Quantum mechanics without the measurement axiom.

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Abstract.

We present the axiomatization of quantum mechanics which does not contain axioms concerning the measurement. Instead of the concept of measurement this axiomatization uses the concept of the observation of the individual state of the measuring system after the run of the experiment. It is proved that the resulting theory is empirically equivalent to the standard quantum mechanics but it is also shown that these two theories are (theoretically) different.

1. Introduction.

The measurement problem exists from the beginning of quantum mechanics (QM). This problem has many facets: the superposition of measurement states, the von Neumann's infinite chain of measuring instruments, the problem of definite values etc.

The special problem is connected with the axiomatic description of QM: it is a problem that measurement makes a part of axioms. For example, John Bell has written an article named "Against measurement" (see [5]) and Mermin has required that "the concept of measurement should play no fundamental role" (see [6]). The Mermins's requirements of the so-called Ithaca interpretation of QM ([6])[1] contains this requirement. We agree with both authors – measurement should be one of possible processes in QM and it should be considered at the same footing as other processes in QM.

[1] Mermin's requiremnts for the Ithaca interpretation of QM are:
(1) Is unambiguous about objective reality.
(2) Uses no prior concept of measurement.
(3) Applies to individual systems.
(4) Applies to (small) isolated systems.
(5) Satisfies generalized Einstein–locality.
(6) Rests on prior concept of objective probability.
It is interesting that our reformulation of QM (the modified QM) satisfies requirements (1) – (5) but it surely does not satisfy the requirement (6).

In this paper we shall give the axiomatic reformulation of QM not containing the concept of a measurement. Then we shall show that the empirical predictions of a proposed reformulation of QM are identical to the empirical predictions of the standard QM. These two theories are empirically indistinguishable, but they are different as theories (see [4]).

In fact, what we surely do in QM is the observation of the individual state of the measuring system in one run of the experiment. Thus instead of the observation of the value of some observable we use the concept of the observation of the individual state of the measuring system in one run of the experiment.

The origin of the proposed reformulation of QM lies in the probabilistic approach to QM developed in [1], while the reformulation proposed here is only slightly different from the modified QM presented in [2] and [3].

In part 2 our reformulation of QM is given. In part 3 we rewrite our formulation into the operator language. In part 4 we describe the internal measurement process in the modified QM in all details. In part 5 we compare these two formulations and in part 6 we give conclusions.

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2. Axioms for quantum mechanics not using the measurement concept.

We shall consider only systems with the finite-dimensional Hilbert spaces (for simplicity).

Axiom 1. To each system S there is associated a finite set $D_S = \{s_1, \dots , s_n\}$, $n \geq 2$. These states $s_1, \dots , s_n$ are considered as a set of possible individual states of the system S. It is assumed that at each time the system S is in some individual state

$$\mathrm{IndSt}\ (S; t) \in D_S .$$

Definition. The ensemble of systems

$$\mathbf{E} = \{S_1, \dots, S_N\} , \qquad N \to \infty .$$

is the set of systems based on the same D_S which is generated by some preparation procedure[2].

The possible states of ensembles are defined in the next axiom.

Axiom 2. *The state space* St (D_S) of possible states of an ensemble is the set of all functions q : $D_S \times D_S \to C$ satisfying[3]

[2] In the classical probability theory the concept of an ensemble is absolutely necessary. The individual state of an Brownian particle is the point in the space, but the prediction of the future state is the probability distribution over the space and it is associated with the ensemble of Brownian particles.

(i) $q(s', s) = q(s, s')^*$

(ii) $\sum q(s, s')\, \psi(s)\, \psi(s')^* \geq 0$ for each function $\psi : D_S \to C$

(iii) $\sum \{ q(s, s) \mid s \in D_S \} = 1$

A state q is called a *pure state* if it can be written as $q(s, s') = \psi(s)\, \psi(s')^*$ for some function $\psi : D_S \to C$.

A state q is called a deterministic state if there exists $r \in D_S$ such that q has the form $q(r, r) = 1$, $q = 0$ otherwise.

Axiom 3. *The time evolution* of the state q^t (t = time), is given by the one-parameter group of unitary matrices $\{ U^t_{rs} \}$, such that

$$q^t(r, r') = \sum U^t_{rs}\, U^{t*}_{r's'}\, q^0(s, s') .$$ [4]

Axiom 4 . For *the composed system* $T = M \oplus S$ we have

$$D_T = D_M \times D_S .$$

For independent systems we have [5]

$$q_T((m, s), (m', s')) = q_M(m, m')\, q_S(s, s') \ , \quad m, m' \in D_M, \quad s, s' \in D_S .$$

Axiom 5. Let the system S be at the time t in the state q^t. Then the probability to find the system S in the individual state $s \in D_S$ at the time t is given by

$$\text{prob}\ (\text{IndSt}\ (S; t) = s \mid q^t) = q^t(s, s)$$ [6] .

We simply obtain for $A \subset D_S$ that

$$\text{prob}\ (\text{IndSt}\ (S; t) \in A \mid q^t) = \sum \{ q^t(s, s) \mid s \in A \} .$$

Axiom 6. Let us assume that we have obtained an information that the individual state of the system S is in the set $A \subset D_S$. Then the state q of the system S has to be up-dated. The new state q' of the system S will be given by the formula [7]

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[3] In the classical case, the probability distribution depends on one variable while in our case the probability distribution q depends on two variables – this is the essential difference between these two cases. The state q of an ensemble **E** should be considered as a generalized probability distribution (see [1]).

[4] The evolution is the evolution of the state of an ensemble, this is not the evolution of the individual state of an individual system. It is the same situation as for Brownian particle, where the evolution is the evolution of the state of an ensemble of Brownian particles. There is a fundamental difference between these two cases consisting in the fact that in our case the time evolution is time-reversible and it is fundamentally different from the classical (dissipative) evolution of the distribution function.

[5] This is similar to the classical probability theory.

[6] Thus the probability depends only on the diagonal part of q but in the evolution of q the main role is played by the non-diagonal part of q.

$$q'(s, s') = q(s, s')\, \chi(A; s)\, \chi(A; s')\, N_A^{-1}$$

where $\chi(A; s)$ denotes the characteristic function of the set A and N_A denotes the normalization factor $N_A = \sum \{ q(s, s) \mid s \in A \}$.

Axiom 7. For each n = 2, 3, … there exists at least one system M satisfying

(i) The set D_M contains n elements
(ii) When experiment containing the system M is finished then the individual state of the system M can be observed. (Such systems will be called the observable systems.)

3. The reformulation of our axioms in the operator language.

Almost all content of these axioms can be reformulated using the language of Hilbert spaces and operators.

We shall define the Hilbert space associated with the system S by

$$H_S = \{ \psi \mid \psi : D_S \to C \}, \qquad \langle \psi, \psi' \rangle = \sum \{ \psi(s)\, \psi'(s)^* \mid s \in D_S \} .$$

For each s, s' $\in D_S$ we set, as usual, $\delta_{ss'} = 1$ if s = s' and $\delta_{ss'} = 0$ otherwise.

Then we define the function $\delta_s \in H_S$, $s \in D_S$, by $\delta_s(s') = \delta_{ss'}$. The equivalent notation for δ_s will be $\delta(s)$. Evidently, the set $\{ \delta(s) \mid s \in D_S \}$ is the orthogonal base of the Hilbert space H_S .

For each $r \in D_S$ we shall define the state q_r by

$$q_r(s, s') = \delta_{rs}\, \delta_{rs'} .$$

For each state $q : D_S \times D_S \to C$ we shall define the corresponding operator $\rho : H_S \to H_S$ by

$$\rho = \sum q(s', s)\, \delta_s \otimes \delta_{s'}^* \qquad \text{or equivalently as} \quad q(s, s') = \langle \delta_{s'}, \rho(\delta_s) \rangle .$$

Using the group of unitary matrices $\{ U^t_{rs} \}$ it is possible to define the group of unitary operators $\{ U^t \}$ by

$$U^t = \sum U^t_{rs}\ \delta_r \otimes \delta_s^* .$$

One can represent states q^t and q^0 by operators ρ^t and ρ^0 resp. and then we obtain that

$$\rho^t = U^t \rho^0 U^{t*} .$$

For the pure state $\rho^0 = \psi^0 \otimes \psi^{0*}$ we obtain $\rho^t = (U^t \psi^0) \otimes (U^t \psi^0)^* = \psi^t \otimes \psi^{t*}$ where $\psi^t = U^t \psi^0$.

[7] This updating formula is analogous to the updating formula in the classical probability theory.

For the composition $T = M \oplus S$ of two systems we obtain $H_T = H_M \otimes H_S$ and $\rho_T = \rho_M \otimes \rho_S$.

For the probability we have

$$\text{prob}\,(s \mid \rho) = q\,(s, s) = \langle \delta_s, \rho(\delta_s) \rangle$$

$$\text{prob}\,(s \in A \mid \rho) = \sum \{\langle \delta_s, \rho(\delta_s) \rangle \mid \delta_s \in [A]\} = \text{Tr}_{[A]}\,(\rho)$$

where [A] is the subspace of H_S generated by δ_s , $s \in A$.

Evidently we have the update rule for the density operator

$$\rho' = P_{[A]}\, \rho\, P_{[A]}\, N_A^{-1} .$$

4. The intrinsic measurement process.

The standard measurement is specified by the orthogonal base $\{ \phi_i \mid i = 0, 1, \ldots, n\text{-}1 \}$ of the Hilbert space H_S of the system S.

We shall consider a measuring system M (it exists on the base of Axiom 7) satisfying dim H_M = n. In the set D_M we shall choose an element m_0 which will be the initial state of the measuring system.

We shall assume that the initial state of the measured system S will be

$$\Phi = \sum b_i\, \phi_i \in H_S, \quad \text{i.e. } \rho_S = \Phi \otimes \Phi^* .$$

Then the state of the total system $T = M \oplus S$ will be

$$\Psi = \delta(m_0) \otimes \Phi , \quad \text{i.e. } \rho_T = \rho_M \otimes \rho_S , \text{ where } \rho_M = \delta(m_0) \otimes \delta(m_0)^* .$$

The internal measurement process consists in two steps:

(i) The unitary map U is applied to the state Ψ such that this map transforms the vector $\delta(m_0) \otimes \phi_i$ onto vector $\delta(m_i) \otimes \phi_i$, $i = 0, 1, \ldots, n\text{-}1$ and then the state ρ_T will be transformed onto the state $\rho_T' = U\, \rho_T\, U^*$. The detailed description of the map U will be given below.

(ii) We observe the system T in the state ρ_T' and we found that the individual state of the measuring system M is $\delta(m_k)$. We shall introduce the set

$$A_k = \{ (m, s) \in D_T \mid m = m_k \} = \{ m_k \} \times D_S .$$

We see that the observation of the system M in the state $\delta(m_k)$ is equivalent to the observation stating that the individual state of the system T lies in the set A_k .

Using Axiom 5 we can calculate the probability of an event that we observe the measuring system M in the individual state m_k . Using then Axiom 6 we can calculate what it will be the new state of the system T: it has the form $\delta(m_k) \otimes \phi_k$.

Now we shall describe the full definition of the map U. We can assume that the set D_M can be written in the form

$$D_M = \{ m_0, m_1, \ldots, m_n \} .$$

Then the unitary transformation U is defined by

$$U (\delta(m_j) \otimes \phi_i) = \delta(m_{i\oplus j}) \otimes \phi_i , \quad i, j = 0, \ldots, n\text{-}1$$

where $i\oplus j = i+j$ if $i+j \leq n-1$ and $i\oplus j = i+j-n$ if $i+j \geq n$.

5. The comparison between the standard QM and the modified QM.

We assert that the standard QM and the modified QM defined above produce the same empirical predictions. To prove this it is necessary to show that

(i) The probability to obtain the output a_k in the standard QM is equal to the probability of finding the measuring system in the individual state m_k
(ii) The new state of the measured system S obtained after the measurement is the same in both variants of QM

Both these assertions are proved in the paper [3] by doing the explicit calculations.

Thus both theories are empirically equivalent. But they are surely theoretically different (see [4]) since these theories contain different sets of theorems.

We shall now describe the fundamental differences between these theories:

(i) The main difference consists in the concept of the individual states. In the standard QM it is (automatically) assumed that each pure state describes the possible state of some individual system. This is not true in the modified QM since here individual states form only certain orthogonal base of the Hilbert space. As a consequence the superposition principle for individual states is true in the standard QM while it is false in the modified QM.
(ii) In the standard QM the evolution of an individual state is deterministic, while in the modified QM the evolution of an individual state is non-deterministic, i.e. random. This is a consequence of the different concepts of an individual state in these theories.

(iii) The origin of randomness in these two theories is different. In the modified QM the randomness is concentrated in the evolution (as in the Brownian motion), while in the standard QM the randomness is concentrated in the axiom of measurement.

(iv) The basic difference is given by the definition: in the modified QM there is no measurement axiom. This means that there is also no concept of the observable. But there is a new concept of an individual state of the measuring system. The concept of the measurement is replaced by the concept of the observation of an individual state of the measuring system[8].

(v) The unitary transformation U (see the description of the internal measurement process) of the total system T changes the state of T and as a consequence the state of the measured system S is changed. This explains naturally the collapse of the state of the measured system S.

(vi) The von Neumann's infinite chain of measurements (M_2 measures M+S, M_3 measures M_2+M+S, etc.) is cut by the observation of the individual state of the measuring system.

(vii) The problem of the superposition of states of the measuring system is solved since in the modified QM the individual superposition principle is false[9] - in fact, no nontrivial superposition of individual states is an individual state (see [2]).

The difference between the above QM and the modified QM from [3] consists in Axiom 7 which is weaker than the corresponding axiom in [3].

6. Conclusions.

The axiomatization of QM which does not use the axioms on the measurement is presented. The concept of an observation of the individual state of a measuring apparatus replaces the concept of the measurement. It was proved that this new theory is empirically equivalent to the standard QM. The internal measurement process inside this new QM was described in all details. It was shown that the probabilities of outputs of the measurement are the same as in the standard QM.

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[8] We think that the concept of the observable is the most problematic concept in QM. It stays at the origin of most problems in QM. At the end what we really observe is the individual state of the individual measuring system – this is the only fact which is sure. We do not observe directly the probability of an output, but we observe individual outputs from which these probabilities are calculated.

[9] There may perhaps exist an ensemble of alive and dead cats, but the individual superposition between alive and dead cats does not exist in the modified QM.

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