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Application of the Variational Iteration Method and the Homotopy Perturbation Method to the Fisher Type Equation

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ABSTRACT

To obtain the exact solution of the Fisher Type equation, the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM) are used. Obtained results by using the methods are numerically compared to present accuracy of the algorithms in solving the equation. The results prove that the methods are effective and powerful algorithm in order to solve the Fisher type equation as a non-linear differential equation.

Keywords: Fisher Type equation, Variational Iteration Method, Homotopy Perturbation Method, Nonlinear Differential Equations.

Mathematics Subject Classification: 35D35, 35D05

1. INTRODUCTION

Effects and behaviors of many phenomena in various fields of physical science and engineering ranging from gravitation to fluid dynamics can be described by nonlinear differential equations. Obtaining exact and approximate solutions of the equations is an important and active area of research in the field of mathematical sciences.

Semi-analytical methods such as the Variational Iteration Method (VIM) and Homotopy Perturbation Method (HPM) are introduced by He (1999; 1999) to obtain exact solutions of the equations. Then, application of the HPM in solving the non-linear non-homogeneous partial differential equations is presented by He (2005). In order to solve autonomous ordinary differential equation and delay differential equation, the VIM is applied by He (2000; 1997).

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The VIM and HPM are powerful and efficient algorithms which are used in solving various kinds of linear and nonlinear equations to obtain the exact solutions. The homotopy perturbation method is used by Nourazar et al. (2011; 2015; 2015) in order to obtain exact solution of nonlinear differential equations. Barari et al. (2009) used the homotopy perturbation method for solving tenth order boundary value problems. Application of the Variational Iteration Method to the Fisher's equations is presented by Matinfar and Ghanbari (2009). Ağırseven and Turgut (2010) presented an analytical study for Fisher type equations by using homotopy perturbation method.
The Fisher type equation was first introduced by Fisher (1937) as Eq. (1.1) in order to describe the propagation of a mutantgene where u denotes to the density of an advantageous. Then, the equation is used to model several chemical and physical contexts such as large number of the chemical kinetics, logistic population growth, flame propagation, population in one-dimensional habitual, neurophysiology, branching Brownian motion and autocatalytic chemical reactions.
The Fisher type equation is written as:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha\left(1-u^{\beta}\right)(u-a), \tag{1.1}$$

Where $\alpha,\beta\in[0,1]$ are real constants.

In the present research work, the Variational Iteration Method (VIM) and the Homotopy Perturbation Method (HPM) are applied to obtain the closed form solution of the non-linear Fisher type equation. The trend of rapid convergence of the sequences constructed by the VIM and HPM toward the exact solution of the equation are also numerically presented. As a result, a comparison between obtained results by the methods is presented in order to show their accuracy and reliability in solving the equation.
The ideas of variational iteration method as well as homotopy perturbation method are presented in the section 2 and section 3 respectively. Application of the variational iteration method and the homotopy perturbation method to the exact solution of Fisher type equation is presented in the section 4.

2. THE IDEA OF VARIATIONAL ITERATION METHOD

The idea of the variational iteration method is based on constructing a correction functional by a general Lagrange multiplier. The multiplier is chosen in such a way that its correction solution is improved with respect to the initial approximation or to the trial function. To illustrate the basic idea of the variational iteration method, consider the following nonlinear equation:

$$Lu(t)+Nu(t)=g(t), \tag{2.1}$$

Where L is a linear operator, N is a nonlinear operator, and $g(t)$ is a known analytic function. According to the variational iteration method, we can construct the following correction functional:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(Lu_n(\xi)+N\tilde{u}_n(\xi)-g(t)\right)d\xi, \tag{2.2}$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory and $\tilde{u}_n$ is considered as a restricted variation which means $\delta\tilde{u}_n=0$.

$u_0(t)$ is an initial approximation with possible unknowns. We first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. With λ determined, then several approximations $u_n(t), n \geq 0$ follow immediately. Consequently, the exact solution may be obtained as:

$$u(t) = \lim_{n \to \infty} u_n(t), \tag{2.3}$$

The correction functional of the Eq. (2.1) gives several approximations. Therefore, the exact solution can be obtained as the limit of resulting successive approximations.

3. THE IDEA OF HOMOTOPY PERTURBATION METHOD

The homotopy perturbation method (HPM) is originally initiated by He (1999). This is a combination of the classical perturbation technique and homotopy technique. The basic idea of the HPM for solving nonlinear differential equations is as follow; consider the following differential equation:

$$E(u) = 0, \tag{3.1}$$

Where E is any differential operator. We construct a homotopy as follow:

$$H(u, p) = (1-p)F(u) + p(E(u) - F(u)). \tag{3.2}$$

Where $F(u)$ is a functional operator with the known solution v_0. It is clear that when p is equal to zero then $H(u,0) = F(u) = 0$, and when p is equal to 1, then $H(u,1) = E(u) = 0$. It is worth noting that as the embedding parameter p increases monotonically from zero to unity the zero order solution v_0 continuously deforms into the original problem $E(u) = 0$. The embedding parameter, $p \in [0,1]$ is considered as an expanding parameter (He, 2005). In the homotopy perturbation method the embedding parameter p is used to get series expansion for solution as:

$$u = \sum_{i=0}^{\infty} p^i v_i = v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots \tag{3.3}$$

When, $p \to 1$ then Eq. (3.2) becomes the approximate solution to Eq. (3.1) as:

$$u = v_0 + v_1 + v_2 + v_3 + \cdots \tag{3.4}$$

The series Eq. (3.4) is a convergent series and the rate of convergence depends on the nature of Eq. (3.1) (He, 1999; He, 2005). It is also assumed that Eq. (3.2) has a unique solution and by comparing the like powers of p the solution of various orders is obtained. These solutions are obtained using the Maple package.

4. THE FISHER TYPE EQUATION

The Fisher type equation for $\alpha = 3, \beta = 1, a = 0,$ is constructed as Eq. (4.1) in order to present the capability and reliability of the methods.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 3u(1-u), \tag{4.1}$$

Subject to initial condition:

$$u(x,0)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}, \tag{4.2}$$

4.1. THE VARIATIONAL ITERATION METHOD

The correction functional for Eq. (4.1) is in the following form:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-3u_n(x,\xi)+3u_n^2(x,\xi)\right)d\xi, \tag{4.3}$$

Where u_n is restricted variation $\partial u_n=0$, λ is a Lagrange multiplier and u_0 is an initial approximation or trial function.

With above correction functional stationary we have:

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-3u_n(x,\xi)+3u_n^2(x,\xi)\right)d\xi,$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)\delta\int_0^t \lambda(\xi)\left(\frac{\partial u_n(x,\xi)}{\partial \xi}\right)d\xi,$$

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)\big(1+\lambda(\xi)\big)-\delta\int_0^t \lambda'(\xi)u_n(x,\xi)d\xi, \tag{4.4}$$

By using the following stationary conditions:

$$\delta u_n : 1+\lambda(\xi)=0, \tag{4.5}$$

$$\delta u_n : \lambda'(\xi)=0, \tag{4.6}$$

This gives the Lagrange multiplier $\lambda(\xi)=-1,$ therefore the following iteration formula becomes as:

$$u_{n+1}(x,t)= u_n(x,t)-\int_0^t \left(\frac{\partial u_n(x,\xi)}{\partial \xi}-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-3u_n(x,\xi)+3u_n^2(x,\xi)\right)d\xi, \tag{4.7}$$

We can select $u_0(x,y)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2},$ from the given condition.

Using this selection into the Eq. (4.7), the following successive approximation can be obtained as:

$$u_0(x,y)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2},$$

$$u_1(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}-\int_0^t\left(\frac{\partial u_0(x,\xi)}{\partial\xi}-\frac{\partial^2 u_0(x,\xi)}{\partial x^2}-3u_0(x,\xi)+3u_0^2(x,\xi)\right)d\xi$$

$$=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}+5\frac{e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t,$$

$$u_2(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}+5\frac{e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t-\int_0^t\left(\frac{\partial u_1(x,\xi)}{\partial\xi}-\frac{\partial^2 u_1(x,\xi)}{\partial x^2}-3u_1(x,\xi)+3u_1^2(x,\xi)\right)d\xi$$

$$=\frac{1}{5}\frac{5e^{\frac{\sqrt{2}}{2}x}+5\left(e^{\frac{\sqrt{2}}{2}x}\right)^2-1}{e^{\frac{\sqrt{2}}{2}x}\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}+\frac{5e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t+\frac{25}{4}\frac{e^{\frac{\sqrt{2}}{2}x}+\left(2e^{\frac{\sqrt{2}}{2}x}-1\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^4}t^2-25\frac{\left(e^{\frac{\sqrt{2}}{2}x}\right)^2}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^6}t^3, \tag{4.8}$$

As a result, the series of the exact solution of the Eq. (4.1) can be constructed as:

$$u_n(x,t)=\frac{1}{5}\frac{5e^{\frac{\sqrt{2}}{2}x}+5\left(e^{\frac{\sqrt{2}}{2}x}\right)^2-1}{e^{\frac{\sqrt{2}}{2}x}\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}+\frac{5e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t+\frac{25}{4}\frac{e^{\frac{\sqrt{2}}{2}x}+\left(2e^{\frac{\sqrt{2}}{2}x}-1\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^4}t^2-25\frac{\left(e^{\frac{\sqrt{2}}{2}x}\right)^2}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^6}t^3+\ldots \tag{4.9}$$

Using the identity,

$$u(x,t)=\lim_{n\to\infty}u_n(x,t), \tag{4.10}$$

We can write Eq. (4.9) in the closed form as:

$$\upsilon(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x-\frac{5}{2}t}\right)^2}, \tag{4.11}$$

This is the exact solution of the problem, Eq. (4.1).

4.2. THE HOMOTOPY PERTURBATION METHOD

We construct a homotopy for Eq. (4.1) in the following form:

$$H(\upsilon,p)=(1-p)\left[\frac{\partial\upsilon}{\partial t}-\frac{\partial u_0}{\partial t}\right]+p\left[\frac{\partial\upsilon}{\partial t}-\frac{\partial^2\upsilon}{\partial x^2}-3\upsilon(1-\upsilon)\right]. \tag{4.12}$$

The solution of Eq. (3.10) can be written as a power series in p as:

$$v=v_0+pv_1+p^2v_2+... \tag{4.13}$$

Substituting Eq. (4.13) and Eq. (4.2) into Eq. (4.12) and equating the term with identical powers of p, leads to:

$$p^0:\frac{\partial v_0}{\partial t}=\frac{\partial u_0}{\partial t}, \qquad v_0(x,0)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2},$$

$$p^1:\frac{\partial\upsilon_1}{\partial t}+\frac{\partial u_0}{\partial t}=\frac{\partial^2\upsilon_0}{\partial x^2}+3\upsilon_0(1-\upsilon_0)\,, \qquad v_1(x,0)=0,$$

$$p^2:\frac{\partial\upsilon_2}{\partial t}=\frac{\partial^2\upsilon_1}{\partial x^2}+3\upsilon_1(1-\upsilon_0)-3\upsilon_0\upsilon_1\,, \qquad v_2(x,0)=0,$$

$$p^3:\frac{\partial\upsilon_3}{\partial t}=\frac{\partial^2\upsilon_2}{\partial x^2}-3\upsilon_0\upsilon_2-3\upsilon_1^2+3\upsilon_2(1-\upsilon_0), \qquad v_3(x,0)=0. \tag{4.14}$$

Using the Maple package to solve recursive sequences, Eq. (4.14), we obtain the followings:

$$v_0(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2},$$

$$v_1(x,t)=\frac{5e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t,$$

$$v_2(x,t)=\frac{25}{4}\frac{e^{\frac{\sqrt{2}}{2}x}\left(-1+2e^{\frac{\sqrt{2}}{2}x}\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^4}t^2,$$

$$v_3(x,t)=\frac{125}{24}\frac{e^{\frac{\sqrt{2}}{2}x}\left(1-7e^{\frac{\sqrt{2}}{2}x}+4\left(e^{\frac{\sqrt{2}}{2}x}\right)^2\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^5}t^3, \tag{4.15}$$

By setting $p=1$ in Eq. (4.13), the solution of Eq. (4.1) can be obtained as $v=v_0+v_1+v_2+v_3+\ldots$. Therefore the solution of Eq. (4.1) is written as:

$$v(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}+\frac{5e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t+\frac{25}{4}\frac{e^{\frac{\sqrt{2}}{2}x}\left(-1+2e^{\frac{\sqrt{2}}{2}x}\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^4}t^2+\frac{125}{24}\frac{e^{\frac{\sqrt{2}}{2}x}\left(1-7e^{\frac{\sqrt{2}}{2}x}+4\left(e^{\frac{\sqrt{2}}{2}x}\right)^2\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^5}t^3+\ldots \tag{4.16}$$

The Taylor series expansion for $\left(\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x-\frac{5}{2}t}\right)^2}\right)$ is written as:

$$\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x-\frac{5}{2}t}\right)^2}$$

$$=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^2}+\frac{5e^{\frac{\sqrt{2}}{2}x}}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^3}t+\frac{25}{4}\frac{e^{\frac{\sqrt{2}}{2}x}\left(-1+2e^{\frac{\sqrt{2}}{2}x}\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^4}t^2+\frac{125}{24}\frac{e^{\frac{\sqrt{2}}{2}x}\left(1-7e^{\frac{\sqrt{2}}{2}x}+4\left(e^{\frac{\sqrt{2}}{2}x}\right)^2\right)}{\left(1+e^{\frac{\sqrt{2}}{2}x}\right)^5}t^3+\ldots \tag{4.17}$$

Combining Eq. (4.17) with Eq. (4.16), we get as follow:

$$v(x,t)=\frac{1}{\left(1+e^{\frac{\sqrt{2}}{2}x-\frac{5}{2}t}\right)^2}, \tag{4.18}$$

This is the exact solution of the problem, Eq. (4.1).

4.3. COMPARISON

In order to compare accuracy of the obtained results by VIM and HPM, Table 1 is presented. Table1 shows the trend of rapid convergence of the results of $S_0(x,t)=u_0(x,t)$ to $S_6(x,t)=u_6(x,t)$ by using the VIM and $S_0(x,t)=v_0(x,t)$ to $S_6(x,t)=\sum_{i=0}^{5}v_i(x,t)$ by using the HPM.

		Percentage of relative error (%RE)					
		$x=1$		$x=2$		$x=3$	
		VIM	HPM	VIM	HPM	VIM	HPM
$t=0.1$	$S_1(x,t)$	0.02103243774	0.01837463774	0.031765859	0.0297463238	0.0057843948	0.005482013
	$S_3(x,t)$	0.000035932334	0.00000475987	0.00000052769	0.00000046298	3.0013823842e-7	8.328473643 e-7
	$S_5(x,t)$	2.09837466 e-10	3.2987754 e-11	3.03760876e-10	3.52024313e-10	3.84203193e-11	3.604923847e-11
	$S_6(x,t)$	3.447832245e-11	8.364087789e-12	7.348343102e-12	2.94635493e-12	5.310645210e-12	6.20348112 e-11
$t=0.3$	$S_1(x,t)$	0.0489340238	0.0523342321	0.0318743343	0.030323848	0.0132334433	0.0149932384
	$S_3(x,t)$	0.000059963485	0.000061212343	0.000004635293	0.000004240384	0.00000198876	0.000002032123
	$S_5(x,t)$	6.740957454 e-7	7.003454344 e-8	8.983209903 e-8	9.012388231 e-8	6.02349384 e-9	6.00548374 e-8
	$S_6(x,t)$	1.006345487 e-9	1.007576557 e-8	1.003128833 e-10	1.000423138 e-9	2.004344349e-10	2.03495584 e-9
$t=0.5$	$S_1(x,t)$	0.07193302232	0.07287744432	0.0495322332	0.04699094434	0.0254554948	0.0244545434
	$S_3(x,t)$	0.000013433432	0.00001573238	0.000005833244	0.000005648577	0.00005765565	0.000054584778
	$S_5(x,t)$	5.766606975e-7	4.965857643e-7	6.048774543e-8	4.003847343e-7	2.434439846e-8	2.34398656 e-7
	$S_6(x,t)$	7.345560056 e-7	6.234454342 e-8	8.381223943 e-9	6.340495440 e-9	3.02123951 e-9	2.32332543 e-9

Table 1 shows the percentage of relative errors of the results of $S_0(x,t)=u_0(x,t)$ to $S_6(x,t)=u_6(x,t)$ by using the VIM and $S_0(x,t)=v_0(x,t)$ to $S_6(x,t)=\sum_{i=0}^{5}v_i(x,t)$ by using the HPM.

5. CONCLUSION

In the present research work, the VIM and HPM are used in order to obtain the exact solution of the Fisher type nonlinear diffusion equation. The validity and effectiveness of the VIM and HPM are shown by solving the non-homogenous non-linear differential equations and the very rapid convergence to the exact solutions is numerically demonstrated. Also, a comparison between obtained results by the methods is presented in order to show their accuracy and reliability in solving the equation. As shown in the Table 1, the maximum relative error of less than 0.000074% and

0.0000063% are achieved by using the VIM and HPM respectively. The study proves accuracy and reliability of the methods in solving the Fisher type equation. Moreover, validity and great potential of the methods in obtaining the exact solution of nonlinear differential equations are presented.

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