# Distributional Spacetime in Classical and Quantum Cosmology.

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**Abstract**:Distributional spacetime in classical and quantum cosmology are considered. Distributional quantum spacetime with distributional Ashtekar connection also are in detail considered.

# 1.Introduction

# 2. Distributional Spacetime in Classical FRW Cosmology

## 2.1.The Problem of Initial Singularity in Classical Cosmology

Let us consider now the classical Robertson-Walker simplest generic metric

$$ds^2 = -dt^2 + a^2(t)dx^2. \tag{2.1.1}$$

Note that in classical cosmology one usually assumes space to be homogeneous and isotropic, which is an excellent approximation on large scales today. The metric of space is then solely determined by the scale factor $a(t)$ which gives the size of the universe at any given time $t$. The function $a(t)$ describes the expansion or contraction of space in a way dictated by the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(a) \tag{2.1.2}$$

which is the reduction of Einstein's equations under the assumption of isotropy. In this equation, $G$ is the gravitational constant and $\rho(a)$ the energy density of whatever matter we have in the universe. Once the matter content is chosen and $\rho(a)$ is known, one can solve the Friedmann equation in order to obtain $a(t)$.

As an example we consider the case of radiation which can be described phenomenologically by the energy density $\rho(a) \propto a^{-4}$. This is only a phenomenological description since it ignores the fundamental formulation of electrodynamics of the Maxwell field. Instead of using the Maxwell Hamiltonian in order to define the energy density, which would complicate the situation by introducing the electromagnetic fields with new field equations coupled to the Friedmann equation, one uses the fact that on large scales the energy density of radiation is diluted by the expansion and in addition red-shifted. This leads to a behavior proportional $a^{-3}$ from dilution times $a^{-1}$ from redshift. In this example we then solve the Friedmann equation $\dot{a} \propto a^{-1}$ by

$$a(t) \propto \sqrt{t-t_0} \tag{2.3}$$

with a constant of integration $t_0$. Then Robertson-Walker generic metric (2.1) takes the form

$$ds^2 = -dt^2 + (t-t_0)dx^2. \tag{2.4}$$

**Remark 2.1**. Note that the generic metric (2.4) is degenerate at instant $t = t_0$.

**Remark 2.2**. This solution given by Eq.(2.3) demonstrates the occurrence of singularities
in classical Friedmann cosmology:

(**I**) For any solution there is a time $t = t_0$ where the size of space vanishes and the
energy density $\rho(a(t_0))$ diverges, i.e.

$$\rho(t_0) = \infty. \tag{2.5}$$

(**II**) At this point not only the matter system becomes unphysical, but also the gravitational
evolution breaks down: when the right hand side of Eq.(2.2) diverges at some time $t = t_0$,
we cannot follow the evolution further by setting up an initial value problem there and
integrating the equation.

**Remark 2.3**. Note that the occurrence of singularity at some time $t = t_0$ in respect to
degeneracy of generic metric (2.4) at the same time $t = t_0$.

We can thus only learn that there is a singularity in the classical theory, but do not obtain any information as to what is happening there and beyond. These are the two related but not identical features of a singularity: energy densities diverge and the evolution breaks down.

One could think that the problem comes from too strong idealizations such as symmetry assumptions or the phenomenological description of matter. That this is not the case follows from the singularity theorems which do not depend on these assumptions. One can also illustrate the singularity problem with a field theoretic rather than phenomenological description of matter. For simplicity we now assume that matter is provided by a scalar $\phi$ whose energy density then follows from the Hamiltonian

$$\rho(a) = a^{-3}H(a) = a^{-3}\left(\frac{1}{2}a^{-3}p_\phi^2 + a^3V(\phi)\right) \tag{2.6}$$

with the scalar momentum $p_\phi$ and potential $V(\phi)$. At small scale factors $a$, there still is a diverging factor $a^{-3}$ in the kinetic term which we recognized as being responsible for the singularity before. Since this term dominates over the non-diverging potential term, we still cannot escape the singularity by using this more fundamental description of matter. This is true unless we manage to arrange the evolution of the scalar in such a way that $p_\phi \to 0$ when $a \to 0$ in just the right way for the kinetic term not to diverge. This is difficult to arrange in general, but is exactly what is attempted in slow-roll inflation (though with a different motivation, and not necessarily all the way up to the classical singularity).

For the evolution of $p_\phi$ we need the scalar equation of motion, which can be derived from the Hamiltonian $H$ in Eq.(2.3) via $\dot{\phi} = \{\phi, H\}$ and $\dot{p}_\phi = \{p_\phi, H\}$. This results in the isotropic Klein–Gordon equation in a time-dependent background determined by $a(t)$,

$$\ddot{\phi} + 3\dot{a}a^{-1}\dot{\phi} + V'(\phi) = 0\,. \tag{2.7}$$

In an expanding space with positive $\dot{a}$ the second term implies friction such that, if we assume the potential $V'(\phi)$ to be flat enough, $\phi$ will change only slowly (slow-roll). Thus, $\dot{\phi}$ and $p_\phi = a^3\dot{\phi}$ are small and at least for some time we can ignore the kinetic term in the energy density. Moreover, since $\phi$ changes only slowly we can regard the potential $V(\phi)$ as a constant $\Lambda$ which again allows us to solve the Friedmann equation with $\rho(a) = \Lambda$. The solution $a \propto \exp(\sqrt{8\pi G\Lambda/3}\; t)$ is inflationary since $\ddot{a} > 0$ and non-singular: $a$ becomes zero only in the limit $t \to -\infty$.

Thus, we now have a mechanism to drive a phase of accelerated expansion important for observations of structure. However, this expansion must be long enough, which means that the phase of slowly rolling $\phi$ must be long. This can be achieved only if the potential is very flat and $\phi$ starts sufficiently far away from its potential minimum. Flatness means that the ratio of $V(\phi_{\text{initial}})$ and $\phi_{\text{initial}}$ must be of the order $10^{-10}$, while $\phi_{\text{initial}}$ must be huge, of the order of the Planck mass. These assumptions are necessary for agreement with observations, but are in need of more fundamental explanations.

Moreover, inflation alone does not solve the singularity problem. The non-singular solution we just obtained was derived under the approximation that the kinetic term can be ignored when $\dot{\phi}$ is small. This is true in a certain range of $a$, depending on how small $\dot{\phi}$ really is, but never very close to $a = 0$. Eventually, even with slow-roll conditions, the diverging $a^{-3}$ will dominate and again lead to a singularity.

## 2.2. Distributional Spacetime in Classical FRW Cosmology

the term ensures its non-degeneracy

As pointed out above (see Remark 2.3) the problem of the singularity comes from degeneracy of the generic metric (2.4) at instant $t = t_0$.

Thus in order to avoiding the singularity in classical cosmology one can applay

Let us consider now the regularized metric (2.4) under condition $t \geq t_0$ :

$$
\begin{gathered}
ds_{\varepsilon}^{2} = -dt^{2} + (t - t_{0} + \varepsilon)dx^{2}, \\
\varepsilon \in (0,1], t \geq t_{0}.
\end{gathered} \tag{2.8}
$$

From metric (2.8) one obtains the Colombeau distributional metric

Let us consider now the Colombeau distributional RW metric

$$
\begin{gathered}
(ds_{\varepsilon}^{2})_{\varepsilon} = -dt^{2} + \{(a_{\varepsilon}^{2}(t))_{\varepsilon}\}dx^{2}, \\
\varepsilon \in (0,1], (a_{\varepsilon}(t))_{\varepsilon} \in G(\mathbb{R}).
\end{gathered} \tag{2.5}
$$

If we now insert Eq.(4.8) into the generalized Einstein Equations (3.39), then we obtain, after some simple algebra, two equations:

$$
\begin{aligned}
\left(\left(\frac{\dot{a}_{\varepsilon}}{a_{\varepsilon}}\right)_{\varepsilon}\right)^{2} &= \frac{8\pi G}{3}(\rho_{\varepsilon}(a_{\varepsilon}))_{\varepsilon} - kc^{2}(a_{\varepsilon}^{-2}), \\
\left(\frac{\ddot{a}_{\varepsilon}}{a_{\varepsilon}}\right)_{\varepsilon} &= \frac{4\pi G}{3}(\rho_{\varepsilon}(a_{\varepsilon}) + 3c^{-2}p_{\varepsilon}(a_{\varepsilon}))_{\varepsilon}.
\end{aligned} \tag{2.6}
$$

are the generalized Friedmann Equations for modeling a homogeneous, isotropic universe by using Colombeau distributional geometry. For $k = 0$ one obtains

$$
\left(\left(\frac{\dot{a}_{\varepsilon}}{a_{\varepsilon}}\right)_{\varepsilon}\right)^{2} = \frac{8\pi G}{3}(\rho_{\varepsilon}(a_{\varepsilon}))_{\varepsilon}). \tag{2.7}
$$

As an example we consider the case of radiation which can be described by the energy density $\rho_{\varepsilon}(a_{\varepsilon}) \propto a^{-4}$.

# 3. Distributional Spacetime in Wheeler–DeWitt Quantum Cosmology.

## 3.1.The Problem of Initial Singularity in Wheeler–DeWitt Quantum Cosmology.

Since the isotropic reduction of general relativity leads to a system with finitely many degrees of freedom, one can in a first attempt try quantum mechanics to quantize it. Starting with the Friedmann equation (ref: Friedmann) and replacing $\dot{a}$ by its momentum $p_a = 3a\dot{a}/8\pi G$ gives a Hamiltonian which is quadratic in the momentum and can be quantized easily to an operator acting on a wave function depending on the gravitational variable $a$ and possibly matter fields $\phi$. The usual Schrödinger representation yields classical the Wheeler–DeWitt equation

$$
\frac{3}{2}\left(-\frac{1}{9}\ell_{\mathrm{P}}^{4}a^{-1}\frac{\partial}{\partial a}a^{-1}\frac{\partial}{\partial a}\right)a\psi(a,\phi) = 8\pi G\hat{H}_{\phi}(a)\psi(a,\phi) \tag{3.1}
$$

with the matter Hamiltonian $\hat{H}_{\phi}(a)$. This system is different from usual quantum mechanics in that there are factor ordering ambiguities in the kinetic term, and that there is no derivative with respect to coordinate time $t.$ The latter fact is a consequence of general covariance: the Hamiltonian is a constraint equation restricting allowed states $\psi(a,\phi)$, rather than a Hamiltonian generating evolution in coordinate time. Nevertheless,vone can interpret Eq.(3.1) as an evolution equation in the scale factor $a$, which is then called internal time. The left hand side thus becomes a second order time derivative, and it means that the evolution of matter is measured relationally. Straightforward quantization thus gives us a quantum evolution equation, and we can now check what this implies for the singularity. If we look at the equation for $a = 0$, we notice first that the matter Hamiltonian still leads to diverging energy densities. If we quantize (ref: Hscalar), we replace $p_\phi$ by a derivative, but the singular dependence on $a$ does not change; $a^{-3}$ would simply become a multiplication operator acting on the wave function. Moreover, $a = 0$ remains a singular point of the quantum evolution equation in internal time. There is nothing from the theory which tells us what physically happens at the singularity or beyond.

# 4. Distributional Quantum Spacetime in Loop Quantum cosmology

## 4.1. Distributional Bianchi, LRS and Isotropic Models

The setting for implementing a (quantum) symmetry reduction is a symmetry group $S$ acting on a principal fiber bundle $P(\Sigma, G, \pi)$ over the space manifold $\Sigma$ which is here assumed to be compact (this is only for ease of presentation, otherwise the framework has to be adapted appropriately). The structure group is $G = SU(2)$ for gravity in the real Ashtekar formulation cite: AshVar,AshVarReell. A classical symmetry reduction can be done, in the most general framework, by using the classification of invariant connections cite: KobNom, which shows that for a transitive symmetry group each invariant connection can be expressed by some scalar fields (collectively called Higgs field) subject to a Higgs constraint. This constraint is empty for a free action of the group $S$, and depends on a homomorphism $\lambda: F \to G$ (more precisely, its conjugacy class) if the isotropy subgroup $F$ (for a fixed but arbitrary base point $x_0$ in $\Sigma$) of $S$ is nontrivial. The space manifold $\Sigma$ can be identified with $S/F$ or an appropriate compactification thereof. This framework is specialized to cosmological models in reference[], and its results will now be recalled briefly.

The models of interest are Bianchi class A models with a freely acting symmetry group, i.e. $F = \{1\}$, and LRS and isotropic models, for which the symmetry group can be written as a semidirect product $S = N \rtimes_\rho F$ with the translation subgroup $N$ and the isotropy subgroup $F$. The representation $\rho: F \to \mathbf{Aut}(\mathrm{N})$ describes how the isotropy subgroup acts on the tangent space $\mathcal{L}N$ of a point in $\Sigma$. For LRS models we have $F = U(1)$ and for isotropic models $F = SU(2)$, $\rho$ acting in both cases by rotations. An invariant connection can always be written as $A = \phi_I^i \omega^I \tau_i$, where $\tau_j = -\frac{i}{2}\sigma_j$ (using the Pauli matrices $\sigma_j$) are generators of $G = SU(2)$ and $\omega^I$ left invariant one-forms on $N$ (for Bianchi models we set $N := S$). The components $\phi_I^i$ of a linear map $\phi$: $\mathcal{L}N \to \mathcal{L}G$ are collectively denoted as Higgs field. For Bianchi

models these components are unrestricted, whereas they are restricted by the Higgs constraint to be of the form

$$(\phi^i_{\varepsilon,1})_\varepsilon = 2^{-\frac{1}{2}}(a\Lambda^i_{\varepsilon,1} + b\Lambda^i_{\varepsilon,2})_\varepsilon, (\phi^i_{\varepsilon,2})_\varepsilon = 2^{-\frac{1}{2}}(-b\Lambda^i_{\varepsilon,1} + a\Lambda^i_{\varepsilon,2})_\varepsilon, (\phi^i_{\varepsilon,3})_\varepsilon = (c\Lambda^i_{\varepsilon,3})_\varepsilon$$

for LRS models and $\phi_I^i = c\Lambda^i_{\varepsilon,I}$ for isotropic models, respectively, with a fixed but arbitrary dreibein $(\Lambda_\varepsilon)_\varepsilon, \varepsilon \in (0,1]$. The dreibein $(\Lambda_\varepsilon)_\varepsilon$ depends on the homomorphism $\lambda: F \to G$ chosen in its conjugacy class. Fixing such a homomorphism and, therefore, $(\Lambda_\varepsilon)_\varepsilon$ amounts to a partial gauge fixing which will be undone in the quantum theory. Without gauge fixing $(\Lambda_\varepsilon)_\varepsilon$ is arbitrary but pure gauge.The momenta conjugate to the connections above are invariant (with respect to the $S$-action) density-weighted dreibeine given by

$(E_i^a(\varepsilon))_\varepsilon = \left(\sqrt{g_0(\varepsilon)}\, p_i^I X^a_{\varepsilon,I}\right)_\varepsilon$ in terms of left invariant vector fields $(X_{\varepsilon,I})_\varepsilon$ obeying $(\omega^I_\varepsilon(X_{\varepsilon,J}))_\varepsilon = \delta^I_J, \varepsilon \in (0,1]$. For Bianchi models the $(p^I_{\varepsilon,i})_\varepsilon$ are arbitrary and conjugate to $(\phi^i_{\varepsilon,I})_\varepsilon$, whereas they are restricted to be of the form

$$(p^1_{\varepsilon,i})_\varepsilon = 2^{-\frac{1}{2}}(p_a\Lambda^i_{\varepsilon,1} + p_b\Lambda^i_{\varepsilon,2})_\varepsilon, (p^2_{\varepsilon,i})_\varepsilon = 2^{-\frac{1}{2}}(-p_b\Lambda^i_{\varepsilon,1} + p_a\Lambda^i_{\varepsilon,2})_\varepsilon, (p^3_{\varepsilon,i})_\varepsilon = (p_c\Lambda^i_{\varepsilon,3})_\varepsilon$$

for LRS, and $(p^I_{\varepsilon,i})_\varepsilon = (p\Lambda^I_{\varepsilon,i})_\varepsilon$ for isotropic models. The density weight is provided by the determinant $(g_0(\varepsilon))_\varepsilon$ of the left invariant Colombeau generalized metric on $\Sigma$ defined by

$$\left(\omega^1_\varepsilon \wedge \omega^2_\varepsilon \wedge \omega^3_\varepsilon\right)_\varepsilon = \left(\sqrt{g_0(\varepsilon)}\, \mathrm{d}^3x\right)_\varepsilon, \varepsilon \in (0,1].$$

From these momenta expressions for the volume are built as follows:

$$\begin{gathered}(V_\varepsilon)_\varepsilon = \left(\int_\Sigma \mathrm{d}^3x \sqrt{\frac{1}{6}|\epsilon^{ijk}\epsilon_{abc}E_i^aE_j^bE_k^c|}\right)_\varepsilon = \\ \left(\int_\Sigma \mathrm{d}^3x \sqrt{\frac{1}{6}g_0^{\frac{3}{2}}(\varepsilon)|\epsilon^{ijk}\epsilon_{abc}p_i^Ip_j^Jp_k^KX_I^aX_J^bX_K^c|}\right)_\varepsilon = \\ \left(\int_\Sigma \mathrm{d}^3x \sqrt{\frac{1}{6}g_0^{\frac{3}{2}}(\varepsilon)|\epsilon^{ijk}\epsilon_{IJK}p_i^Ip_j^Jp_k^K \det(X_L^a)|}\right)_\varepsilon = \\ \left(V_0(\varepsilon)\sqrt{\frac{1}{6}|\epsilon^{ijk}\epsilon_{IJK}p_i^Ip_j^Jp_k^K|}\right)_\varepsilon\end{gathered} \tag{4.1.1}$$

using $(\det X^a_{\varepsilon.I})_\varepsilon = (\det\omega^I_{\varepsilon.a})^{-1}_\varepsilon = \left(g_0^{-1/2}(\varepsilon)\right)_\varepsilon$ and $(V_0(\varepsilon))_\varepsilon \triangleq \left(\int_\Sigma \mathrm{d}^3x\sqrt{g_0(\varepsilon)}\right)_\varepsilon$. For LRS models this leads to

$$(V_\varepsilon)_\varepsilon = \left[\sqrt{\frac{1}{2}(p_a^2 + p_b^2)|p_c|}\right](V_0(\varepsilon))_\varepsilon, \tag{4.1.2}$$

and for isotropic models to

$$(V_\varepsilon)_\varepsilon = \left[|p|^{\frac{3}{2}}\right](V_0(\varepsilon))_\varepsilon. \tag{4.1.3}$$

The basic ingredient for a quantum symmetry reduction is a pull back map from the space of functions on the space of connections on $\Sigma$, which is the auxiliary Hilbert space of the full theory, to a space of functions on the space of fields classifying invariant connections, i.e. to functions on spaces of Higgs fields. In quantum theory one uses certain extensions of the spaces of

## 4.2.The Problem of Initial Singularity in Loop Quantum cosmology

Singularities are physically extreme and require special properties of any theory aimed at tackling them. First, there are always strong fields (classically diverging) which requires a non-perturbative treatment. Moreover, classically we expect space to degenerate at the singularity, for instance a single point in an isotropic model. This means that we cannot take the presence of a classical geometry to measure distances for granted, which is technically expressed as background independence. A non-perturbative and background independent quantization of gravity is available in the form of loop quantum gravity, which by now is understood well enough in order to be applicable in physically interesting situations.Here, we only mention salient features of the theory which will turn out to be important for cosmology. The first one is the kind of basic variables used, which are the Ashtekar connection describing the curvature of space and a densitized triad describing the metric by a collection of three orthonormal vectors in each point. These variables are important since they allow a background independent representation of the theory, where the Colombeau generalized connection $(A^i_{\varepsilon,a})_\varepsilon = (A^i_a(\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ is integrated to holonomies

$$(h_e(A_\varepsilon))_\varepsilon = \left(\mathcal{P}\exp\int_e A^i_{\varepsilon,a}\tau_i \dot{e}^a \mathrm{d}t\right)_\varepsilon \tag{4.2.1}$$

along curves $e$ in space and the Colombeau generalized densitized triad $(E^a_{\varepsilon,i})_\varepsilon$ to fluxes

$$(F_S(E_\varepsilon))_\varepsilon = \left(\int_S E^a_{\varepsilon,i}\tau^i n_a \mathrm{d}^2y\right)_\varepsilon \tag{4.2.2}$$

along surfaces $S$. (In these expressions, $\dot{e}^a$ denotes the tangent vector to a curve and $n_a$ the conormal to a surface, both of which are defined without reference to a background metric. Moreover, $\tau_j = -\frac{1}{2}i\sigma_j$ in terms of Pauli matrices). While usual quantum field theory techniques rest on the presence of a background metric, for instance in order to decompose a field in its Fourier modes and define a vacuum state and particles, this is no longer available in quantum gravity where the metric itself must be turned into an operator. On the other hand, some integration is necessary since the fields themselves are distributional in quantum field theory and do not allow

Before discussing the quantum level we reformulate isotropic cosmology in connection and triad variables instead of $(a)_\varepsilon$. The role of the scale factor is now played by the densitized triad component $(p_\varepsilon)_\varepsilon$ with $(|p_\varepsilon|)_\varepsilon = (a_\varepsilon^2)_\varepsilon$ whose canonical momentum is the isotropic connection component $(c_\varepsilon)_\varepsilon = -\frac{1}{2}(\dot{a})_\varepsilon$ with $[(\{c_\varepsilon, p_\varepsilon\})_\varepsilon] = \{[(c_\varepsilon)_\varepsilon], [(p_\varepsilon)_\varepsilon]\} = 8\pi G/3$. The main difference to metric variables is the fact that $(p_\varepsilon)_\varepsilon$, unlike $a$, can take both signs with $\operatorname{sgn}[(p_\varepsilon)_\varepsilon]$ being the orientation of space. This is a consequence of having to use triad variables which not only know about the size of space but also its orientation (depending on whether the set of orthonormal vectors is left or right handed).

States in the full theory are usually written in the connection representation as functions of holonomies. Following the reduction procedure for an isotropic symmetry group leads to orthonormal states which are functions of the isotropic connection component $(c_\varepsilon)_\varepsilon$ and given by

$$(\langle c_\varepsilon|\mu_\varepsilon\rangle)_\varepsilon = (\exp(i\mu_\varepsilon c_\varepsilon/2))_\varepsilon, (\mu_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}. \tag{4.2.3}$$

On these states the basic variables $(p_\varepsilon)_\varepsilon$ and $(c_\varepsilon)_\varepsilon$ are represented by

$$(\widehat{p}_\varepsilon|\mu_\varepsilon\rangle)_\varepsilon = \tfrac{1}{6}(\ell_\varepsilon^2\mu_\varepsilon|\mu_\varepsilon\rangle)_\varepsilon, \tag{4.2.4}$$

$$\left(\widehat{\exp(i\mu'_\varepsilon c_\varepsilon/2)|\mu_\varepsilon\rangle}\right)_\varepsilon = (|\mu_\varepsilon + \mu'_\varepsilon\rangle)_\varepsilon \tag{4.2.5}$$

with the properties:

**1**. $\left(\left[\widehat{\exp(i\mu'_\varepsilon c_\varepsilon/2)}, \widehat{p}_\varepsilon\right]\right)_\varepsilon = -\dfrac{1}{6}\left(\ell_\varepsilon^2\mu'_\varepsilon\widehat{\exp(i\mu'_\varepsilon c_\varepsilon/2)}\right)_\varepsilon = i\hbar\left(\widehat{\{\exp(i\mu'_\varepsilon c_\varepsilon/2), p_\varepsilon\}}\right)_\varepsilon$,

**2**. $(\widehat{p}_\varepsilon)_\varepsilon$ has a discrete spectrum in Colombeau sense and

**3**. only exponentials $(\exp(i\mu'_\varepsilon c_\varepsilon/2))_\varepsilon$ of $(c_\varepsilon)_\varepsilon$ are represented, not $(c_\varepsilon)_\varepsilon$ directly.

These statements deserve further explanation: First, the classical Poisson relations between the basic variables are indeed represented correctly, turning the Poisson brackets into commutators divided by $i\hbar$. On this representation, the set of eigenvalues of $(\widehat{p}_\varepsilon)_\varepsilon$ is the full Colombeau algebra $\widetilde{\mathbb{R}}$ (Colombeau real line) since $(\mu_\varepsilon)_\varepsilon$ can take arbitrary values from $\widetilde{\mathbb{R}}$. Nevertheless, the spectrum of $(\widehat{p}_\varepsilon)_\varepsilon$ is discrete in the technical sense that eigenstates of $(\widehat{p}_\varepsilon)_\varepsilon$ are normalizable. This is indeed the case in this non-separable Hilbert space where (4.2.3) defines an orthonormal basis. The last property follows since the exponentials are not continuous in the label $(\mu'_\varepsilon)_\varepsilon$, for otherwise one could simply take the derivative with respect to $\mu'$ at $\mu' = 0$ and obtain an operator for $c$. The discontinuity can be seen, e.g., from

$$(\langle\mu_\varepsilon|(\exp(i\mu'_\varepsilon c_\varepsilon/2))_\varepsilon|\mu_\varepsilon\rangle)_\varepsilon = (\delta_{0,\mu'_\varepsilon})_\varepsilon$$

which is not continuous.

These properties are quite unfamiliar from quantum mechanics, and indeed the representation is inequivalent to the Schrödinger representation (the discontinuity of the $c$-exponential evading the Stone–von Neumann theorem which usually implies uniqueness of the representation). In fact, the loop representation is inequivalent to the Wheeler-DeWitt quantization which just assumed a Schrödinger like quantization. In view of the fact that the phase space of our system is spanned by $(c_\varepsilon)_\varepsilon$ and $(p_\varepsilon)_\varepsilon$ with $(\{c_\varepsilon,p_\varepsilon\})_\varepsilon \propto 1$ just as in classical mechanics, the question arises how such a difference in the quantum formulation arises. As a mathematical problem the basic step of quantization occurs as follows: given the calassical Poisson algebra of Colombeau generalized observables $(Q_\varepsilon)_\varepsilon$ and $(P_\varepsilon)_\varepsilon$ with $(\{Q_\varepsilon,P_\varepsilon\})_\varepsilon = 1,$ how can we define a representation of the observables on a Hilbert space such that the Poisson relations become commutator relations and complex conjugation, meaning that $(Q_\varepsilon)_\varepsilon$ and $(P_\varepsilon)_\varepsilon$ are real, becomes adjointness? The problem is mathematically much better defined if one uses the bounded expressions $(e^{isQ_\varepsilon})_\varepsilon$ and $\left(e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon$ instead of the unbounded $(Q_\varepsilon)_\varepsilon$ and $(P_\varepsilon)_\varepsilon$, which still allows us to distinguish any two points in the whole phase space. The basic objects $(e^{isQ_\varepsilon})_\varepsilon$ and $\left(e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon$ upon quantization will then not commute but fulfill the commutation relation (Weyl algebra)

$$\left(e^{isQ_\varepsilon}e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon = e^{ist}\left(e^{it\hbar^{-1}P_\varepsilon}e^{isQ_\varepsilon}\right)_\varepsilon \tag{4.2.4}$$

as unitary operators on a Hilbert space. In the Schrödinger representation this is done by using a Hilbert space $L^2(\widetilde{\mathbb{R}},\mathrm{d}q)$ of square integrable Colombeau generalized functions $(\psi_\varepsilon(q))_\varepsilon$ with $\left(\int_{\mathbb{R}} \mathrm{d}q|\psi_\varepsilon(q)|^2\right)_\varepsilon < \widetilde{\infty}, \varepsilon \in (0,1]$. The representation of basic Colombeau generalized operators is

$$(e^{isQ_\varepsilon}\psi_\varepsilon(q))_\varepsilon = e^{isq}\left(\psi_\varepsilon(q), e^{it\hbar^{-1}P_\varepsilon}\psi_\varepsilon(q)\right)_\varepsilon = (\psi_\varepsilon(q+t))_\varepsilon \tag{4.2.5}$$

which indeed are unitary and fulfill the required commutation relation. Moreover, the operator families as functions of $s$ and $t$ are continuous and we can take the derivatives in $s = 0$ and $t = 0$, respectively:

$$\begin{gathered} -i\left(\frac{\mathrm{d}}{\mathrm{d}s}e^{isQ_\varepsilon}\right)_\varepsilon\Big|_{s=0} = (q_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}, \\ -i\hbar\left(\frac{\mathrm{d}}{\mathrm{d}t}e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon\Big|_{t=0} = (\hat{p}_\varepsilon)_\varepsilon = -i\hbar\left(\frac{\mathrm{d}}{\mathrm{d}q_\varepsilon}\right)_\varepsilon. \end{gathered} \tag{4.2.6}$$

This is the familiar representation of quantum mechanics which, according to the Stone–von Neumann theorem is unique under the condition that $(e^{isQ_\varepsilon})_\varepsilon$ and $\left(e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon$ are indeed continuous in both $s$ and $t$. The latter condition is commonly taken for granted in quantum mechanics, but in general there is no underlying physical or mathematical reason. It is easy to define representations violating continuity in $s$ or $t$, for instance if we use a Hilbert space $\ell^2(\widetilde{\mathbb{R}})$ where states are again maps $(\psi_{\varepsilon,q_\varepsilon})_\varepsilon, \varepsilon \in (0,1]$ from the Colombeau real line $\widetilde{\mathbb{R}}$ to Colombeau algebra $\widetilde{\mathbb{C}}$ of Colombeau complex numbers but with norm

$$\left(\sum_{q_\varepsilon}|\psi_{\varepsilon,q_\varepsilon}|^2\right)_\varepsilon$$

which implies that normalizable $(\psi_{\varepsilon,q})_\varepsilon$ can be non-zero for at most countably many

$[(q_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$. We obtain a representation with basic operators

$$(e^{is_\varepsilon Q}\psi_{\varepsilon,q_\varepsilon})_\varepsilon = (e^{is_\varepsilon q_\varepsilon}\psi_{\varepsilon,q_\varepsilon})_\varepsilon, \left(e^{it\hbar^{-1}P_\varepsilon}\psi_{q_\varepsilon}\right)_\varepsilon = (\psi_{q_\varepsilon+t_\varepsilon})_\varepsilon$$

which is of the same form as before. However, due to the different Hilbert space the second operator $\left[\left(e^{it\hbar^{-1}P_\varepsilon}\right)_\varepsilon\right]$ is no longer continuous in $t = [(t_\varepsilon)_\varepsilon] \in \widetilde{\mathbb{R}}$ which can be checked as in the case of $(e^{i\mu_\varepsilon c_\varepsilon/2})_\varepsilon$. In fact, the representation for $Q$ and $P$ is isomorphic to that of $p$ and $c$ used before, where a general state $(|\psi_\varepsilon\rangle)_\varepsilon = \left(\sum_{\mu_\varepsilon}\psi_{\varepsilon,\mu_\varepsilon}|\mu_\varepsilon\rangle\right)_\varepsilon$ has coefficients $(\psi_{\varepsilon,\mu_\varepsilon})_\varepsilon$ in $\ell^2(\widetilde{\mathbb{R}})$. This explains mathematically why different, inequivalent representations are possible, but what are the physical reasons for using different representations in quantum mechanics and quantum cosmology? In quantum mechanics it turns out that the choice of representation is not that important and is mostly being done for reasons of familiarity with the standard choice. Physical differences between inequivalent representations can only occur at very high energies which are not probed by available experiments and do not affect characteristic quantum effects related to the ground state or excited states. Thus, quantum mechanics as we know it can well be formulated in an inequivalent representation, and also in quantum field theory this can be done and even be useful.

In quantum cosmology we have a different situation where it is the high energies which are essential. We do not have direct observations of this regime, but from conceptual considerations such as the singularity issue we have learned which problems we have to face. The classical singularity leads to the highest energies one can imagine, and it is here where the question of which representation to choose becomes essential. As shown by the failure of the Wheeler–DeWitt quantization in trying to remove the singularity, the Schrödinger representation is inappropriate for quantum cosmology. The representation underlying loop quantum cosmology, on the other hand, implies very different properties which become important at high energies and can shed new light on the singularity problem.

Moreover, by design of the symmetric models as derived from the full theory, we have the same basic properties of a loop representation in cosmological models and the full situation where they were recognized as being important for a background independent quantization: discrete fluxes $\left(\hat{F}_S(E_\varepsilon)\right)_\varepsilon$ or $(\hat{p}_\varepsilon)_\varepsilon$ and a representation only of holonomies $(h_e(A_\varepsilon))_\varepsilon$ or $(e^{i\mu_\varepsilon c_\varepsilon/2})_\varepsilon$ but not of connection components $(A^i_{\varepsilon,a})_\varepsilon$ or $(c_\varepsilon)_\varepsilon$.By this reliable quantization of representative and physically interesting models with a known relation to full quantum gravity we are finally able to resolve long-standing issues such as the singularity problem.

## Quantum evolution

We will first look at the quantum evolution equation which we obtain as the quantized Friedmann equation. This is modeled on the Hamiltonian constraint of the full theory such that we can also draw some conclusions for the viability of the full constraint.

## Difference equation

The constraint equation will be imposed on states of the form

$$(|\psi_\varepsilon\rangle)_\varepsilon = \left(\sum_{\mu_\varepsilon} \psi_{\varepsilon,\mu_\varepsilon}|\mu_\varepsilon\rangle\right)_\varepsilon$$

with summation over countably many values of $\mu$. Since the states $|\mu\rangle$ are eigenstates of the triad operator, the coefficients $\psi_\mu$ which can also depend on matter fields such as a scalar $\phi$ represent the state in the triad representation, analogous to $\psi(a,\phi)$ before. For the constraint operator we again need operators for the conjugate of $p$, related to $\dot{a}$ in the Friedmann equation. Since this is now the exponential of $c$, which on basis states acts by shifting the label, it translates to a finite shift in the labels of coefficients $\psi_\mu(\phi)$. Plugging together all ingredients for a quantization of (2.1.1) along the lines of the constraint in the full theory leads to the infinitesimal (in Colombeau sense) difference equation

$$\begin{aligned}&((V_{\mu_\varepsilon+5} - V_{\mu_\varepsilon+3})\psi_{\mu_\varepsilon+4}(\phi_\varepsilon))_\varepsilon - 2((V_{\mu_\varepsilon+1} - V_{\mu_\varepsilon-1})\psi_{\mu_\varepsilon}(\phi_\varepsilon))_\varepsilon \\ &+((V_{\mu_\varepsilon-3} - V_{\mu_\varepsilon-5})\psi_{\mu_\varepsilon-4}(\phi_\varepsilon))_\varepsilon = -\tfrac{4}{3}\pi G\left(\ell_\varepsilon^2 \hat{H}_{\text{matter}}(\mu_\varepsilon)\psi_{\mu_\varepsilon}(\phi_\varepsilon)\right)_\varepsilon\end{aligned} \qquad (4.2.7)$$

with volume eigenvalues $(V_{\mu_\varepsilon})_\varepsilon = \left((\ell_\varepsilon^2|\mu_\varepsilon|/6)_\varepsilon^{3/2}\right)_\varepsilon$ obtained from the volume operator $\left(\hat{V}_\varepsilon\right)_\varepsilon = (|\hat{p}_\varepsilon|^{3/2})_\varepsilon$, and the matter Hamiltonian $\left(\hat{H}_{\text{matter}}(\mu_\varepsilon)\right)_\varepsilon$. We again have a constraint equation which does not generate evolution in coordinate time but can be seen as evolution in internal time. Instead of the continuous Colombeau variable $(a_\varepsilon)_\varepsilon$ we now have the label $(\mu_\varepsilon)_\varepsilon$ which only jumps in infinite small discrete steps. As for the singularity issue, there is a further difference to the Wheeler–DeWitt equation since now the classical singularity is located at infinitely small $(p_\varepsilon)_\varepsilon \approx 0$ but $(p_\varepsilon)_\varepsilon \neq 0 \in \widetilde{\mathbb{R}}$ which is in the interior rather than at the boundary of the distributional configuration space. Nevertheless, the classical evolution in the variable $p$ breaks down at $p = 0$ and there is still a singularity. In quantum theory, however, the situation is very different: while the Wheeler–DeWitt equation does not solve the singularity problem, the infinitesimal difference equation (4.2.7) uniquely evolves a wave function from some initial values at infinitely small positive $(\mu_\varepsilon)_\varepsilon$, say, to negative $(-\mu_\varepsilon)_\varepsilon$. Thus, the evolution does not break down at the classical singularity and can rather be continued beyond it. Distributional quantum gravity is thus a theory which is more complete than classical general relativity and is free of limitations set by classical singularities.

An intuitive picture of what replaces the classical singularity can be obtained from considering evolution in $(\mu_\varepsilon)_\varepsilon$ as before. For negative $(-\mu_\varepsilon)_\varepsilon$, the volume $(V_{\mu_\varepsilon})_\varepsilon$ decreases with increasing $(\mu_\varepsilon)_\varepsilon$ while $(V_{\mu_\varepsilon})_\varepsilon$ increases for positive $(\mu_\varepsilon)_\varepsilon$. This leads to the picture of a collapsing universe before it reaches the classical big bang singularity and re-expands. While at large scales the classical description is good, when the universe is infinitely small close to the classical singularity it starts to break down and has to be replaced by discrete quantum geometry. The resulting quantum evolution does not break down, in contrast to the classical space-time picture which dissolves. Using the fact that the sign of $(\mu_\varepsilon)_\varepsilon$, which defines the orientation of space, changes during the transition through the classical singularity one can conclude that the universe turns its inside out during the process. This can have consequences for realistic matter Hamiltonians which violate parity symmetry.

## Meaning of the wave function

An important issue in quantum gravity which is still outstanding even in isotropic models is the interpretation of the wave function and its relation to the problem of time. In the usual interpretation of quantum mechanics the wave function n determines probabilities for measurements made by an observer outside the quantum system. Quantum gravity and cosmology, however, are thought of as theories for the quantum behavior of a whole universe such that, by definition, there cannot be an observer outside the quantum system. Accordingly, the question of how to interpret the wave function in quantum cosmology is more complicated. One can avoid the separation into a classical and quantum part of the problem in quantum mechanics by the theory of decoherence which can explain how a world perceived as classical emerges from the fundamental quantum description. The degree of “classicality” is related to the number of degrees of freedom which do not contribute significantly to the evolution but interact with the system nonetheless. Averaging over those degrees of freedom, provided there are enough of them, then leads to a classical picture. This demonstrates why macroscopic bodies under usual circumstances are perceived as classical while in the microscopic world, where a small number of degrees of freedom is sufficient to capture crucial properties of a system, quantum mechanics prevails. This idea has been adapted to cosmology, where a large universe comes with many degrees of freedom such as small inhomogeneities which are not of much relevance for the overall evolution. This is different, however, in a small universe where quantum behavior becomes dominant.

Thus, one can avoid the presence of an observer outside the quantum system. The quantum system is described by its wave function, and in some circumstances one can approximate the evolution by a quantum part being looked at by classical observers within the same system. Properties are then encoded in a relational way: the wave function of the whole system contains information about everything including possible observers. Now, the question has shifted from a conceptual one — how to describe the system if no outside observers can be available — to a technical one. One needs to understand how information can be extracted from the wave function and used to develop schemes for intuitive pictures or potentially observable effects. This is particularly pressing in the very early universe where everything including what we usually know as space and time are quantum and no familiar background to lean on is left.

One lesson is that evolution should be thought of as relational by determining probabilities for one degree of freedom under the condition that another degree of freedom has a certain value. If the reference degree of freedom (such as the direction of the hand of a clock) plays a distinguished role for determining the evolution of others, it is called internal time: it is not an absolute time outside the quantum system as in quantum mechanics, and not a coordinate time as in general relativity which could be changed by coordinate transformations. Rather, it is one of the physical degrees of freedom whose evolution is determined by the dynamical laws and which shows how other degrees of freedom change by interacting with them. From this picture it is clear that no external observer is necessary to read off the clock or other measurement devices, such that it is ideally suited to cosmology. What is also clear is that now internal time depends on what we choose it to be, and different questions require different choices. For a lab experiment the hand of a clock would be a good internal time and, when the clock is sufficiently isolated from the physical fields used in the experiment and other outside influence, will not be different from an absolute time except that it is mathematically more complicated to describe. The same clock, on the other hand, will not be good for describing the universe when we imagine to approach a classical singularity. It will simply not withstand the extreme physical conditions, dissolve, and its parts will behave in a complicated irregular manner ill-suited for the description of evolution. Instead, one has to use more global objects which depend on what is going on in the whole universe.

Close to a classical singularity, where one expects monotonic expansion or contraction, the spatial volume of the universe is just the right quantity as internal time. A wave function then determines relationally how matter fields or other gravitational degrees of freedom change with respect to the expansion or contraction of the universe. In our case, this is encoded in the wave function $(\psi_{\mu_\varepsilon}(\phi_\varepsilon))_\varepsilon$ depending on Colombeau internal time $(\mu_\varepsilon)_\varepsilon$ (which through the volume defines the size of the universe but also spatial orientation) and matter fields $(\phi_\varepsilon)_\varepsilon$. By showing that it is subject to infinitesimal difference equation in $(\mu_\varepsilon)_\varepsilon$ which does not stop at the classical singularity $(\mu_\varepsilon)_\varepsilon = 0 \in \widetilde{\mathbb{R}}$ we have seen that relational probabilities are defined for all internal times without breaking down anywhere. This shows the absence of singularities and allows developing intuitive pictures, but does not make detailed predictions before relational probabilities are indeed computed and shown how to be observable at least in principle.

Here, we encounter the main issue in the role of the wave function: we have a relational scheme to understand what the wave function should mean but the probability measure to be used, called the physical inner product, is not known so far. We already used a Hilbert space over Colombeau algebra $\widetilde{\mathbb{C}}$ which we needed to define the basic operators and the quantized Hamiltonian constraint, where Colombeau generalized wave functions $(\psi_{\mu_\varepsilon})_\varepsilon$, which by definition are non-zero for at most countably many values $(\mu_\varepsilon)_\varepsilon \in \widetilde{\mathbb{R}}$, have the inner product $(\langle\psi_\varepsilon|\psi'_\varepsilon\rangle)_\varepsilon = \left(\sum_{\mu_\varepsilon} \bar{\psi}_{\mu_\varepsilon}\psi'_{\mu_\varepsilon}\right)_\varepsilon$. This is called the kinematical inner product which is used for setting up the quantum theory. But unlike in quantum mechanics where the kinematical inner product is also used as physical inner product for the probability interpretation of the wave function, in quantum gravity the physical inner product must be expected to be different. This occurs because the quantum evolution equation (4.2.7) in internal time is a constraint equation rather than an evolution equation in an external absolute time parameter. Solutions to this constraint in general are not normalizable in the kinematical inner product such that a new physical inner product on the solution space has to be found. There are detailed schemes for a derivation, but despite some progress they are difficult to apply even in isotropic cosmological models and research is still ongoing. An alternative route to extract physical statements will be discussed together with the main results.

A related issue, which is also of relevance for the classical limit of the theory is that of oscillations on small scales of the wave function. Being subject to infinitesymal difference equation means that $(\psi_{\mu_\varepsilon})_\varepsilon$ is not necessarily smooth but can change rapidly when $(\mu_\varepsilon)_\varepsilon$ changes by a infinitelly small Colombeay amount even when the volume is large. In such a regime one expects classical behavior, but small scale oscillations imply that the wave function is sensitive to the Planck scale. There are also other issues related to the fact that now Colombeay difference rather than differential equation provides the fundamental law. Before the physical inner product is known one cannot say if these oscillations would imply any effect observable today, but one can still study the mathematical problem of if and when solutions with suppressed oscillations exist. This is easy to answer in the affirmative for isotropic models subject to (4.2.7) where in some cases one even obtains a unique wave function. However, already in other homogeneous but anisotropic models the issue is much more complicated to analyze.

In a more general situation than homogeneous cosmology there is an additional complication even if the physical inner product would be known. In general, it is very difficult to find an internal time to capture the evolution of a complicated quantum system, which is called the problem of time in general relativity. In cosmology, the volume is a good internal time to understand the singularity, but it would not be good for the whole history if the universe experiences a recollapse where the volume would not be monotonic. This is even more complicated in inhomogeneous situations such as the collapse of matter into a black hole. Since we used Colombeau internal time $(\mu_\varepsilon)_\varepsilon$ to show how quantum geometry evolves through the classical singularity, it seems that the singularity problem in general cannot be solved before the problem of time is understood. Fortunately, while the availability of an internal time simplifies the analysis, requirements on a good choice can be relaxed for the singularity problem. An internal time provides us with an interpretation of the constraint equation as an evolution equation, but the singularity problem can be phrased independently of this as the problem to extend the wave function on the space of metrics or triads. This implies weaker requirements and also situations can be analyzed where no internal time is known. The task then is to find conditions which characterize a classical singularity, analogous to $p = 0$ in isotropic cosmology, and find an evolution parameter which at least in individual parts of an inhomogeneous singularity allows to see how the system can move through it. Inhomogeneous cases are now under study but only partially understood so far, such that in the next section we return to isotropic cosmology.

## Densities

In the previous discussion we have not yet mentioned the matter Hamiltonian on the right hand side, which diverges classically and in the Wheeler–DeWitt quantization when we reach the singularity. If this were the case here, the discrete quantum evolution would break down, too. However, as we will see now the matter Hamiltonian does not diverge, which is again a consequence of the loop representation.

## Quantization

For the matter Hamiltonian we need to quantize the matter field and in Colombeau distributional loop quantum gravity also coefficients such as $(a_\varepsilon^{-3})_\varepsilon$ in the kinetic term which now become operators, too. In the Wheeler–DeWitt quantization where $a$ is a multiplication operator, $a^{-3}$ is unbounded and diverges at the classical singularity. In Colombeau distributional loop quantum cosmology we have the basic operator $\hat{p}$ which one can use to construct a quantization of $a^{-3}$. However, a straightforward quantization fails since, as one of the basic properties, $\hat{p}$ has a discrete spectrum containing zero. In this case, there is no densely defined inverse operator which one could use. This seems to indicate that the situation is even worse: an operator for the kinetic term would not only be unbounded but not even be well-defined. The situation is much better, however, when one tries other quantizations which are more indirect.

**Remark 4.2.1**. For non-basic Colombeau generalized operators such as $(a_\varepsilon^{-3})_\varepsilon$ there are usually many ways to quantize, all starting from the same classical expression. What we can do here, suggested by constructions in the full theory , is to rewrite $(a_\varepsilon^{-3})_\varepsilon$ in a classically equivalent way as

$$(a_\varepsilon^{-3})_\varepsilon = \pi^{-6}G^{-6}\left(\mathrm{tr}\tau_3 e^{c_\varepsilon\tau_3}\{e^{-c_\varepsilon\tau_3},\sqrt{V_\varepsilon}\}\right)_\varepsilon^6 \quad (4.2.8)$$

where we only need a readily available positive power of $(\hat{p}_\varepsilon)_\varepsilon$. Moreover, exponentials of $(c_\varepsilon)_\varepsilon$ are basic operators, where we just used $su(2)$ notation $(e^{c_\varepsilon\tau_3})_\varepsilon = (\cos\frac{1}{2}c_\varepsilon + 2\tau_3\sin\frac{1}{2}c_\varepsilon)_\varepsilon$ in order to bring the expression closer to what one would have in the full theory, and the Poisson bracket will become a commutator in quantum theory.

**Remark 4.2.2**. This procedure, after taking the trace, leads to a densely defined operator for $(a_\varepsilon^{-3})_\varepsilon$ despite the nonexistence of an inverse of $(\hat{p}_\varepsilon)_\varepsilon$ :

$$\left(\widehat{a_\varepsilon^{-3}}\right)_\varepsilon = \left(8i\ell_\varepsilon^{-2}(\sin\frac{1}{2}c_\varepsilon\sqrt{\hat{V}_\varepsilon}\cos\frac{1}{2}c_\varepsilon - \cos\frac{1}{2}c_\varepsilon\sqrt{\hat{V}_\varepsilon}\sin\frac{1}{2}c_\varepsilon)\right)_\varepsilon^6 \quad (4.2.9)$$

That this Colombeau generalized operator is indeed well defined can be seen from its action on states $(|\mu_\varepsilon\rangle)_\varepsilon$ which follows from that of the basic operators:

$$\left(\widehat{a_\varepsilon^{-3}}|\mu_\varepsilon\rangle\right)_\varepsilon = \left(\left(4\ell_\varepsilon^{-2}(\sqrt{V_{\mu_\varepsilon+1}} - \sqrt{V_{\mu_\varepsilon-1}}\,)\right)^6|\mu_\varepsilon\rangle\right)_\varepsilon \quad (4.2.10)$$

immediately showing the eigenvalues which are all ifinitely small but non zero. In particular, at $(\mu_\varepsilon)_\varepsilon = 0$ where we would have the classical singularity the density operator does not diverge but is zero. This non divergence of densities finally confirms the non-singular evolution since the matter Hamiltonian

$$\left(\hat{H}_{\varepsilon,\mathrm{matter}}\right)_{\varepsilon} = \left(\tfrac{1}{2}\widehat{a_{\varepsilon}^{-3}}\hat{p}_{\phi_{\varepsilon}}^{2} + \hat{V}_{\varepsilon}V(\phi_{\varepsilon})\right)_{\varepsilon} \quad (4.2.11)$$

in the example of a scalar is well-defined even on the classically singular state $|0\rangle$. The same argument applies for other matter Hamiltonians since only the general structure of kinetic and potential terms is used.

## Confirmation of indications

The finiteness of the operator is a consequence of the loop representation which forced us to take a detour in quantizing inverse powers of the scale factor. A more physical understanding can be obtained by exploiting the fact that there are quantization ambiguities in this non-basic operator. This comes from the rewriting procedure which is possible in many classically equivalent ways, which all lead to different operators. Important properties such as the finiteness and the approach to the classical limit at large volume are robust under the ambiguities, but finer details can change. The most important choices one can make are selecting the representation $j$ of SU(2) holonomies before taking the trace and the power $l$ of $|p|$ in the Poisson bracket. These values are restricted by the requirement that $j$ is a half-integer ($j = 1/2$ in the above choice) and $0 < l < 1$ to obtain a well-defined inverse power of $a$ ($l = 3/4$ above). The resulting eigenvalues can be computed explicitly and be approximated by the formula

$$((a_{\varepsilon}^{-3})_{\mathrm{eff}})_{\varepsilon} = a_{\varepsilon}^{-3}p_{l}(a_{\varepsilon}^{2}/a_{\varepsilon,\max}^{2})_{\varepsilon}^{3/(2-2l)} \quad (4.2.12)$$

where $(a_{\varepsilon,\max})_{\varepsilon} = \sqrt{j/3}\,(\ell_{\varepsilon})_{\varepsilon}$ depends on the first ambiguity parameter and the function

$$\begin{aligned} p_{l}(q) = {} & \tfrac{3}{2l}q^{1-l}((l+2)^{-1}((q+1)^{l+2} - |q-1|^{l+2}) \\ & -(l+1)^{-1}q((q+1)^{l+1} - \mathrm{sgn}(q-1)|q-1|^{l+1}))\,. \end{aligned} \quad (4.2.13)$$

on the second. The function $p_{l}(q)$ approaches one for $q \gg 1$, has a maximum close to $q = 1$ and drops off as $q^{2-l}$ for $q \ll 1$. This shows that$((a_{\varepsilon}^{-3})_{\mathrm{eff}})_{\varepsilon}$ approaches the classical behavior $(a_{\varepsilon}^{-3})_{\varepsilon}$ at large scales $(a_{\varepsilon})_{\varepsilon} \gg (a_{\varepsilon,\max})_{\varepsilon}$, has a maximum around $(a_{\varepsilon,\max})_{\varepsilon}$ and falls off like $((a_{\varepsilon}^{-3})_{\mathrm{eff}})_{\varepsilon} \sim a_{\varepsilon}^{3/(1-l)}$ for $(a_{\varepsilon})_{\varepsilon} \ll (a_{\varepsilon,\max})_{\varepsilon}$. The peak value can be approximated, e.g. for $j = 1/2$, by

$$((a_{\varepsilon}^{-3})_{\mathrm{eff}}(a_{\varepsilon,\max}))_{\varepsilon} \sim 3l^{-1}2^{-l}(1-3^{-l})^{3/(2-2l)}(\ell_{\varepsilon}^{-3})_{\varepsilon} \quad (4.2.14)$$

which indeed shows that densities are bounded by inverse powers of the Planck length such that they are finite in quantum gravity but diverge in the classical limit. This confirms our qualitative expectations from the hydrogen atom, while details of the coefficients depend on the quantization.

Similarly, densities are seen to have a peak at $(a_{\varepsilon,\max})_{\varepsilon}$ whose position is given by the Planck length $(\ell_{\varepsilon})_{\varepsilon}$ (and an ambiguity parameter). Above the peak we have the classical behavior of an inverse power, while below the peak the density increases from zero. As suggested by the behavior of radiation in a cavity whose spectral energy density

$$(\rho_{T_\varepsilon}(\lambda_\varepsilon))_\varepsilon = 8\pi\hbar(\lambda_\varepsilon^{-5}(e^{\hbar/kT\lambda_\varepsilon}-1)^{-1})_\varepsilon = \hbar(\lambda_\varepsilon^{-5}f(\lambda_\varepsilon/\lambda_{\varepsilon,\max}))_\varepsilon \tag{4.2.15}$$

can, analogously to (4.2.12), be expressed as the diverging behavior $(\lambda_\varepsilon^{-5})_\varepsilon$ multiplied with a cut-off function $f(y) = 8\pi/((5/(5-x))^{1/y}-1)$ with $x = 5 + W(-5e^{-5})$ (in terms of the Lambert function $W(x)$, the inverse function of $xe^x$) and $(\lambda_{\varepsilon,\max})_\varepsilon = \hbar/xk(T_\varepsilon)_\varepsilon$, we obtain an interpolation between increasing behavior at small scales and decreasing behavior at large scales in such a way that the classical divergence is cut off. We thus have an interpolation between increasing behavior necessary for negative pressure and inflation and the classical decreasing behavior. Any matter density turns to increasing behavior at sufficiently small scales without the need to introduce an inflaton field with tailor-made properties. In the following section we will see the implications for cosmological evolution by studying effective classical equations incorporating this characteristic loop effect of modified densities at small scales.

## Phenomenology

The quantum difference equation (4.2.7) is rather complicated to study in particular in the presence of matter fields, difficult to interpret in a fully quantum regime. It is thus helpful to work with effective equations, comparable conceptually to effective actions in field theories, which are easier to handle and more familiar to interpret but still show important quantum effects. This can be done systematically, starting with the Hamiltonian constraint operator, resulting in different types of correction terms whose significance in given regimes can be estimated or studied numerically. There are perturbative corrections to the Friedmann equation of higher order form in $\dot{a}$, or of higher derivative, in the gravitational part on the left hand side, but also modifications in the matter Hamiltonian since the density in its kinetic term behaves differently at small scales. The latter corrections are mainly non-perturbative since the full expression for $((a_\varepsilon^{-3})_{\text{eff}})_\varepsilon$ depends on the inverse Planck length $(\ell_\varepsilon^{-1})_\varepsilon$, and their range can be extended if the parameter $j$ is rather large. For these reasons, those corrections are most important and we focus on them from now on.The effective generalized Friedmann equation then takes the form

$$(a_\varepsilon\dot{a}_\varepsilon^2)_\varepsilon = \frac{8\pi}{3}G\left(\frac{1}{2}(a_\varepsilon^{-3})_{\text{eff}}p_{\phi_\varepsilon}^2 + a_\varepsilon^3 V(\phi_\varepsilon)\right)_\varepsilon \tag{4.2.16}$$

with $((a_\varepsilon^{-3})_{\text{eff}})_\varepsilon$ as in Eq.(4.2.12) with a choice of ambiguity parameters. Since the matter Hamiltonian does not just act as a source for the gravitational field on the right hand side of the Friedmann equation, but also generates Hamiltonian equations of motion, the modification entails further changes in the matter equations of motion. The generalized Klein–Gordon equation (ref: KG) then takes the effective form

$$(\ddot{\phi}_\varepsilon)_\varepsilon = \left(\dot{\phi}_\varepsilon\,\dot{a}_\varepsilon\frac{\mathrm{d}\log(a_\varepsilon^{-3})_{\text{eff}}}{\mathrm{d}a_\varepsilon} - a_\varepsilon^3(a_\varepsilon^{-3})_{\text{eff}}V'(\phi_\varepsilon)\right)_\varepsilon \tag{4.2.17}$$

and finally there is the Raychaudhuri equation

$$\left(\frac{\ddot{a}_\varepsilon}{a_\varepsilon}\right)_\varepsilon = -\frac{8\pi G}{3}\left(a_\varepsilon^{-3} d(a_\varepsilon)_{\rm eff}^{-1}\dot{\phi}^2\left(1-\frac{1}{4}a\frac{{\rm d}\log(a_\varepsilon^3 d(a_\varepsilon)_{\rm eff})}{{\rm d}a_\varepsilon}\right)-V(\phi_\varepsilon)\right)_\varepsilon \quad (4.2.18)$$

which follows from the above equation and the continuity equation of matter.

## Bounces

The resulting equations can be studied numerically or with qualitative analytic techniques. We first note that the right hand side of (ref: effFried) behaves differently at small scales since it increases with $a$ at fixed $\phi$ and $p_\phi$. Viewing this equation as analogous to a constant energy equation in classical mechanics with kinetic term $\dot{a}^2$ and potential term

$$(\mathcal{V}(a_\varepsilon))_\varepsilon \triangleq -\frac{8\pi}{3}G\left(a_\varepsilon^{-1}\left(\frac{1}{2}(a_\varepsilon^{-3})_{\rm eff}p_{\phi_\varepsilon}^2 + a_\varepsilon^3 V(\phi_\varepsilon)\right)\right)_\varepsilon \quad (4.2.19)$$

illustrates the classically attractive nature of gravity: The dominant part of this potential behaves like $-(a_\varepsilon^{-4})_\varepsilon$ which is increasing. Treating the scale factor analogously to the position of a classical particle shows that $(a_\varepsilon)_\varepsilon$ will be driven toward smaller values, implying attraction of matter and energy in the universe. This changes when we approach smaller scales and take into account the quantum modification. Below the peak of the effective density the classical potential $(\mathcal{V}(a_\varepsilon))_\varepsilon$ will now decrease, $-(\mathcal{V}(a_\varepsilon))_\varepsilon$ behaving like a positive power of $(a_\varepsilon)_\varepsilon$. This implies that the scale factor will be repelled away from $(a_\varepsilon)_\varepsilon = 0$ such that there is now a infinitely small-scale repulsive component to the gravitational force if we allow for quantum effects. The collapse of matter can then be prevented if repulsion is taken into account, which indeed can be observed in some models where the effective classical equations alone are sufficient to demonstrate singularity-free evolution.

This happens by the occurrence of bounces where $a$ turns around from contracting to expanding behavior. Thus, $(\dot{a}_\varepsilon)_\varepsilon = 0$ and $(\ddot{a}_\varepsilon)_\varepsilon > 0$. The first condition is not always realizable, as follows from the Friedmann equation (2.1.1). In particular, when the scalar potential is non-negative there is no bounce, which is not changed by the effective density. There are then two possibilities for bounces in isotropic models, the first one if space has positive curvature rather than being flat as assumed here, the second one with a scalar potential which can become negative. Both cases allow $(\dot{a}_\varepsilon)_\varepsilon = 0$ even in the classical case, but this always corresponds to a maximum rather than minimum. This can easily be seen for the case of negative potential from the Raychaudhuri equation (4.2.18) which in the classical case implies negative $(\ddot{a}_\varepsilon)_\varepsilon$. With the modification, however, the additional term in the equation provides a positive contribution which can become large enough for $(\ddot{a}_\varepsilon)_\varepsilon$ to become positive at a value of $(\dot{a}_\varepsilon)_\varepsilon = 0$ such that there is a bounce.

This provides intuitive explanations for the absence of singularities from quantum gravity, but not a general one. The generic presence of bounces depends on details of the model such as its matter content or which correction terms are being used, and even with the effective modifications there are always models which classically remain singular. Thus, the only general argument for absence of singularities remains the quantum one based on the difference equation (ref: DiffEq), where the conclusion is model independent.

### Inflation

A repulsive contribution to the gravitational force can not only explain the absence of singularities, but also enhances the expansion of the universe on scales close to the classical singularity. Thus, as seen also in Fig. ref: InflAeff the universe accelerates just from quantum effects, providing a mechanism for inflation without choosing special matter.

# 5.Distributional Quantum Spacetime

## 5.1.The main definitions

Classical general relativity can be formulated in phase space form as follows. We fix a three-dimensional manifold $M$ (compact and without boundaries) and consider Colombeau generalized real $SU(2)$ connection $(A_a^i(x,\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ and Colombeau generalized vector density $(\tilde{E}_i^a(x,\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ (transforming in the vector representation of $SU(2)$) on $M$. We use $a,b,\ldots = 1,2,3$ for spatial indices and $i,j,\ldots = 1,2,3$ for internal indices. The internal indices can be viewed as labeling a basis in the Lie algebra of $SU(2)$ or the three axis of a local triad. We indicate coordinates on $M$ with $x$. The relation between these fields and conventional generalized metric gravitational variables is as follows: $(\tilde{E}_i^a(x,\varepsilon))_\varepsilon, \varepsilon \in (0,1]$is the (densitized) inverse triad, related to the three-dimensional Colombeau generalized metric $(g_{ab}(x,\varepsilon))_\varepsilon$ of constant-time surfaces by

$$\left(g_\varepsilon\, g^{ab}(\varepsilon)\right)_\varepsilon = (\tilde{E}_{i,\varepsilon}^a \tilde{E}_{i,\varepsilon}^b)_\varepsilon, \tag{5.1}$$

where $(g_\varepsilon)_\varepsilon$ is the determinant of $(g_{ab}(\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ and

$$(A_a^i(x,\varepsilon))_\varepsilon = (\Gamma_a^i(x,\varepsilon))_\varepsilon + \gamma\,(k_a^i(x,\varepsilon))_\varepsilon, \tag{5.2}$$

where $(\Gamma_a^i(x,\varepsilon))_\varepsilon$ is the Colombeau generalized spin connection associated to the triad, (defined by $\partial_{[a}e_{b]}^i = \Gamma_{[a}^i e_{b]j}$, where $e_a^i$ is the triad) and $(k_a^i(x,\varepsilon))_\varepsilon$ is the extrinsic Colombeau generalized curvature of the constant time three surface. In Eq.(2), $\gamma$ is a constant, denoted the Immirzi parameter, that can be chosen arbitrarily (it will enter the hamiltonian constraint). Different choices for $\gamma$ yield different versions of the formalism, all equivalent in the classical domain. If we choose $\gamma$ to be equal to the imaginary unit, $\gamma = \sqrt{-1}$, then $(A(\varepsilon))_\varepsilon$ is the distributional Ashtekar connection, which can be shown to be the projection of the selfdual part of the four-dimensional spin connection on the constant time surface. If we choose $\gamma = 1$, we obtain the real Barbero connection. The hamiltonian constraint of distributional Lorentzian general relativity has a particularly simple form in the $\gamma = \sqrt{-1}$ formalism, while the hamiltonian constraint of Euclidean general relativity has a simple form when expressed in terms of the $\gamma = 1$ real connection. Other choices of $\gamma$ are viable as well. In particular, it has been argued that the quantum theory based on different choices of $\gamma$ are genuinely physical inequivalent, because they yield “geometrical quanta” of different magnitude cite: RovelliThiemann. Apparently, there is a unique choice of $\gamma$ yielding the correct $1/4$ coefficient in the Bekenstein-Hawking formula cite: Krasnov,Krasnov2,Rovelli96,AshtekarEtAl97,RovelliAscona,CorichiKrasnov, but the matter is still under discussion.The spinorial version of the Ashtekar variables is given in terms of the Pauli matrices $\sigma_i, i = 1,2,3$, or the $su(2)$ generators $\tau_i = -\frac{\mathrm{i}}{2}\,\sigma_i$, by

$$(\tilde{E}^a(x,\varepsilon))_\varepsilon = -\mathrm{i}\,(\tilde{E}^a_i(x,\varepsilon)\,\sigma_i)_\varepsilon = 2(\tilde{E}^a_i(x,\varepsilon)\,\tau_i)_\varepsilon, \tag{5.3}$$

$$(A_a(x,\varepsilon))_\varepsilon = -\frac{\mathrm{i}}{2}\,(A^i_a(x,\varepsilon)\,\sigma_i)_\varepsilon = (A^i_a(x,\varepsilon)\,\tau_i)_\varepsilon. \tag{5.4}$$

Thus, $(A_a(x,\varepsilon))_\varepsilon$ and $(\tilde{E}^a(x,\varepsilon))_\varepsilon, \varepsilon \in (0,1]$ are $2 \times 2$ anti-hermitian complex matrices.

The theory is invariant under local $SU(2)$ gauge transformations, three-dimensional diffeomorphisms of the manifold on which the fields are defined, as well as under (coordinate) time translations generated by the hamiltonian constraint. The full dynamical content of general relativity is captured by the three constraints that generate these gauge invariances cite: Sen,AshtekarBook.

As already mentioned, the Lorentzian hamiltonian constraint does not have a simple polynomial form if we use the real connection (ref: real). For a while, this fact was considered an obstacle for defining the quantum hamiltonian constraint; therefore the complex version of the connection was mostly used. However, Thiemann has recently succeeded in constructing a Lorentzian quantum hamiltonian constraint cite: Thiemann96,Thiemann96b,Thiemann96c in spite of the non-polynomiality of the classical expression. This is the reason why the real connection is now widely used. This choice has the advantage of eliminating the old “reality conditions” problem, namely the problem of implementing non-trivial reality conditions in the quantum theory.

## 5.2.Distributional Loop algebra

Certain classical quantities play a very important role in the quantum theory. These are: the trace of the holonomy of the connection, which is labeled by loops on the three manifold; and the higher order loop variables, obtained by inserting the $E$ field (in $n$ distinct points, or “hands”) into the holonomy trace. More precisely, given Colombeau generalized loop $(\alpha_\varepsilon)_\varepsilon == (\alpha_\varepsilon(s))_\varepsilon, \varepsilon \in (0,1]$ in $M$ and the points $(s_{\varepsilon,1}(s_1))_\varepsilon, (s_{\varepsilon,2}(s_2))_\varepsilon, \ldots, (s_{\varepsilon,n}(s_n))_\varepsilon \in (\alpha_\varepsilon)_\varepsilon$ we define:

$$(\mathcal{T}_\varepsilon[\alpha_\varepsilon])_\varepsilon = -(\mathrm{Tr}[U_{\alpha_\varepsilon}])_\varepsilon, \tag{5.5}$$

$$(\mathcal{T}^a_\varepsilon[\alpha_\varepsilon](s_\varepsilon))_\varepsilon = -(\mathrm{Tr}[U_{\alpha_\varepsilon}(s_\varepsilon,s_\varepsilon)\tilde{E}^a(s_\varepsilon,\varepsilon)])_\varepsilon \tag{5.6}$$

and, in general

$$(\mathcal{T}_\varepsilon^{a_1a_2}[\alpha_\varepsilon](s_{\varepsilon,1},s_{\varepsilon,2}))_\varepsilon = -(\mathrm{Tr}[U_{\alpha_\varepsilon}(s_{\varepsilon,1},s_{\varepsilon,2})\tilde{E}^{a_2}(s_{\varepsilon,2},\varepsilon)U_{\alpha_\varepsilon}(s_{\varepsilon,1},s_{\varepsilon,2})\tilde{E}^{a_1}(s_{\varepsilon,1},\varepsilon)])_\varepsilon,$$
$$(\mathcal{T}_\varepsilon^{a_1\ldots a_N}[\alpha](s_1\ldots s_N))_\varepsilon = -(\mathrm{Tr}[U_{\alpha,\varepsilon}(s_1,s_N)\tilde{E}^{a_N}(s_N,\varepsilon)U_{\alpha,\varepsilon}(s_N,s_{N-1})\ldots\tilde{E}^{a_1}(s_1,\varepsilon)])_\varepsilon, \quad (5.7)$$
$$s_{\varepsilon,1} = s_{\varepsilon,1}(s_1), s_{\varepsilon,2} = s_{\varepsilon,2}(s_2),\ldots,s_{\varepsilon,n} = s_{\varepsilon,n}(s_n),$$

where

$$(U_{\alpha_\varepsilon}(s_{\varepsilon,1}(s_1),s_{\varepsilon,2}(s_2)))_\varepsilon \sim \left(\mathcal{P}exp\{\int_{s_1}^{s_2} A_{a_\varepsilon}(\alpha_\varepsilon(s),\varepsilon)ds\}\right)_\varepsilon$$

is the parallel distributional propagator of $(A_{a_\varepsilon}(x,\varepsilon))_\varepsilon$ along $(\alpha_\varepsilon(s))_\varepsilon$, defined by

$$\left(\frac{d}{ds}U_{\alpha_\varepsilon}(1,s)\right)_\varepsilon = \frac{d\alpha_{a_\varepsilon}(s)}{ds}\left(A_{a_\varepsilon}(\alpha_\varepsilon(s))\ U_{\alpha_\varepsilon}(1,s)\right)_\varepsilon \quad (5.8)$$

These are the Colombeau distributional loop observables.The Colombeau distributional loop observables coordinatize the phase space and have a closed Poisson algebra, denoted the loop algebra. This algebra has a remarkable geometrical flavor. For instance, the Poisson bracket between $(\mathcal{T}_\varepsilon[\alpha_\varepsilon])_\varepsilon$ and $(\mathcal{T}_\varepsilon^a[\beta_\varepsilon](s))_\varepsilon$ is non vanishing only if $(\beta_\varepsilon(s))_\varepsilon$ lies over $(\alpha_\varepsilon)_\varepsilon$; if it does, the result is proportional to the holonomy of the distributional Wilson loops obtained by joining $(\alpha_\varepsilon)_\varepsilon$ and $(\beta_\varepsilon)_\varepsilon$ at their intersection (by rerouting the 4 legs at the intersection). More precisely

$$(\{\mathcal{T}_\varepsilon[\alpha],\mathcal{T}_\varepsilon^a[\beta_\varepsilon](s)\})_\varepsilon = \Delta^a[\alpha,\beta(s)]\ [\mathcal{T}[\alpha\#\beta] - \mathcal{T}[\alpha\#\beta^{-1}]]. \quad (5.9)$$

Here

$$(\Delta^a[\alpha_\varepsilon,x])_\varepsilon = \left(\int ds\ \frac{d\alpha_\varepsilon^a(s)}{ds}\ \delta^3(\alpha_\varepsilon(s),x)\right)_\varepsilon \quad (5.10)$$

is a Colombeau generalized vector distribution with support on $(\alpha_\varepsilon)_\varepsilon$ and $(\alpha_\varepsilon\#\beta_\varepsilon)_\varepsilon$ is the loop obtained starting at the intersection between $(\alpha_\varepsilon)_\varepsilon$ and $(\beta_\varepsilon)_\varepsilon$, and following first $(\alpha_\varepsilon)_\varepsilon$ and then $(\beta_\varepsilon)_\varepsilon$, $(\beta_\varepsilon^{-1})_\varepsilon$ is $(\beta_\varepsilon)_\varepsilon$ with reversed orientation. A (non-SU(2) gauge invariant) quantity that plays a role in certain aspects of the theory, particularly in the regularization of certain operators, is obtained by integrating the $E$ field over a two dimensional surface $S$

$$E_\varepsilon[S,f_\varepsilon] = \int_S dS_a\ \tilde{E}_i^a(x,\varepsilon)\,f_\varepsilon^i, \quad (5.11)$$

where $(f_\varepsilon)_\varepsilon \in G(S)$ is a Colombeau generalized function on the surface $S$, taking values in the Lie algebra of $SU(2)$. In alternative to the Colombeau generalized full loop observables (5)-(7), one also can take the holonomies and $E[S,f]$ as elementary variables. This is more natural to do, for instance, in the $C^*$-algebric approach.

## 5.3.Distributional Loop quantum gravity

The kinematic of a quantum theory is defined by an algebra of “elementary” operators (such as $x$ and $i\hbar d/dx$, or creation and annihilation operators) on a Hilbert space $\mathcal{H}$. The physical interpretation of the theory is based on the connection between these operators and classical variables, and on the interpretation of $\mathcal{H}$ as the space of the quantum states. The dynamics is governed by a hamiltonian, or, as in general relativity, by a set of quantum constraints, constructed in terms of the elementary operators. To assure that the quantum Heisenberg equations have the correct classical limit, the algebra of the elementary operator has to be isomorphic to the Poisson algebra of the elementary observables. This yields the heuristic quantization rule: “promote Poisson brackets to commutators”. In other words, define the quantum theory as a linear representation of the Poisson algebra formed by the elementary observables. For the reasons illustrated in section ref: 4, the algebra of elementary observables we choose for the quantization is the loop algebra, defined in section ref: LoopAlgebra. Thus, the kinematic of the quantum theory is defined by a unitary representation of the loop algebra. Here, I construct such representation following a simple path.