Thompson’s Approach and Quantum Mechanics: The Harmonic Oscillator

P. R. Silva – Retired associate professor – Departamento de Física – ICEx – Universidade Federal de Minas Gerais (UFMG) – email: prsilvafis@gmail.com

Abstract- Thompson’s Method is applied to a variational principle in order to treat the harmonic oscillator in one dimension. It is also used to link the de Broglie frequency to a harmonic oscillator model.

1 – Introduction

As can be found in Landau and Lifchitz [1], the Schroedinger’s equation can be deduced from the variational principle

$$\delta \int d^d x\, \Psi^*(H_{op} - \varepsilon)\Psi = 0. \qquad (1)$$

In (1) H_{op} is the Hamiltonian operator, Ψ is the wave function and ε the energy. For our proposal and in the case of the time-independent equation, we can write the action

$$A = \int d^d x\, [(-\tfrac{1}{2}\hbar^2/m)\,(\partial\Psi/\partial\mathbf{x})^2 - V(x)\,\Psi^*\Psi + \varepsilon\,\Psi^*\Psi]. \qquad (2)$$

Imposing the stationary condition over A, namely taking δA = 0, we get

$$(-\tfrac{1}{2}\hbar^2/m)\,\Delta\Psi + V(x)\,\Psi = \varepsilon\,\Psi. \qquad (3)$$

Naturally, we got in (3) the time-independent Schroedinger equation.

Thompson [2] has proposed formalism as a means to describe the critical behavior of cooperative systems. The main hypothesis is that in neighborhood of the critical point each term of certain action must be of the same order of magnitude. Making the requirement for this action keeps stationary, leads to a non-linear Schroedinger equation which occurs in the study of solitons [3], for instance. A φ_4 field theory, within the same universality class of the Ising model, was used by Thompson [2] to develop his formalism. The same lagrangian was also used by Wilson [4] to illustrate the renormalization group (RG) approach. It seems that both quantum mechanics and critical phenomena are theories where fluctuations play an important role. This gives an indication that Thompson's formalism, can be an alternative tool to deal with Quantum Mechanics. Next we use Thompson's approach to treat the harmonic oscillator in one-dimension. A connection will also be made with the de Broglie frequency.

2 – Application to the harmonic oscillator

Let us consider the action given by

$$A = \int d^d x \, [(- \tfrac{1}{2}\, \hbar^2 / m)\, (\partial\Psi / \partial \mathbf{x})^2 - (\tfrac{1}{2}\, k\, x^2)\, \Psi^*\Psi + \varepsilon\, \Psi^*\Psi]. \qquad (4)$$

In (4), $V(x) = \tfrac{1}{2}\, k\, x^2$, being k the spring constant.

Now we apply Thompson's recipe which says: the absolute value of each term of the action given by (4) is separately of the order of the unity. We have

$$\int (\tfrac{1}{2}\, \hbar^2 / m)\, (\partial\Psi/\partial x)^2 \, d^d x = (\tfrac{1}{2}\, \hbar^2 / m) <\Psi^2> \ell^{d-2} = 1. \qquad (5)$$

In (5), ℓ is a characteristic length, d the dimension and $<\Psi^2>$ is the average of the wave function, here taken as a real quantity. Solving for $<\Psi^2>$, we obtain

$$<\Psi^2> = 2m\, \ell^{2-d} / \hbar^2. \tag{6}$$

Applying the Tompson's recipe to the second term of (4), we get

$$\int \tfrac{1}{2}\, k\, x^2\, \Psi^2\, d^d x = \tfrac{1}{2}\, k <x^2> <\Psi^2> \ell^d = 1. \tag{7}$$

Taken $<x^2> = \ell^2$, using the result of (6) into (7), and solving for ℓ, we find

$$\ell = k^{-1/4}\, (m / \hbar^2)^{-1/4}. \tag{8}$$

The equality between the second and the third terms of (4) leads to

$$\int \tfrac{1}{2}\, k\, x^2\, \Psi^2\, d^d x = \tfrac{1}{2}\, k\, \ell^2 \int \Psi^2\, d^d x = \varepsilon \int \Psi^2\, d^d x. \tag{9}$$

Relation (9) implies

$$\varepsilon = \tfrac{1}{2}\, k\, k^{-1/2}\, (\hbar^2 / m)^{1/2} = \tfrac{1}{2}\, (k / m)^{1/2}\, \hbar = \tfrac{1}{2}\, \hbar\, \omega. \tag{10}$$

In obtaining (10), we also have used (8). We notice that ε corresponds to the ground state energy of the one-dimensional harmonic oscillator.

Taking in account that the first and the second terms of the action (4) are separately equal to the unity, we can write

$$\int(\tfrac{1}{2}\,\hbar^2/m)\,(\partial\Psi/\partial x)^2\,dx = \int\tfrac{1}{2}\,k\,x^2\,\Psi^2\,dx. \quad (11)$$

Relation (11) implies that

$$(\tfrac{1}{2}\,\hbar^2/m)\,(d\Psi/dx)^2 = \tfrac{1}{2}\,k\,x^2\,\Psi^2. \quad (12)$$

Extracting the square root of (12) and choosing the signal which leads the wave function make sense, namely, the squared wave function gives an integral finite. Doing this we obtain

$$(\hbar/m^{1/2})\,(d\Psi/dx) = -\,k^{1/2}\,x\,\Psi. \quad (13)$$

Performing the integration of (13), we find

$$\Psi = \Psi_0\,\exp[\,-\,(km)^{1/2}\,x^2/(2\hbar)]. \quad (14)$$

The wave function we just have obtained corresponds to that of the ground-state of the one-dimensional harmonic oscillator.

3 – The de Broglie frequency

Let us take the characteristic length ℓ as the reduced Compton length (λ_C) of a particle of mass m. From relation (8) and from the definition of the Compton length we can write

$$\ell^4 = \hbar^2/(mk) = \hbar^4/(m^4\,c^4). \quad (15)$$

Solving (15) for k, we find

$$k = (m^3 c^4)/\hbar^2. \qquad (16)$$

$$\omega = (k/m)^{1/2} = m c^2/\hbar \equiv \omega_{dB}. \qquad (17)$$

Therefore the choice of the characteristic length as the Compton length, leads to a harmonic oscillator model for the de Broglie frequency.

References

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