Fidelity between two Hamiltonians

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(Dated: Sep. 26, 2014)

Suppose a Hamiltonian depends on some parameter g. For two different values of g, g_1 and g_2, the Hamiltonian has two different sets of eigenstates:

$$\{\psi_1^{(1)}, \psi_2^{(1)}, \ldots, \psi_N^{(1)}\} \tag{1}$$

and

$$\{\psi_1^{(2)}, \psi_2^{(2)}, \ldots, \psi_N^{(2)}\}. \tag{2}$$

Here N is the size of the Hilbert space.

If g_1 is very close to g_2, one might expect that $\psi_i^{(1)} \sim \psi_i^{(2)}$ for $1 \leq i \leq N$. That is, the eigenstates of $H(g_1)$ are not so much mixed to form the eigenstates of $H(g_2)$. However, if g_1 is far away from g_2, say, they are on the opposite side of each other to some critical point, one would expect each state in the second spectrum is a superposition of "a lot" (to be quantified somehow below) of states in the first spectrum. Intuitively, we say, the spectrum is greatly twisted as g is varied from g_1 to g_2.

We want to characterize the extent of twisting. The strategy is like this. We want to find a one-to-one mapping between the two spectra, such that the sum of the overlaps inside each pair is maximized. That is, we want

$$I = \max_{p \in S_N} \left(\sum_{i=1}^{N} |\langle \psi_i^{(1)} | \psi_{p(i)}^{(2)} \rangle|^2 \right). \tag{3}$$

Apparently, the answer is independent of the representation.

Mathematically, this problem can be reformulated as follows. Let us first construct the $N \times N$ matrix, whose elements are defined as

$$M_{ij} = |\langle \psi_i^{(1)} | \psi_j^{(2)} \rangle|^2. \tag{4}$$

We want to choice N elements of M, under the constraint that there is one and only one in each row and in each column, and maximize their sum.

For this combinatorial problem, there is an efficient, polynomial algorithm, i.e., the so-called Hungarian algorithm. Therefore, the problem can be efficiently solved numerically (like diagonalizing a matrix).

The rest problem is physical: for which model would this idea yield some interesting results?