# Pi Formulas , Part 16

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abstract

In this note we show some formulas related with the constant Pi

# Colección de fórmulas para la constante Pi

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## Resumen-Abstract

En esta nota mostramos una colección de fórmulas que involucran a la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265\ldots$$

## Introducción

En esta nota mostramos una colección de fórmulas, que corresponden a casos particulares de las siguientes fórmulas generales:

$$A = \sum_{n=1}^{\infty}(-1)^{n-1}\, H_n\, r^n \cos(n\,\alpha)$$

$$B = \sum_{n=1}^{\infty}(-1)^{n-1}\, H_n\, r^n \sin(n\,\alpha)$$

donde

$$A = u\,\pi + v\ , \qquad B = x\,\pi + y\ , \qquad H_n = \sum_{k=1}^{n}\frac{1}{k}$$

$$u, v, x, y \in \mathbb{R}$$

Los números $v, y$ , involucran logaritmos.

| $\alpha$ | $r$ | $A$ | $B$ |
|---|---|---|---|
| $\frac{7\pi}{12}$ | $\sqrt{\frac{2}{3}}$ | $\frac{(3-\sqrt{3})}{8}\pi+\frac{(3-\sqrt{3})}{8}\ln\left(\frac{7+4\sqrt{3}}{9}\right)$ | $\frac{(3-\sqrt{3})}{8}\pi+\frac{(3-\sqrt{3})}{4}\ln(6-3\sqrt{3})$ |
| $\frac{2\pi}{3}$ | $\sqrt{3}-1$ | $\frac{(3+\sqrt{3})}{24}\pi+\frac{(3+\sqrt{3})}{24}\ln(63-36\sqrt{3})$ | $\frac{(3+\sqrt{3})}{24}\pi+\frac{(3+\sqrt{3})}{24}\ln\left(\frac{7+4\sqrt{3}}{9}\right)$ |
| $\frac{3\pi}{4}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\pi}{4}-\frac{1}{2}\ln 2$ | $\frac{\pi}{4}+\frac{1}{2}\ln 2$ |
| $\frac{5\pi}{6}$ | $\sqrt{3}-1$ | $\frac{(1+\sqrt{3})}{8}\pi+\frac{(1+\sqrt{3})}{8}\ln(7-4\sqrt{3})$ | $\frac{(1+\sqrt{3})}{8}\pi-\frac{(1+\sqrt{3})}{8}\ln(7-4\sqrt{3})$ |
| $\frac{11\pi}{12}$ | $\sqrt{\frac{2}{3}}$ | $\frac{(3+\sqrt{3})}{8}\pi+\frac{(3+\sqrt{3})}{8}\ln\left(\frac{7-4\sqrt{3}}{9}\right)$ | $\frac{(3+\sqrt{3})}{8}\pi+\frac{(3+\sqrt{3})}{8}\ln(63+36\sqrt{3})$ |

| $\alpha$ | $r$ | $A$ | $B$ |
|---|---|---|---|
| $\frac{5\pi}{12}$ | $\frac{1}{\sqrt{2}}$ | $\frac{(\sqrt{3}-1)}{12}\pi-\frac{(3-\sqrt{3})}{4}\ln(4-2\sqrt{3})$ | $\frac{(3-\sqrt{3})}{12}\pi+$ $\frac{1}{4}\left(\sqrt{3}\,\ln(4-2\sqrt{3})+\ln\left(\frac{2+\sqrt{3}}{2}\right)\right)$ |
| $\frac{\pi}{2}$ | $\frac{1}{\sqrt{3}}$ | $\frac{\sqrt{3}}{24}\pi+\frac{3}{8}\ln\left(\frac{4}{3}\right)$ | $\frac{\pi}{8}-\frac{\sqrt{3}}{8}\ln\left(\frac{4}{3}\right)$ |
| $\frac{7\pi}{12}$ | $\frac{\sqrt{2}}{1+\sqrt{3}}$ | $\frac{\pi}{12}$ | $\frac{\pi}{4\sqrt{3}}$ |
| $\frac{2\pi}{3}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{18}\pi-\frac{1}{2}\ln\left(\frac{4}{3}\right)$ | $\frac{\pi}{6}+\frac{\sqrt{3}}{6}\ln\left(\frac{4}{3}\right)$ |
| $\frac{3\pi}{4}$ | $\frac{\sqrt{2}}{1+\sqrt{3}}$ | $\frac{(1+\sqrt{3})\,\pi}{24}+\frac{(3+\sqrt{3})\,\ln(4-2\sqrt{3})}{8}$ | $\frac{(3+\sqrt{3})\,\pi}{24}-\frac{(1+\sqrt{3})\,\ln(4-2\sqrt{3})}{8}$ |
| $\frac{5\pi}{6}$ | $\frac{1}{\sqrt{3}}$ | $\frac{\sqrt{3}}{12}\pi-\frac{3}{4}\ln 3$ | $\frac{\pi}{4}+\frac{\sqrt{3}}{4}\ln 3$ |
| $\frac{11\pi}{12}$ | $\frac{1}{\sqrt{2}}$ | $\frac{(1+\sqrt{3})}{12}\pi+\frac{(3+\sqrt{3})}{4}\ln\left(1-\frac{\sqrt{3}}{2}\right)$ | $\frac{(3+\sqrt{3})}{12}\pi+\frac{(1+\sqrt{3})}{4}\ln(4+2\sqrt{3})$ |

| $\alpha$ | $r$ | $A$ |
|---|---|---|
| $\frac{\pi}{3}$ | $(\sqrt{2}-1)\left(1+\sqrt{2-\sqrt{3}}\right)$ | $\frac{\pi(3\sqrt{2}-2\sqrt{3}+\sqrt{6})}{96}+\frac{(6+3\sqrt{2}-\sqrt{6})}{24}\ln\left(\frac{3}{2}(8-5\sqrt{2}-4\sqrt{3}+3\sqrt{6})\right)$ |
| $\frac{5\pi}{12}$ | $\frac{2}{1+\sqrt{2}+\sqrt{3}}$ | $\frac{\pi(\sqrt{3}+\sqrt{2}-1)}{16(2+\sqrt{2})}+\frac{(2-\sqrt{2}+\sqrt{6})}{8}\ln\left(2-\sqrt{\frac{3}{2}+\frac{1}{\sqrt{2}}}\right)$ |
| $\frac{\pi}{2}$ | $\sqrt{2}-1$ | $\frac{\pi}{16\sqrt{2}}+\frac{(2+\sqrt{2})}{8}\ln(4-2\sqrt{2})$ |
| $\frac{7\pi}{12}$ | $\frac{2}{1+\sqrt{3}+\sqrt{6}}$ | $\frac{\pi\ (4-\sqrt{2}-2\sqrt{3}+\sqrt{6})}{32}+\frac{\left(1+\sqrt{6-3\sqrt{3}}\right)}{4}\ln\left(14+8\sqrt{3}-\sqrt{362+209\sqrt{3}}\right)$ |
| $\frac{2\pi}{3}$ | $\frac{2}{1+\sqrt{3}+\sqrt{6}}$ | $\frac{\pi\ (\sqrt{2}-1)\left(3+\sqrt{3(2+\sqrt{3})}\right)}{48}+\frac{\left(3+\sqrt{3(2+\sqrt{3})}\right)}{12}\ln\left(\frac{3}{2}(8-5\sqrt{2}+4\sqrt{3}-3\sqrt{6})\right)$ |
| $\frac{3\pi}{4}$ | $\sqrt{2}-1$ | $\frac{\pi}{16}+\frac{(1+\sqrt{2})}{4}\ln(2-\sqrt{2})$ |
| $\frac{5\pi}{6}$ | $\frac{2}{1+\sqrt{2}+\sqrt{3}}$ | $\frac{\pi(\sqrt{2}+2\sqrt{3}-\sqrt{6})}{32}+\frac{(2+\sqrt{2}+\sqrt{6})}{8}\ln\left(4-3\sqrt{\frac{3}{2}+\frac{5}{\sqrt{2}}}-2\sqrt{3}\right)$ |
| $\frac{11\pi}{12}$ | $(\sqrt{2}-1)\left(1+\sqrt{2-\sqrt{3}}\right)$ | $\frac{\pi\left(2+\sqrt{3}-\sqrt{2+\sqrt{3}}\right)}{16}+\frac{\left(1+\sqrt{3(2+\sqrt{3})}\right)}{4}\ln\left(14-8\sqrt{3}+\sqrt{362-209\sqrt{3}}\right)$ |

| $\alpha$ | $r$ | $B$ |
|---|---|---|
| $\frac{\pi}{3}$ | $(\sqrt{2}-1)\left(1+\sqrt{2-\sqrt{3}}\right)$ | $\frac{\pi\left(3+\sqrt{6-3\sqrt{3}}\right)}{48}-\frac{(3\sqrt{2}-2\sqrt{3}+\sqrt{6})}{24}\ln\left(\frac{3}{2}(8-5\sqrt{2}-4\sqrt{3}+3\sqrt{6})\right)$ |
| $\frac{5\pi}{12}$ | $\frac{2}{1+\sqrt{2}+\sqrt{3}}$ | $\frac{\pi(2-\sqrt{2}+\sqrt{6})}{32}-\frac{(\sqrt{3}+\sqrt{2}-1)}{4(2+\sqrt{2})}\ln\left(2-\sqrt{\frac{3}{2}+\frac{1}{\sqrt{2}}}\right)$ |
| $\frac{\pi}{2}$ | $\sqrt{2}-1$ | $\frac{\pi(2+\sqrt{2})}{32}-\frac{1}{4\sqrt{2}}\ln(4-2\sqrt{2})$ |
| $\frac{7\pi}{12}$ | $\frac{2}{1+\sqrt{3}+\sqrt{6}}$ | $\frac{\pi\left(1+\sqrt{6-3\sqrt{3}}\right)}{16}-\frac{(4-\sqrt{2}-2\sqrt{3}+\sqrt{6})}{8}\ln\left(14+8\sqrt{3}+\sqrt{362+209\sqrt{3}}\right)$ |
| $\frac{2\pi}{3}$ | $\frac{2}{1+\sqrt{3}+\sqrt{6}}$ | $\frac{\pi\sqrt{2}\,(3+\sqrt{3}-\sqrt{6})\left(2+\sqrt{2+\sqrt{3}}\right)}{96}-\frac{\sqrt{3}\,(\sqrt{2}-1)\left(-1+\sqrt{2+\sqrt{3}}\right)}{12}\ln\left(\frac{3}{2}(8-5\sqrt{2}+4\sqrt{3}-3\sqrt{6})\right)$ |
| $\frac{3\pi}{4}$ | $\sqrt{2}-1$ | $\frac{\pi\,(1+\sqrt{2})}{16}-\frac{1}{4}\ln(2-\sqrt{2})$ |
| $\frac{5\pi}{6}$ | $\frac{2}{1+\sqrt{2}+\sqrt{3}}$ | $\frac{\pi(2+\sqrt{2}+\sqrt{6})}{32}-\frac{(\sqrt{2}+2\sqrt{3}-\sqrt{6})}{8}\ln\left(4-3\sqrt{\frac{3}{2}+\frac{5}{\sqrt{2}}}-2\sqrt{3}\right)$ |
| $\frac{11\pi}{12}$ | $(\sqrt{2}-1)\left(1+\sqrt{2-\sqrt{3}}\right)$ | $\frac{\pi\left(1+\sqrt{3(2+\sqrt{3})}\right)}{16}-\frac{\left(2+\sqrt{3}-\sqrt{2+\sqrt{3}}\right)}{4}\ln\left(14-8\sqrt{3}+\sqrt{362-209\sqrt{3}}\right)$ |

| $\alpha$ | $r$ | $A$ | $B$ |
|---|---|---|---|
| $\frac{\pi}{4}$ | $\sqrt{2-\sqrt{3}}$ | $\frac{\pi(\sqrt{3}-1)}{48}+\frac{(1+\sqrt{3})}{8}\ln(2)$ | $\frac{\pi(1+\sqrt{3})}{48}-\frac{(\sqrt{3}-1)}{8}\ln(2)$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}-1}{2}$ | $\frac{\pi(3-\sqrt{3})}{72}+\frac{(3+\sqrt{3})}{12}\ln\left(\frac{3}{2}\right)$ | $\frac{\pi(3+\sqrt{3})}{72}-\frac{(3-\sqrt{3})}{12}\ln\left(\frac{3}{2}\right)$ |
| $\frac{5\pi}{12}$ | $\sqrt{\frac{2}{3}-\frac{1}{\sqrt{3}}}$ | $\frac{\pi\,(2\sqrt{3}-3)}{24}+\frac{\sqrt{3}}{4}\ln\left(\frac{2}{3}+\frac{1}{\sqrt{3}}\right)$ | $\frac{\pi}{8\sqrt{3}}-\frac{(2\sqrt{3}-3)}{4}\ln\left(\frac{2}{3}+\frac{1}{\sqrt{3}}\right)$ |
| $\frac{\pi}{2}$ | $2-\sqrt{3}$ | $\frac{\pi}{48}+\frac{(2+\sqrt{3})}{8}\ln(8-4\sqrt{3})$ | $\frac{\pi\,(2+\sqrt{3})}{48}-\frac{1}{8}\ln(8-4\sqrt{3})$ |
| $\frac{7\pi}{12}$ | $\frac{\sqrt{2-\sqrt{3}}}{2}$ | $\frac{\pi(2-\sqrt{3})}{12}-\frac{1}{2}\ln(8-4\sqrt{3})$ | $\frac{\pi}{12}-\frac{(2-\sqrt{3})}{2}\ln\left(\frac{2+\sqrt{3}}{4}\right)$ |
| $\frac{2\pi}{3}$ | $2-\sqrt{3}$ | $\frac{\pi}{24\sqrt{3}}+\frac{(3+2\sqrt{3})}{12}\ln(6-3\sqrt{3})$ | $\frac{\pi(2\sqrt{3}+3)}{72}-\frac{\sqrt{3}}{12}\ln(6-3\sqrt{3})$ |
| $\frac{3\pi}{4}$ | $\sqrt{\frac{2}{3}-\frac{1}{\sqrt{3}}}$ | $\frac{\pi\,(3-\sqrt{3})}{48}-\frac{(3+\sqrt{3})}{8}\ln\left(\frac{3}{2}\right)$ | $\frac{\pi\,(3+\sqrt{3})}{48}+\frac{(3-\sqrt{3})}{8}\ln\left(\frac{3}{2}\right)$ |
| $\frac{5\pi}{6}$ | $\frac{\sqrt{3}-1}{2}$ | $\frac{\pi(\sqrt{3}-1)}{24}-\frac{(1+\sqrt{3})}{4}\ln(2)$ | $\frac{\pi(1+\sqrt{3})}{24}+\frac{(\sqrt{3}-1)}{4}\ln(2)$ |
| $\frac{11\pi}{12}$ | $\sqrt{2-\sqrt{3}}$ | $\frac{\pi}{24}+\frac{(2+\sqrt{3})}{4}\ln(2-\sqrt{3})$ | $\frac{\pi(2+\sqrt{3})}{24}-\frac{1}{4}\ln(2-\sqrt{3})$ |

## Otros Ejemplos

$$\pi = 2\sum_{n=1}^{\infty}(-1)^{n-1}\,H_n\left(\frac{1}{\sqrt{2}}\right)^{n}\left(\cos\left(\frac{3\pi n}{4}\right)+\sin\left(\frac{3\pi n}{4}\right)\right)$$

$$\pi = 2\sum_{n=1}^{\infty}(-1)^{n-1}\,H_n\left(\sqrt{3}-1\right)^{n+1}\left(\cos\left(\frac{5\pi n}{6}\right)+\sin\left(\frac{5\pi n}{6}\right)\right)$$

$$\pi = 4\sqrt{2}\sum_{n=1}^{\infty}(-1)^{n-1} H_n \left(\sqrt{2-\sqrt{2}}\right)^n \cos\left(\frac{5\pi n}{8}\right)$$

$$\pi = 4\sqrt{2}\sum_{n=1}^{\infty}(-1)^{n-1} H_n \left(\sqrt{2-\sqrt{2}}\right)^n \sin\left(\frac{5\pi n}{8}\right)$$

$$\frac{7}{24}\pi + \frac{7}{12}\ln\left(\frac{49}{18}\right) = \sum_{n=1}^{\infty}(-1)^{n-1} H_n \left(\frac{5}{7}\right)^n \sin\left(n\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{4}{5}\right)\right)\right) = \sum_{n=1}^{\infty} H_n \left(\frac{5}{7}\right)^n \sin\left(n\cos^{-1}\left(\frac{4}{5}\right)\right)$$
$$= \sum_{n=1}^{\infty} H_n \left(\frac{1}{7}\right)^n \left(\frac{(4+3i)^n - (4-3i)^n}{2i}\right) = \frac{3}{5}\sum_{n=1}^{\infty} H_n \left(\frac{1}{7}\right)^n c_n$$

donde

$$c_{n+2} = 8c_{n+1} - 25c_n \ , c_1 = 5, c_2 = 8$$

$$\frac{7}{24}\pi - \frac{7}{12}\ln\left(\frac{49}{18}\right) = \sum_{n=1}^{\infty}(-1)^{n-1} H_n \left(\frac{5}{7}\right)^n \cos\left(n\left(\frac{\pi}{2} + \sin^{-1}\left(\frac{4}{5}\right)\right)\right) = -\sum_{n=1}^{\infty} H_n \left(\frac{5}{7}\right)^n \cos\left(n\cos^{-1}\left(\frac{4}{5}\right)\right)$$
$$= -\sum_{n=1}^{\infty} H_n \left(\frac{1}{7}\right)^n \left(\frac{(4-3i)^n + (4+3i)^n}{2}\right) = -\sum_{n=1}^{\infty} H_n \left(\frac{1}{7}\right)^n c_n$$

donde

$$c_{n+2} = 8c_{n+1} - 25c_n \ , c_1 = 4, c_2 = 7$$

$$\pi = \frac{12}{7}\sum_{n=1}^{\infty} H_n \left(\frac{1}{7}\right)^n c_n$$

donde

$$c_{n+2} = 8c_{n+1} - 25c_n \ , c_1 = -1, c_2 = 17$$

$$c_n = \left(\frac{-1+i}{2}\right)((4-3i)^n + i(4+3i)^n)\ , n \in \mathbb{N}$$

# Referencias

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