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\frac{n^{\alpha}}{S(2)S(3)...S(n)} < \frac{n^{\alpha}}{p_1p_2...p_k} < \frac{p_{k+1}^{\alpha}}{p_1p_2...p_k}$$

where $p_i \leq n$ for $i = \overline{1,k}$, and $p_{k+1} > n$.

Therefore

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{S(1)S(2)...S(n)} < 1 + 2^{\alpha-1} + \sum_{n=1}^{\infty} \frac{(p_{k+1}-p_k)\cdot p_{k+1}^{\alpha}}{p_1p_2...p_k} <$$
$$< 1 + 2^{\alpha-1} + \sum_{n=1}^{\infty} \frac{p_{k+1}^{\alpha+1}}{p_1p_2...p_k}$$

Because it exists $k_0 \in N^*$ such that for any $k \geq k_0$ we have $p_1p_2...p_k > p_{k+1}^{\alpha+3}$, one have

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{S(1)S(2)...S(n)} < 1 + 2^{\alpha-1} + \sum_{k=1}^{k_0-1} \frac{p_{k+1}^{\alpha+1}}{p_1p_2...p_k} + \sum_{k=k_0}^{\infty} \frac{1}{p_{k+1}^2}$$

and so our series is convergents.

Consequences. 1) There exists $n_0 \in N^*$ such that $S(1)S(2)S(3)...S(n) > n^\alpha$ for every $n \geq n_0$. Indeed,

$$\lim_{n \to \infty} \frac{n^\alpha}{S(1)S(2)S(3)...S(n)} = 0 \Longrightarrow \frac{n^\alpha}{S(1)S(2)S(3)...S(n)} < 1 \text{ for } n \geq n_0$$

2) It exists $n_0 \in N^*$ such that

$$S(1) + S(2) + S(3) + ... + S(n) > n^{\frac{\alpha+n}{n}} \quad \text{for } n \geq n_0$$

Indeed, we have:

$$S(1) + S(2) + ... + S(n) > n\sqrt[n]{S(1)S(2)...S(n)} > n \cdot n^{\frac{\alpha}{n}} = n^{\frac{\alpha+n}{n}}$$

for $n \geq n_0$.

1.7 Diophantine Equations Involving the Smarandache Function

The formula (1.21) may be used to solve certain diophantine equations involving the Smarandache function.

1) The equation

$$S(x \cdot y) = S(x) + S(y) \tag{1.96}$$

has an infinity of solutions.

Indeed, from (1.16) it results that if x_0 and y_0 are solutions of the above equation then $x_0 \underset{d}{\wedge} y_0 \neq 1$. That because

$$S(x_0 \cdot y_0) - S(x_0 \overset{d}{\vee} y_0) = \max\{S(x_0), S(y_0)\}$$

Let now $x = p^a A, y = p^b B$ be such that

$$S(x) = S(p^a), \quad S(y) = S(p^b)$$

Then $S(x \cdot y) = S(p^{a+b})$ and the equation becomes $p((a+b)_{[p]})_{(p)} = p(a_{[p]})_{(p)} + p(b_{[p]})_{(p)}$, or

$$((a+b)_{[p]})_{(p)} = (a_{[p]})_{(p)} + (b_{[p]})_{(p)} \tag{1.97}$$

There are infinitely many values of a and b satisfying this equality. For instance, $a = a_3(p) = 100_{[p]}, b = a_2(p) = 10_{[p]}$, for which (1.97) becomes:

$$(110_{[p]})_{(p)} = (100_{[p]})_{(p)} + (10_{[p]})_{(p)}$$

2) The equation

$$S(x \cdot y) = S(x) \cdot S(y) \tag{1.98}$$

has no solutions $x, y > 2$.

Indeed, let us note $m = S(x)$ and $n = S(y)$. It is sufficient to prove that $S(x \cdot y) \neq m \cdot n$. But it is said that $m! \cdot n!$ divides $(m+n)!$, so

$$(m \cdot n)! \underset{d}{\leq} (m+n)! \underset{d}{\leq} m! \cdot n! \underset{d}{\leq} x \cdot y$$

and consequently $S(x \cdot y) \leq m \cdot n$. This is a strict inequality if $m \cdot n > m + n$, so it is for $m, n > 2$.

Consequently the equation (1.98) has as solutions only the numbers $x, y \leq 2$.

3 The equation:

$$x \underset{d}{\wedge} y = S(x) \underset{d}{\wedge} S(y) \tag{1.99}$$

also has infinitely many solutions.

Indeed, because $x \geq S(x)$, and the equality holds if and only if x is a prime or $x = 4$, it results that the equation (1.99) has as solution every paire of prime numbers, as well as every paire of square free numbers.

Let now x and y be such that $x \underset{d}{\wedge} y = d > 1$ and

$$S(x) = p(a_{[p]})_{(p)};\ S(y) = q(b_{[q]})_{(q)}$$

Because $p \underset{d}{\wedge} q = 1$, noting $a_1 = (a_{[p]})_{(p)}$ and $b_1 = (b_{[q]})_{(q)}$, if we have $p \underset{d}{\wedge} b_1 = a_1 \underset{d}{\wedge} q = 1$, the equation becomes: $a_1 \underset{d}{\wedge} b_1 = d$. This equality is satisfied for many values of a and b. For instance, if $x = 2 \cdot 3^a$ and $y = 2 \cdot 5^b$ it results $d = 2$ and we have

$$(a_{[3]})_{(3)} \underset{d}{\wedge} (b_{[5]})_{(5)} = 2$$

for many vralues of a and b.

4) Let now consider the equation:

$$x \overset{d}{\vee} y = S(x) \overset{d}{\vee} S(y)$$

Every pair of primes is a solution of this equation, and if x, y are composite numbers, we observe that if we note

$$S(x) = S(p_i^{a_i})\ ;\ S(y) = S(p_j^{a_j}),\ \text{with } p_i \neq p_j$$

it results that the pair (x, y) is not a solution of the equation, because:

$$x \overset{d}{\vee} y > p_i^{a_i} \cdot p_j^{b_j} \geq S(x) \cdot S(y) \geq S9x) \overset{d}{\vee} S(y)$$

Finally, if $x = p^a A, y = p^b B$, with $S(x) = S(p^a)$ and $S(y) = S(p^b)$. it results

$$S(x) \overset{d}{\vee} S(y) = p(a_{[p]})_{(p)} \overset{d}{\vee} p(b_{[p]})_{(p)} = p((a_{[p]})_{(p)} \overset{d}{\vee} (b_{[p]})_{(p)})$$

and $x \overset{d}{\vee} y = p^{\max\{a,b\}}(A \overset{d}{\vee} B)$, consequently the equation als has many other solutions, which are not relatively prime.

5) The equation

$$S(x) + y = x + S(y) \tag{1.100}$$

has as solution every pair of prime numbers, but also every composit numbers $x = y$ are solution. It may be found other kind of composit numbers as solution for this equation. For instance, if p and q are consecutive primes and we note

$$q - p = h \tag{1.101}$$

taking $x = pA, y = qB$, the equation becomes:

$$y - x = S(y) - S(x) \tag{1.102}$$

Considering the diophantine equation $qB - pA = h$, it results from (1.100) that $A_0 = B_0 = 1$ is a particular solution for this equation, and then the general solution is

$$A = 1 + rq, \quad B = 1 + rp, \qquad \text{for arbitrary } r \in N$$

Taking $r = 1$ it results $x = p(1 + q), y = q(1 + p)$, and $y - x = h$. In addition, because p and q are consecutive primes, of course $p + 1$ and $q + 1$ are composite numbers and then

$$S(x) = p, \ S(y) = q, \ S(y) - S(x) = h$$

so the equation (1.102) is verified.

6) To solve the equation

$$S(m \cdot x) = m \cdot S(x) \tag{1.103}$$

let us observe that $S(m \cdot x) \leq S(x) + m$. This fact results from the equality

$$(S(x) + m)! = S(x)!(S(x) + 1).....(S(x) + m)$$

taking into account that $S(x)!$ is divisible by x and the product of m consecutive integers is divisible by m.

If x is a solution of the equation it results $m \cdot S(x) \leq S(x)+m$, so

$$(m-1)(S(x)-1) \leq 1 \tag{1.104}$$

Then we have to analyse the following cases:

(a) If $m = 1$, the equation becomes $S(x) = x$ and has as solution every positive integer.

(b) If $m = 2$, it results we can have $S(x) \in \{1,2\}$, and then $x \in \{1,2\}$.

(c) If $m \geq 3$, we must have $S(x) = 1$, so $x = 1$.

7) For the equation

$$S(x^y) = y^x \tag{1.105}$$

let us observe that $S(x^y) \leq y \cdot x$, because $(yx)! = 1 \cdot 2 \ldots . x \ldots . (2x) \ldots . (yx)$. Then, if the pair (x, y) is a solution for the equation, we must have $y^x \leq yx$. That is

$$y^{x-1} \leq x \tag{1.106}$$

If $x = 1$, the above condition is satisfied, and the equation becomes $S(1) = y$. Consequently, the pair $(1,1)$ is a solution of the equation.

For $x \geq 2$, only the pair $(2,2)$ verifies the inequality (1.106), so it is a solution of the equation.

Indeed, for $x \geq 3$ we have $x < 2^{x-1} \Longleftrightarrow \ln x < (x-1)\ln 2$, and considering the function

$$f(x) = (x-1)\ln 2 - \ln x$$

it results $f'(x) = (x \ln 2 - 1)/x$, so $f'(x) = 0 \Longleftrightarrow x = 1/\ln 2$.

For $x > [1/\ln 2] + 1$, hence for $x \geq 2$, this function is increasing, and in addition $f(2) = \dot{0}$. Then for $x \geq 3$ the inequality is strict.

Let us now consider the equation

$$\frac{S(n)}{n} = k \tag{1.107}$$

where $k \in (0, 1]$ is a rational number. In [48] there are answered the following questions:

(q_1) For every $k \in (0, 1]$ there exists solutions of the equation (1.107) ?

(q_2) Find the values of k for which the equation has infinitely many solutions in N^*.

The answer to (q_1) is negative, and the values of k for which the equation has an infinity of solutions are the following:

$k = \frac{1}{r}$ with $r \in N^*$ and
$k \in Q \cap (0, 1], k = \frac{p}{q}$, with $p, q \in N^*$, $0 < q \leq p$, $p \underset{d}{\wedge} q = 1$

Indeed, if n is a solution of our equation, let

$$\frac{S(n)}{n} = \frac{p}{q}$$

and let $d = n \underset{d}{\wedge} S(n)$. Then, from the definition of d and from the fact that p and q are relatively prime, it results that $S(n) = qd$, $n = pd$ and we have

$$S(pd) = qd \tag{1.108}$$

Using the definition of S it results $(qd)! = M(pd)$ and

$$(qd - 1)! = \frac{(qd)!}{qd} = \frac{M(pd)}{qd} = \frac{M(p)}{q}$$

Because p and q are relatively prime, it results that $(qd-1)!$ is divisible by p and consequently

$$S(p) \leq qd - 1$$

Let us prove also that $S(p) \geq (q-1)d$.

But, if the inequality $S(p) < (q-1)d$ holds, it results $((q-1)d-1)!$ divisible by p. Then from $d \underset{d}{\leq} (q-1)d$, it results $pd \underset{d}{\leq} ((q-1)d)!$, and so $S(pd) < (q-1)d$. This inequality is a contradiction of the fact that $S(pd) = qd > (q-1)d$.

So, we have

$$(q-1)d \leq S(p) \leq qd - 1 \tag{1.109}$$

Taking $q \geq 2$, from the first of the above inequalities, it results $d \leq S(p)/(q-1)$, and from the second it results that $(S(p+1)/q) \leq d$, hence

$$\frac{S(p+1)}{q} \leq d \leq \frac{S(p)}{q-1} \tag{1.110}$$

For $q \geq 2$ and $k = p/q$ it results a necessary condition for the existence of at least a solution of the equation (1.107), namely the existence of an integer between $S(p+1)/q$ and $S(p)/(q-1)$.

But this condition is not a sufficient condition, as we can see from the examples listed bellow.

Examples. 1) For $k = 4/5$ we have $S(p+1)/q = 3/2$ and $S(p)/(q-1) = 5/3$, so the equation has no solution.

2) For $k = 3/10$ we have $S(p+1)/q = 11/3$ and $S(p)/(q-1) = 5/2$, with the same conclusion as in the preceding example.

3) For $k = 3/29$ it results $S(p+1)/q = 5/3$ and $S(p)/(q-1) = 14.5$, so between $S(p+1)/q$ and $S(p)/(q-1)$ there exist more than one integer. However, the equation

$$\frac{S(n)}{n} = \frac{3}{29}$$

has no solutions. Indeed, the number of the solutions equals the number of values of d for which (1.110) and then (1.108) holds. But it does not exist any integer between 2 and14 satisfying these conditions.

Let us study now the equation (1.107) for $k = 1/p$, with $p \in N^*$. We shall prove in this case that the equation has infinitely many solutions.

Indeed, let p_0 be a prime number greater than p and let $n = pp_0$. It results $S(n) = S(pp_0) = p_0$, and $S(n)/n = 1/p = k$.

In [48] it is also answered the following question, posed by F. Smarandache:

(q_3) There exists infinitely many positive integers x such that

$$0 \overset{(1)}{<} \{\frac{x}{S(x)}\} \overset{(2)}{<} \{\frac{S(x)}{x}\} \tag{1.111}$$

where $\{x\} = x - \{x\}$?

The system (1.111) of inequations has only one solution, namely $x = 9$. To prove this we shall prove first that the inequation

$$\{\frac{x}{S(x)}\} < \{\frac{S(x)}{x}\} \tag{1.112}$$

has infinitely many solutions.

The inequality holds for $x = 9$, because

$$\{\frac{9}{S(9)}\} = \{\frac{9}{6}\} = \frac{1}{2} \quad \text{and} \quad \{\frac{S(9)}{9}\} = \frac{2}{3}$$

At the same time one observe that any prime p is not a solution of the inequation.

Let now x be of the form:

$$x = p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_t^{\alpha_t}, \text{ with } t \geq 2$$

We have

$$S(x) = \max_{1 \leq k \leq t} S(p_k^{\alpha_k})$$

and let us put $S(x) = S(p^\alpha)$, where p^α is one of $p_i^{\alpha_i}$, for $i = \overline{1,t}$. Then if x is a solution for (1.112) the number $\{\frac{x}{S(x)}\}$ may take one of the following values:

$$\frac{1}{S(x)}, \frac{2}{S(x)}, \ldots, \frac{S(x)-1}{S(x)}$$

For such an x we have

$$\frac{S(x)}{x} \geq \frac{1}{S(x)}, \text{ so } (S(p^\alpha))^2 > x > p^\alpha \tag{1.113}$$

It is said that from Legendre's formula (1.15) it results $S(p^\alpha) \leq \alpha p$. Then using (1.112) we deduce $\alpha^2 p^2 > p^\alpha$, so

$$\alpha^2 > p^{\alpha-2} \tag{1.114}$$

If $p \geq 2$ then the last inequality holds only for integers $\alpha \leq \alpha_0$.

Indeed, we have $p^{\alpha-2} \geq 2^{\alpha-2}$ and $2^{\alpha-2} \geq \alpha^2$ holds for $\alpha \geq 8$ (the function $f(\alpha) = 2^{\alpha-2} - \alpha^2$ is increasing and $f(8) = 0$)

We have to prove only that for $\alpha \in \{1, 2, ..., 7\}$ the system (1.111) has no solutions.

(a) If $\alpha = 1$ it results $S(x) = S(p) = p$, and because p divides x we have $x/p \in Z$, first of the considered inequalities is not satisfied.

Let us observe that there exist solutions for the second inequality.Indeed, noting $p = p_1$, the number x is of the form $x = p_1 \cdot p_2^{\alpha_2} ... p_t^{\alpha_t}$, so

$$\{\tfrac{x}{S(x)}\} = \{\tfrac{x}{p_1}\} = \{p_2^{\alpha_2} \cdot p_3^{\alpha_3} ... p_t^{\alpha_t}\} = 0 \text{ and}$$
$$\{\tfrac{S(x)}{x}\} = \{\tfrac{1}{p_2^{\alpha_2} ... p_t^{\alpha_t}}\} = \tfrac{1}{p_2^{\alpha_2} ... p_t^{\alpha_t}} > 0$$

Example. For $x = 23 \cdot 2^{19} \cdot 3^9$, we have $S(x) = 23$ and

$$\{\frac{x}{S(x)}\} = \{2^{19} \cdot 3^9\} = 0 \; ; \; \{\frac{S(x)}{x}\} = \frac{1}{2^{19} \cdot 3^9}$$

(b) For $\alpha = 2$ let us note $x = p^\alpha \cdot x_1$. Then $S(x) = S(p^2) = 2p$ and

$$\{\frac{x}{S(X)}\} = \{\frac{px_1}{2}\} \in \{0, \frac{1}{2}\}$$

so we must have

$$\{\frac{px_1}{2}\} = \frac{1}{2} < \{\frac{S(x)}{x}\} = \frac{2}{px_1}$$

and it results $px_1 < 4$, that is $p \in \{2, 3\}$.

If $p = 2$, it results $x_1 = 1$ and so $x = 4$, which is not a solution for the inequation (1) from (1.111) because $S(4) = 4$.

If $p = 3$, it results also $x_1 = 1$, so $x = p^2 = 9$.

Lfet us observe that the second inequation from (1.111) has also solutions. Indeed, with the notation $p = p_1$ we have:

$$\{\frac{x}{S(x)}\} = \{\frac{p_2^{\alpha_2} \cdot p_3^{\alpha_3} ... p_t^{\alpha_t}}{2}\} \text{ and } \{\frac{S(x)}{x}\} = \frac{2}{p_2^{\alpha_2} \cdot p_3^{\alpha_3} ... p_t^{\alpha_t}}$$

consequently the inequation is verified for $x > 2$ even number.

Example. For $x = 2^5 \cdot 3^7 \cdot 11^2$ we have $S(x) = 19$ and

$$\{\frac{x}{S(x)}\} = \{\frac{2^5 \cdot 3^7 \cdot 11^2}{2 \cdot 11}\} = 0; \; \{\frac{S(x)}{x}\} = \frac{1}{2^4 \cdot 3^7 \cdot 11}$$

(c) Let now be $\alpha = 3$. We have seen that in this case if $S(x) = S(p^\alpha)$, it results $p \leq 7$.

If $p = 2$ it results $S(x) = S(2^3) = 4$ and then

$$\{\frac{x}{S(x)}\} = \{\frac{2^3 \cdot x_1}{4}\} \in Z$$

consequently the inequation (1) from (1.111) has no solutions. However, there exist solutions of the second inequation. Indeed, considering for instance x of the form

$$x = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \tag{1.115}$$

with $a, b, c, d \in N^*$ such that $d = a_n(7) = (7^n - 1)/(7 - 1)$ and $S(x) = S(7^d)$ it results $S(x) = 7^n$ and so $x/S(x)$ is an integer.

If $p = 3$, we have $S(x) = S(3^3) = 9$ and also

$$\{\frac{x}{S(x)}\} \in Z \tag{1.116}$$

The inequation (2) has solutions in this case too. For instance $x = 3^3 \cdot x_1$ are solutions, because

$$\{\frac{S(x)}{x}\} = \{\frac{9}{3^3 x_1}\} = \frac{1}{3x_1}$$

If $p = 5$, we have $S(x) = S(5^3) = 15$ and (1.111) becomes:

$$0 < \{\frac{5^2 x_1}{3}\} < \{\frac{3}{5^2 x_1}\}, \quad \text{with} \quad x_1 \underset{d}{\wedge} 5 = 1$$

From the first of these inequalities it results:

$$\{\frac{5^2 x_1}{3}\} \in \{\frac{1}{3}, \frac{2}{3}\}$$

so we must have $1/3 < 3/(5^2 x_1)$. That is $5^2 x_1 < 9$, which is an imposibility.

If $p = 7$, it results $S(x) = S(7^3) = 21$ and

$$0 < \{\frac{7^2 x_1}{3}\} < \frac{3}{7^2 x_1}$$

so

$$\left\{\frac{7^2x_1}{3}\right\} \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$$

Analogously it results the contradiction $3/(7^2x_1) > 1/3$.

If $\alpha = 4$ one obtain $p \in \{2, 3\}$. For $p = 2$ it results $S(x) = S(2^4) = 6$ and because $x = 2^4x_1$, with $2 \underset{d}{\wedge} x_1 = 1$, the system (1.111) becomes:

$$0 < \left\{\frac{8x_1}{3}\right\} < \frac{3}{8x_1}$$

From the condition $3/(8x_1) > 1/3$ it results $x_1 = 1$, so $x = 16$. But for this value of x we have

$$\left\{\frac{x}{S(x)}\right\} = \frac{2}{3} > \frac{3}{8} = \left\{\frac{S(x)}{x}\right\}$$

For $p = 3$, we have $S(x) = S(3^4) = 9$ and one arrive at the condition (1.115).

For $\alpha \in \{5, 6, 7\}$ we get only $p = 2$ satisfying the condition (1.114), so $x = 2^\alpha x_1$ and because $S(2^5) = S(2^6) = S(2^7)$ it results for all the cases $S(x) = 8$. The condition (1.116) is verified again and the system has no solutions.

1.8 Solved and Unsolved Problems

In the sequel we indicate by a star (*) the unsolved problems. For the solutions of solved problems see the collection of *Smarandache Function Journal* and its extension *The Smarandache Notion Journal*.

1*) Find a formula for the calculus of $S(n)$, containing instead of prime divisors of n the number n himself.

2) Prove that $S(p^{p+1}) = p^2$.

3) Inddicate the number of solutions of the equation

$$S(x) = n!.$$

4) Prove that the equation $S(x) = p$, where p is a given prime, has exactly $d((p-1)!)$ solutions, all of them between p and $p!$, where $d(x)$ is the number of divisors of x. (*A. Stuparu*)

Generalisation: The number of solutions of the equation $S(x) = n$ is $d(n!) - d((n-1)!)$.

5) Prove that $\max\{\frac{S(n)}{n} \,/\, n \geq 4$ is a composite number$\} = \frac{2}{3}$. (*T. Yau*)

6) Let q be a prime number and k be an exponent such that $S(q^k) = n!$. Let $p_1, p_2, ... p_r$ be the list of primes less than q. Then the number of solutions of the equation $S(x) = n!$, where x contains exactly k instance of the prime q, is at least $(k+1)^r$. (*Ch. Asbacher*)

7) For every prime p and $k \geq 1$ prove that

$$\frac{S(p^k)}{p^k} \geq \frac{S(p^{k+1})}{p^{k+1}} \quad (\textit{Ch. Asbacher})$$

8) Is the number $r = 0.1234574651...$, where the digits are the values of $S(n)$ for $n \geq 1$, an irational number ? (*F. Smarandache*)

9) Find the largest *strictly increasing* series of integers for which the Smarandache function is *strictly decreasing.* (*J. Rodriguez*)

10) Find a strictly increasing series of integer numbers such that for any consecutive three of them the Smarandache function is neither increasing nor decreasing. (*J. Rodriguez*)

11) Are the points $p(n) = \frac{S(n)}{n}$ uniformly distributed in the interval $(0, 1]$?

12) Prove that

$$\lim_{i \longrightarrow \infty} \frac{S(p_i^m)}{p_i} = m$$

where $p_1 < p_2 < ...p_k...$ is the sequence of prime numbers. (*P. Melendez*)

13) For every composite integer $n \geq 48$, between $S(n)$ and n there exist at least five prime numbers. (*L. Seagull*)

14*) Calculate $\sum_{i=1}^{n} \sigma_{[p]}(i)$ using $\sum_{i=1}^{n} \sigma_{(p)}(i)$.

15) If we note

$$T(n) = 1 - \ln S(n) + \sum_{i=1}^{n} \frac{1}{S(i)}$$

prove that

$$\lim_{i \longrightarrow \infty} T(n) = \infty$$

16) If $(p_n)_{n \in N^*}$ denote the sequence of all the prime numbers then the sequence $\{\frac{p_n - 1}{S(p_n - 1)}\}$ is unbounded. (*M. Popescu. P. Popescu*)

17) For every $k \in N$ there exists a sequence $n_1 < n_2 < ...n_i...$ of positive integers such that

$$\lim_{n \longrightarrow \infty} \frac{n_i}{S(n_i)} > k \quad (\textit{Th. Martin})$$

18*) Solve the following equations:

$$\begin{array}{ll} (i)\ S(x_1^{x_1}) \cdot S(x_2^{x_2}) ....S(x_n^{x_n}) = S(x_{n+1}^{x_{n+1}}) & \\ (ii)\ S(x_1^{x_2}) \cdot S(x_2^{x_3}) ...S(x_{n-1}^{x_n}) = S(x_n^{x_1}) & \end{array} \quad (\textit{Bencze})$$

19) Solve the equations:

$$\begin{array}{l} x^{S(x)} = S(x)^x \\ x^{S(y)} = S(y)^x \qquad \text{(L. Tutescu, E. Burton)} \\ x^{S(x)} + S(x)S(x)^x + x \end{array}$$

20) For all positive integers m, n, r, s holds:

(i) $S(mn) \leq mS(n)$
(ii) $S(mn) \geq \max\{S(m), S(n)\}$
(iii) $\max\{S(m), S(n)\} \leq mS(n)$ (S. Jozsef)
(iv) $m \underset{d}{\leq} n \Longrightarrow \frac{S(m)}{m} \geq \frac{S(n)}{n}$
(v) $S(mn) + S(rs) \geq \max\{S(m) + S(r), S(n) + S(s)\}$

Consequence. For all composite numbers $m, n > 4$ holds

$$\frac{S(mn)}{mn} \leq \frac{S(m) + S(n)}{m + n} \leq \frac{2}{3} \quad (S.\ Jozsef)$$

21*) Find n such that the sum

$$1^{S(n-1)} + 2^{S(n-1)} + \ldots + (n-1)^{S(n-1)} + 1$$

is divisible by n. (*M. Bencze*)

22*) May be written every positive integer n as

$$n = (S(x))^3 + 2(S(y))^3 + 3(S(z))^3\ ? \quad (M.\ Bencze)$$

23*) Prove that

$$\sum_{k=1}^{\infty} \frac{1}{(S(k))^2 - S(k) + 1}$$

is irrational. (*M. Bencze*)

24*) Solve the equation $S(x) = S(x+1)$.

25) Prove that

$$\sum_{n=1}^{\infty} \frac{S(n)}{n^{p+1}}$$

is convergent, for every $p > 1$.

Chapter 2

Generalisations of Smarandache Function

2.1 Extension to the Set Q of Rational Nnumbers

To obtain such a generalisation we shall define first a dual function for the Smarandache function.

In [15] and [17] it is make evident a duality principle by means of which, starting from a given lattice on the unit interval [0, 1], there may be constructed some other lattices on the same interval. We mention that the results of these papers have been used to construct a kind of bitopological spaces and to introduce a new point of view in the study of fuzzy sets.

In [16] the method to construct new lattices on the unit interval, proposed in [17] has been extended to a general lattice.But the main ideas from these papers may be used in various domains of mathematics. We shall use here to construct a generalisation of Smarandache function to the set Q of all rational numbers.

In the sequel we adopt a method from [16] permitting to construct all the functions linked, in a certain sens of duality, with the Smarandache function.

One observe that if we note

$$\mathcal{R}_d(n) = \{m \;/\; n \underset{d}{\leq} m!\}, \quad \mathcal{L}_d = \{m \;/\; m! \underset{d}{\leq} n\}$$
$$\mathcal{R}(n) = \{m \;/\; n \leq m!\}, \quad \mathcal{L}(n) = \{m \;/\; m! \leq n\}$$

we can say that the Smarandache function is defined by means of the triplet $(\wedge, \in, \mathcal{R}_d)$, because one can write:

$$S(n) = \wedge\{m \;/\; m \in \mathcal{R}_d(n)\}$$

We may also create all the functions defined using the triplets (a, b, c), where:

- a is one of the symbols: $\vee$, $\wedge$, $\underset{d}{\wedge}$, and $\overset{d}{\vee}$
- b is one of the symbols: $\in$ and $\notin$
- c is one of the sets: $\mathcal{R}_d(n)$, $\mathcal{L}_d(n)$, $\mathcal{R}(n)$, $\mathcal{L}(n)$ defined above

Not all of these functions are not-trivial. As we have already seen the triplet $(\wedge, \in, \mathcal{R}_d)$ defines the function $S_1(n) = S(n)$, but the triplet $(\wedge, \in, \mathcal{L}_d)$ defines the function

$$S_2(n) = \wedge\{m \;/\; m! \underset{d}{\leq} n\}$$

which is the identity.

Many of the functions obtained using this method are step functions. For instance if we note by S_3 the function obtained from the triplet $(\wedge, \in, \mathcal{R})$, we have:

$$S_3(n) = \{m \;/\; n \leq m!\}$$

so $S_3(n) = m$ if and only if $n \in [(m-1)! + 1, m!]$.

In the following we focus the attention on the function S_4, defined by the triplet $(\vee, \in, \mathcal{L}_d)$:

$$S_4(n) = \vee\{m \;/\; m! \underset{d}{\leq} n\} \tag{2.1}$$

which is, in a certain sense, a dual of Smarandache function.

2.1.1 Proposition. The function S_4 satisfies:

$$S_4(n_1 \underset{d}{\wedge} n_2) = S_4(n_1) \wedge S_4(n_2) \tag{2.2}$$

so is a morphisme from $(N^*, \underset{d}{\wedge})$ to $(N^*, \wedge)$.

Proof. If $p_1, p_2, ..., p_i, ...$is the increasing sequence of all the primes and

$$n_1 = \Pi p_i^{\alpha_i}\ ,\quad n_2 = \Pi p_i^{\beta_i} \text{ with } \alpha_i, \beta_i \in N$$

only a finite number of α_i and β_i being non-nulls, we get:

$$n_1 \underset{d}{\wedge} n_2 = \Pi p_i^{\min(\alpha_i, \beta_i)}$$

If we note $S_4(n_1, n_2) = m$, $S_4(n_i) = m_i$, for $i = 1, 2$,and supposing $m_1 \leq m_2$, it results that the right hand side of (2.2) is $m_1 \wedge m_2$.

From the definition of S_4 we get for the exponent $e_{p_i}(m)$ of the prime p_i in the factorisation of $m!$ the following inequality:

$$e_{p_i}(m) \leq \min(\alpha_i, \beta_i) \text{ for } i \geq 1$$

and at the same time it exists $j \geq 1$ such that

$$e_{p_j}(m+1) > \min(\alpha_j, \beta_j)$$

Then it results:

$$\alpha_i \geq e_{p_i}(m),\quad \beta_i \geq e_{p_i}(m) \quad \text{for } i \geq 1$$

We also have:

$$e_{p_i}(m_1) \leq \alpha_i, \quad e_{p_i}(m_2) \leq \alpha_i$$

and in addition it exists h and k such that:

$$e_{p_h}(m_1 + 1) > \alpha_h, \quad e_{p_k}(m_2 + 1) > \alpha_k$$

So, because $m_1 \leq m_2$, it results

$$\min(\alpha_i, \beta_i) \geq \min(e_{p_i}(m_1), e_{p_i}(m_2)) = e_{p_i}(m_1)$$

and then $m_1 \leq m$. If we suppose the inequality is stricte, it results $m! \leq n_1$, so it exists h such that $e_{p_h}(m) > \alpha_h$ and we get the contradiction:

$$e_{p_h}(m) > \min(\alpha_h, \beta_h)$$

Remark. For many positive integers n we have $S_4(n) = 1$. For instance, $S_4(2n+1) = 1$ for all $n \in N$ and $S_4(n) > 1$ if and only if n is an even number.

2.1.2 Proposition. Let $p_1, p_2, ..., p_i, ...$ the sequence of all consecutive primes and

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_k^{\alpha_k} \cdot q_1^{\beta_1} \cdot q_2^{\beta_2} ... q_r^{\beta_r}$$

the decomposition into primes of a given number $n \in N^*$, such that the first part of the decomposition is formed by the (eventually) first consecutive primes. If we note:

$$t_i = \begin{cases} S(p_i^{\alpha_i}) - 1 & \text{if } e_{p_i}(S(p_i^{\alpha_i})) > \alpha_i \\ S(p_i^{\alpha_i}) + p_i - 1 & \text{if } e_{p_i}(S(p_i^{\alpha_i})) = \alpha_i \end{cases} \tag{2.3}$$

then

$$S_4(n) = \min\{t_1, t_2, ..., t_k, p_{k+1} - 1\} \tag{2.4}$$

Proof. If $e_{p_i}(S(p_i^{\alpha_i})) > \alpha_i$, from the definition of Smarandache function we deduce that $S(p_i^{\alpha_i}) - 1$ is the greatest positive integer m such that $e_{p_i}(m) \leq \alpha_i$. Also, if $e_{p_i}(S(p_i^{\alpha_i}) = \alpha_i$ then $S(p_i^{\alpha_i}) + p_i - 1$ is the greatest positive integer m such that $e_{p_i}(m) = \alpha_i$.

It results the number $\min\{t_1, t_2, ..., t_k, p_{k+1} - 1\}$ is the greatest positive integer m for which $e_{p_i}(m) \leq \alpha_i$ for all $i = 1, 2, ..., k$.

2.1.3 Proposition. The function S_4 satisfies:

$$S_4(n_1 + n_2) \wedge S_4(n_1 \overset{d}{\vee} n_2) = S_4(n_1) \wedge S_4(n_2)$$

for every $n_1, n_2 \in N^*$.

Proof. The equality results from (2.2) taking into account that:

$$(n_1 + n_2) \underset{d}{\wedge} (n_1 \overset{d}{\vee} n_2) = n_1 \underset{d}{\wedge} n_2$$

Before to construct the extension of the Smarandache function to the set Q_+ of all positive rationals we shall make evident some morphism properties of any functions defined by the triplets (a, b, c).

2.1.4 Proposition. (i) The function $S_5 : N^* \longrightarrow N^*$, where

$$S_5(n) = \overset{d}{\vee} \{ m \;/\; m! \underset{d}{\leq} n \}$$

satisfies:

$$S_5(n_1 \underset{d}{\wedge} n_2) = S_5(n_1) \underset{d}{\wedge} S_5(n_2) = S_5(n_1) \wedge S_5(n_2) \qquad (2.5)$$

(ii) The function $S_6 : N^* \longrightarrow N^*$, defined by:

$$S_6(n) = \overset{d}{\vee} \{m \;/\; n \underset{d}{\leq} m!\}$$

satisfies:

$$S_6(n_1 \overset{d}{\vee} n_2) = S_6(n_2) \overset{d}{\vee} S_6(n_2) \tag{2.6}$$

(iii) The function $S_7 : N^* \longrightarrow N^*$, defined by:

$$S_7(n) = \overset{d}{\vee} \{m \;/\; m! \leq n\} \tag{2.7}$$

satisfies:

$$S_7(n_1 \wedge n_2) = S_7(n_1) \wedge S_7(n_2), \; S_7(n_1 \vee n_2) = S_7(n_2) \vee S_7(n_2) \tag{2.8}$$

Proof. (i) Let

$$A = \{a_i \;/\; a_i! \underset{d}{\leq} n_1\}, \; B = \{b_j \;/\; b_j! \underset{d}{\leq} n_2\}, \; C = \{c_k \;/\; c_k \underset{d}{\leq} n_1 \underset{d}{\wedge} n_2\}$$

Then we have $A \subset B$ or $B \subset A$. Indeed, let

$$A = \{a_1, a_2, ..., a_h\}, \; B = \{b_1, b_2, ..., b_r\}$$

be the elements of A and B writen in increasing order. That is $a_i < a_{i+1}$ and $b_j < b_{j+1}$ for $i = \overline{1, h-1}$ and $j = \overline{1, r-1}$. Then if $a_h \leq b_r$, it results $a_i \leq b_r$ for $i = \overline{1, h}$, so $a_i! \underset{d}{\leq} b_r! \underset{d}{\leq} n_2$. Consequently $A \subset B$.

Analogously, if $b_r \leq a_h$, it results $B \subset A$, and of course we have $C = A \cap B$. So, if $A \subset B$ it results

$$S_5(n_1 \underset{d}{\wedge} n_2) = \overset{d}{\vee} c_k = \overset{d}{\vee} a_i = S_5(n_1) =$$
$$\min\{S_5(n_1), S_5(n_2)\} = S_5(n_1) \underset{d}{\wedge} S_5(n_2)$$

Considering the function S_5 defined on the lattice $\mathcal{N}_d$, from (1.100) it results that it is order preserving. But if we consider this function defined on the lattice $\mathcal{N}_o$ it is not order preserving, because

$$m! < m! + 1 \text{ but } S_5(m!) = [1, 2, ..., m] \text{ and } S_5(m! + 1) = 1$$

(*ii*) Let us observe that

$$S_6(n) = \overset{d}{\vee} \{m \;/\; (\exists)\, i \in \overline{1,t} \text{ such that } e_{p_i}(m) < \alpha_i\}$$

If we note $a = \vee\{m \;/\; n \underset{d}{\leq} m!\}$ then $n \underset{d}{\leq} (a+1)!$ and

$$a + 1 = \wedge\{m \;/\; n \underset{d}{\leq} m!\} = S(n)$$

so

$$S_6(n) = [1, 2, ..., S(n) - 1]$$

and then

$$S_6(n_1 \overset{d}{\vee} n_2) = [1, 2, ..., S(n_1 \overset{d}{\vee} n_2) - 1] = S_7(n_1 \vee n_2) = S_7(n_2) \vee S_7(n_2)$$

Also, we have:

$$S_6(n_1) \overset{d}{\vee} S_6(n_2) = [[1, 2, ..., S(n_1) - 1], [1, 2, ..., S(n_2) - 1]] = [1, 2, ..., S(n_1) \vee S(n_2) - 1]$$

(*iii*) The equalities results from the fact that if m is given by (2.7) then

$$S_7(n) = [1, 2, ..., m] \Longleftrightarrow n \in [m!, (m+1)! - 1]$$

Let us now define the extension of the Smarandache function to the set Q_+ of positive rationals.

It is said [25] that every positive rational a may be written under the form

$$a = \prod_p p^{\alpha_p} \tag{2.9}$$

with p a prime, $\alpha_p \in Z$ and only a finite number of the exponents are non-nulls. Taking into account this equality one may define the divisibility of rational numbers as follows:

2.1.5 Definition. The rational number $a = \prod_p p^{\alpha_p}$ divides the rational number $b = \prod_p p^{\beta_p}$ if $\alpha_p \leq \beta_p$ for all prime p.

The equality (2.9) implies that the multiplication of rational numbers is reduced to the addition of some exponents. Consequently the problems on the divisibility of these numbers are reduced to order problems between exponents.

The greatest common divisor d and the smallest common multiple e for rational numbers are defined [25] by:

$$d = (a, b, ...) = \prod_p p^{\min\{\alpha_p, \beta_p, ...\}}, \quad e = [a, b, ...] = \prod_p p^{\max\{\alpha_p, \beta_p, ...\}} \tag{2.10}$$

Moreover, between the greatest common divisor d and the smallest common multiple of any rational numbers there exists the relation:

$$[a, b, ...] = \frac{1}{\left(\frac{1}{a}, \frac{1}{b}, ...\right)} \tag{2.11}$$

Of course, every positive rational a may be written under the form:

$$a = \frac{n}{n_1} \text{ with } n \in N, n_1 \in N^*, \text{ and } (n, n_1) = 1$$

2.1.6 Definition. The extension $S : Q_+^* \longrightarrow Q_+^*$ of the Smarandache function to the positive rationals is:

$$S(\frac{n}{n_1}) = \frac{S_1(n)}{S_4(n_1)} \tag{2.12}$$

A consequence of this definition is that if n_1 and n_2 are positive integers then:

$$S(\frac{1}{n_1} \overset{d}{\vee} \frac{1}{n_2}) = S(\frac{1}{n_1}) \vee S(\frac{1}{n_2}) \tag{2.13}$$

Indeed,

$$S(\tfrac{1}{n_1} \overset{d}{\vee} \tfrac{1}{n_2}) = S(\tfrac{1}{n_1 \underset{d}{\wedge} n_2}) = \tfrac{1}{S_4(n_1 \underset{d}{\wedge} n_2)} = \tfrac{1}{S_4(n_1) \wedge S_4(n_2)} =$$
$$= \tfrac{1}{S_4(n_1)} \vee \tfrac{1}{S_4(n_2)} = S(\tfrac{1}{n_1}) \vee S(\tfrac{1}{n_2})$$

For two arbitrary positive rationals we have:

$$S(\frac{n}{n_1} \overset{d}{\vee} \frac{m}{m_1}) = (S(n) \vee S(m)) \cdot (S(\frac{1}{n_1}) \vee S(\frac{1}{m_1})) \tag{2.14}$$

This formula generalise the equality (1.16).

2.1.7 Definition. The function $\tilde{S} : Q_+^* \longrightarrow Q_+^*$ defined by:

$$\tilde{S}(a) = \frac{1}{S(\frac{1}{a})} \tag{2.15}$$

is called *the dual of Smarandache function.*

2.1.8 Proposition. The dual $\tilde{S}$ of the function S satisfies:

$$(i)\ \tilde{S}(n_1 \underset{d}{\wedge} n_2) = \tilde{S}(n_1) \wedge \tilde{S}(n_2)$$
$$(ii)\ \tilde{S}(\tfrac{1}{n_1} \underset{d}{\wedge} \tfrac{1}{n_2}) = \tilde{S}(\tfrac{1}{n_1}) \wedge \tilde{S}(\tfrac{1}{n_2})$$

for all positive integers n_1 and n_2. Moreover, we also have

$$\tilde{S}(\frac{n}{n_1} \underset{d}{\wedge} \frac{m}{m_1}) = (\tilde{S}(n) \wedge \tilde{S}(m)) \cdot (\tilde{S}(\frac{1}{n_1}) \wedge \tilde{S}(\frac{1}{m_1}))$$

The proof is evident.

Remarks. 1) The restriction of the function $\tilde{S}$ to the set of the positive integers coincide with the function S_4.

2) The extension of the function $S : Q_+^* \longrightarrow Q_+^*$ to the set $Q^*$of all non-nulls rationals may be made for instance by the equality:

$$S(-a) = S(a) \text{ for all } a \in Q_+^*$$

2.2 Numerical Functions Inspired from the Definition of Smarandache Function

In this section we shall utilise the equalities (2.1) and (1.58) to define, by analogy, other numerical functions.

Let us observe that if n is any positive integer then $n!$ is the product of all positive integers not greater than n in the lattice $\mathcal{L}$. Analogously the product ρ_m of all divisors of a given m, including 1 and m, is the product of all positive integers not greater than m in the lattice $\mathcal{L}_d$. So we can consider functions of the form:

$$\theta(n) = \wedge \{ m \,/\, n \underset{d}{\leq} \rho(m) \}$$

It is said that if

$$m = p_1^{x_1} \cdot p_2^{x_2} ... p_t^{x_t}$$

is the decomposition into primes of a given number m, then the product of all the divisors of m is

$$\rho(m) = \sqrt{m^{\tau(m)}} \tag{2.16}$$

where $\tau(m) = (x_1+1)(x_2+1)...(x_t+1)$ is the number of divisors of m.

If n has the decomposition

$$n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_t^{\alpha_t} \tag{2.17}$$

then the inequality $n \underset{d}{\leq} \rho(m)$ is equivalent with:

$$\begin{array}{l} g_1(x) \equiv x_1(x_1+1)...(x_t+1) - 2\alpha_1 \geq 0 \\ g_2(x) \equiv x_2(x_1+1)...(x_t+1) - 2\alpha_2 \geq 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ g_t(x) \equiv x_t(x_1+1)...(x_t+1) - 2\alpha_t \geq 0 \end{array} \tag{2.18}$$

So, $\theta(n)$ may be deduced solving the following non-linear programming problem:

$$(\min)\ f(x) = p_1^{x_1} \cdot p_2^{x_2} ... p_t^{x_t} \tag{2.19}$$

under the restrictions (2.18).

The solution of this problem may be obtained applying for instance the algorithm $SUMT$ (Sequential Unconstrained Minimisation Techniques) does to Fiacco and Mc. Cormick [18].

Examples. 1) For $n = 3^4 \cdot 5^{12}$ the equalities (2.18) and (2.19) become:

$$(\min) f(x) = 3^{x_1} \cdot 5^{x_2}$$

with the restrictions

$$\begin{cases} g_1(x) \equiv x_1(x_1+1)(x_2+1) \geq 8 \\ g_2(x) \equiv x_2(x_1+1)(x_2+1) \geq 24 \end{cases}$$

Using the algorithm $SUMT$ we consider the function

$$u(x,n) = f(x) - r\sum_{i=1}^{t} \ln g_i(x)$$

and the system

$$\begin{cases} \frac{\partial u}{\partial x_1} = 0 \\ \frac{\partial u}{\partial x_2} = 0 \end{cases} \tag{2.20}$$

In [18] it is shown that if the solution $x_1(r), x_2(r)$ of this system can't be found explicitely from the system, we can take $r \longrightarrow 0$. Then the system becomes:

$$\begin{cases} x_1(x_1+1)(x_2+1) = 8 \\ x_2(x_1+1)(x_2+1) = 24 \end{cases}$$

and has the solution $x_1 = 1, x_2 = 3$. So we have:

$$\min\{\, m \;/\; 3^4 \cdot 5^{12} \leq \rho(m)\} = m_0 = 3 \cdot 5^3$$

Indeed, $\rho(m_0) = \sqrt{m_0^{\tau(m_0)}} = m_0^4 = 3^4 \cdot 5^{12} = n$.

2) For $n = 3^2 \cdot 5^7$, from (2.20) it results for x_2 the equation

$$2x_2^3 + 9x_2^2 + 7x_2 - 98 = 0$$

with a real solution in the interval $(2,3)$. It results $x_1 \in (4/7, 5/7)$.

Considering $x_1 = 1$ we observe that for $x_2 = 2$ the pair (x_1, x_2) is not an admisible solution of the problem, but $x_2 = 3$ give $\theta(3^2 \cdot 5^7) = 3^4 \cdot 5^{12}$.

3) In general, for $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2}$ it results from the system (2.20) the equation:

$$\alpha_1 x_2^3 + (\alpha_1 + \alpha_2)x_2^2 + \alpha_2 x_2 - 2\alpha_2^2 = 0$$

with the solution given by the formula of Cartan.

Remark. Using *"the method of triplets"* we may attache to the function θ defined above many other functions.

Starting from the function ν, given by (1.58), we may also obtain numerical functions by the same method.

In the following we shall study the analogous of Smarandache function and its dual in this second case.

2.2.1 Proposition. If n has the decomposition (2.17) then:

$$(i)\ \nu(n) = \max_{i=\overline{1,t}} p_i^{\alpha_i}\ ,\quad (ii)\ \nu(n_1 \overset{d}{\vee} n_2) = \nu(n_1) \vee \nu(n_2)$$

Proof. (i) Let be $p_u^{\alpha_u} = \max p_i^{\alpha_i}$. Then $p_i^{\alpha_i} \leq p_u^{\alpha_u}$ for all $i = \overline{1,t}$, so

$$p_i^{\alpha_i} \underset{d}{\leq} [1, 2, ..., p_u^{\alpha_u}]$$

But $(p_i^{\alpha_i}, p_j^{\alpha_j}) = 1$ for $i \neq j$ and then

$$n \underset{d}{\leq} [1, 2, ..., p_u^{\alpha_u}]$$

If for some $m < p_u^{\alpha_u}$ we have $n \underset{d}{\leq} [1, 2, ...m]$, it results the contradiction

$$p_u^{\alpha_u} \underset{d}{\leq} [1, 2, ...m]$$

(ii) If

$$n_1 = \Pi p^{\alpha_p}\ ,\quad n_2 = \Pi p^{\beta_p}$$

then

$$n_1 \overset{d}{\vee} n_2 = \Pi p^{\max\{\alpha_p, \beta_p\}}$$

so $\nu(n_1 \overset{d}{\vee} n_2) = \max p^{\max\{\alpha_p,\beta_p\}} = \max\{\max p^{\alpha_p}, \max p^{\beta_p}\}$and the property is proved.

Of course, we can say that the function $\nu_1 = \nu$ is defined by the triplet $(\wedge, \in, \mathcal{R}_{[d]})$, where

$$\mathcal{R}_{[d]} = \{m \;/\; n \underset{d}{\leq} [1, 2, ..., m]\}$$

Its dual, in the sense defined in the preceding section, is the function defined by the triplet $(\vee, \in, \mathcal{L}_{[d]})$, where

$$\mathcal{L}_{[d]} = \{m \;/\; [1, 2, ..., m] \underset{d}{\leq} n\}.$$

Let us note by ν_4 this function:

$$\nu_4(n) = \vee\{m \;/\; [1, 2, ..., m] \underset{d}{\leq} n\}$$

Then $\nu_4(n)$ is the greatest positive integer having the property that all positive integers $m \leq \nu_4(n)$ divide n.

Let us observe now that a necessary and sufficient condition to have $\nu_4(n) > 1$ is the existence of $m > 1$ such that every primes $p \leq m$ divide n.

From the definition of ν_4 it also results

$$\nu_4(n) = m \Longleftrightarrow n \text{ is divisible by every } i \leq m\text{,but not by } m + 1$$

2.2.2 Proposition. The function ν_4 satisfies:

$$\nu_4(n_1 \underset{d}{\wedge} n_2) = \nu_4(n_1) \wedge \nu_4(n_2)$$

Proof. Let us note

$$n = n_1 \underset{d}{\wedge} n_2, \;\; \nu_4(n) = m, \;\; \nu_4(n_i) = m_i \;\text{ for } i = 1, 2$$

If $m_1 = m_1 \wedge m_2$, we prove that $m = m_1$. Indeed, from the definition of ν_4 it results:

$$\nu_4(n_i) = m_i \iff$$
$$\iff \{(\forall)\, i \leq m_i \Longrightarrow n \text{ is divisible by } m_i \text{ but not by } m_i + 1\ \}$$

If we have $m < m_1$, then $m+1 \leq m_1 \leq m_2$, so $m+1$ divides n_1 and n_2, and so $m+1$ divides n.

If $m > m_1$, then $m_1 + 1 \leq m$, so $m_1 + 1$ divides n.

But n divides n_1, so $m_1 + 1$ divides n_1, and the proposition is proved.

Let us observe that if we note

$$t_0 = \max\{i \;/\; j \leq i \Longrightarrow n \text{ is divisible by } j\}$$

then $\nu_4(n)$ may be obtained solving the linear programming problem

$$(\max) f(x) = \sum_{i=1}^{t_0} x_i \ln p_i;$$
$$x_i \leq \alpha_i \text{ for } i = \overline{1, t_0};\ \sum_{i=1}^{t_0} x_i \ln p_i \leq \ln p_{t_0+1}$$

If f_0 is the maximum of f from this problem, then $\nu_4(n) = e^{f_0}$.

For instance $\nu_4(2^3 \cdot 3^2 \cdot 5 \cdot 11) = 6$.

Of course, the function ν may be extended to the set of all rational numbers by the same method as Smarandache function.

2.3 Smarandache Functions of First, Second and Third Kind

Let X be an arbitrary nonvoid set, $r \subset X \times X$ an equivalence relation, $\widehat{X}$ the corresponding quotient set and $(I, \leq)$ a total ordered set.

2.3.1 Definition. If $g : \widehat{X} \longrightarrow I$ is an arbitrary injective function then the function

$$f : X \longrightarrow I \ , \ \text{defined by} \ f(x) = g(\widehat{x}) \qquad (2.21)$$

is said to be a standardisation. About the set X we shall say in this case that it is $(r, (I, \leq), f)$ *standardised.*

2.3.2 Definition. If r_1 and r_2 are two equivalence relation on X, the relation $r = r_1 \wedge r_2$ is given by:

$$x \, r \, y \Longleftrightarrow x \, r_1 \, y \quad \text{and} \quad x \, r_2 \, y \qquad (2.22)$$

Of course, r defined as above is an equivalence relation.

2.3.3Definition. The functions $f_i : X \longrightarrow I$, $i = \overline{1, s}$ are of the same monotonicity if for every $x, y \in X$ we have:

$$f_k(x) \leq f_k(y) \Longleftrightarrow f_j(x) \leq f_j(y) \quad \text{for} \ k, j = \overline{1, s} \qquad (2.23)$$

2.3.4 Theorem. If the standardisations $f_i : X \longrightarrow I$, corresponding to the equivalence relations r_i (for $i = \overline{1, s}$) are of the same monotonicity then the function

$$f = \max f_i$$

is a standardisation, corresponding to $r = \bigwedge\limits_{i=1}^{s} r_i$, and it is of the same monotonicity as the functions f_i.

Proof. We give here the proof when $s = 2$. For an arbitrary value of s the assertion results then by induction.

Let $\widehat{x}_{r_1}$, $\widehat{x}_{r_2}$ and $\widehat{x}_r$ be the classes of equivalence of x corresponding to the relations r_1, r_2 and $r = r_1 \wedge r_2$. If $\widehat{X}_{r_1}, \widehat{X}_{r_2}, \widehat{X}_r$ denote the quotient sets induced by these relations then:

$$f_i(x) = g_i(\widehat{x}_{r_i}) \, , \text{for} \ i = 1, 2, \text{where} \ g_i : \widehat{X}_{r_i} :\longrightarrow I \ \text{are injective}$$

The function $g : \widehat{X}_r \longrightarrow I$ defined by $g(\widehat{x}_r) = \max(g_1(\widehat{x}_{r_1}), g_2(\widehat{x}_{r_2}))$ is injective. Indeed, if $\widehat{x}_r^1 \neq \widehat{x}_r^2$ and

$$\max(g_1(\widehat{x}_{r_1}^1), g_2(\widehat{x}_{r_2}^1)) = \max(g_1(\widehat{x}_{r_1}^2), g_2(\widehat{x}_{r_2}^2))$$

then from the injectivity of g_1 and g_2 it results for instance:

$$\max(g_1(\widehat{x}_{r_1}^1), g_2(\widehat{x}_{r_2}^1)) = g_1(\widehat{x}_{r_1}^1) = g_2(\widehat{x}_{r_2}^2) = \max(g_1(\widehat{x}_{r_1}^2), g_2(\widehat{x}_{r_2}^2))$$

and we have a contradiction, because

$$\begin{aligned} f_1(x^2) &= g_1(\widehat{x}_{r_1}^2) < g_1(\widehat{x}_{r_1}^1) = f_1(x^1) \\ f_2(x^1) &= g_2(\widehat{x}_{r_1}^1) < g_2(\widehat{x}_{r_2}^2) = f_2(x^2) \end{aligned}$$

That is f_1 and f_2 are not of the same monotonicity.

From the injectivity of g it results that the function

$$f : X \longrightarrow I \quad f(x) = g(\widehat{x}_r)$$

is a standardisation. Moreover, we have:

$$\begin{aligned} f(x^1) \leq f(x^2) &\Longleftrightarrow g(\widehat{x}_r^1) \leq g(\widehat{x}_r^2) \Longleftrightarrow \max(g_1(\widehat{x}_{r_1}^1), g_2(\widehat{x}_{r_2}^1)) \leq \\ &\leq \max(g_1(\widehat{x}_{r_1}^2), g_2(\widehat{x}_{r_2}^2)) \Longleftrightarrow \max(f_1(x^1), f_2(x^1)) \leq \\ &\leq \max(f_1(x^2), f_2(x^2)) \Longleftrightarrow f_1(x^1) \leq f_1(x^2) \text{ and } f_2(x^1) \leq f_2(x^2) \end{aligned}$$

because f_1 and f_2 are of the same monotonicity.

Let us now consider two algebraic lows $\top$ and $\perp$ on X respectively on I.

2.3.5 Definition. The standardisation $f : X \longrightarrow I$ is said to be Σ-compatible with the lows $\top$ and $\perp$ if for every $x, y \in X$, the triplet $(f(x), f(y), f(x \top y))$ satisfies the condition Σ. In this case we shall also say that the function f Σ-standardise the structure $(X, \top)$on the structure $(I, \leq, \perp)$.

Example. If the function f is the Smarandache function $S : N^* \longrightarrow N^*$, one can make evident the following standardisations:

(a) The function S, Σ_1-standardise $(N^*, \cdot)$ on $(N^*, \leq, +)$ because we have:

$$(\Sigma_1) : S(a \cdot b) \leq S(a) + S(b)$$

(b) The function S also satisfies:

$$(\Sigma_2) : \max(S(a), S(b)) \leq S(a \cdot b) \leq S(a), S(b)$$

so this function Σ_2-standardise the structure $(N^*, \cdot)$ on the structure $(N^*, \leq, \cdot)$.

Now we may define the Smarandache function of first kind. We have already seen (section 1.2) that the Smarandache function is defined by means of the functions S_p. We remember that for every prime number p the function $S_p : N^* \longrightarrow N^*$ is defined by the conditions:

1) $S_p(n)!$ is divisible by p^n,

2) $S_p(n)$ is the smallest positive integer with the property 1).

Using the definition of a standardisation in [2] there are given three generalisations of the functions S_p.

To present these generalisations let us note by $M(n)$ any multiple of the integer n.

2.3.6 Definition. The relation $r_n \subset N^* \times N^*$is defined for every $n \in N^*$ by the conditions:

(i) If $n = u^i$, with $u = 1$ or $u = p$ (a prime) and $i, a, b \in N^*$, then:

$$a\, r_n\, b \Longleftrightarrow (\exists)\, K \in N^*,\ \text{such that}\ \ k! = M(u^{ia}),\ k! = M(u^{ib})$$

and k is the smallest positive integer with this property.

(ii) If

$$n = p_1^{i_1} \cdot p_2^{i_2} ... p_s^{i_s} \tag{2.24}$$

is the decomposition of n onto primes, then:

$$r_n = r_{p_1^{i_1}} \wedge r_{p_2^{i_2}} \wedge \ldots \wedge r_{p_s^{i_s}}$$

2.3.7 Definition. For every $n \in N^*$ the Smarandache function of first kind is the function $S_n : N^* \longrightarrow N^*$ satisfying the conditions:

(i) If $n = u^i$, with $u = 1$ or $u = p$, then $S_n(a)$ is the smallest positive integer k having the property $k! = M(u^{ia})$.

(ii) If $n = p_1^{i_1} \cdot p_2^{i_2} \ldots p_s^{i_s}$ then

$$S_n(a) = \max_{1 \leq i \leq s} (S_{p_j^{i_j}}(a))$$

Remarks. 1. The functions S_n are standardisations corresponding to equivalence relations r_n defined above. If $n = 1$, it results $\widehat{x}_{r_1} = N^*$, for every $x \in N^*$, and $S_1(n) = 1$ for every $n \in N^*$.

2. If $n = p$ is a prime number then S_n is just the function S_p defined by F. Smarandache.

3. All the functions S_n are increasing and so are of the same monotonicity, in the sense of definition 2.3.3.

2.3.8 Theorem. The functions S_n have the properties that Σ_1-standardise $(N^*, +)$ on $(N^*, \leq, +)$ by the relation:

$$(\Sigma_1): \max(S_n(a), S_n(b)) \leq S_n(a+b) \leq S_n(a) + S_n(b)$$

for every $a, b \in N^*$, and also Σ_2-standardise the structure $(N^*, +)$ on the structure $(N^*, \leq, \cdot)$ by:

$$(\Sigma_2): \max(S_n(a), S_n(b)) \leq S_n(a+b) \leq S_n(a) \cdot S_n(b)$$

for every $a, b \in N^*$.

Proof. Let p be a prime and $n = p^i$, with $i \in N^*$. Let also be $a^* = S_{p^i}(a)$, $b^* = S_{p^i}(b)$, $k = S_{p^i}(a+b)$. Then from the definition of S_n it results that a^*, b^* and k are the the smallest positive integers satisfying the properties:

$$a^*! = M(p^{ia}),\ b^*! = M(p^{ib}),\ k! = M(p^{i(a+b)})$$

From $k! = M(p^{ia}) = M(p^{ib})$ it results $a^* \leq k$ and $b^* \leq k$, so $\max(a^*, b^*) \leq k$ and the first inequality from (Σ_1), as from (Σ_2), is proved.

Because

$$(a^* + b^*)! = a^*!(a^* + 1)...(a^* + b^*) = M(a^*!b^*!) = M(p^{i(a+b)})$$

it results $k \leq a^* + b^*$, so (Σ_1) is satisfied.

If $n = p_1 i_1 \cdot p_2^{i_2}...p_s^{i_s}$, taking into account the above considerations we get:

$$(\Sigma_1):\ \max(S_{p_j^{i_j}}(a), S_{p_j^{i_j}}(b)) \leq S_{p_j^{i_j}}(a+b) \leq S_{p_j^{i_j}}(a) + S_{p_j^{i_j}}(b)$$

for $j = \overline{1,s}$ and consequently:

$$\max(\max_j S_{p_j^{i_j}}(a), \max_j S_{p_j^{i_j}}(b)) \leq \max_j S_{p_j^{i_j}}(a+b) \leq$$
$$\leq \max_j S_{p_j^{i_j}}(a) + \max_j S_{p_j^{i_j}}(b)$$

for $\overline{1,s}$,so

$$\max(S_n(a), S_n(b)) \leq S_n(a+b) \leq S_n(a) + S_n(b)$$

To prove the second inequality from (Σ_2) we remember that $(a+b)! \leq (ab)!$ if and only if $a > 1$ and $b > 1$. Our inequality is satisfied for $n = 1$, because

$$S_1(a+b) = S_1(a) = S_1(b) = 1$$

Let now be $n > 1$. It results for $a^* = S_n(a)$ that $a^* > 1$. Indeed, if n has the decomposition (2.24) then:

$$a^* = 1 \Longleftrightarrow S_n(a) = \max S_{p_j^{i_j}}(a) = 1$$

and that implies $p_1 = p_2 = ... = p_s = 1$, so $n = 1$.

Consequently for every $n > 1$ we have

$$S_n(a) = a^* > 1 \text{ and } S_n(b) = b^* > 1$$

Then $(a^* + b^*)! \leq (a^* \cdot b^*)!$ and we get:

$$S_n(a + b) \leq S_n(a) + S_n(b) \leq S_n(a) \cdot S_n(b)$$

In the sequel we present some results on the monotonicity of Smarandache functions of the first kind.

2.3.9 Proposition. For every positive integer n the Smarandache function of first kind is increasing.

Proof. If n is a prime and $k_1 < k_2$ from $(S_n(k_2))! = M(n^{k_2}) = M(n^{k_1})$ it results $S_n(k_1) \leq S_n(k_2)$.

If n is an arbitrary positive integer let

$$S_{p_m}(i_m k_1) = \max_{1 \leq j \leq k} S_{p_j}(i_j k_1) = S_n(k_1)$$
$$S_{p_t}(i_t k_2) = \max_{1 \leq j \leq r} S_{p_j}(i_j k_2) = S_n(k_2)$$

From

$$S_{p_m}(i_m k_1) \leq S_{p_m}(i_m k_2) \leq S_{p_t}(i_t k_2)$$

it results $S_n(k_1) \leq S_n(k_2)$ and the proposition is proved.

2.3.10 Proposition. The sequence of functions $(S_{p^i})_{i \in N^*}$ is monotonously increasing, for every prime number p.

Proof. For every $i_1, i_2 \in N^*$, with $i_1 < i_2$ and for every $n \in N^*$ we have:

$$S_{p^{i_1}}(n) = S_p(i_1 \cdot n) \leq S_p(i_2 \cdot n) = S_{p^{i_2}}(n)$$

so $S_{p^{i_1}} \leq S_{p^{i_2}}$ and the proposition is proved.

2.3.11 Proposition. Let p and q be two given primes. Then:

$$p < q \Longrightarrow S_p(k) < S_q(k) \quad \text{for every} \quad k \in N^*$$

Proof. The arbitrary integer $k \in N^*$ may be written in the scale $[p]$ as:

$$k = t_1 a_s(p) + t_2 a_{s-1}(p) + \dots + t_s a_1(p) \tag{2.25}$$

It is said that $0 \leq t_i \leq p-1$ for $i = \overline{1,s}$ and the last non-null digit may also be p.

Passing from k to $k+1$ in (2.25) we can make evident the following algorithm:

(i) t_s increases with unit.

(ii) if t_s can't increase with unit, then t_{s-1} increase with an unit and t_s take the value zero.

(iii) if neither t_s nor t_{s-1} can increase with an unit then t_{s-2} increase and t_s as well as t_{s-1} become zero.

The processus is continued until we get the expression of $k+1$.

Noting

$$\Delta_k(S_p) = S_p(k+1) - S_p(k) \tag{2.26}$$

the increment of function S_p when we pass from k to $k+1$, following the above algorithm one obtain:

- if (i) holds then $\Delta_k(S_p) = p$,
- if (ii) holds then $\Delta_k(S_p) = 0$,
- if (iii) holds then $\Delta_k(S_p) = 0$.

and it results

$$S_p(n) = \sum_{k=1}^{n} \Delta_k(S_p) + S_p(1)$$

Analogously:

$$S_q(n) = \sum_{k=1}^{n} \Delta_k(S_q) + S_q(1)$$

Taking into account that $S_p(1) = p < q = S_q(1)$ and using the algorithm mentioned above it results that the number of increments of value zero of the function S_p is greatest than the number of increments of value zero for the function S_q, and the increments with value p of S_p are smaller than the increments of value q of S_q. So:

$$\sum_{k=1}^{n} \Delta_k(S_p) + S_p(1) < \sum_{k=1}^{n} \Delta_k(S_q) + S_q(1) \qquad (2.27)$$

and then $S_p(n) < S_q(n)$ for every $n \in N^*$.

Example. The values of S_2 and S_3 are listed bellow.

k	1	2	3	4	5	6	7	8	9	10
increment		2	0	2	2	0	0	2	2	0
$S_2(k)$	2	4	4	6	8	8	8	10	12	12
increment		3	3	0	3	3	3	0	3	3
$S_3(k)$	3	6	9	9	12	15	18	18	21	24

k	11	12	13	14	15	16	17	18	19	20
increment	2	2	0	0	0	2	0	2	2	2
$S_2(k)$	14	16	16	16	16	18	18	20	22	24
increment	3	0	0	3	3	3	0	3	3	3
$S_3(k)$	27	27	27	30	33	36	36	39	42	45

and one observe that $S_2(k) < S_3(k)$, for $k = \overline{1, 20}$.

Remark. For every increasing sequence

$$p_1 < p_2 < \ldots < p_n < \ldots$$

of prime numbers it results:

$$S_1 < S_{p_1} < S_{p_2} < ... < S_{p_n} < ...$$

and if $n = (p_1 \cdot p_2 ... p_t)^i$ with $p_1 < p_2 < ... < p_t$, then

$$S_n(k) = \max_{1 \leq j \leq k} S_{p_j^i}(k) = S_{p_t^i}(k) = S_{p_t}(ik)$$

2.3.12 Proposition. If p and q are prime numbers and $p \cdot i < q$, then $S_{p^i} < S_q$.

Proof. From $p \cdot i < q$ it results:

$$S_{p^i}(1) \leq p \cdot i < q = S_q(1) \text{ and } S_{p^i}(k) = S_p(ik) \leq iS_p(k) \quad (2.28)$$

Passing from k to $k+1$, from (2.28) one deduce:

$$\Delta_k(S_{p^i}) \leq \Delta_k(S_p) \quad (2.29)$$

The proposition (2.311) and the equality (2.29) imply that passing from k to $k+1$ we get:

$$\Delta_k(S_{p^i}) \leq \Delta_k(S_p) \leq i \cdot p < q, \; i \sum_{k=1}^{n} \Delta_k(S_p) \leq \sum_{k=1}^{n} \Delta_k(S_q) \quad (2.30)$$

Because we have

$$S_{p^i}(n) = S_{p^i}(1) + \sum_{k=1}^{n} \Delta_k(S_{p^i}) \leq S_{p^i}(1) + i \sum_{k=1}^{n} \Delta_k(S_p)$$

and

$$S_q(n) = S_q(1) + \sum_{k=1}^{n} \Delta_k(S_q)$$

from (2.28) and (2.30) it results $S_{p^i}(n) \leq S_q(n)$ for every $n \in N^*$, and the property is proved.

2.3.13 Proposition. If p is a prime number, then $S_n < S_p$ for every $n < p$.

Proof. If n is a prime, from $n < p$ and the proposition (2.3.11) it results $S_n(k) < S_p(k)$ for every $k \in N^*$. If

$$n = p_1^{i_1} \cdot p_2^{i_2} \dots p_t^{i_t}$$

is a composit number then:

$$S_n(k) = \max_{1 \leq j \leq t} S_{p_j^{i_j}}(k) = S_{p_r^{i_r}}(k)$$

and from $n < p$ it results $p_r^{i_r} < p$. So, using the preceding proposition and the inequality $p_r \leq p_r^{i_r} \leq p$, one obtain

$$S_{p_r^{i_r}}(k) \leq S_p(k)$$

That is $S_n(k) < S_p(k)$ for every $k \in N^*$.

We shall present now the Smarandache function of second kind, defined in [2].

2.3.14 Definition. The Smarandache functions of second kind are the functions

$$S^k : N^* \longrightarrow N^* \text{ , defined by } \quad S^k(n) = S_n(k)$$

for every fixed $k \in N^*$, where S_n is a Smarandache function of first kind.

From this definition it results that for $k = 1$, S^k is just the function S. Indeed, for $n > 1$ we have

$$S^1(n) = S_n(1) = \max_j S_{p_j^{i_j}}(1) = \max S_{p_j}(i_j) = S(n)$$

2.3.15 Theorem. Every Smarandache functions of second kind Σ_3-standardise the structure $(N^*, \cdot)$ on the structure $(N^*, \leq, +)$ by:

$$(\Sigma_3): \ \max(S^k(a), S^k(b)) \leq S^k(a \cdot b) \leq S^k(a) + S^k(b)$$

for every $a, b \in N^*$. At the same time these functions Σ_4-standardise the structure $(N^*, \cdot)$ on $(N^*, \leq, \cdot)$ by:

$$(\Sigma_4): \ \max(S^k(a), S^k(b)) \leq S^k(a \cdot b) \leq S^k(a) \cdot S^k(b)$$

for every $a, b \in N^*$.

Proof. The equivalence relation r^k corresponding to S^k is defined by:

$$a\, r^k\, b \iff (\exists)\, a^* \in N^* \ a^*! = M(a^*), \ a^*! = M(b^k) \qquad (2.31)$$

and a^* is the smallest positive integer satisfying (2.31). Consequently we may say that S^k is a standardisation attached to the equivalence relation r^k.

Let us observe that the Smarandache functions of second kind are not of the same monotonicity, because, for instance, $S^2(a) \leq S^2(b) \iff S(a^2) \leq S(b^2)$ and from this it does not result $S^1(a) \leq S^1(b)$.

For every $a, b \in N^*$ let us note $a^* = S^k(a)$, $b^* = S^k(b)$, $c^* = S^k(a \cdot b)$. Then a^*, b^*, c^* are the smallest positive integers with the properties:

$$a^*! = M(a^k), \ b^*! = M(b^k), \ c^*! = M(a^k \cdot b^k)$$

and so $c^*! = M(a^*) = M(b^*)$. It results $a^* \leq c^*$, $b^* \leq c^*$, and then $\max(a^*, b^*) \leq c^*$. That is:

$$\max(S^k(a), S^k(b)) \leq S^k(a \cdot b) \tag{2.32}$$

But from $(a^* + b^*)! = M(a^*!b^*!) = M(a^k b^k)$, it results $c^* \leq a^* + b^*$, so

$$S^k(a \cdot b) \leq S^k(a) + S^k(b) \tag{2.33}$$

From (2.32) and (2.33) one obtain:

$$\max(S^k(a), S^k(b)) \leq S^k(a) + S^k(b)$$

so (Σ_3) is verified.

Finaly, because $(a^* b^*)! = M(a^*!b^*!)$, we have also:

$$S^k(a \cdot b) \leq S^k(a) \cdot S^k(b)$$

and (Σ_4) is proved.

2.3.16 Proposition. For every $k, n \in N^*$ we have

$$S^k(n) < n \cdot k \tag{2.34}$$

Proof. Let us consider $n = p_1^{i_1} \cdot p_2^{i_2} \ldots p_t^{i_t}$ and

$$S(n) = \max_{1 \leq j \leq t}(S_{p_j}(i_j) = S(p_m^{i_m})$$

Then because

$$S^k(n) = S(n^k) = \max_{1 \leq j \leq t} S_{p_j}(i_j \cdot k) = S(p_r^{i_r \cdot k}) \leq kS(p_r^{i_r}) \leq \\ \leq kS(p_m^{i_m}) = kS(n)$$

and $S(n) \leq n$, it results (2.34).

2.3.17 Theorem. Every prime number $p \geq 5$ is a local maximum for the functions S^k, and

$$S^k(p) = p(k - i_p(k))$$

where i_p are the functions defined by the equality (1.33).

Proof. If $p \geq 5$ is a prime, the first part of the theorem results from the inequalities

$$S_{p-1}(k) < S_p(k) \quad \text{and} \quad S_{p+!}(k) < S_p(k)$$

satisfied by the Smarandache function of first kind.

The second part of the theorem results from the definition of functions S^k:

$$S^k(p) = S_p(k) = p(k - i_p(k))$$

and the theorem is proved.

Remark. For $p \geq k$ we have $S^k(p) = pk$.

2.3.18 Theorem. All the numbers kp, with p a prime and $p > k$ are fixed points for the function S^k.

Proof. Let $m = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_t^{\alpha_t}$ be the decomposition of a given m into primes and $p > 4$ be a prime number. Then $p_i \cdot \alpha_i \leq p_i^{\alpha_i} < p$ for $i = \overline{1,t}$, so we have

$$S^k(mp) = S((mp)^k) = \max_{1 \leq i \leq t}(S_{p_i}(\alpha_i), S_p(k)) = S_p(k) = kp$$

For $m = k$ it results $S^k(kp) = kp$, so kp is a fixed point for S^k.

2.3.19 Theorem. The Smarandache function of second kind has the properties:

$$(i)\ S^k(n) = o(n^{1+\epsilon}) \quad \text{for every } \epsilon > 0$$
$$(ii)\ \lim_{n \to \infty} \sup \frac{S^k(n)}{n} = k$$

Proof. We have

$$0 \leq \lim_{n \to \infty} \frac{S^k(n)}{n^{1+\epsilon}} = \lim_{n \to \infty} \frac{S(n^k)}{n^{1+\epsilon}} \leq \lim_{n \to \infty} \frac{kS(n)}{n^{1+\epsilon}} = k \lim_{n \to \infty} \frac{S(n)}{n^{1+\epsilon}} = 0$$

and (i) is proved.

Also,

$$\lim_{n\longrightarrow\infty}\sup\frac{S^k(n)}{n}=\lim_{n\longrightarrow\infty}\sup\frac{S(n^k)}{n}=\lim_{n\longrightarrow\infty}\frac{S(p_n^k)}{p_n}=k$$

where $(p_n)_{n\in N^*}$ is the increasing sequence of all the primes.

2.3.20 Theorem. The Smarandache functions of second kind are generaly increasing, in the sense that

$$(\forall)\ n\in N^*\ (\exists)\ m_0\in N^*\ (\forall)\ m\geq_0\Longrightarrow S^k(m)\geq S^k(n)$$

Proof. It is said [44] that the Smarandache function is generally increasing, in the following sense

$$(\forall)\ t\in N^*\ (\exists)\ r_0\in N^*\ (\forall)\ r\geq r_0\Longrightarrow S(r)\geq S(t)\qquad(2.35)$$

Let $t=n^k$ and r_0 be such that $S(r)\geq S(n^k)$, for every $r\geq r_0$. Let also $m_0=[\sqrt[k]{r_0}]+1$. Of course, $m_0\geq\sqrt[k]{r_0}\Longleftrightarrow m_0^k\geq r_0$, and $m\geq m_0\Longleftrightarrow m^k\geq m_0^k$.

From $m^k\geq m_0^k\geq r_0$, it results $S(m^k)\geq S(n^k)$, so $S^k(m)\geq S^k(n)$.

Then we have:

$$(\forall)\ n\in N^*\ (\exists)\ m_0=[\sqrt[k]{r_0}]+1\ (\forall)\ m\geq m_0\Longrightarrow S^k(m)\geq S^k(n)$$

where $r_0=r_0(n^k)$ is given by (2.35).

2.3.21 Theorem. If $p\geq\max(3,k)$ is any prime number, then $n=p!$ is a local minimum for S^k.

Proof. Let $p!=p_1^{i_1}\cdot p_2^{i_2}...p_m^{i_m}\cdot p$ the factorisation of $p!$, such that $2=p_1<p_2<...,p_m<p$. Because $p!$ is divisible by $p_j^{i_j}$, it results $S(p_j^{i_j})\leq p=S(p)$ for every $j=\overline{1,m}$.

Of course,

$$S^k(p!) = S((p!)^k) = \max_{1\leq j\leq m}(S(p_j^{k\cdot i_j}), S(p^k))$$

and

$$S(p_j^{k\cdot i_j}) \leq S(p_j^{i_j}) < kS(p) = kp = S(p^k)$$

for $k \leq p$. Consequently,

$$S^k(p!) = S(p^k) = kp \text{ for } k \leq p \tag{2.36}$$

If the decomposition of $p! - 1$ into primes is

$$p! - 1 = q_1^{i_1} \cdot q_2^{i_2} \ldots q_t^{i_t}$$

then we have $q_j > p$ for $j = \overline{1,t}$.

It results:

$$S(p! - 1) = \max_{1\leq j\leq t}(S(q_j^{i_j})) = S(q_m^{i_m})$$

with $q_m > p$, and because $S(q_m^{i_m}) > S(p) = S(p!)$ it also results

$$S(p! - 1) > S(p!)$$

Analogously it can be proved that $S(p!) + 1 > S(p!)$.

Of course,

$$S^k(p! - 1) = S((p! - 1)^k) \geq S(q_m^{k\cdot i_m}) \geq S(q_m^k) > S(p^k) = kp \tag{2.37}$$

and

$$S^k(p! + 1) = S((p! + 1)^k) > kp \tag{2.38}$$

From (2.36), (2.37) and (2.38) it results the assertion.

Now we present the Smarandache function of third kind [2].

Let us consider two sequences:

$$(a): \ 1 = a_1, a_2, ..., a_n, ...$$
$$(b): \ 1 = b_1, b_2, ..., b_n, ...$$

satisfying the properties:

$$a_{k \cdot n} = a_k \cdot a_n, \ \ b_{k \cdot n} = b_k \cdot b_n \tag{2.39}$$

Of course there exist infinitely many such sequences, because chosing an arbitrary value for a_2, the next terms of the sequence (a) are determined by the recurrence relation (2.39).

Let now be the function

$$f_a^b : N^* \longrightarrow N^* \text{ defined by } \ f_a^b(n) = S_{a_n}(b_n)$$

where S_{a_n} is the Smarandache function of first kind.

One observe easily that

$$(i): \text{if } a_n = 1, \text{ and } b_n = n \text{ for every } n \in N^*, \text{ then } f_a{}^b = S_1$$
$$(ii): \text{ if } a_n = n \text{ and } b_n = 1 \text{ for every } n \in N^*, \text{ then } f_a{}^b = S^1 \tag{2.40}$$

2.3.22 Definition. The Smarandache functions of third kind are the functions defined by any sequences (a) and (b), different from those of (2.40), such that:

$$S_a^b = f_a{}^b$$

2.3.23 Theorem. All function $f_a{}^b$, Σ_5 - standardise the structure $(N^*, \cdot)$ on the structure $(N^*, \leq, +, \cdot)$ by:

$$(\Sigma_5): \ \max(f_a{}^b(k), f_a{}^b(n)) \leq f_a{}^b(k \cdot n) \leq b_n f_a{}^b(k) + b_k f_a{}^b(n)$$

Proof. Let us note

$$f_a^b(k) = S_{a_k}(b_k) = k^*, \quad f_a^b(n) = S_{a_n}(b_n) = n^*,$$
$$f_a^b(nk) = S_{a_{k\cdot n}}(b_{k\cdot n}) = t^*$$

Then k^*, n^* and t^* are the smallest positive integers for which

$$k^*! = M(a_k^{b_k})\ ,\ n^*! = M(a_n^{b_n})\ ,\ t^*! = M(a_{n\cdot k}^{b_{n\cdot k}}) = M((a_k \cdot a_n)^{b_k \cdot b_n})$$

so

$$\max(k^*, n^*) \le t^* \tag{2.41}$$

Moreover, because $(b_k \cdot n^*)! = M((n^*!)^{b_k})$, $(b_n \cdot k^*)! = M((k^*!)^{b_n})$ and

$$(b_k \cdot n^* + b_n \cdot k^*)! = M((b_k \cdot n^*)!(b_n \cdot k^*)!) =$$
$$= M((n^*!)^{b_k} \cdot (k^*!)^{b_n}) = M((a_n^{b_n})^{b_k} \cdot (a_k^{b_k})^{b_n}) = M((a_k \cdot a_n)^{b_k \cdot b_n})$$

it results

$$t^* \le b_n \cdot k^* + b_k \cdot n^* \tag{2.42}$$

From (2.41) and (2.42) one obtain:

$$\max(k^*, n^*) \le t^* \le b_n \cdot k^* + b_k \cdot n^* \tag{2.43}$$

From the last inequality it results (Σ_5), so any Smarandache function of third kind satisfies:

$$(\Sigma_6):\ \max(S_a^b(k), S_a^b(n)) \le S_a^b(kn) \le b_n S_a^b(k) + b_k S_a^b(n)$$

for every $k, n \in N^*$.

Example. If the sequences (a) and (b) are determined by the condition $a_n = b_n = n$, for $n \in N^*$, then the Smarandache function of third kind is:

$$S_a^a : N^* \longrightarrow N^*, \ \ S_a^a(n) = S_n(n)$$

and (Σ_6) becomes:

$$\max(S_k(k), S_n(n)) \leq S_{k\cdot n}(k \cdot n) \leq nS_k(k) + kS_n(n)$$

for every $n \in N^*$. This relation is equivalent with the following relation, written using the Smarandache function:

$$\max(S(k^k), S(n^n)) \leq S((kn)^{kn}) \leq nS(k^k) + kS(n^n)$$

2.4 Connections with Fibonacci Sequence

In the Introduction of the Proceedings of the Conferences *"Applications of Fibonacci numbers "* [3], [36], [38], it is mentioned that the sequence:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 55,\ 89, \ldots\ldots\ldots\ldots \tag{2.44}$$

known as the Fibonacci sequence, was named by the nineteenth-century French mathematician Edouard Lucas, after *Leonard Fibonacci of Pisa*, one of the best mathematicians of the Middle Ages, who referred to him in this book *Liber Abaci* (1202) in connection with his rabbit problem.

TheGerman astronomer Johann Kepler rediscovered Fibonacci numbers, independently, and since then several renowned mathematicians, as J. Binet, B. Lamé and E. Cartan, have dealt with them.

Edouard Lucas studied Fibonacci numbers extensively, and the simple generalisation:

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ \ldots\ldots \tag{2.45}$$

bears his name.

It said that there exists a strong connection between the Fibonacci sequence and the gold number:

$$\Phi = \frac{1+\sqrt{5}}{2}$$

For instance noting by $F(n)$ the $n-th$ term of Fibonacci sequence (2.44) one has:

$$\lim_{n\longrightarrow\infty} \frac{F(n+1)}{F(n)} = \Phi \tag{2.46}$$

and so,

$$\lim_{n\longrightarrow\infty} \sqrt[n]{F(n)} = \Phi$$

Let us now remember some of the properties of Fibonacci sequence.

It is said that Fibonacci sequence satisfies the recurrence relation

$$F(n+2) = F(n+1) + F(n), \text{ with } F(1) = F(2) = 1 \tag{2.47}$$

and also the properties:

(φ_1) $F(n) = \frac{1}{\sqrt{5}}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$
(φ_2) $F(1) + F(2) + ... + F(n) = F(n+2) - 1$
(φ_3) $F(1) + F(3) + ... + F(2n-1) = F(2n)$
(φ_4) $F(2) + F(4) + ... + F(2n) = F(2n+1) - 1$
(φ_5)$F(2) - F(3) + F(4) - ... + (-1)^n F(n) = (-1)^n F(n-1)$
(φ_6) $F^2(1) + F^2(2) + ... + F^2(n) = F(n) \cdot F(n+1)$
(φ_7) $F(n) \cdot F(n+2) = F^2(n+1) + (-1)^{n+1}$
(φ_8) $F(2n) = F^2(n) + F^2(n-1)$
(φ_9) $F(2n+1) = F^2(n) + F^2(n+1)$
(φ_{10}($F(n-1) \cdot F(n+1) - F^2(n) = (-1)^n$
(φ_{11}) $F(n-2) \cdot F(n+2) - F^2(n) = (-1)^{n+1}$
(φ_{12}) $F(n-1) \cdot F(n+1) - F^2(n-2) \cdot F(n+2) = 2(-1)^n$

T. Yau [50] has posed first a problem concerning a connection between Fibonacci sequence and the Smarandache function. Namely, for whath triplets $(n-2, n-1, n)$ of positive integers the Smarandache function verifies a Fibonacci-like equality:

$$S(n-2) + S(n-1) = S(n) \tag{2.48}$$

Calculating the values of $S(n)$ for the first 1200 positive integers he found twosuch triplets, namely $(9, 10, 11)$ and $(119, 120, 121)$. Indeed, we have:

$$S(9) + S(10) = S(11), \text{ and } S(119) + S(120) = S(121)$$

More recently H. Ibstedt [26] showed that the following numbers generating such triplets are:

$$n = 4,902; n = 26,245; n = 32,112; n = 64,010;$$
$$n = 368,140; n = 415,664$$

and proved the existence of infinitely many positive integers satisfying the equality (2.48).

Indeed, excepting the triplet generated by $n = 26,245$ the other triplets $(S(n-2), S(n-1), S(n))$ satisfy the property that one of terms is the duble of a prime number, and the other two are prime numbers. For instance taking $n = 4902 = 2 \cdot 3 \cdot 19 \cdot 43$ we have $n-1 = 4901 = 13^2 \cdot 29$, $n-2 = 4900 = 2^2 \cdot 5^2 \cdot 7^2$ and the equality (2.48) becomes $2 \cdot 7 + 29 = 43$. Also, for $n = 32,112 = 2^4 \cdot 3 \cdot 223$ it results $n-1 = 32,111 = 163 \cdot 198$, $n-2 = 32,110 = 2 \cdot 3 \cdot 13^2 \cdot 19$, so (2.48) becomes $2 \cdot 13 + 197 = 223$.

Using this remark, H. Ibstedt proposed [26] the following algorithm:

Let us consider the triplets $(n-2, n-1, n)$ satisfying the relations:

$$n = x \cdot p^a, \text{ with } a \leq p \text{ and } S(x) < a \cdot p \tag{2.49}$$

$$n - 1 = y \cdot q^b, \text{ with } b \leq q \text{ and } S(y) < bq \tag{2.50}$$

$$n - 2 = z \cdot r^c, \text{ with } c \leq r \text{ and } S(z) < c \cdot r \tag{2.51}$$

where p, q, r are prime numbers. In these conditions it results:

$$S(n) = a \cdot p, S(n-1) = b \cdot q, S(n-2) = c \cdot r$$

Substracting (2.50) from (2.49), and (2.51) from (2.50) we get the system:

$$x \cdot p^a - y \cdot q^b = 1 \tag{2.52}$$

$$y \cdot q^b - z \cdot r^c = 1 \tag{2.53}$$

$$a \cdot p = b \cdot q + c \cdot r \tag{2.54}$$

Every solution of the equation (2.54) generate an infinity of solutions for (2.53) which may be written under the form

$$x = x_0 + q^b \cdot t, \quad y = y_0 - p^a \cdot t \tag{2.55}$$

where t is an integer parameter and (x_0, y_0) is a particular solution (such a solution may be found by means of the algorithm of Euclid).

The solutions (2.55) are then introduced in the equality

$$z = \frac{y \cdot q^b - 1}{r^c}$$

for obtaining integer values of z.

H. Ibstedt in [26] give a very large list of triplets $(n-2, n-1, n)$ for which (2.48) is verified. These solutions have been generated for

$$(a, b, c) = (2, 1, 1),\ (a, b, c) = (1, 2, 1) \text{ and } (a, b, c) = (1, 1, 2)$$

with the parameter t restrained only to the interval $-9 \leq t \leq 10$.

To make now in evidence an other connection between the Smarandachr function and Fibonacci sequence we return to the twoo latticeal structures defined on the set N^* of positive integers.

We have already seen that the Smarandache function etablishe a connection of these lattices by the equality:

$$S(n_1 \overset{d}{\vee} n_2) = S(n_1) \vee S(n_2)$$

and so we are conducted to consider $S : \mathcal{N}_d \longrightarrow \mathcal{N}_o$.

2.4.1 Definition. The sequence $\sigma : \mathcal{N}_o \longrightarrow \mathcal{N}_d$ is said to be multiplicatively convergent to zero $(m.c.z)$ if:

$$(\forall)\, n \in N^* \ (\exists)\, m_n \in N^* \ (\forall)\, m \geq m_n \Longrightarrow \sigma(m) \underset{d}{\geq} n \qquad (2.56)$$

In [10] a sequence $\sigma : N \longrightarrow N$ satisfying (2.56) is named *multiplicatively convergent to infinity.* We prefered the above definition which is in connection with the fact that zero is the last element in the lattice $\mathcal{N}_d$.

The $(m.c.z)$ sequences having also the property of monotonicity are used in [10] to obtain a generalisation of $p-adic$ numbers.

The set Z_p of $p - adic$ numbers may be considered as an inverse limit (see [10]) of the rings $E_n = Z/p^n Z$ of *integers "modulo p^n"*, where p is a prime number.

Considering, instead of the sequence $(p^n)_{n\in N}$ an arbitrary $(m.c.z)$ and monotonous sequence $(\sigma(n))_{n\in N}$ there are obtained the sets $E_n = Z/\sigma(n)Z$ whose inverse limit is a generalisation of $p - adic$ numbers.

Let us observe that the monotonicity for a sequence $\sigma : \mathcal{N}_o \longrightarrow \mathcal{N}_d$ is expressed by the condition

$$(m_{od}) \ \ n \leq m \Longrightarrow \sigma(n) \underset{d}{\leq} \sigma(m)$$

The sequence $\sigma(n) = n!$ is a $(m.c.z)$ sequence and for every fixed $n \in N^*$ the smallest m_n given by (2.56) is exactly the value $S(n)$ of the Smarandache function. So, we can pose the problem of generalisation of Smarandache function in the following sense:

To each $(m.c.z)$ sequence $\sigma : \mathcal{N}_o \longrightarrow \mathcal{N}_d$ one may attach a function

$$f_\sigma : N^* \longrightarrow N^*, \ \ f_\sigma(n) = \text{ the smallest } m_n \text{ given by } (2.56)$$

and we observe that if $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_t^{\alpha_t}$ is the decomposition of $n \in N^*$ into primes then:

$$f_\sigma(n) = \max f_\sigma(p_i^{\alpha_i}) \tag{2.57}$$

This formula generalise the formula (1.16) of the calculus of $S(n)$. But the efective calculus of $f_\sigma(p_i^{\alpha_i})$ depends on the particular expression of the sequence σ.

We have also the properties:

$$\begin{array}{l} (f_1)\ \ f_\sigma(n_1 \overset{d}{\vee} n_2) = f_\sigma(n_1) \vee f_\sigma(n_2) \\ (f_2)\ \ n_1 \underset{d}{\leq} n_2 \Longrightarrow f_\sigma(n_1) \leq f_\sigma(n_2) \end{array}$$

which entitle us to consider

$$f_\sigma : \mathcal{N}_d \longrightarrow \mathcal{N}_o$$

Now, we may also consider the sequence

$$S \circ \sigma : \mathcal{N}_o \longrightarrow \mathcal{N}_o$$

or, more general, if σ and θ are two $(m.c.z)$ sequences, then there exist the sequences:

$$\begin{array}{ll} f_\sigma \circ \theta : \mathcal{N}_o \longrightarrow \mathcal{N}_o, & f_\theta \circ \sigma : \mathcal{N}_o \longrightarrow \mathcal{N}_o \\ \theta \circ f_\sigma : \mathcal{N}_d \longrightarrow \mathcal{N}_d & \sigma \circ f_\theta : \mathcal{N}_d \longrightarrow \mathcal{N}_d \end{array} \tag{2.58}$$

2.4.2 Proposition. If the sequences $\sigma, \theta : \mathcal{N}_o \longrightarrow \mathcal{N}_d$ are monotonous, then the sequences defined by (2.58) are also monotonous, in $\mathcal{N}_o$ and $\mathcal{N}_d$ respectively.

Proof. For an arbitrary $n \in N^*$ one has $\theta(n) \underset{d}{\leq} \theta(n+1)$ and f_σ satisfies (f_2), so:

$$(f_\sigma \circ \theta)(n) = f_\sigma(\theta(n)) \leq f_\sigma(\theta(n+1)) = (f_\sigma \circ \theta)(n+1)$$

For the second kind of sequences let $n_1 \underset{d}{\leq} n_2$. Then $f_\sigma(n_1) \leq f_\sigma(n_2)$ and so

$$(\theta \circ f_\sigma)(n_1) = \theta(f_\sigma(n_1)) \underset{d}{\leq} \theta(f_\sigma(n_2)) = (\theta \circ f_\sigma)(n_2)$$

The two latticeal structures considered on N^* justify the consideration of the following kind of sequences:

$$(i) \qquad (o,o)sequences: \quad \sigma_{oo} : \mathcal{N}_o \longrightarrow \mathcal{N}_o$$

$$(ii) \qquad (o,d)sequences: \quad \sigma_{od} : \mathcal{N}_o \longrightarrow \mathcal{N}_d:$$

$$(iii) \qquad (d,o)sequences: \quad \sigma_{do} : \mathcal{N}_d \longrightarrow \mathcal{N}_o$$

$$(iv) \qquad (d,d)sequences: \quad \sigma_{dd} : \mathcal{N}_d \longrightarrow \mathcal{N}_d:$$

For each of these sequences one may adapt the definition of monotonicity and of the limit. We have so the following situations:

1) For an (o,o) sequence σ_{oo} the condition of monotonicity is:

$$(m_{oo}) \ (\forall)\, n_1, n_2 \in N^*,\ n_1 \leq n_2 \Longrightarrow \sigma_{oo}(n_1) \leq \sigma_{oo}(n_2)$$

an this sequence tends to *infinity* if:

$$(c_{oo}) \ (\forall)\, n \in N^* \ (\exists)\, m_n \in N^* \ (\forall)\, m \geq m_n \Longrightarrow \sigma_{oo}(m) \geq n$$

2) The (o,d) sequence σ_{od} is monotonous if:

$$(m_{od}) \ (\forall)\, n_1, n_2 \in N^*, n_1 \leq n_2 \Longrightarrow \sigma_{od}(n_1) \underset{d}{\leq} \sigma_{od}(n_2)$$

and it is (multiplicatively) *convergent to zero* if

$$(c_{od})\ \ (\forall)\, n \in N^*\ (\exists)\, m_n \in N^*\ (\forall)\, m \geq m_n \Longrightarrow \sigma_{od}(m) \underset{d}{\geq} n$$

3) If σ_{do} is a (d, o) sequence, it is monotonous if

$$(m_{do})\ \ (\forall)\, n_1, n_2 \in N^*, n_1 \underset{d}{\leq} n_2 \Longrightarrow \sigma_{do}(n_1) \leq \sigma_{do}(n_2)$$

and tends to *infinity* if

$$(c_{do})\ \ (\forall)\, n \in N^*\ (\exists)\, m_n \in N^*\ (\forall)\, m \underset{d}{\geq} m_n \Longrightarrow \sigma_{do}(m) \geq n$$

From the properties of the Smarandache function it results that the sequence $(S(n))_{\in N^*}$ is a (d, o) sequence, satisfying the conditions (m_{do}) and (c_{do}).

4) The condition of monotonicity for a (d, d) sequence σ_{dd} is

$$(m_{dd})\ \ (\forall)\, n_1, n_2 \in N^*, n_1 \underset{d}{\leq} n_2 \Longrightarrow \sigma_{dd}(n_1) \underset{d}{\leq} \sigma_{dd}(n_2)$$

N. Jensen in [5] named *divisibility sequence* a sequence satisfying the condition (m_{dd}). This concept has been introduced by M. Ward [51], [52].

Moreover, the sequence σ_{dd} is said to be strong divisibility sequence (shortly (sds), see [5] pg. 181) if the equality

$$\sigma_{dd}(n_1 \underset{d}{\wedge} n_2) = \sigma_{dd}(n_1) \underset{d}{\wedge} \sigma_{dd}(n_2) \qquad (2.59)$$

holds for every $n_1, n_2 \in N^*$.

The term of (sds) has been used first in [28]. It is easely to see that if a sequence is (sds) then it is also a divisibility sequence (shortly, (ds)).

It is proved [12] that the Fibonacci sequence is (sds).

On the sequence σ_{dd} we shall say that it is (multiplicatively) *convergent to zero* if:

$$(c_{dd})\ (\forall)\, n \in N^*\ (\exists)\, m_n \in N^*\ (\forall)\, m \underset{d}{\geq} m_n \Longrightarrow \sigma_{do}(m) \underset{d}{\geq} n$$

To each sequence σ_{ij}, with $i, j \in \{o, d\}$, satisfying the conditions (m_{ij}) and (c_{ij}) we may attach a sequence f_{ij} defined by:

$$f_{ij}(n) = \min\{m_n \ /\ m_n \text{ is defined by } (c_{ij})\} \tag{2.60}$$

2.4.3 Proposition. Each function f_{oo} defined by (2.60) has the properties:

(i) f_{oo} satisfies the condition (m_{oo}) of monotonicity

(ii) $f_{oo}(n_1 \vee n_2) = f_{oo}(n_1) \vee f_{oo}(n_2)$

(iii) $f_{oo}(n_1 \wedge n_2) = f_{oo}(n_1) \wedge f_{oo}(n_2)$

Proof. (i) We have:

$$f_{oo}(n_1) = \min\{\ m_{n_1}\ /\ (\forall)\ m \geq m_{n_1} \Longrightarrow \sigma_{oo}(m) \geq n_1\ \}$$
$$f_{oo}(n_2) = \min\{\ m_{n_2}\ /\ (\forall)\ m \geq m_{n_2} \Longrightarrow \sigma_{oo}(m) \geq n_2\ \}$$

so, for every m$\geq f_{oo}(n_2)$ it results: $\sigma_{oo}(m) \geq n_1 \geq n_1$.

The assertions (ii) and (iii) are consequences of (i).

2.4.4 Proposition. Each function f_{od} has the properties:

(iv) f_{od} satisfies the condition (m_{od}) of monotonicity

(v) $f_{od}(n_1 \overset{d}{\vee} n_2) \geq f_{od}(n_1) \vee f_{od}(n_2)$

(vi) $f_{od}(n_1 \underset{d}{\wedge} n_2) \leq f_{od}(n_1) \wedge f_{od}(n_2)$

Proof. (iv) Let be $n_1 \underset{d}{\leq} n_2$. Then from

$$f_{od}(n_i) = \min\{\ m_{n_i}\ /\ (\forall)\ m \geq m_{n_i} \Longrightarrow \sigma_{od}(m) \underset{d}{\geq} n_i \text{ for } i = 1, 2$$

it results $\sigma_{od}(m) \underset{d}{\geq} n_2 \underset{d}{\geq} n_1$, for $m \geq f_{od}(n_2)$. So, $f_{od}(n_1) \leq f_{od}(n_2)$.

The properties (v) and (vi) result from (iv).

2.4.5 Proposition. Every function f_{do} has the properties:

$$(vii) \text{ is (only) } (o, o) \text{ monotonous}$$
$$(viii)\ \ f_{do}(n_1 \vee n_2) \underset{d}{\leq} f_{do}(n_1) \overset{d}{\vee} f_{do}(n_2)$$
$$(ix)\ \ f_{do}(n_1 \wedge n_2) \underset{d}{\geq} f_{do}(n_1) \underset{d}{\wedge} f_{do}(n_2)$$

Proof. (vii) If $n_1 \leq n_2$ then for every $m \underset{d}{\geq} m_{n_2}$ we have $\sigma_{do}(m) \geq n_2 \geq n_1$, and so $f_{do}(n_1) \leq f_{do}(n_2)$.

(viii) For $i = 1, 2$ one has:

$$f_{do}(n_i) = \min\{\ m_{n_i}\ /\ (\forall)\ m \underset{d}{\geq} m_{n_i} \Longrightarrow \sigma_{do}(m) \geq n_i$$

Let us suppose $n_1 \leq n_2$, so $n_1 \vee n_2 = n_2$ and $f_{do}(n_1 \vee n_2) = f_{do}(n_2)$. Then if we note

$$m_0 = f_{do}(n_1) \overset{d}{\vee} f_{do}(n_2)$$

for $m \underset{d}{\geq} m_0$ it results $\sigma_{do}(m) \geq n_i$, for $i = 1, 2$, so $\sigma_{do}(m) \geq n_1 \vee n_2$ and so

$$f_{do}(n_1 \vee n_2) = f_{do}(n_2) \underset{d}{\leq} f_{do}(n_1) \overset{d}{\vee} f_{do}(n_2)$$

Consequences. From (vii) it result the following properties:

$$f_{do}(n_1 \vee n_2) = f_{do}(n_1) \vee f_{do}(n_2)$$
$$f_{do}(n_1 \wedge n_2) = f_{do}(n_1) \wedge f_{do}(n_2)$$

and so:

$$f_{do}(n_1) \underset{d}{\wedge} f_{do}(n_2) \leq f_{do}(n_1) \wedge f_{do}(n_2) = f_{do}(n_1 \wedge n_2) \leq$$
$$\leq f_{do}(n_1) \vee f_{do}(n_2) = f_{do}(n_1 \vee n_2) \leq f_{do}(n_1) \overset{d}{\vee} f_{do}(n_2)$$

2.4.6 Proposition. The functions f_{dd} satisfie:

(x) $f_{dd}(n_1 \overset{d}{\vee} n_2) \leq f_{dd}(n_1) \overset{d}{\vee} f_{dd}(n_2)$

(xi) If $n_1 \underset{d}{\leq} n_2$ or $n_2 \underset{d}{\leq} n_1$ then $f_{dd}(n_1 \overset{d}{\vee} n_2) =$
$= f_{dd}(n_1) \vee f_{dd}(n_2)$

(xii) $f_{dd}(n_1 \underset{d}{\wedge} n_2) \leq f_{dd}(n_1) \wedge f_{dd}(n_2)$

Proof. It is analogous with the proof of above propositions.

2.4.7 Theorem. If the sequence σ_{dd} is (sds) and satisfies the condition (c_{dd}), then:

$$(a)\ f_{dd}(n_1 \overset{d}{\vee} n_2) = f_{dd}(n_1) \overset{d}{\vee} f_{dd}(n_2)$$
$$(b)\ n_1 \underset{d}{\leq} n_2 \Longrightarrow f_{dd}(n_1) \underset{d}{\leq} f_{dd}(n_2)$$

Proof. (a) It is sufficient to prove the inequality

$$f_{dd}(n_i) \underset{d}{\leq} f_{dd}(n_1 \overset{d}{\vee} n_2) \quad \text{for} \quad i = 1, 2 \tag{2.61}$$

If, for instance, this inequality does not hold for n_1, it results:

$$f_{dd}(n_1) \underset{d}{\wedge} f_{dd}(n_1 \overset{d}{\vee} n_2) = d_0 < f_{dd}(n_1)$$

and we have

$$\sigma_{dd}(d_0) = \sigma_{dd}(f_{dd}(n_1) \underset{d}{\wedge} f_{dd}(n_1 \overset{d}{\vee} n_2)) =$$
$$= \sigma_{dd}(f_{dd}(n_1)) \underset{d}{\wedge} \sigma_{dd}(f_{dd}(n_1 \overset{d}{\vee} n_2) \underset{d}{\geq} n_1 \underset{d}{\wedge} n_2 = n_1$$

because $\sigma_{dd}(f_{dd}(n_1)) \underset{d}{\geq} n_1$ and $n_1 \underset{d}{\leq} n_1 \overset{d}{\vee} n_2 \underset{d}{\leq} \sigma_{dd}(f_{dd}(n_1 \overset{d}{\vee} n_2))$. So, one obtain the contradiction $f_{dd}(n_1) \leq d_0 < f_{dd}(n_1)$.

(*b*) This condition is the (d, d) monotonicity. If $n_1 \underset{d}{\leq} n_2$ then $n_2 = n_1 \overset{d}{\vee} n_2$, and using the property (a) it results:

$$f_{dd}(n_2) = f_{dd}(n_1 \overset{d}{\vee} n_2) = f_{dd}(n_1) \overset{d}{\vee} f_{dd}(n_2)$$

so $f_{dd}(n_1) \underset{d}{\leq} f_{dd}(n_2)$.

Remarks. 1) Even if σ_{dd} is (sds), does not result the surjectivity of f_{dd}, in general. Indeed, the function f_{dd} attached to Fibonacci sequence is not surjective, because, for instance, $f_{dd}^{-1}(2) = \emptyset$. We also remember that the Smarandache function is the function f_{od} corresponding to the (o, d) sequence $\sigma_{od}(n) = n!$, and it is surjective.

2) One of the most interesting diophantine equations associated to a function f_{ij}, for $i, j \in \{1, 2\}$, is that giving its fixed points:

$$f_{ij}(x) = x \tag{2.62}$$

The function f_{ij} attached to Fibonacci sequence has $n = 5$ and $n = 12$ as fixed points, but the problem of finding the general solution of the equation (2.62) corresponding to this famous sequence is an open problem,until now.

In the section 1.6 there has been studied the convergence of some numerical series involving the Smarandache function. Such kind of series may be attached to all (generalised) sequences f_{ij}.

In the sequel we focus the attention on the analogous of the series

$$\sum_{k=1}^{\infty} \frac{1}{S(k)!} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{S(k)^{\alpha} \cdot \sqrt{S(k)!}}$$

in the case when the function S is replaced by an arbitrary function f_{d_o}, corresponding to a $(m.c.z)$ sequence.

2.4.8 Theorem. If σ is a $(m.c.z)$ sequence satisfying the condition (m_{od}), let us denote by f_σ the corresponding f_{od} sequence and by g_σ the sequence $\sigma \circ f_\sigma$. Then for every $\alpha > 1$ the series

$$(i) \ \sum_{k=1}^{\infty} \frac{1}{(f_\sigma(k))^{\alpha} \cdot \sqrt{g_\sigma(k)}} \quad (ii) \ \sum_{k=1}^{\infty} \frac{1}{g_\sigma(k)}$$

are convergent.

Proof. To prove these assertions we use the same method as for the series (1.90) and (1.91).

(i) We have:

$$\sum_{t=1}^{\infty} \frac{1}{(f_\sigma(k))^{\alpha} \cdot \sqrt{g_\sigma(k)}} = \sum_{k=1}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{\sigma(t)}}$$

where $m_t = card\{k \ / \ f_\sigma(k) = t\}$. But

$$k \underset{d}{\leq} \sigma(t) \Longrightarrow m_t \leq d(\sigma(t))$$

where $d(n)$ is the number of divisors of n.

From the inequality $d(\sigma(t)) < 2\sqrt{\sigma(t)}$ it results

$$\sum_{t=1}^{\infty} \frac{m_t}{t^{\alpha}\sqrt{\sigma(t)}} \leq \sum_{t=1}^{\infty} \frac{2\sqrt{\sigma(t)}}{t^{\alpha}\sqrt{\sigma(t)}} = 2\sum_{t=1}^{\infty} \frac{1}{t^{\alpha}}$$

(ii) If we note $\sigma(n+1)/\sigma(n) = k_{n+1}$, it results successively:

$$\sum_{t=1}^{\infty} \frac{1}{g_\sigma(k)} = \sum_{t=1}^{\infty} \frac{m_t}{\sigma(t)} \leq \sum_{t=1}^{\infty} \frac{2\sqrt{\sigma(t)}}{\sigma(t)} = 2\sum_{t=1}^{\infty} \frac{1}{\sqrt{\sigma(t)}}$$

and putting $x_t = 1/\sigma(t)$, it results $x_{t+1}/x_t = 1/\sqrt{k_{t+1}}$.

As $m_t = 0$ if $k_t = 1$, it results that when $m_t \neq 0$ we have $k_t > 1$, so the series $\sum_{t=1}^{\infty} (1/\sqrt{\sigma(t)})$ is convergent, as well as the series (ii).

Example.Let the sequence σ be defined in the following way: $\sigma(t) = k!$ if and only if $k! < t \leq (k+1)!$.

It results that σ is a $(m.c.z)$ sequence satisfying the condition (m_{od}) and we have:

$$\sigma(1) = 1, \sigma(2) = 2!, \sigma(3) = \sigma(4) = 3!, \sigma(5) = ... = \sigma(10) = 4!$$
$$\sigma(11) = \sigma(12) = ... = \sigma(26) = 5!, ...$$

Then

$$f_\sigma(1) = 1, f_\sigma(2) = 2, f_\sigma(3) = 3, f_\sigma(4) = 5, f_\sigma(5) = 11,$$
$$f_\sigma(6) = 3, f_\sigma(7) = 71, f_\sigma 98) == 5, ...$$

and so

$$\sum_{t=1}^{\infty} \frac{1}{g_\sigma(k)} = \frac{1}{\sigma(1)} + \frac{1}{\sigma(2)} + \frac{1}{\sigma(3)} + \frac{1}{\sigma(5)} + \frac{1}{\sigma(11)} +$$
$$+ \frac{1}{\sigma(3)} + \frac{1}{\sigma(71)} + ... = \sum_{t=1}^{\infty} \frac{m_t}{\sigma(t)}$$

From the fact that

$$m_4 = 0, m_6 = m_7 = ... = m_{10} = 0, m_{12} = m_{13} = ...m_{26} = 0$$

it results:

$$\sum_{t=1}^{\infty} \frac{m_t}{\sigma(t)} = \frac{m_2}{\sigma(2)} + \frac{m_3}{\sigma(3)} + \frac{m_5}{\sigma(5)} + \frac{m_{11}}{\sigma(11)} + \frac{m_{27}}{\sigma(27)} + \ldots.. =$$
$$= \frac{m_2}{2!} + \frac{m_3}{3!} + \frac{m_5}{4!} + \frac{m_{11}}{5!} + \frac{m_{27}}{6!} + \ldots.. \leq$$
$$\leq \sum_{t=1}^{\infty} \frac{d(t!)}{t!} = 2 \sum_{t=1}^{\infty} \frac{1}{\sqrt{t!}}$$

which is a convergent series.

Remark. As one can see from the above example, the functions f_σ are, in general, neither one-to-one, nor onto.

d

2.5 Solved and Unsolved Problems

As in the section 1.8 we note by a star (*) the unsolved problems. By $p_1 < p_2 < \ldots < p_k \ldots$ is denoted the increasing sequence of all the prime numbers. For the solutions of solved problems see the collection of *Smarandache Function Journal*.

1) Prove that the Smarandache function does not verify the Liepschitz condition

$$(\exists)\ M > 0\ (\forall)\ m, n \in N^* \implies /S(m) - S(n)/ < M/m - n/$$

2) The functions $S^{(1)}$ and $S^{(2)}$ defined by:

$$S^{(1)}(n) = \frac{1}{S(n)}\ ;\ S^{(2)}(n) = \frac{S(n)}{n}$$

verify the Liepschitz condition, but the function $S^{(3)}(n) = \frac{n}{S(n)}$ does not verify this condition. (*M. Popescu. P. Popescu*)

3) If

$$\sigma_S(x) = \sum_{\substack{d \le x \\ d}} S(d), \text{ and}$$

$$T(n) = 1 - \ln \sigma_S(n) + \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{1}{\sigma_S(p_i^k)}$$

then $\lim_{n \longrightarrow \infty} T(n) = -\infty$.

4) If $\Pi(x) = card\{p \mid p \text{ is a prime}, p \le x\}$, prove that the following numerical functions:

$$(i) \ \ F_S : N^* \longrightarrow N, \ \ F_S(x) = \sum_{i=1}^{\Pi(x)} S(p_i^x),$$

$$(ii) \ \ \theta : N^* \longrightarrow N, \ \ \theta(x) = \sum_{\substack{p \le x \\ d}} S(p_i^x),$$

$$(iii) \ \ \tilde{\theta} : N^* \longrightarrow N, \ \ \tilde{\theta}(x) = \sum_{\substack{p_i \le x \\ p_i \text{ not divides } x}} S(p_i^x)$$

which involve the Smarandache function, do not verify the Liepschitz condition. (*M. Popescu,P. Popescu, V. Seleacu*)

5) Let $a : N^* \longrightarrow N^*$ be the function defined by:

$a(n) = k \iff k$ is the smallest positive integer such that nk is a perfect square.

Prove that: (i) If n has the factorisation $n = q_1^{\alpha_1} \cdot q_2^{\alpha_2} \ldots q_r^{\alpha_r}$, then $a(n) = q_1^{\beta_1} \cdot q_2^{\beta_2} \ldots q_r^{\beta_r}$, with

$$\beta_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is odd number} \\ 0 & \text{if } \alpha_i \text{ is even number} \end{cases}$$

(ii) The function a is multiplicative, that is $a(xy) = a(x)a(y)$ for all $x, y \in N^*$ such that $x \underset{d}{\wedge} y = 1$.

(iii) The series $\sum_{n \ge 1} \frac{a(n)}{n}$ diverges.(*I. Balacenoiu, M. Popescu, V. Seleacu*)

6) For the function a defined in the preceding problem prove that: (i) if $x, y > 1$ are not perfect squares and $x \underset{d}{\wedge} y = 1$, then the diophantine equation $a(x) = a(y)$ has no solution.

(ii) $a(xy^2) = a(x)$, for $x, y \geq 1$.

(iii) $a(x^n) = 1$ if n is even and $a(x^n) = a(x)$ if n is odd.

(iv) for every perfect square $m \in N^*$ the equation $xa(x) = m$ has 2^k different solutions, where k is the number of prime factors of m.

(v) solve the equations:

$$xa(x) + ya(y) = za(z),$$
$$\frac{1}{xa(x)} + \frac{1}{ya(y)} = \frac{1}{za(z)}$$
$$Aa(x) + Ba(y) + Ca(z) = 0, \ Aa(x) + Ba(y) = C$$

(I. Balacenoiu, M. Popescu, V. Seleacu)

7) For the same function a defined above prove that if F_a^d denote the generating function associated to this function by means of the lattice $\mathcal{N}_d$, then:

$$(i) \ \ F_a^d(q^\alpha) = \begin{cases} \frac{\alpha}{2}(q+1) = 1 & \text{if } \alpha \text{ is even} \\ ([\frac{\alpha}{2}]+1)(q+1) & \text{if } \alpha \text{ is odd} \end{cases}$$
$$(ii) \ \ F_a^d(n) = \prod_{j=1}^{r} \left(H(\alpha_j)(q+1) + \frac{1+(-1)^{\alpha_j}}{2}\right.$$

where $n = q_1^{\alpha_1} \cdot q_2^{\alpha_2} ... q_r^{\alpha_r}$ is the decomposition of n into primes and $H(\alpha) = card\{x \ / \ x \leq \alpha, \ x \text{ is odd}\}$.*(I. Balacenoiu, M. Popescu, V. Seleacu)*

8) The Smarandache no-square digits sequence is defined as follows: 2, 3, 5, 6, 7, 8, 2, 3, 5, 6, 7, 8, 2, 2, 22, 23, 2, 25, 26, 27, 28, 2, 3, 3, 32, 33, 3, 35, 36, 37, 38, ... (take out all square digits of n). It is any number that occurs infinitely many time in this sequence ?

9*) Let n be a positive integer with not all digits the same, and let n' its digital reverse. Then let $n_1 = /n - n'/$, and n_1' its digital reverse. Again, let $n_2 = /n_1 - n_1'/$, and n_2' be its digital reverse. After a finite number of steps one finds an n_j which is equal to a previous n_i, therefore the sequence is periodical (because if n has, say, k digits, all other integers n_i following it will have k digits or less, hence their number is limited and one applies the Dirichlet's box principle).

Find the length of the period (with its corresponding numbers) and the length of the sequence'till the first repetition occurs for the integers of three digits and the integers of four digits. Generalisation. (*M. R. Popov*)

10) Let $\sigma : N \longrightarrow N$ be a second order recurrence sequence, defined by:

$$\sigma(n) = A\sigma(n-1) + B\sigma(n-2)$$

where A and B are fixed non-zero coprime integers and $\sigma(1) = 1, \sigma(2) = A$. We shall denote the roots of the characteristic polynomial

$$P(x) = x^2 + Ax + B$$

by α and β. Prove that:

(i) if the sequence is non-degenerate (that is $AB \neq 0, A^2 + 4B \neq 0$ and $\frac{\alpha}{\beta}$ is not a root of unity) then the terms $\sigma(n)$ can be expressed as:

$$\sigma(n) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

for all $n \in N^*$, and if p is a prime such that $p \underset{d}{\wedge} B = 1$ then there are terms in the sequence σ divisible by p. (The least positive index of these terms is called the rank of apparition of p in the sequence and it is denoted by $r(p)$. Thus $r(p) = n$ if $p \underset{d}{\leq} \sigma(n)$ holds, but $p \underset{d}{\leq} (n+1)$ does not hold).

(ii) there is no term of the sequence σ, divisible by the prime p if p divide B and $A \underset{d}{\wedge} B = 1$.

(iii) if p does not divides B and we note: $D = A^2 + 4B$ and $(D/p) =$ the Legendre symbol, with $(D/p) = 0$ if $p \underset{d}{\leq} D$, then

$$1)\ r(p) \underset{d}{\leq} (p - (D/p))$$
$$2)\ p \underset{d}{\leq} \sigma(n) \Longleftrightarrow r(p) \underset{d}{\leq} n$$

11*) Find a formula for the calculus of Smarandache generalised function f_σ corresponding to Fibonacci sequence.

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The function named in the title of this book is originated from the exiled Romanian mathematician Florentin Smarandache, who has significant contributions not only in mathematics, but also in literature. He is the father of *The Paradoxist Literary Movement* and is the author of many stories, novels, dramas, poems.

The Smarandache function, say S, is a numerical function defined such that for every positive integer n, its image $S(n)$ is the smallest positive integer whose factorial is divisible by n.

The results already obtained on this function contain some surprises . . .

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