Double Integrals , Numerical Integration , Euler Constant

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abstract

In this note we show the numerical integration of some double integrals related with the Euler constant:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = 0.5772\ldots$$

We use *Mathematica* tools.

Keyword:Double Integrals,Numerical Integration,Euler Constant

I. Introducción

Integrales dobles para la constante de Euler

Recordamos algunas integrales dobles para la constante gamma de Euler :

$$\gamma = -\int_0^\infty \int_0^\infty \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{1}$$

$$\gamma = \int_0^1 \int_0^1 \frac{\ln(-\ln(x\,y))}{\ln(x\,y)}\, dx\, dy \tag{2}$$

$$\gamma = \int_0^1 \int_0^y \frac{\ln(-\ln x)}{y \ln x}\, dx\, dy \tag{3}$$

$$\gamma = 2\int_0^1 \int_0^y \frac{\ln(-\ln(x\,y))}{\ln(x\,y)}\, dx\, dy = 2\int_0^1 \int_0^x \frac{\ln(-\ln(x\,y))}{\ln(x\,y)}\, dx\, dy \tag{4}$$

$$\gamma = -\int_0^\infty \int_y^\infty \frac{\ln x}{x} e^{-x}\, dx\, dy \tag{5}$$

$$\gamma = -\int_1^\infty \int_1^\infty \frac{\ln(\ln(x\,y))}{x^2\,y^2 \ln(x\,y)}\, dx\, dy \tag{6}$$

$$\gamma = -1 - \int_0^\infty \int_0^\infty e^{-y\,e^x} \ln y\, dx\, dy \tag{7}$$

$$\gamma = -1 - \int_0^\infty \int_{-\infty}^\infty x\, e^{x-e^{x+y}}\, dx\, dy \tag{8}$$

$$\gamma = -1 + \int_0^\infty \int_0^\infty x\, e^{-x-e^{-x+y}}\, dx\, dy - \int_0^\infty \int_0^\infty x\, e^{x-e^{x+y}}\, dx\, dy \tag{9}$$

$$\gamma = -2\int_0^\infty \int_0^y \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{10}$$

$$\gamma = -\int_0^1 \int_0^y \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int_1^\infty \int_0^\infty \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int_0^1 \int_y^\infty \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{11}$$

$$\gamma = -\int_0^1 \int_0^{1-y} \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int_1^\infty \int_0^\infty \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int_0^1 \int_{1-y}^\infty \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{12}$$

$$\gamma = \left\{ \int_0^1 \int_0^{e^{-1}} \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy + \int_{e^{-1}}^1 \int_0^{(e\, y)^{-1}} \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy \right\} + \int_{e^{-1}}^1 \int_{(e\, y)^{-1}}^1 \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy \tag{13}$$

$$\gamma = \int\int_{R1} \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy + \int\int_{R2} \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy \tag{14}$$

donde

$$R1 = \left\{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1,\ x\, y < e^{-1}\right\} \tag{15}$$

$$R2 = \left\{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1,\ x\, y > e^{-1}\right\} \tag{16}$$

$$\gamma = -\int\int_{R1} \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int\int_{R2} \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{17}$$

donde

$$R1 = \left\{(x,\ y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ x + y < 1\right\} \tag{18}$$

$$R2 = \left\{(x,\ y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ x + y > 1\right\} \tag{19}$$

$$\gamma = \int\int_{R1} \frac{\ln(-\ln(x\, y))}{\ln(x\, y)}\, dx\, dy - \int\int_{R2} \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy \tag{20}$$

donde

$$R1 = \left\{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1,\ x\, y < e^{-1}\right\} \tag{21}$$

$$R2 = \left\{(x,\ y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1,\ x + y < 1\right\} \tag{22}$$

$$\gamma = -\int\int_{R1} \frac{\ln(x+y)}{x+y} e^{-x-y}\, dx\, dy - \int\int_{R2} \frac{1}{(x+y)\, x\, y} e^{-x-y} \ln\left(\frac{1}{x} + \frac{1}{y}\right) dx\, dy \tag{23}$$

donde

$$R1 = \left\{(x,\ y) \in \mathbb{R}^2 : x > 0,\ y > 0,\ x + y > 1\right\} \tag{24}$$

$$R2 = \left\{(x,\ y) \in \mathbb{R}^2 : x > 1,\ y > 1,\ \frac{1}{x} + \frac{1}{y} < 1\right\} \tag{25}$$

II. Integración Numérica

Ejemplos de Integración numérica vía *Mathematica*

```
γ = N[EulerGamma, 50]
```

```
0.57721566490153286060651209008240243104215933593992
```

Ejemplo 1: Integral (1)

$$I1 = \text{NIntegrate}\left[-\frac{\text{Log}[x+y]}{x+y}\, e^{-x-y}, \{x, 0, \infty\}, \{y, 0, \infty\}\right]$$

0.577216

Abs[I1 - γ]

1.4174×10^{-9}

Ejemplo 2: Integral (1)

$$I2 = \text{NIntegrate}\left[-\frac{\text{Log}[x+y]}{x+y}\, e^{-x-y}, \{x, 0, \infty\}, \{y, 0, \infty\}, \text{WorkingPrecision}\to 20\right]$$

0.57721566490155422889

Abs[I2 - γ]

2.136829×10^{-14}

Ejemplo 3: Integral (2)

$$I3 = \text{NIntegrate}\left[\frac{\text{Log}[-\text{Log}[x\,y]]}{\text{Log}[x\,y]}, \{x, 0, 1\}, \{y, 0, 1\}\right]$$

0.577216

Abs[I3 - γ]

9.63804×10^{-10}

Ejemplo 4: Integral (2)

$$I4 = \text{NIntegrate}\left[\frac{\text{Log}[-\text{Log}[x\,y]]}{\text{Log}[x\,y]}, \{x, 0, 1\}, \{y, 0, 1\}, \text{WorkingPrecision}\to 20\right]$$

0.57721566490151770870

Abs[I4 - γ]

1.515191×10^{-14}

Ejemplo 5: Integral (1)

$$I5 = \text{NIntegrate}\left[-\frac{\text{Log}[x+y]}{x+y}\, e^{-x-y}, \{x, 0, \infty\}, \{y, 0, \infty\}, \text{Method}\to \text{"GaussKronrodRule"}, \text{WorkingPrecision}\to 40\right]$$

0.5772156649015328606065120900824024310395

```
Abs[I5 - γ]
```

2.7×10^{-39}

Ejemplo 6: Integral (2)

```
I6 = NIntegrate[Log[-Log[x y]]/Log[x y], {x, 0, 1}, {y, 0, 1}, Method → "GaussKronrodRule",
  WorkingPrecision → 40]
```

0.5772156649015328606065120900824024310422

```
Abs[I6 - γ]
```

$0. \times 10^{-41}$

Ejemplo 7: Integral (1)

```
I7 = NIntegrate[-Log[x + y]/(x + y) e^(-x-y), {x, 0, ∞}, {y, 0, ∞},
  Method → "GaussBerntsenEspelidRule", WorkingPrecision → 40]
```

0.5772156649015328606065120900824024310422

```
Abs[I7 - γ]
```

$0. \times 10^{-41}$

Ejemplo 8: Integral (2)

```
I8 = NIntegrate[Log[-Log[x y]]/Log[x y], {x, 0, 1}, {y, 0, 1}, Method → "GaussBerntsenEspelidRule",
  WorkingPrecision → 40]
```

0.5772156649015328606065120900824024310422

```
Abs[I8 - γ]
```

$0. \times 10^{-41}$

Ejemplo 9: Integral (1)

```
I9 = NIntegrate[-Log[x + y]/(x + y) e^(-x-y), {x, 0, ∞}, {y, 0, ∞}, Method → "ClenshawCurtisRule",
  WorkingPrecision → 20]
```

0.57721566490153170957

```
Abs[I9 - γ]
```

1.15104×10^{-15}

Ejemplo 10: Integral (2)

```
I10 = NIntegrate[Log[-Log[x y]]/Log[x y], {x, 0, 1}, {y, 0, 1}, Method → "ClenshawCurtisRule",
  WorkingPrecision → 20]
```

0.57721566490153205135

```
Abs[I10 - γ]
```

8.0926×10^{-16}

Ejemplo 11: Integral (1)

```
I11 = NIntegrate[-Log[x + y]/(x + y) e^(-x-y), {x, 0, ∞}, {y, 0, ∞}, Method → "LobattoKronrodRule",
  WorkingPrecision → 30]
```

0.577215664901532860606512090086

```
Abs[I11 - γ]
```

$3.\times10^{-30}$

Ejemplo 12: Integral (2) ,

```
I12 = NIntegrate[Log[-Log[x y]]/Log[x y], {x, 0, 1}, {y, 0, 1}, Method → "LobattoKronrodRule",
  WorkingPrecision → 30]
```

0.577215664901532860606512089857

```
Abs[I12 - γ]
```

2.25×10^{-28}

Ejemplo 13: Integral (1)

```
I13 = NIntegrate[-Log[x + y]/(x + y) e^(-x-y), {x, 0, ∞}, {y, 0, ∞},
  Method → {"NewtonCotesRule", "Type" → "Open"}, WorkingPrecision → 15]
```

0.577215765287158

```
Abs[I13 - γ]
```

1.00385625×10^{-7}

Ejemplo 14: Integral (2)

```
I14 = NIntegrate[Log[-Log[x y]]/Log[x y], {x, 0, 1}, {y, 0, 1},
  Method → {"NewtonCotesRule", "Type" → "Open"}, WorkingPrecision → 15]
```

0.577215664675593

```
Abs[I14 - γ]
```

2.25940×10^{-10}

Ejemplo 15: Integral (6)

```
I15 = NIntegrate[-Log[Log[x y]]/(x^2 y^2 Log[x y]), {x, 1, ∞}, {y, 1, ∞}, Method → "GaussKronrodRule",
  WorkingPrecision → 20]
```

0.57721566490153291264

```
Abs[I15 - γ]
```

5.204×10^{-17}

Ejemplo 16: Integral (6)

```
I16 = NIntegrate[-Log[Log[x y]]/(x^2 y^2 Log[x y]), {x, 1, ∞}, {y, 1, ∞},
  Method → "GaussBerntsenEspelidRule", WorkingPrecision → 20]
```

0.57721566490153284209

```
Abs[I16 - γ]
```

1.851×10^{-17}

Ejemplo 17: Integral (7)

```
I17 = NIntegrate[-e^(-y e^x) Log[y], {x, 0, ∞}, {y, 0, ∞}, Method → "GaussKronrodRule",
  WorkingPrecision → 30]
```

1.57721566490153286060651217054

```
Abs[-1 + I17 - γ]
```

8.046×10^{-26}

Ejemplo 18: Integral (8)

```
I18 = NIntegrate[-x e^(x - e^(x+y)), {x, -∞, ∞}, {y, 0, ∞}, Method → "GaussKronrodRule",
  WorkingPrecision → 30]
```

1.57721566490153286060651209012

```
Abs[-1 + I18 - γ]
```

$4.\times10^{-29}$

Ejemplo 19: Integral (20)

```
I19R1 = NIntegrate[Log[-Log[x y]] / Log[x y],
  {x, y} ∈ ImplicitRegion[x y < e^-1 && 0 < x < 1 && 0 < y < 1, {x, y}], WorkingPrecision → 20]
```

-0.21938393439552917744

```
I19R2 = NIntegrate[-Log[x + y] / (x + y) e^(-x-y),
  {x, y} ∈ ImplicitRegion[x + y < 1 && 0 < x < 1 && 0 < y < 1, {x, y}], WorkingPrecision → 20]
```

0.79659959929705244131

```
Abs[I19R1 + I19R2 - γ]
```

9.5967×10^{-15}

Ejemplo 20: Integral (23)

```
I20R1 = NIntegrate[-Log[x + y] / (x + y) e^(-x-y),
  {x, y} ∈ ImplicitRegion[x + y > 1 && 0 < x && 0 < y, {x, y}], WorkingPrecision → 20]
```

-0.21938393439552505132

```
I20R2 = NIntegrate[-Log[1/x + 1/y] / ((x + y) x y) e^(-1/x - 1/y),
  {x, y} ∈ ImplicitRegion[1/x + 1/y < 1 && 1 < x && 1 < y, {x, y}], WorkingPrecision → 20]
```

0.79659959929705824559

```
Abs[I20R1 + I20R2 - γ]
```

3.337×10^{-16}

III. La Integral de Jonathan Sondow

La integral de J. Sondow para la constante gamma de Euler:

$$\gamma = \int_0^1 \int_0^1 \frac{1-x}{(1-x\,y)\,(-\ln(x\,y))}\,dx\,dy \tag{26}$$

mediante cambio de variables , se obtienen otras integrales alternativas:

$$\gamma = \int_1^\infty \int_1^\infty \frac{x-1}{x^2\,y\,(x\,y-1)\,\ln(x\,y)}\,dx\,dy \tag{27}$$

$$\gamma = \int_0^\infty \int_0^\infty \frac{1-e^{-x}}{(x+y)\,(e^{x+y}-1)}\,dx\,dy \tag{28}$$

$$\gamma = \int\int_{R1} \frac{1-e^{-x}}{(x+y)\,(e^{x+y}-1)}\,dx\,dy + \int\int_{R2} \frac{1-e^{-1/x}}{x\,y\,(x+y)\left(e^{(1/x)+(1/y)}-1\right)}\,dx\,dy \tag{29}$$

donde

$$R1 = \left\{(x,\ y)\in \mathbb{R}^2 : x+y>1,\ x>0,\ y>0\right\} \tag{30}$$

$$R2 = \left\{(x,\ y)\in \mathbb{R}^2 : \frac{1}{x}+\frac{1}{y}<1,\ x>1,\ y>1\right\} \tag{31}$$

$$\gamma = \int_1^\infty \int_0^\infty (1-e^{-x})\,f(x,\ y)\,dx\,dy \tag{32}$$

donde

$$f(x,\ y) = \frac{1}{y(1+x\,y)\left(e^{x+(1/y)}-1\right)} + \frac{1}{(x+y)\,(e^{x+y}-1)} \tag{33}$$

IV. Integración Numérica para Integrales de III

Ejemplo 21: Integral (26)

```
I21 = NIntegrate[(1 - x)/((1 - x y) (-Log[x y])), {x, 0, 1}, {y, 0, 1}, WorkingPrecision → 20]
```

0.57721566490156001010

```
Abs[I21 - γ]
```

2.714949×10^{-14}

Ejemplo 22: Integral (26)

```
I22 = NIntegrate[(1 - x)/((1 - x y) (-Log[x y])), {x, 0, 1}, {y, 0, 1},
  Method → "MultidimensionalRule", WorkingPrecision → 20]
```

0.57721566490156001010

```
Abs[I22 - γ]
```

2.714949×10^{-14}

Ejemplo 23: Integral (26)

```
I23 = NIntegrate[(1 - x)/((1 - x y) (-Log[x y])), {x, 0, 1}, {y, 0, 1}, Method → "GaussKronrodRule",
  WorkingPrecision → 40]
```

0.5772156649015328606065120900824024330085

```
Abs[I23 - γ]
```

1.9664×10^{-36}

Ejemplo 24: Integral (26)

```
I24 = NIntegrate[(1 - x)/((1 - x y) (-Log[x y])), {x, 0, 1}, {y, 0, 1}, Method → "ClenshawCurtisRule",
  WorkingPrecision → 30]
```

0.577215664901532860606512090095

```
Abs[I24 - γ]
```

1.3×10^{-29}

Ejemplo 25: Integral(26)

```
I25 = NIntegrate[(1 - x)/((1 - x y) (-Log[x y])), {x, 0, 1}, {y, 0, 1},
  Method → {"NewtonCotesRule", "Type" → "Open"}, WorkingPrecision → 15]
```

0.577215663130707

```
Abs[I25 - γ]
```

1.770826×10^{-9}

Ejemplo 26: Integral (27)

```
I26 = NIntegrate[(x - 1)/(x^2 y (x y - 1) Log[x y]), {x, 1, ∞}, {y, 1, ∞}, Method → "GaussKronrodRule",
  WorkingPrecision → 40]
```

0.5772156649015328606065120900824024330085

```
Abs[I26 - γ]
```

1.9664×10^{-36}

Ejemplo 27: Integral (27)

```
I27 = NIntegrate[(x - 1)/(x^2 y (x y - 1) Log[x y]), {x, 1, ∞}, {y, 1, ∞},
  Method → "GaussBerntsenEspelidRule", WorkingPrecision → 40]
```

0.5772156649015328606065120900824024327291

```
Abs[I27 - γ]
```

1.6869×10^{-36}

Ejemplo 28: Integral (28)

```
I28 = NIntegrate[(1 - e^-x)/((x + y) (e^(x+y) - 1)), {x, 0, ∞}, {y, 0, ∞}, Method → "GaussKronrodRule",
  WorkingPrecision → 20]
```

0.57721566490153665602

```
Abs[I28 - γ]
```

3.79541×10^{-15}

Ejemplo 29: Integral (28)

```
I29 = NIntegrate[(1 - e^-x)/((x + y) (e^(x+y) - 1)), {x, 0, ∞}, {y, 0, ∞},
  Method → "GaussBerntsenEspelidRule", WorkingPrecision → 20]
```

0.57721566490153566117

```
Abs[I29 - γ]
```

2.80056×10^{-15}

Ejemplo 30: Integral (28)

```
I30 = NIntegrate[(1 - e^-x)/((x + y) (e^(x+y) - 1)), {x, 0, ∞}, {y, 0, ∞},
  Method → "GaussBerntsenEspelidRule", Exclusions → (x + y == 0), WorkingPrecision → 30]
```

0.577215664901532860606512090689

```
Abs[I30 - γ]
```

6.06×10^{-28}

Ejemplo 31: Integral (29)

```
I31R1 = NIntegrate[(1 - e^-x)/((x + y) (e^(x+y) - 1)),
  {x, y} ∈ ImplicitRegion[x + y > 1 && x > 0 && y > 0, {x, y}], WorkingPrecision → 20]
```

0.23929121099157198884

```
I31R2 = NIntegrate[(1 - e^(-1/x))/(x y (x + y) (e^((1/x)+(1/y)) - 1)),
  {x, y} ∈ ImplicitRegion[1/x + 1/y < 1 && x > 1 && y > 1, {x, y}], WorkingPrecision → 20]
```

0.33792445391012083328

```
Abs[I31R1 + I31R2 - γ]
```

$1.5996152 \times 10^{-13}$

Ejemplo 32: Integral (32)

```
I32 = NIntegrate[(1 - e^-x) (1/((x + y) (e^(x+y) - 1)) + 1/(y (1 + x y) (e^(x+(1/y)) - 1))), {x, 0, ∞},
  {y, 1, ∞}, Method → "GaussKronrodRule", WorkingPrecision → 20]
```

0.57721566490153652387

```
Abs[I32 - γ]
```

3.66326×10^{-15}

Comentarios

Los métodos con mejor performance son: GaussKronrod , GaussBerntsenEspelid .

Referencias

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