Generators of Quantum Fields
Gravitation and Fermion Sector

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Abstract

G matrices are matrices with single entries 1,i. It is shown that linear sums of G matrices generate the SU(n), SO(n) generators. The Kronecka sums of the unitary representations of so(n) and su(n) results in an expression for the dimension of a SU(N) group. The spatial dimension n is found to be restricted to 3 dimensions. It is shown that the allowable unitary representations of so(3), are the spin spaces of spin ½. It is shown that there are 3 generations of chiral electroweak doublets of quarks and leptons. Exponentiation of differential operators acting on spinors results in the Lagrangians of Gravitation with Dark Energy and the Fermion sector of the Standard Model.

Introduction

The dimension of Space-Time and the Standard Model (SM) gauge groups require explanation. The approach here is to not assume the existence of physical quantities but to translate mathematical quantities to physical quantities. N dimensional manifolds are also not assumed to be the basis for a description of nature.

It is recognised that any matrix can be expanded as a sum of single entry matrices: G matrices with elements $\pm 1, \pm i$ hence the following axiom is proposed:

Axiom

The set of matrices $G = \{G_a\}$ called generators with single-entries $\pm 1, \pm i$ over the field $\mathbb{C}$ of complex scalars which are assumed to be random variables form the basis for a fundamental description of nature.

1 Space-Time & Standard Model Lie groups

The generators G_a satisfy the anti-commutation relation

$$\tfrac{1}{2}\{G_a^\dagger, G_b\} = \delta_{ab} M_{ab} \quad \textbf{(1)}$$

M are single-entry matrices with entry +1. The general state is $G = \beta_b G_b$

Expectation values for a random matrix S, where $S = \beta_a G_a$ from the general state G is

$$\langle S \rangle = \langle \beta_a G_a | \beta_b G_b \rangle \quad \textbf{(2)}$$

Let $\langle S \rangle$ be a generator of the Lie algebras $so(n)$ or $su(n)$. A generalised Kronecka sum of the $so(n)$ and $su(n)$ generators are the set of matrices

$$\{F_k\} = \{so(n) \otimes su(n)\} \quad \textbf{(3)}$$

$$\{F_k\} = \{\beta_{ab} G_a \otimes I_p + I_q \otimes \gamma_{cd} G_c\}$$

The generators of $so(n)$ and $su(n)$ are expanded using the G matrices.

The matrices I_p and I_p are by definition such that $p, q \geq 2$. The special case where F_k are linear combinations of the generators of $SU(N)$ for unitary I_p and I_p

$$F_k = a_{ki} T_i \tag{4}$$

The set of matrices $\{F_k\}$ are elements of the Lie algebra $su(N)$ with dimension $d = \dim su(N)$:

$$d = \dim(so(n)) \dim(su(n)) \tag{5}$$

$$d = \frac{1}{2} n(n-1)(n^2-1) \tag{6}$$

Only 1 solution to **(6)**, $n = 3$. It is conjectured that there are no other solutions to **(6)** with $d = \dim su(N)$ hence Space is a maximum of 3d.

$\boldsymbol{n = 3}$, **(6)** results in $d = 24$ and $N = 5$. The $su(3)$ generators are 3×3 matrices and since $su(5)$ generators are 5×5 matrices it follows that the $so(3) \otimes I_p$ and $I_q \otimes su(3)$ matrices are 5×5. The unitary representations of $so(3)$ are $m \times m$ matrices, thus $m + p = 5$, $m \in \{2,3\}$, thus the allowable spin representations of $so(3)$ are $s \in \{½, 1\}$ The $su(3)$ generators form the Lie algebra of QCD.

$$\begin{array}{c} so(3) \otimes I_p - spin \frac{1}{2}\ ,8\ \ 3 \times 3\ \ su(3) \\ so(3) \otimes I_p\ spin\ 1\ 8\ 2 \times 2\ is\ 2\ u(1) \times su(2) \end{array} \tag{7}$$

The generators of the $u(1) \times su(2)$ are $\{T_1^a, T_2^a\}$ form a doublet under SU(2). Let e_g be the basis vector for su(2) space.

$$T = \begin{pmatrix} T_1^a \\ T_2^a \end{pmatrix}_g e_g \tag{8}$$

Since su(2) is 3d then it follows that there are 3 electroweak chiral doublets of quarks and leptons. Hence there are 3 generations of quarks and leptons.

The coefficients $\beta_{ab}\ \gamma_{cd}$ are random, so the matrices F can transit to the generators of spin space $su(2)$ and Lie algebras $su(3)$, $u(1) \times su(2)$

$$F \rightarrow su(2), su(3), u(1) \times su(2) \tag{9}$$

The 5x5 su(5) matrices act on 5d spinors. The largest gamma matrices that can form from the F matrices are 4x4. Since Space is 3d the extra gamma matrix is orthogonal to Space ie basis for Time. Hence

$$so(3) \rightarrow so(1,3) \quad \gamma^\mu = \{\gamma^0, \gamma^i\} \tag{10}$$

where γ^μ are the 4x4 Dirac-Gamma matrices.

2 Dynamics

Exponential map of a matrix A is

$$exp: t, A \to e^{tA} \tag{11}$$

where t is a parameter. Extend (11) to the differential operators $L\nabla_{\mu_a}$ where ∇_μ is the gauge covariant derivative D_μ

$$\begin{gathered} \nabla_\mu(e_\lambda\psi) = e_\lambda D_\mu\psi - \Gamma^\sigma_{\lambda\mu}e_\sigma\psi \\ D_\mu = \partial_\mu + i\partial_\mu(\omega T) \end{gathered} \tag{12}$$

Let $\nabla_{\mu_a} \equiv L\nabla_{\mu_a}$ the exponential differential operator $\mathbb{D}$ is

$$\begin{gathered} \mathbb{D} = e^{\epsilon\prod i_a e_{\mu_a} + i_a\nabla_{\mu_a}} \\ [i_a, i_b] = 0,\, i_a i_a = i_a i_b = 1 \\ \epsilon \in \mathbb{R} \end{gathered} \tag{13}$$

For 2 operators expanding (13) using the Baker-Campbell-Hausdoff formula gives

$$\mathbb{D} = \left\{\epsilon\prod i_a e_{\mu_a} + i_a\nabla_{\mu_a} + \frac{i_1 i_2}{2}\left[\nabla_\mu, \nabla_\nu\right] + \cdots\right\} \tag{14}$$

$$\mathbb{D}(e_\lambda\psi) = \left(\epsilon e_\mu e_\nu e_\lambda\psi + i_a\nabla_{\mu a}(e_\lambda\psi) + \frac{1}{2}R^\sigma_{\lambda\mu\nu}e_\sigma\psi + \frac{i}{2}F_{\mu\nu}e_\lambda\psi + \cdots\right) \tag{15}$$

$$\left(\epsilon e_\sigma + i_a\nabla^{\mu a}(e_\mu.) + e_\sigma R + ie^\mu F_{\mu\sigma}\right)\psi \tag{16}$$

(16) is obtained from (15) by transforming (15) to a covariant expression and using the contravariant metric to contract the tensor indices. The operator $\mathbb{D}(e_\lambda\psi^\dagger)$ with $\nabla_\mu \to \nabla^{\dagger\mu}$ applied to the conjugate spinor $\psi^\dagger$, the conjugate of (16) is

$$\left(\epsilon e^\sigma + i_a\nabla^{\dagger\mu a}(e_\mu.) + e^\sigma R - ie^\mu F^\sigma_\mu\right)\psi^\dagger \tag{17}$$

The product of (17) and (16) after manipulating tensor indices and noting that

$$i_a\nabla^{\dagger\mu a}(\psi^\dagger)i_a\nabla_{\mu a}(\psi) = P^{\mu a}P_{\mu a} = 0 \tag{18}$$

Results in the following Lagrangian

$$\mathcal{L} = \psi^\dagger\left\{\epsilon^2 + 2\epsilon R + \left(R^2 + \frac{1}{N}F^{\mu\nu}F_{\mu\nu}\right)\right\}\psi \tag{19}$$

Taking account that It $\nabla_{\mu_a} \equiv L\nabla_{\mu_a}$ follows that $\Lambda = \frac{\epsilon}{L^2}$ and $\kappa = \frac{\Lambda L^4}{\epsilon\hbar c} = \frac{L^2}{\hbar c}$

The Lagrangian for the spinor ψ is

$$\mathcal{L} = \bar{\psi} i_a \nabla^{\mu a} (e_\mu \psi) \tag{20}$$

The spinor ψ is a mixed state of 3 generations as a consequence of (7).
(19) and (20) are the Lagrangians for the Gravitational and Fermion sectors.

Conclusion

The G matrices form the generators of the SO(n) and SU(n) groups. The Kronecker sums of the so(n) and su(n) leads to the dimension of space to be 3. The result is 3 generations of doublets of chiral quarks and leptons. Exponentiation of the differential operators enables the Lagrangian of Fermions coupled to gravity and Lagrangian of Gravitation with dark energy to be formulated.