Pi Formulas

Part 7: Machin Formulas

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abstract

In this note we give some formulas for the constant Pi

Pi Fórmulas , Sumas de Arcotangentes

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Resumen-Abstract

En esta nota mostramos una colección de fórmulas del tipo Machin:

$$\pi = \sum_{k=1}^{n} a_k \tan^{-1}\left(x_k\right) \quad , a_k \in \mathbb{Q}^+, x_k \in \mathbb{R}, n \in \mathbb{N}$$

donde

$$\pi = 4\sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{2n+1} = 3.1415926535...$$

1. Introducción

Recordamos algunas fórmulas típicas del tipo Machin:

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) \tag{1}$$

$$\pi = 4\tan^{-1} x + 4\tan^{-1}\left(\frac{1-x}{1+x}\right) \quad ,0 \le x \le 1 \tag{2}$$

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) \tag{3}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{7}\right) + 8\tan^{-1}\left(\frac{79}{3}\right) \tag{4}$$

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{1}{7}\right) \tag{5}$$

$$\pi = 16\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{70}\right) + 4\tan^{-1}\left(\frac{1}{99}\right) \tag{6}$$

$$\pi = 24\tan^{-1}\left(\frac{1}{8}\right) + 8\tan^{-1}\left(\frac{1}{57}\right) + 4\tan^{-1}\left(\frac{1}{239}\right) \tag{7}$$

$$\pi = 48\tan^{-1}\left(\frac{1}{18}\right) + 32\tan^{-1}\left(\frac{1}{57}\right) - 20\tan^{-1}\left(\frac{1}{239}\right) \tag{8}$$

La función $\tan^{-1}(x)$ se define por:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad , x^2 \le 1 \tag{9}$$

$$\tan^{-1}(x) = \frac{x}{\left(1+x^2\right)^{1/2}} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}}{2n+1} \left(\frac{x^2}{1+x^2}\right)^n \quad , x^2 < \infty \tag{10}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!} \frac{x^{2n+1}}{\left(1+x^2\right)^{n+1}} \quad , x^2 < \infty \tag{11}$$

2. Fórmulas

2.1. Sea $n \in \mathbb{N}$, sean $p_n, q_n \in \mathbb{N}$, definidas como sigue:

$$p_{n+1} = p_n + 2q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 3, q_1 = 2 \tag{12}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(3,2),(7,5),(17,12),(41,29),...\} \tag{13}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n - q_n}{q_n}\right) + 4\tan^{-1}\left(\frac{2q_n - p_n}{p_n}\right) \quad , n \in \mathbb{N} \tag{14}$$

Ejemplos:

$$n = 1; \quad \pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) \tag{15}$$

$$n = 2; \quad \pi = 4\tan^{-1}\left(\frac{2}{5}\right) + 4\tan^{-1}\left(\frac{3}{7}\right) \tag{16}$$

$$n = 3; \quad \pi = 4\tan^{-1}\left(\frac{5}{12}\right) + 4\tan^{-1}\left(\frac{7}{17}\right) \tag{17}$$

$$n = 4\,; \quad \pi = 4\tan^{-1}\left(\frac{12}{29}\right) + 4\tan^{-1}\left(\frac{17}{41}\right) \tag{18}$$

2.2. Algunas fórmulas equivalentes a (14):

$$\pi = 4\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{p_{n+1} - p_n}{p_{n+1} + p_n}\right) \quad , n \in \mathbb{N} \tag{19}$$

$$p_{n+2} = 2\,p_{n+1} + p_n \quad , p_1 = 3, p_2 = 7 \tag{20}$$

$$\pi = 4\tan^{-1}\left(\frac{q_n}{q_{n+1}}\right) + 4\tan^{-1}\left(\frac{q_{n+1} - q_n}{q_{n+1} + q_n}\right) \quad , n \in \mathbb{N} \tag{21}$$

$$q_{n+2} = 2q_{n+1} + q_n \quad , q_1 = 2, q_2 = 5 \tag{22}$$

$$\pi = 4\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{q_n}{q_{n+1}}\right) \quad , n \in \mathbb{N} \tag{23}$$

2.3. Propiedad de la fórmula (14):

$$\lim_{n\to\infty} \frac{p_n - q_n}{q_n} = \lim_{n\to\infty} \frac{2q_n - p_n}{p_n} = \sqrt{2} - 1 \tag{24}$$

$$\pi = 8\tan^{-1}\left(\sqrt{2} - 1\right) \tag{25}$$

2.4. Otras fórmulas relacionadas con las sucesiones p_n, q_n :

$$\pi = 8\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\left(-1\right)^n \tan^{-1}\left(\frac{1}{2\,p_n\,p_{n+1} + \left(-1\right)^n}\right) \quad , n \in \mathbb{N} \tag{26}$$

$$\pi = 8\tan^{-1}\left(\frac{q_n}{q_{n+1}}\right) - 4\left(-1\right)^n \tan^{-1}\left(\frac{1}{4q_n q_{n+1} + \left(-1\right)^{n-1}}\right) \quad , n \in \mathbb{N} \tag{27}$$

Observación: $2\,p_n\,p_{n+1} + \left(-1\right)^n = 4q_n q_{n+1} + \left(-1\right)^{n-1}$, $n \in \mathbb{N}$.

Ejemplos:

$$\pi = 8\tan^{-1}\left(\frac{3}{7}\right) - 4\tan^{-1}\left(\frac{1}{41}\right) \tag{28}$$

$$\pi = 8\tan^{-1}\left(\frac{7}{17}\right) + 4\tan^{-1}\left(\frac{1}{239}\right) \tag{29}$$

$$\pi = 8\tan^{-1}\left(\frac{17}{41}\right) - 4\tan^{-1}\left(\frac{1}{1393}\right) \tag{30}$$

$$\pi = 8\tan^{-1}\left(\frac{41}{99}\right) + 4\tan^{-1}\left(\frac{1}{8119}\right) \tag{31}$$

$$\pi = 8\tan^{-1}\left(\frac{99}{239}\right) - 4\tan^{-1}\left(\frac{1}{47321}\right) \tag{32}$$

$$\pi = 8\tan^{-1}\left(\frac{239}{577}\right) + 4\tan^{-1}\left(\frac{1}{275807}\right) \tag{33}$$

$$\pi = 8\tan^{-1}\left(\frac{2}{5}\right) + 4\tan^{-1}\left(\frac{1}{41}\right) \tag{34}$$

$$\pi = 8\tan^{-1}\left(\frac{5}{12}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) \tag{35}$$

$$\pi = 8\tan^{-1}\left(\frac{12}{29}\right) + 4\tan^{-1}\left(\frac{1}{1393}\right) \tag{36}$$

$$\pi = 8\tan^{-1}\left(\frac{29}{70}\right) - 4\tan^{-1}\left(\frac{1}{8119}\right) \tag{37}$$

$$\pi = 8\tan^{-1}\left(\frac{70}{169}\right) + 4\tan^{-1}\left(\frac{1}{47321}\right) \tag{38}$$

$$\pi = 8\tan^{-1}\left(\frac{169}{408}\right) - 4\tan^{-1}\left(\frac{1}{275807}\right) \tag{39}$$

$$\pi = 4\tan^{-1}\left(\frac{q_n}{p_n}\right) + 2\tan^{-1}\left(\frac{q_n}{2p_n}\right) + 2(-1)^{n-1}\tan^{-1}\left(\frac{2p_n}{q_n\left(5p_n^2 - q_n^2\right)}\right), n \in \mathbb{N} \tag{40}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{2}{3}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) + 2\tan^{-1}\left(\frac{3}{41}\right) \tag{41}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{7}\right) + 2\tan^{-1}\left(\frac{5}{14}\right) - 2\tan^{-1}\left(\frac{7}{550}\right) \tag{42}$$

$$\pi = 4\tan^{-1}\left(\frac{12}{17}\right) + 2\tan^{-1}\left(\frac{6}{17}\right) + 2\tan^{-1}\left(\frac{17}{7806}\right) \tag{43}$$

$$\pi = 4\tan^{-1}\left(\frac{29}{41}\right) + 2\tan^{-1}\left(\frac{29}{82}\right) - 2\tan^{-1}\left(\frac{41}{109678}\right) \tag{44}$$

$$\pi = 4\tan^{-1}\left(\frac{70}{99}\right) + 2\tan^{-1}\left(\frac{35}{99}\right) + 2\tan^{-1}\left(\frac{99}{1543675}\right) \tag{45}$$

$$\pi = 4\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{p_{n+1}}{p_{n+2}}\right) + 4(-1)^n \tan^{-1}\left(\frac{1}{p_{2n+3}}\right) \quad , n \in \mathbb{N} \tag{46}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{3}{7}\right) + 4\tan^{-1}\left(\frac{7}{17}\right) - 4\tan^{-1}\left(\frac{1}{99}\right) \tag{47}$$

$$\pi = 4\tan^{-1}\left(\frac{7}{17}\right) + 4\tan^{-1}\left(\frac{17}{41}\right) + 4\tan^{-1}\left(\frac{1}{577}\right) \tag{48}$$

$$\pi = 4\tan^{-1}\left(\frac{17}{41}\right) + 4\tan^{-1}\left(\frac{41}{99}\right) - 4\tan^{-1}\left(\frac{1}{3363}\right) \tag{49}$$

$$\pi = 4\tan^{-1}\left(\frac{q_n}{q_{n+1}}\right) + 4\tan^{-1}\left(\frac{q_{n+1}}{q_{n+2}}\right) + 4(-1)^{n-1} \tan^{-1}\left(\frac{1}{p_{2n+3}}\right) \quad , n \in \mathbb{N} \tag{50}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{2}{5}\right) + 4\tan^{-1}\left(\frac{5}{12}\right) + 4\tan^{-1}\left(\frac{1}{99}\right) \tag{51}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{12}\right) + 4\tan^{-1}\left(\frac{12}{29}\right) - 4\tan^{-1}\left(\frac{1}{577}\right) \tag{52}$$

$$\pi = 4\tan^{-1}\left(\frac{12}{29}\right) + 4\tan^{-1}\left(\frac{29}{70}\right) + 4\tan^{-1}\left(\frac{1}{3363}\right) \tag{53}$$

2.5. Sea p_n , la sucesión definida por:

$$p_{n+2} = 4p_{n+1} - p_n \quad , p_1 = 3, p_2 = 12 \tag{54}$$

$$\{p_n : n \in \mathbb{N}\} = \{3, 12, 45, 168, 627, 2340, ...\} \tag{55}$$

Se tiene:

$$\pi = 12\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{p_n^3 - 3p_{n+1}^2 p_n - 3p_{n+1}p_n^2 + p_n^3}{p_n^3 + 3p_{n+1}^2 p_n - 3p_{n+1}p_n^2 - p_n^3}\right) \quad , n \in \mathbb{N} \tag{56}$$

Una forma equivalente de (56) es:

$$\pi = 12\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{u_n}{v_n}\right) \quad , n \in \mathbb{N} \tag{57}$$

donde

$$u_{n+2} = 4u_{n+1} - u_n \quad , u_1 = 5, u_2 = 19 \tag{58}$$

$$\begin{aligned} &v_{n+4} = 56v_{n+3} - 210v_{n+2} + 56v_{n+1} - v_n \\ &v_1 = 99, v_2 = 5291, v_3 = 275561, v_4 = 14325849 \end{aligned} \tag{59}$$

Ejemplos:

$$\pi = 12\tan^{-1}\left(\frac{3}{12}\right) + 4\tan^{-1}\left(\frac{5}{99}\right) \tag{60}$$

$$\pi = 12\tan^{-1}\left(\frac{12}{45}\right) + 4\tan^{-1}\left(\frac{19}{5291}\right) \tag{61}$$

$$\pi = 12\tan^{-1}\left(\frac{45}{168}\right) + 4\tan^{-1}\left(\frac{71}{275561}\right) \tag{62}$$

$$\pi = 12\tan^{-1}\left(\frac{168}{627}\right) + 4\tan^{-1}\left(\frac{265}{14325849}\right) \tag{63}$$

$$\pi = 12\tan^{-1}\left(\frac{627}{2340}\right) + 4\tan^{-1}\left(\frac{989}{744675931}\right) \tag{64}$$

2.6. Sea p_n , la sucesión definida por:

$$p_{n+2} = 4p_{n+1} - p_n \quad , p_1 = 2, p_2 = 7 \tag{65}$$

$$\{p_n : n \in \mathbb{N}\} = \{2, 7, ...\} \tag{66}$$

Se tiene:

$$\pi = 12\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{p_n^3 - 3p_{n+1}^2 p_n - 3p_{n+1}p_n^2 + p_n^3}{p_n^3 + 3p_{n+1}^2 p_n - 3p_{n+1}p_n^2 - p_n^3}\right) \quad , n \in \mathbb{N} \tag{67}$$

Una forma equivalente de (67) es:

$$\pi = 12\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) - 4\tan^{-1}\left(\frac{u_n}{v_n}\right) \quad , n \in \mathbb{N} \tag{68}$$

donde

$$u_{n+2} = 4u_{n+1} - u_n \quad , u_1 = 27, u_2 = 99 \tag{69}$$

$$\begin{aligned} &v_{n+4} = 56v_{n+3} - 210v_{n+2} + 56v_{n+1} - v_n \\ &v_1 = 545, v_2 = 27607, v_3 = 1432283, v_4 = 74440885 \end{aligned} \tag{70}$$

Ejemplos:

$$\pi = 12\tan^{-1}\left(\frac{2}{7}\right) - 4\tan^{-1}\left(\frac{27}{545}\right) \tag{71}$$

$$\pi = 12\tan^{-1}\left(\frac{7}{26}\right) - 4\tan^{-1}\left(\frac{99}{27607}\right) \tag{72}$$

$$\pi = 12\tan^{-1}\left(\frac{26}{97}\right) - 4\tan^{-1}\left(\frac{369}{1432283}\right) \tag{73}$$

$$\pi = 12\tan^{-1}\left(\frac{97}{362}\right) - 4\tan^{-1}\left(\frac{1377}{74440885}\right) \tag{74}$$

$$\pi = 12\tan^{-1}\left(\frac{362}{1351}\right) - 4\tan^{-1}\left(\frac{5139}{3869455577}\right) \tag{75}$$

2.7. Sean p_n, q_n, u_n, v_n , definidas por:

$$p_{n+1} = -p_n + 3q_n \quad , \quad q_{n+1} = -p_n + 7q_n \quad , p_1 = 1, q_1 = 3 \tag{76}$$

$$u_{n+1} = 3u_n + v_n \quad , \quad v_{n+1} = 3u_n + 4v_n \quad , u_1 = 2, v_1 = 5 \tag{77}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{u_n}{v_n}\right) + 4\tan^{-1}\left(\frac{(q_n - p_n)v_n - (q_n + p_n)u_n}{(q_n + p_n)v_n + (q_n - p_n)u_n}\right) \ , n \in \mathbb{N} \tag{78}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{2}{5}\right) + 4\tan^{-1}\left(\frac{1}{12}\right) \tag{79}$$

$$\pi = 4\tan^{-1}\left(\frac{2}{5}\right) + 4\tan^{-1}\left(\frac{11}{26}\right) + 4\tan^{-1}\left(\frac{1}{215}\right) \tag{80}$$

$$\pi = 4\tan^{-1}\left(\frac{13}{33}\right) + 4\tan^{-1}\left(\frac{59}{137}\right) + 4\tan^{-1}\left(\frac{13}{3741}\right) \tag{81}$$

$$\pi = 4\tan^{-1}\left(\frac{43}{109}\right) + 4\tan^{-1}\left(\frac{314}{725}\right) + 4\tan^{-1}\left(\frac{61}{65462}\right) \tag{82}$$

$$\pi = 4\tan^{-1}\left(\frac{71}{180}\right) + 4\tan^{-1}\left(\frac{1667}{3842}\right) + 4\tan^{-1}\left(\frac{361}{1146045}\right) \tag{83}$$

Observación: $\lim_{n\to\infty} \frac{p_n}{q_n} = 4 - \sqrt{13} \quad , \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\sqrt{13} - 1}{6}$

La fórmula (78) se puede escribir como:

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{u_n}{v_n}\right) + 4\tan^{-1}\left(\frac{x_n}{y_n}\right) \quad , n \in \mathbb{N} \tag{84}$$

donde

$$x_{n+2} = 4x_{n+1} + 9x_n \quad , x_1 = 1, x_2 = 1 \tag{85}$$

$$y_{n+4} = 21y_{n+3} - 50y_{n+2} - 189y_{n+1} - 81y_n \quad , y_1 = 12, y_2 = 215, y_3 = 3741, y_4 = 65462 \tag{86}$$

2.8. Sea p_n definida por:

$$p_{n+4} = 4p_{n+3} + 6p_{n+2} - 4p_{n+1} - p_n \quad , p_1 = 1, p_2 = 4, p_3 = 22, p_4 = 108 \tag{87}$$

Se tiene:

$$\pi = 16\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(f\left(\frac{p_n}{p_{n+1}}\right)\right) \quad , n \in \mathbb{N} \tag{88}$$

donde

$$f(x) = \frac{1 - 4x - 6x^2 + 4x^3 + x^4}{1 + 4x - 6x^2 - 4x^3 + x^4} \tag{89}$$

Ejemplos:

$$\pi = 16\tan^{-1}\left(\frac{1}{4}\right) - 4\tan^{-1}\left(\frac{79}{401}\right) \tag{90}$$

$$\pi = 16\tan^{-1}\left(\frac{2}{11}\right) + 4\tan^{-1}\left(\frac{1457}{22049}\right) \tag{91}$$

2.9. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 3q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 5, q_1 = 3 \tag{92}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(5,3), (14,8), (38,22), (104,60), (284,164), \ldots\} \tag{93}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{q_n}{p_n}\right) + 4\tan^{-1}\left(\frac{2q_n - p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{p_n^2 - 3q_n^2}{4p_nq_n - p_n^2 - q_n^2}\right) \ , n \in \mathbb{N} \tag{94}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{3}{5}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) - 4\tan^{-1}\left(\frac{1}{13}\right) \tag{95}$$

$$\pi = 4\tan^{-1}\left(\frac{4}{7}\right) + 4\tan^{-1}\left(\frac{1}{4}\right) + 4\tan^{-1}\left(\frac{1}{47}\right) \tag{96}$$

$$\pi = 4\tan^{-1}\left(\frac{11}{19}\right) + 4\tan^{-1}\left(\frac{3}{11}\right) - 4\tan^{-1}\left(\frac{1}{177}\right) \tag{97}$$

$$\pi = 4\tan^{-1}\left(\frac{15}{26}\right) + 4\tan^{-1}\left(\frac{4}{15}\right) + 4\tan^{-1}\left(\frac{1}{659}\right) \tag{98}$$

$$\pi = 4\tan^{-1}\left(\frac{41}{71}\right) + 4\tan^{-1}\left(\frac{11}{41}\right) - 4\tan^{-1}\left(\frac{1}{2461}\right) \tag{99}$$

2.10. Sea $F_n \in \mathbb{N}$ definida por:

$$F_{n+2} = F_{n+1} + F_n \quad , F_1 = 1, F_2 = 1 \tag{100}$$

$$\{F_n : n \in \mathbb{N}\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\} \tag{101}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{F_n}{F_{n+1}}\right) + 4\tan^{-1}\left(\frac{2F_n - F_{n+1}}{F_{n+1}}\right) + 4(-1)^n \tan^{-1}\left(\frac{1}{F_{2n}}\right) \quad , n \in \mathbb{N} \tag{102}$$

Observación: $F_{2n} = 2F_nF_{n+1} - F_n^2 = u_n$, $u_{n+2} = 3u_{n+1} - u_n$, $u_1 = 1, u_2 = 3$.

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{2}{3}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) - 4\tan^{-1}\left(\frac{1}{8}\right) \tag{103}$$

$$\pi = 4\tan^{-1}\left(\frac{3}{5}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{1}{21}\right) \tag{104}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{8}\right) + 4\tan^{-1}\left(\frac{1}{4}\right) - 4\tan^{-1}\left(\frac{1}{55}\right) \tag{105}$$

$$\pi = 4\tan^{-1}\left(\frac{8}{13}\right) + 4\tan^{-1}\left(\frac{3}{13}\right) + 4\tan^{-1}\left(\frac{1}{144}\right) \tag{106}$$

$$\pi = 4\tan^{-1}\left(\frac{13}{21}\right) + 4\tan^{-1}\left(\frac{5}{21}\right) - 4\tan^{-1}\left(\frac{1}{377}\right) \tag{107}$$

2.11. Sea $F_n \in \mathbb{N}$ definida por (100) , se tiene:

$$\pi = 4\tan^{-1}\left(\frac{F_n}{F_{n+1}+F_n}\right)+4\tan^{-1}\left(\frac{F_n}{2F_{n+1}-F_n}\right) \\ +4(-1)^n\tan^{-1}\left(\frac{1}{F_{n+1}^2+2F_{n+1}F_n-F_n^2}\right) \quad ,n\in\mathbb{N} \tag{108}$$

Observación: $u_n = F_{n+1}^2+2F_{n+1}F_n-F_n^2$, $u_{n+3}=2u_{n+2}+2u_{n+1}-u_n$, $u_1=2,u_2=7,u_3=17$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{2}{5}\right)+4\tan^{-1}\left(\frac{1}{2}\right)-4\tan^{-1}\left(\frac{1}{17}\right) \tag{109}$$

$$\pi = 4\tan^{-1}\left(\frac{3}{8}\right)+4\tan^{-1}\left(\frac{3}{7}\right)+4\tan^{-1}\left(\frac{1}{46}\right) \tag{110}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{13}\right)+4\tan^{-1}\left(\frac{5}{11}\right)-4\tan^{-1}\left(\frac{1}{119}\right) \tag{111}$$

$$\pi = 4\tan^{-1}\left(\frac{8}{21}\right)+4\tan^{-1}\left(\frac{4}{9}\right)+4\tan^{-1}\left(\frac{1}{313}\right) \tag{112}$$

$$\pi = 4\tan^{-1}\left(\frac{13}{34}\right)+4\tan^{-1}\left(\frac{13}{29}\right)-4\tan^{-1}\left(\frac{1}{818}\right) \tag{113}$$

2.12. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1}=2p_n+6q_n \quad , \quad q_{n+1}=p_n+2q_n \quad , p_1=5, q_1=2 \tag{114}$$

$$\{(p_n,q_n):n\in\mathbb{N}\}=\{(5,2),(22,9),(98,40),(436,178),...\} \tag{115}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n-2q_n}{q_n}\right)+4\tan^{-1}\left(\frac{2p_n-3q_n}{5q_n}\right) \\ +4\tan^{-1}\left(\frac{6q_n^2-p_n^2}{7p_nq_n-p_n^2-7q_n^2}\right) \quad ,n\in\mathbb{N} \tag{116}$$

Observación:

$6q_n^2-p_n^2=(-1)^n 2^{n-1}$, $u_n=7p_nq_n-p_n^2-7q_n^2$, $u_{n+3}=18u_{n+2}+36u_{n+1}-8u_n$,
$u_1=17, u_2=335, u_3=6636$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{2}{5}\right) - 4\tan^{-1}\left(\frac{1}{17}\right) \tag{117}$$

$$\pi = 4\tan^{-1}\left(\frac{4}{9}\right) + 4\tan^{-1}\left(\frac{17}{45}\right) + 4\tan^{-1}\left(\frac{2}{335}\right) \tag{118}$$

$$\pi = 4\tan^{-1}\left(\frac{9}{20}\right) + 4\tan^{-1}\left(\frac{19}{50}\right) - 4\tan^{-1}\left(\frac{1}{1659}\right) \tag{119}$$

$$\pi = 4\tan^{-1}\left(\frac{40}{89}\right) + 4\tan^{-1}\left(\frac{169}{445}\right) + 4\tan^{-1}\left(\frac{2}{32843}\right) \tag{120}$$

$$\pi = 4\tan^{-1}\left(\frac{89}{198}\right) + 4\tan^{-1}\left(\frac{188}{495}\right) - 4\tan^{-1}\left(\frac{1}{162557}\right) \tag{121}$$

2.13. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = 3p_n + q_n \quad , \quad q_{n+1} = 10p_n + 3q_n \quad , p_1 = 1, q_1 = 3 \tag{122}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(1,3),(6,19),(37,117),(228,721),(1405,4443),...\} \tag{123}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{11p_n - 2q_n}{9p_n}\right) + 4(-1)^n \tan^{-1}\left(\frac{1}{11p_nq_n - p_n^2 - q_n^2}\right) \quad , n \in \mathbb{N} \tag{124}$$

Observación:

$u_n = 11p_nq_n - p_n^2 - q_n^2$, $u_{n+3} = 37u_{n+2} + 37u_{n+1} - u_n$
$u_1 = 23, u_2 = 857, u_3 = 32561$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{5}{9}\right) - 4\tan^{-1}\left(\frac{1}{23}\right) \tag{125}$$

$$\pi = 4\tan^{-1}\left(\frac{6}{19}\right) + 4\tan^{-1}\left(\frac{14}{27}\right) + 4\tan^{-1}\left(\frac{1}{857}\right) \tag{126}$$

$$\pi = 4\tan^{-1}\left(\frac{37}{117}\right) + 4\tan^{-1}\left(\frac{173}{333}\right) - 4\tan^{-1}\left(\frac{1}{32561}\right) \tag{127}$$

$$\pi = 4\tan^{-1}\left(\frac{228}{721}\right) + 4\tan^{-1}\left(\frac{533}{1026}\right) + 4\tan^{-1}\left(\frac{1}{1236443}\right) \tag{128}$$

$$\pi = 4\tan^{-1}\left(\frac{1405}{4443}\right) + 4\tan^{-1}\left(\frac{6569}{12645}\right) - 4\tan^{-1}\left(\frac{1}{46952291}\right) \tag{129}$$

2.14. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 2q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 3, q_1 = 2 \tag{130}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(3,2),(7,5),(17,12),(41,29),(99,70),\ldots\} \tag{131}$$

Se tiene:

$$\begin{aligned} \pi = {} & 4\tan^{-1}\left(\frac{p_n - q_n}{2q_n}\right) + 4\tan^{-1}\left(\frac{2q_n - p_n}{2q_n}\right) + 4\tan^{-1}\left(\frac{21q_n - 6p_n}{41q_n}\right) \\ & + 4\tan^{-1}\left(\frac{3p_n^3 + p_n^2 q_n - 6p_n q_n^2 - 2q_n^3}{-3p_n^3 + 40p_n^2 q_n - 105p_n q_n^2 + 206q_n^3}\right) \quad , n \in \mathbb{N} \end{aligned} \tag{132}$$

Ejemplos:

$$\pi = 8\tan^{-1}\left(\frac{1}{4}\right) + 4\tan^{-1}\left(\frac{12}{41}\right) + 4\tan^{-1}\left(\frac{11}{1027}\right) \tag{133}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{3}{10}\right) + 4\tan^{-1}\left(\frac{63}{205}\right) - 4\tan^{-1}\left(\frac{1}{621}\right) \tag{134}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{24}\right) + 4\tan^{-1}\left(\frac{7}{24}\right) + 4\tan^{-1}\left(\frac{25}{82}\right) + 4\tan^{-1}\left(\frac{21}{74303}\right) \tag{135}$$

$$\pi = 4\tan^{-1}\left(\frac{6}{29}\right) + 4\tan^{-1}\left(\frac{17}{58}\right) + 4\tan^{-1}\left(\frac{363}{1189}\right) - 4\tan^{-1}\left(\frac{76}{1573413}\right) \tag{136}$$

2.15. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = 2p_n + 7q_n \quad , \quad q_{n+1} = p_n + 2q_n \quad , p_1 = 8, q_1 = 3 \tag{137}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(8,3), (37,14), (172,65), (799,302), ...\} \tag{138}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{11q_n - 4p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{5q_n - 2p_n}{2p_n - 6q_n}\right) \quad , n \in \mathbb{N} \tag{139}$$

$$\begin{aligned} \pi = 4\tan^{-1}\left(\frac{11q_n - 4p_n}{q_n}\right) + 4\tan^{-1}\left(\frac{p_n - q_n}{4q_n}\right) \\ + 4\tan^{-1}\left(\frac{2p_n^2 - 14q_n^2}{2p_n^2 - 15p_n q_n + 29q_n^2}\right) \quad , n \in \mathbb{N} \end{aligned} \tag{140}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{1}{2}\right) \tag{141}$$

$$\pi = 4\tan^{-1}\left(\frac{3}{7}\right) + 4\tan^{-1}\left(\frac{2}{5}\right) \tag{142}$$

$$\pi = 4\tan^{-1}\left(\frac{27}{65}\right) + 4\tan^{-1}\left(\frac{19}{46}\right) \tag{143}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{5}{12}\right) + 4\tan^{-1}\left(\frac{2}{29}\right) \tag{144}$$

$$\pi = 4\tan^{-1}\left(\frac{3}{7}\right) + 4\tan^{-1}\left(\frac{23}{56}\right) - 4\tan^{-1}\left(\frac{3}{326}\right) \tag{145}$$

$$\pi = 4\tan^{-1}\left(\frac{27}{65}\right) + 4\tan^{-1}\left(\frac{107}{260}\right) + 4\tan^{-1}\left(\frac{18}{13993}\right) \tag{146}$$

$$\pi = 4\tan^{-1}\left(\frac{63}{151}\right) + 4\tan^{-1}\left(\frac{497}{1208}\right) - 4\tan^{-1}\left(\frac{27}{151124}\right) \tag{147}$$

2.16. Sean $p_n, q_n, r_n, s_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 3q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 5, q_1 = 3 \tag{148}$$

$$r_{n+1} = 2r_n + 3s_n \quad , \quad s_{n+1} = r_n + 2s_n \quad , r_1 = 7, s_1 = 4 \tag{149}$$

$$\{(p_n,q_n):n\in\mathbb{N}\}=\{(5,3),(14,8),(38,22),\ldots\} \tag{150}$$

$$\{(r_n,s_n):n\in\mathbb{N}\}=\{(7,4),(26,15),(97,56),\ldots\} \tag{151}$$

Se tiene:

$$\pi=4\tan^{-1}\left(\frac{2q_n-p_n}{q_n}\right)+4\tan^{-1}\left(\frac{2s_n-r_n}{s_n}\right)+4\tan^{-1}\left(\frac{-p_nr_n+3q_nr_n+3p_ns_n-7q_ns_n}{p_n(s_n-r_n)+q_n(s_n+r_n)}\right)\quad ,n\in\mathbb{N} \tag{152}$$

Ejemplos:

$$\pi=4\tan^{-1}\left(\frac{1}{3}\right)+4\tan^{-1}\left(\frac{1}{4}\right)+4\tan^{-1}\left(\frac{2}{9}\right) \tag{153}$$

$$\pi=4\tan^{-1}\left(\frac{1}{4}\right)+4\tan^{-1}\left(\frac{4}{15}\right)+4\tan^{-1}\left(\frac{25}{87}\right) \tag{154}$$

$$\pi=4\tan^{-1}\left(\frac{3}{11}\right)+4\tan^{-1}\left(\frac{15}{56}\right)+4\tan^{-1}\left(\frac{119}{452}\right) \tag{155}$$

2.17. Una fórmula de Euler:

$$\pi=20\tan^{-1}\left(\frac{1}{7}\right)+8\tan^{-1}\left(\frac{79}{3}\right) \tag{156}$$

Dos transformaciones de la fórmula anterior son:

$$\pi=20\tan^{-1}\left(\frac{1}{7}\right)+8\tan^{-1}\left(\frac{1}{27}\right)+4\tan^{-1}\left(\frac{1}{534}\right)+4\tan^{-1}\left(\frac{1}{609094818}\right) \tag{157}$$

$$\pi=20\tan^{-1}\left(\frac{1}{7}\right)+8\tan^{-1}\left(\frac{1}{26}\right)-4\tan^{-1}\left(\frac{1}{1028}\right)+4\tan^{-1}\left(\frac{1}{2115626}\right)-4\tan^{-1}\left(\frac{1}{4601197828405182}\right) \tag{158}$$

2.18. Sea $F_n\in\mathbb{N}$ definida por (100) , se tiene:

$$\pi=4\tan^{-1}\left(\frac{F_{n+1}}{F_n}\right)-4\tan^{-1}\left(\frac{2F_{n+1}-3F_n}{F_n}\right)+4(-1)^n\tan^{-1}\left(\frac{1}{F_{2n-1}}\right)\quad ,n\in\mathbb{N} \tag{159}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{3}{2}\right) - 4\tan^{-1}\left(\frac{1}{5}\right) \tag{160}$$

$$\pi = 4\tan^{-1}\left(\frac{5}{3}\right) - 4\tan^{-1}\left(\frac{1}{3}\right) + 4\tan^{-1}\left(\frac{1}{13}\right) \tag{161}$$

$$\pi = 4\tan^{-1}\left(\frac{8}{5}\right) - 4\tan^{-1}\left(\frac{1}{5}\right) - 4\tan^{-1}\left(\frac{1}{34}\right) \tag{162}$$

$$\pi = 4\tan^{-1}\left(\frac{13}{8}\right) - 4\tan^{-1}\left(\frac{1}{4}\right) + 4\tan^{-1}\left(\frac{1}{89}\right) \tag{163}$$

$$\pi = 4\tan^{-1}\left(\frac{21}{13}\right) - 4\tan^{-1}\left(\frac{3}{13}\right) - 4\tan^{-1}\left(\frac{1}{233}\right) \tag{164}$$

2.19.

$$\begin{aligned}\pi = 28\tan^{-1}\left(\frac{1}{9}\right) + 4\tan^{-1}\left(\frac{5}{463}\right) - 4\tan^{-1}\left(\frac{1}{908170}\right) \\ + 4\tan^{-1}\left(\frac{67}{185573929350115}\right)\end{aligned} \tag{165}$$

2.20. Sean $p_n, q_n, u_n, v_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = 2p_n + 3q_n \quad , \quad q_{n+1} = p_n + 2q_n \quad , p_1 = 3, q_1 = 2 \tag{166}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(3,2),(12,7),(45,26),(168,97),...\} \tag{167}$$

$$\begin{aligned} v_n &= p_n p_{n+1} + q_n q_{n+1} \\ v_{n+3} &= 15v_{n+2} + 15v_{n+1} - v_n \quad , v_1 = 50, v_2 = 722, v_3 = 10082 \end{aligned} \tag{168}$$

$$u_n = p_n + q_n \quad , \quad u_{n+2} = 4u_{n+1} - u_n \quad , u_1 = 5, u_2 = 19 \tag{169}$$

Se tiene:

$$\begin{aligned}\pi &= 3\tan^{-1}\left(\frac{3}{2}\right) + 3\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3}{p_n p_{n+1} + q_n q_{n+1}}\right) = \\ &= 3\tan^{-1}\left(\frac{3}{2}\right) + 3\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3}{v_n}\right) = 3\tan^{-1}\left(\frac{3}{2}\right) + 3\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{3}{2u_n^2}\right)\end{aligned} \tag{170}$$

2.21. Sean $p_n, q_n, u_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 3q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 3, q_1 = 2 \tag{171}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(3,2),(9,5),(24,14),(66,38),...\} \tag{172}$$

$$\begin{aligned} &u_n = 2^{-n+1}\left(p_n p_{n+1} + q_n q_{n+1}\right) \\ &u_{n+3} = 3u_{n+2} + 3u_{n+1} - u_n \quad , u_1 = 37, u_2 = 143, u_3 = 529 \end{aligned} \tag{173}$$

Se tiene:

$$\begin{aligned} \pi &= 3\tan^{-1}\left(\frac{3}{2}\right) + 3\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{3\cdot 2^{n-1}}{p_n p_{n+1} + q_n q_{n+1}}\right) = \\ &= 3\tan^{-1}\left(\frac{3}{2}\right) + 3\sum_{n=1}^{\infty}(-1)^{n-1}\tan^{-1}\left(\frac{3}{u_n}\right) \end{aligned} \tag{174}$$

2.22. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 3q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 3, q_1 = 2 \tag{175}$$

Se tiene:

$$\pi = 3\tan^{-1}\left(\frac{p_n}{q_n}\right) + \tan^{-1}\left(\frac{2p_n^3 q_n^2 + 3p_n q_n^4 - p_n^5}{2p_n^2 q_n^3 + 3p_n^4 q_n - q_n^5}\right) \quad , n \in \mathbb{N} \tag{176}$$

Ejemplos:

$$\pi = 3\tan^{-1}\left(\frac{3}{2}\right) + \tan^{-1}\left(\frac{9}{46}\right) \tag{177}$$

$$\pi = 3\tan^{-1}\left(\frac{9}{5}\right) - \tan^{-1}\left(\frac{27}{545}\right) \tag{178}$$

$$\pi = 3\tan^{-1}\left(\frac{12}{7}\right) + \tan^{-1}\left(\frac{36}{2681}\right) \tag{179}$$

$$\pi = 3\tan^{-1}\left(\frac{33}{19}\right) - \tan^{-1}\left(\frac{99}{27607}\right) \tag{180}$$

2.23. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = 2p_n + q_n \quad , \quad q_{n+1} = 3p_n + 2q_n \quad , p_1 = 1, q_1 = 2 \tag{181}$$

$$\{(p_n,q_n):n\in\mathbb{N}\}=\{(1,2),(4,7),(15,26),(56,97),...\} \tag{182}$$

Se tiene:

$$\begin{aligned}\pi &= 3\tan^{-1}\left(\frac{p_n}{q_n}\right)+3\tan^{-1}\left(\left(p_n^2+q_n^2\right)\sqrt{3}-4p_nq_n\right)= \\ &= 3\tan^{-1}\left(\left(p_n^2+q_n^2\right)\sqrt{3}+4p_nq_n\right)-3\tan^{-1}\left(\frac{p_n}{q_n}\right) \quad ,n\in\mathbb{N}\end{aligned} \tag{183}$$

Observación: $q_n^2-3p_n^2=1$.

Ejemplos:

$$\pi=3\tan^{-1}\left(\frac{1}{2}\right)+3\tan^{-1}\left(5\sqrt{3}-8\right)=3\tan^{-1}\left(5\sqrt{3}+8\right)-3\tan^{-1}\left(\frac{1}{2}\right) \tag{184}$$

$$\pi=3\tan^{-1}\left(\frac{4}{7}\right)+3\tan^{-1}\left(65\sqrt{3}-112\right)=3\tan^{-1}\left(65\sqrt{3}+112\right)-3\tan^{-1}\left(\frac{4}{7}\right) \tag{185}$$

$$\begin{aligned}\pi &= 3\tan^{-1}\left(\frac{15}{26}\right)+3\tan^{-1}\left(901\sqrt{3}-1560\right)= \\ &= 3\tan^{-1}\left(901\sqrt{3}+1560\right)-3\tan^{-1}\left(\frac{15}{26}\right)\end{aligned} \tag{186}$$

2.24. Sea $n\in\mathbb{N}$, se tiene:

$$\pi=4\tan^{-1}\left(\frac{n}{\sqrt{n^2-1}}\right)-4\tan^{-1}\left(\left(n-\sqrt{n^2-1}\right)^2\right) \tag{187}$$

$$\pi=4\tan^{-1}\left(\frac{\sqrt{n^2-1}}{n}\right)+4\tan^{-1}\left(\left(n-\sqrt{n^2-1}\right)^2\right) \tag{188}$$

$$\pi=4\tan^{-1}\left(\frac{n}{\sqrt{n^2+1}}\right)+4\tan^{-1}\left(\left(\sqrt{n^2+1}-n\right)^2\right) \tag{189}$$

$$\pi=4\tan^{-1}\left(\frac{\sqrt{n^2+1}}{n}\right)-4\tan^{-1}\left(\left(\sqrt{n^2+1}-n\right)^2\right) \tag{190}$$

2.25. Sea $n\in\mathbb{N}$, se tiene:

$$\pi = 4\tan^{-1}\left(\left((n+1)(n+2)\right)^{1/2} - \left(n(n+1)\right)^{1/2}\right)$$
$$-4\tan^{-1}\left(\begin{array}{l} 3+4n(n+2)-(6+4n)\left(n(n+1)\right)^{1/2} \\ \qquad -(2+4n)\left((n+1)(n+2)\right)^{1/2}+4(n+1)\left(n(n+2)\right)^{1/2} \end{array}\right) \tag{191}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\sqrt{6}-\sqrt{2}\right) - 4\tan^{-1}\left(15-10\sqrt{2}+8\sqrt{3}-6\sqrt{6}\right) \tag{192}$$

$$\pi = 4\tan^{-1}\left(2\sqrt{3}-\sqrt{6}\right) - 4\tan^{-1}\left(35+24\sqrt{2}-20\sqrt{3}-14\sqrt{6}\right) \tag{193}$$

$$\pi = 4\tan^{-1}\left(2\sqrt{5}-2\sqrt{3}\right) - 4\tan^{-1}\left(63-36\sqrt{3}-28\sqrt{5}+16\sqrt{15}\right) \tag{194}$$

2.26. Sean $p_n, q_n \in \mathbb{N}, u_n, v_n$ definidas por:

$$p_{n+1} = p_n + 2q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 1, q_1 = 1 \tag{195}$$

$$\left\{(p_n, q_n) : n \in \mathbb{N}\right\} = \left\{(1,1),(3,2),(7,5),(17,12),(41,29),\ldots\right\} \tag{196}$$

$$u_n = -1 - p_n + q_n\sqrt{2} \quad , \quad v_n = (-1)^n\left((-1)^{n-1} - 2p_n - 2q_n\sqrt{2}\right) \tag{197}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(u_{2n-1}\right) + 4\tan^{-1}\left(v_{2n-1}\right) \quad , n \in \mathbb{N} \tag{198}$$

$$\pi = -\frac{4}{3}\tan^{-1}\left(u_{2n}\right) - \frac{4}{3}\tan^{-1}\left(v_{2n}\right) \quad , n \in \mathbb{N} \tag{199}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(2\sqrt{2}+1\right) - 4\tan^{-1}\left(2-\sqrt{2}\right) \tag{200}$$

$$\pi = 4\tan^{-1}\left(13+10\sqrt{2}\right) - 4\tan^{-1}\left(8-5\sqrt{2}\right) \tag{201}$$

$$\pi = \frac{4}{3}\tan^{-1}\left(7+4\sqrt{2}\right) + \frac{4}{3}\tan^{-1}\left(4-2\sqrt{2}\right) \tag{202}$$

$$\pi = \frac{4}{3}\tan^{-1}\left(35+24\sqrt{2}\right) + \frac{4}{3}\tan^{-1}\left(18-12\sqrt{2}\right) \tag{203}$$

Observación: $p_n^2 - 2q_n^2 = (-1)^n$.

2.27.

$$\pi = 4\tan^{-1}\left(\sqrt{4-2\sqrt{2}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{4-2\sqrt{2}}\right)^2\left(3+2\sqrt{2}\right)\right) \qquad (204)$$

$$\pi = 4\tan^{-1}\left(\sqrt{2\sqrt{2}-2}\right) + 4\tan^{-1}\left(\left(1-\sqrt{2\sqrt{2}-2}\right)^2\left(3+2\sqrt{2}\right)\right) \qquad (205)$$

$$\pi = 4\tan^{-1}\left(\sqrt{4+2\sqrt{2}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{4+2\sqrt{2}}\right)^2\left(3-2\sqrt{2}\right)\right) \qquad (206)$$

$$\pi = 4\tan^{-1}\left(\sqrt{3-\sqrt{3}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{3-\sqrt{3}}\right)^2\left(2+\sqrt{3}\right)\right) \qquad (207)$$

$$\pi = 4\tan^{-1}\left(\sqrt{\sqrt{3}-1}\right) + 4\tan^{-1}\left(\left(1-\sqrt{\sqrt{3}-1}\right)^2\left(2+\sqrt{3}\right)\right) \qquad (208)$$

$$\pi = 4\tan^{-1}\left(\sqrt{3+\sqrt{3}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{3+\sqrt{3}}\right)^2\left(2-\sqrt{3}\right)\right) \qquad (209)$$

$$\pi = 4\tan^{-1}\left(\sqrt{10-4\sqrt{5}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{10-4\sqrt{5}}\right)^2\left(9+4\sqrt{5}\right)\right) \qquad (210)$$

$$\pi = 4\tan^{-1}\left(\sqrt{4\sqrt{5}-8}\right) + 4\tan^{-1}\left(\left(1-\sqrt{4\sqrt{5}-8}\right)^2\left(9+4\sqrt{5}\right)\right) \qquad (211)$$

$$\pi = 4\tan^{-1}\left(\sqrt{10+4\sqrt{5}}\right) - 4\tan^{-1}\left(\left(1-\sqrt{10+4\sqrt{5}}\right)^2\left(9-4\sqrt{5}\right)\right) \qquad (212)$$

Observación: $\left(\sqrt{2}\pm 1\right)^2 = 3\pm 2\sqrt{2}$, $\left(\sqrt{5}\pm 2\right)^2 = 9\pm 4\sqrt{5}$.

2.28.

$$\pi = 4\tan^{-1}\left(\sqrt{8-5\sqrt{2}}\right) + 4\tan^{-1}\left(\left(1-\sqrt{8-5\sqrt{2}}\right)^2\left(7+5\sqrt{2}\right)\right) \qquad (213)$$

$$\pi = 4\tan^{-1}\left(\sqrt{5\sqrt{2}-6}\right) - 4\tan^{-1}\left(\left(1-\sqrt{5\sqrt{2}-6}\right)^2\left(7+5\sqrt{2}\right)\right) \qquad (214)$$

$$\pi = 4\tan^{-1}\left(\sqrt{8+5\sqrt{2}}\right) + 4\tan^{-1}\left(\left(1-\sqrt{8+5\sqrt{2}}\right)^2\left(7-5\sqrt{2}\right)\right) \tag{215}$$

2.29. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+2} = 5(2n+3)\,p_{n+1} + (5n+1)(5n+9)\,p_n \quad , p_1 = 4, p_2 = 60 \tag{216}$$

$$q_{n+2} = 5(2n+3)\,q_{n+1} + (5n+1)(5n+9)\,q_n \quad , q_1 = 5, q_2 = 84 \tag{217}$$

Se tiene:

$$\pi = 5\tan^{-1}\left(\frac{p_n}{q_n}\right) + 5\tan^{-1}\left(\frac{q_n\sqrt{5-2\sqrt{5}} - p_n}{p_n\sqrt{5-2\sqrt{5}} + q_n}\right) \quad , n \in \mathbb{N} \tag{218}$$

2.30.

$$\pi = 20\tan^{-1}\left(\frac{1}{9}\right) - 20\tan^{-1}\left(\frac{5\sqrt{5-2\sqrt{5}} - 4}{4\sqrt{5-2\sqrt{5}} + 5}\right) \tag{219}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{6}\right) - 20\tan^{-1}\left(\frac{7\sqrt{5-2\sqrt{5}} - 5}{5\sqrt{5-2\sqrt{5}} + 7}\right) \tag{220}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{72}\right) - 20\tan^{-1}\left(\frac{1}{31182}\right) - 20\tan^{-1}\left(\frac{70\sqrt{5-2\sqrt{5}} - 51}{51\sqrt{5-2\sqrt{5}} + 70}\right) \tag{221}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{6}\right) - 20\tan^{-1}\left(\frac{1}{117}\right) - 20\tan^{-1}\left(\frac{11\sqrt{5-2\sqrt{5}} - 8}{8\sqrt{5-2\sqrt{5}} + 11}\right) \tag{222}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{6}\right) - 20\tan^{-1}\left(\frac{1}{124}\right) + 20\tan^{-1}\left(\frac{2}{77133}\right) - 20\tan^{-1}\left(\frac{117\sqrt{5-2\sqrt{5}} - 85}{85\sqrt{5-2\sqrt{5}} + 117}\right) \tag{223}$$

2.31. Sea $p_n \in \mathbb{N}$ definida por:

$$p_{n+4}=8p_{n+3}-2p_{n+2}-8p_{n+1}-p_n \quad , p_1=1, p_2=8, p_3=62, p_4=472 \tag{224}$$

$$\left\{p_n : n\in\mathbb{N}\right\}=\{1,8,62,472,3587,27248,206972,...\} \tag{225}$$

Se tiene:

$$\pi=24\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right)+24\tan^{-1}\left(\frac{p_{n+1}\left(\sqrt{6}-\sqrt{3}+\sqrt{2}-2\right)-p_n}{p_{n+1}+p_n\left(\sqrt{6}-\sqrt{3}+\sqrt{2}-2\right)}\right) \quad ,n\in\mathbb{N} \tag{226}$$

Ejemplo:

$$\begin{aligned}\pi&=24\tan^{-1}\left(\frac{1}{8}\right)+24\tan^{-1}\left(\frac{8\sqrt{6}-8\sqrt{3}+8\sqrt{2}-17}{\sqrt{6}-\sqrt{3}+\sqrt{2}+6}\right)=\\&=24\tan^{-1}\left(\frac{1}{8}\right)+24\tan^{-1}\left(\frac{3055\sqrt{6}-4225\sqrt{3}+5135\sqrt{2}-7426}{193}\right)\end{aligned} \tag{227}$$

2.32.

$$\pi=8\tan^{-1}\left(6-4\sqrt{2}\right)+8\tan^{-1}\left(\frac{1}{9+5\sqrt{2}}\right) \tag{228}$$

$$\pi=8\tan^{-1}\left(6\sqrt{2}-8\right)-8\tan^{-1}\left(\frac{1}{7+7\sqrt{2}}\right) \tag{229}$$

$$\pi=8\tan^{-1}\left(6+6\sqrt{2}\right)-8\tan^{-1}\left(\frac{1}{35\sqrt{2}-49}\right) \tag{230}$$

$$\pi=8\tan^{-1}\left(2-\sqrt{2}\right)-8\tan^{-1}\left(\frac{1}{3+3\sqrt{2}}\right) \tag{231}$$

$$\pi=8\tan^{-1}\left(3\sqrt{2}-4\right)+8\tan^{-1}\left(\frac{1}{5+\sqrt{2}}\right) \tag{232}$$

$$\pi=8\tan^{-1}\left(3\sqrt{2}+2\right)-8\tan^{-1}\left(\frac{1}{19-13\sqrt{2}}\right) \tag{233}$$

2.33.

$$\pi=4\tan^{-1}\left(\frac{1}{4+2\sqrt{2}}\right)+4\tan^{-1}\left(\frac{1}{7-4\sqrt{2}}\right) \tag{234}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{2\sqrt{2}-2}\right) - 4\tan^{-1}\left(\frac{1}{4\sqrt{2}+5}\right) \tag{235}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{4-2\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{1}{7+4\sqrt{2}}\right) \tag{236}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{4\sqrt{2}-5}\right) - 4\tan^{-1}\left(\frac{1}{2\sqrt{2}+2}\right) \tag{237}$$

2.34. Sea $p_n \in \mathbb{N}$ definida por:

$$p_{n+4} = 4p_{n+3} + 14p_{n+2} + 4p_{n+1} - p_n \quad , p_1 = 1,\ p_2 = 4,\ p_3 = 30,\ p_4 = 180 \tag{238}$$

$$\{p_n : n \in \mathbb{N}\} = \{1, 4, 30, 180, 1155, 7256, 45884, ...\} \tag{239}$$

Se tiene:

$$\pi = 20\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 20\tan^{-1}\left(\frac{p_{n+1}\left(1+\sqrt{5}-\sqrt{5+2\sqrt{5}}\right) - p_n}{p_n\left(1+\sqrt{5}-\sqrt{5+2\sqrt{5}}\right) + p_{n+1}}\right) \quad , n \in \mathbb{N} \tag{240}$$

2.35.

$$\pi = 8\tan^{-1}\left(7+5\sqrt{2}\right) - 8\tan^{-1}(2) \tag{241}$$

$$\pi = 8\tan^{-1}\left(\sqrt{3}-1\right) - 8\tan^{-1}\left(5+5\sqrt{2}-4\sqrt{3}-2\sqrt{6}\right) \tag{242}$$

$$\pi = \frac{4}{3}\tan^{-1}\left(19+8\sqrt{5}\right) + \frac{4}{3}\tan^{-1}\left(10-4\sqrt{5}\right) \tag{243}$$

$$\pi = \frac{4}{3}\tan^{-1}\left(17+6\sqrt{7}\right) + \frac{4}{3}\tan^{-1}\left(9-3\sqrt{7}\right) \tag{244}$$

$$\pi = \frac{4}{3}\tan^{-1}\left(21+6\sqrt{11}\right) + \frac{4}{3}\tan^{-1}\left(11-3\sqrt{11}\right) \tag{245}$$

$$\pi = 4\tan^{-1}\left(\sqrt{3}-\sqrt{2}\right) + 4\tan^{-1}\left(\sqrt{2-\sqrt{3}}\right) \tag{246}$$

$$\pi = 4\tan^{-1}\left(\sqrt{2}\right) - 4\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right) \tag{247}$$

$$\pi = 6\tan^{-1}\left(7+3\sqrt{3}\right) - 6\tan^{-1}\left(17\sqrt{3}-28\right) \tag{248}$$

$$\pi = \frac{6}{5}\tan^{-1}\left(17\sqrt{3}+28\right) + \frac{6}{5}\tan^{-1}\left(7-3\sqrt{3}\right) \tag{249}$$

$$\pi = 3\tan^{-1}\left(9\sqrt{3}+16\right) - 3\tan^{-1}\left(4-2\sqrt{3}\right) \tag{250}$$

$$\pi = 3\tan^{-1}\left(4+2\sqrt{3}\right) - 3\tan^{-1}\left(16-9\sqrt{3}\right) \tag{251}$$

2.36.

$$\pi = 4\tan^{-1}\left(\sqrt{\frac{n}{n+1}}\right) + 4\tan^{-1}\left(\left(\sqrt{n+1}-\sqrt{n}\right)^2\right) \quad ,n \in \mathbb{N} \tag{252}$$

2.37. Sea $p_n \in \mathbb{N}$ definida por:

$$p_{n+2} = 10p_{n+1} - 5p_n \quad , p_1 = 1, p_2 = 10 \tag{253}$$

$$\{p_n : n \in \mathbb{N}\} = \{1, 10, 95, 900, 8525, \ldots\} \tag{254}$$

Se tiene:

$$\pi = 5\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + 5\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{\sqrt{5}\left(p_{n+1}-\sqrt{p_n p_{n+2}}\right)}{5\sqrt{p_n p_{n+1}}+\sqrt{p_{n+1}p_{n+2}}}\right) \tag{255}$$

2.38.

$$\pi = \frac{8}{3}\tan^{-1}\left(\sqrt{2}+1\right) \tag{256}$$

$$\pi = 4\tan^{-1}\left(\sqrt{2}+1\right) - 4\tan^{-1}\left(\sqrt{2}-1\right) \tag{257}$$

$$\pi = \frac{16}{3}\tan^{-1}\left(\frac{1}{\sqrt{2}-1+\sqrt{4-2\sqrt{2}}}\right) \tag{258}$$

$$\pi = \frac{16}{5}\tan^{-1}\left(\sqrt{2}-1+\sqrt{4-2\sqrt{2}}\right) \tag{259}$$

$$\pi = 8\tan^{-1}\left(\sqrt{2}-1+\sqrt{4-2\sqrt{2}}\right) - 8\tan^{-1}\left(\frac{1}{\sqrt{2}-1+\sqrt{4-2\sqrt{2}}}\right) \tag{260}$$

$$\pi = \frac{24}{5}\tan^{-1}\left(\frac{1}{2-\sqrt{3}+\sqrt{8-4\sqrt{3}}}\right) \tag{261}$$

$$\pi = \frac{24}{7}\tan^{-1}\left(2-\sqrt{3}+\sqrt{8-4\sqrt{3}}\right) \tag{262}$$

$$\pi = 12\tan^{-1}\left(2-\sqrt{3}+\sqrt{8-4\sqrt{3}}\right) - 12\tan^{-1}\left(\frac{1}{2-\sqrt{3}+\sqrt{8-4\sqrt{3}}}\right) \tag{263}$$

$$\pi = \frac{10}{3}\tan^{-1}\left(\frac{\sqrt{25+10\sqrt{5}}}{5}\right) \tag{264}$$

2.39. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = p_n + 2q_n \quad , \quad q_{n+1} = p_n + q_n \quad , p_1 = 3, q_1 = 2 \tag{265}$$

$$\left\{\left(p_n, q_n\right) : n \in \mathbb{N}\right\} = \left\{(3,2),(7,5),(17,12),\ldots\right\} \tag{266}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n}\right) - 4\tan^{-1}\left(\frac{q_n}{2p_n+3q_n}\right) + 4(-1)^n\tan^{-1}\left(\frac{1}{p_n^2+3p_nq_n+q_n^2}\right) \tag{267}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{3}{2}\right) - 4\tan^{-1}\left(\frac{1}{6}\right) - 4\tan^{-1}\left(\frac{1}{31}\right) \tag{268}$$

$$\pi = 4\tan^{-1}\left(\frac{7}{5}\right) - 4\tan^{-1}\left(\frac{5}{29}\right) + 4\tan^{-1}\left(\frac{1}{179}\right) \tag{269}$$

2.40. Sea $n \in \mathbb{N}$, se tiene:

$$\pi = 4\tan^{-1}\left(\sqrt{n+2\sqrt{n+1}}+\sqrt{n+1}\right) - 4\tan^{-1}\left(\sqrt{n+2\sqrt{n+1}}-\sqrt{n+1}\right) \tag{270}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\sqrt{1+2\sqrt{2}}+\sqrt{2}\right)-4\tan^{-1}\left(\sqrt{1+2\sqrt{2}}-\sqrt{2}\right) \tag{271}$$

$$\pi = 4\tan^{-1}\left(\sqrt{2+2\sqrt{3}}+\sqrt{3}\right)-4\tan^{-1}\left(\sqrt{2+2\sqrt{3}}-\sqrt{3}\right) \tag{272}$$

2.41. Sea $n \in \mathbb{N}$, se tiene:

$$\pi = 6\tan^{-1}\left(\sqrt{n+2\sqrt{3n+3}}+\sqrt{n+1}\right)-6\tan^{-1}\left(\sqrt{n+2\sqrt{3n+3}}-\sqrt{n+1}\right) \tag{273}$$

Ejemplos:

$$\pi = 6\tan^{-1}\left(\sqrt{1+2\sqrt{6}}+\sqrt{2}\right)-4\tan^{-1}\left(\sqrt{1+2\sqrt{6}}-\sqrt{2}\right) \tag{274}$$

$$\pi = 6\tan^{-1}\left(\sqrt{8}+\sqrt{3}\right)-6\tan^{-1}\left(\sqrt{8}-\sqrt{3}\right) \tag{275}$$

2.42. Sea $n \in \mathbb{N}$, se tiene:

$$\pi = 2^{n+1}\tan^{-1}\left(\frac{p_n}{q_n}\right)+2^{n+1}\tan^{-1}\left(\frac{q_n s_n - p_n c_n}{q_n c_n + p_n s_n}\right) \tag{276}$$

donde

$$s_n = \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n-radicales} \tag{277}$$

$$c_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n-radicales} \tag{278}$$

$\frac{p_n}{q_n}$ es una aproximación racional de $\frac{s_n}{c_n}$

Para $n=1$ se tiene el caso trivial $\pi = 4\tan^{-1}(1)$.

Algunas aproximaciones racionales para $\frac{s_n}{c_n}$ son:

$$n=2, \frac{s_2}{c_2}=\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}, \frac{p_2}{q_2}=\left\{\frac{1}{2},\frac{2}{5},\frac{5}{12},\frac{12}{29},\frac{29}{70},\frac{70}{169},...\right\} \tag{279}$$

$$n=3,\frac{s_3}{c_3}=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}},\frac{p_3}{q_3}=\left\{\frac{1}{5},\frac{36}{181},\frac{37}{186},\frac{73}{367},\frac{183}{920},\frac{256}{1287},...\right\} \tag{280}$$

$$n=4,\frac{s_4}{c_4}=\frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}},\frac{p_4}{q_4}=\left\{\frac{1}{10},\frac{6}{61},\frac{7}{71},\frac{13}{132},\frac{111}{1127},\frac{457}{4640},...\right\} \tag{281}$$

Ejemplos:

$$\pi=16\tan^{-1}\left(\frac{1}{5}\right)+16\tan^{-1}\left(\frac{5\sqrt{2-\sqrt{2+\sqrt{2}}}-\sqrt{2+\sqrt{2+\sqrt{2}}}}{5\sqrt{2+\sqrt{2+\sqrt{2}}}+\sqrt{2-\sqrt{2+\sqrt{2}}}}\right) \tag{282}$$

$$\pi=32\tan^{-1}\left(\frac{6}{61}\right)+32\tan^{-1}\left(\frac{61\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}-6\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{61\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}+6\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right) \tag{283}$$

2.43. Sea $n\in\mathbb{N}$,se tiene:

$$\pi=4\tan^{-1}\left(\sqrt{n+1}-\sqrt{n}\right)+4\tan^{-1}\left(\sqrt{\frac{n+2-2\sqrt{n+1}}{n}}\right) \tag{284}$$

2.44. Sean $p_n,q_n\in\mathbb{N}$ definidas por:

$$p_{n+1}=3q_n \quad , \quad q_{n+1}=p_n+6q_n \quad , p_1=1, q_1=2 \tag{285}$$

$$\left\{\left(p_n,q_n\right):n\in\mathbb{N}\right\}=\left\{(1,2),(6,13),(39,84),(252,543),...\right\} \tag{286}$$

Se tiene:

$$\pi=6\tan^{-1}\left(\frac{p_n}{q_n\sqrt{3}}\right)+6\tan^{-1}\left(\frac{(q_n-p_n)\sqrt{3}}{p_n+3q_n}\right) \quad ,n\in\mathbb{N} \tag{287}$$

Ejemplos:

$$\pi=6\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)+6\tan^{-1}\left(\frac{\sqrt{3}}{7}\right) \tag{288}$$

$$\pi = 6\tan^{-1}\left(\frac{6}{13\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{7\sqrt{3}}{45}\right) \tag{289}$$

$$\pi = 6\tan^{-1}\left(\frac{39}{84\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{45\sqrt{3}}{291}\right) \tag{290}$$

2.45. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+1} = 2q_n \quad , \quad q_{n+1} = -p_n + 4q_n \quad , p_1 = 1, q_1 = 2 \tag{291}$$

$$\{(p_n, q_n) : n \in \mathbb{N}\} = \{(1,2),(4,7),(14,24),(48,82),(164,280),...\} \tag{292}$$

Se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p_n}{q_n\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{\left(q_n\sqrt{2} - p_n\right)^2}{q_{2n}}\right) \quad , n \in \mathbb{N} \tag{293}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{\left(2\sqrt{2} - 1\right)^2}{7}\right) \tag{294}$$

$$\pi = 4\tan^{-1}\left(\frac{4}{7\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{\left(7\sqrt{2} - 4\right)^2}{82}\right) \tag{295}$$

2.46.

$$\pi = 6\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{5\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{5}{13\sqrt{3}}\right) \tag{296}$$

$$\pi = 6\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{11\sqrt{3}}\right) \tag{297}$$

$$\pi = 6\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{4\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{5}{29\sqrt{3}}\right) \tag{298}$$

$$\pi = 6\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{5\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{2}{9\sqrt{3}}\right) \tag{299}$$

$$\pi = 6\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{1}{7\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{4}{9\sqrt{3}}\right) \tag{300}$$

$$\pi = 6\tan^{-1}\left(\frac{2}{3\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{2}{5\sqrt{3}}\right) - 6\tan^{-1}\left(\frac{7}{57\sqrt{3}}\right) \tag{301}$$

$$\pi = 6\tan^{-1}\left(\frac{2}{3\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{2}{7\sqrt{3}}\right) - 6\tan^{-1}\left(\frac{1}{79\sqrt{3}}\right) \tag{302}$$

$$\pi = 6\tan^{-1}\left(\frac{2}{7\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{3}{7\sqrt{3}}\right) + 6\tan^{-1}\left(\frac{9}{44\sqrt{3}}\right) \tag{303}$$

2.47.

$$\begin{aligned}\pi &= 4\tan^{-1}\left(\frac{1}{3}\left(\left(17+3\sqrt{33}\right)^{1/3} - 2\left(17+3\sqrt{33}\right)^{-1/3} - 1\right)\right) \\ &\quad + 4\tan^{-1}\left(\frac{1}{3}\left(\left(2\left(13+3\sqrt{33}\right)\right)^{1/3} - 4\cdot 2^{2/3}\left(13+3\sqrt{33}\right)^{-1/3} - 1\right)\right)\end{aligned} \tag{304}$$

2.48. Sean $p_n, q_n \in \mathbb{N}$ definidas por:

$$p_{n+3} = p_{n+2} + p_{n+1} + p_n \quad , p_1 = 1, p_2 = 2, p_3 = 4 \tag{305}$$

$$\{p_n : n \in \mathbb{N}\} = \{1, 2, 4, 7, 13, 24, 44, 81, \ldots\} \tag{306}$$

$$q_{n+3} = 3q_{n+2} + q_{n+1} + q_n \quad , q_1 = 1, q_2 = 3, q_3 = 10 \tag{307}$$

$$\{q_n : n \in \mathbb{N}\} = \{1, 3, 10, 34, 115, 389, 1316, 4452, \ldots\} \tag{308}$$

Se tiene:

$$\begin{aligned}\pi &= 4\tan^{-1}\left(\frac{p_n}{p_{n+1}}\right) + 4\tan^{-1}\left(\frac{q_n}{q_{n+1}}\right) \\ &\quad + 4\tan^{-1}\left(\frac{(p_{n+1}-p_n)q_{n+1} - (p_{n+1}+p_n)q_n}{(p_{n+1}+p_n)q_{n+1} + (p_{n+1}-p_n)q_n}\right) \quad , n \in \mathbb{N}\end{aligned} \tag{309}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{2}\right) + 4\tan^{-1}\left(\frac{3}{10}\right) + 4\tan^{-1}\left(\frac{1}{33}\right) \tag{310}$$

$$\pi = 4\tan^{-1}\left(\frac{4}{7}\right) + 4\tan^{-1}\left(\frac{5}{17}\right) - 4\tan^{-1}\left(\frac{2}{101}\right) \tag{311}$$

$$\pi = 4\tan^{-1}\left(\frac{7}{13}\right) + 4\tan^{-1}\left(\frac{34}{115}\right) + 4\tan^{-1}\left(\frac{5}{1252}\right) \tag{312}$$

2.49.

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right) \tag{313}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt{5}}\right) + 2\tan^{-1}\left(\frac{2}{\sqrt{5}}\right) \tag{314}$$

$$\pi = 4\tan^{-1}\left(\frac{2}{\sqrt{7}}\right) + 2\tan^{-1}\left(\frac{3}{4\sqrt{7}}\right) \tag{315}$$

$$\pi = 4\tan^{-1}\left(\frac{2}{\sqrt{11}}\right) + 2\tan^{-1}\left(\frac{7}{4\sqrt{11}}\right) \tag{316}$$

2.50.

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[n]{2}+1}\right) + 4\tan^{-1}\left(\frac{1}{\sqrt[n]{2^{n-1}}+1}\right) \quad , n \in \mathbb{N} \tag{317}$$

Ejemplo:

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{2}+1}\right) + 4\tan^{-1}\left(\frac{1}{\sqrt[3]{4}+1}\right) \tag{318}$$

2.51.

$$\pi = 4\tan^{-1}\left(\sqrt[n]{2}-1\right) + 4\tan^{-1}\left(\sqrt[n]{2^{n-1}}-1\right) \quad , n \in \mathbb{N} \tag{319}$$

Ejemplo:

$$\pi = 4\tan^{-1}\left(\sqrt[3]{2}-1\right) + 4\tan^{-1}\left(\sqrt[3]{4}-1\right) \tag{320}$$

2.52.

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{n}}\right) + 4\tan^{-1}\left(\frac{n-1+2\sqrt[3]{n}-2\sqrt[3]{n^2}}{n+1}\right) \quad ,n \in \mathbb{N} \tag{321}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{9}}\right) + 4\tan^{-1}\left(\frac{4-3\sqrt[3]{3}+\sqrt[3]{9}}{5}\right) \tag{322}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{\sqrt[3]{14}}\right) + 4\tan^{-1}\left(\frac{13+2\sqrt[3]{14}-2\sqrt[3]{196}}{15}\right) \tag{323}$$

2.53. Sean $0 < q < p$, se tiene:

$$\pi = 8\tan^{-1}\left(\frac{p}{\sqrt{p^2+q^2}+q}\right) - 4\tan^{-1}\left(\frac{p-q}{p+q}\right) \tag{324}$$

Ejemplos:

$$\pi = 8\tan^{-1}\left(\frac{2}{\sqrt{5}+1}\right) - 4\tan^{-1}\left(\frac{1}{3}\right) \tag{325}$$

$$\pi = 8\tan^{-1}\left(\frac{3}{\sqrt{10}+1}\right) - 4\tan^{-1}\left(\frac{1}{2}\right) \tag{326}$$

$$\pi = 8\tan^{-1}\left(\frac{3}{\sqrt{13}+2}\right) - 4\tan^{-1}\left(\frac{1}{5}\right) \tag{327}$$

2.54. Sean $0 < p < q$, se tiene:

$$\pi = 4\tan^{-1}\left(\frac{p}{2q}\right) + 4\tan^{-1}\left(\frac{pq}{p^2+2q^2}\right) + 4\tan^{-1}\left(\frac{q-p}{q+p}\right) \tag{328}$$

$$\pi = 4\tan^{-1}\left(\frac{p}{p+2q}\right) + 4\tan^{-1}\left(\frac{p(p+q)}{p^2+pq+2q^2}\right) + 4\tan^{-1}\left(\frac{q-p}{q+p}\right) \tag{329}$$

$$\pi = 4\tan^{-1}\left(\frac{2p}{p+4q}\right) + 4\tan^{-1}\left(\frac{p(p+2q)}{2p^2+pq+4q^2}\right) + 4\tan^{-1}\left(\frac{q-p}{q+p}\right) \tag{330}$$

Ejemplos:

$$\pi = 4\tan^{-1}\left(\frac{1}{4}\right) + 4\tan^{-1}\left(\frac{2}{9}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) \tag{331}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{5}\right) + 4\tan^{-1}\left(\frac{3}{11}\right) + 4\tan^{-1}\left(\frac{1}{3}\right) \tag{332}$$

$$\pi = 4\tan^{-1}\left(\frac{2}{7}\right) + 4\tan^{-1}\left(\frac{8}{25}\right) + 4\tan^{-1}\left(\frac{1}{5}\right) \tag{333}$$

2.55.

$$\pi = 8\tan^{-1}\left(\frac{\sqrt{2}-1}{2}\right) + 8\tan^{-1}\left(\frac{3\sqrt{2}-1}{17}\right) \tag{334}$$

$$\pi = 12\tan^{-1}\left(\frac{2-\sqrt{3}}{2}\right) + 12\tan^{-1}\left(\frac{6-\sqrt{3}}{33}\right) \tag{335}$$

$$\pi = 5\tan^{-1}\left(\frac{\sqrt{5-2\sqrt{5}}}{2}\right) + 5\tan^{-1}\left(\frac{\sqrt{65+2\sqrt{5}}}{29}\right) \tag{336}$$

$$\pi = 3\tan^{-1}\left(\frac{\sqrt{3}}{2}\right) + 3\tan^{-1}\left(\frac{\sqrt{3}}{5}\right) \tag{337}$$

2.56.

$$\pi = 8\tan^{-1}\left(\frac{1}{\sqrt{n}}\right) + 4\tan^{-1}\left(\frac{\left(n-1-2\sqrt{n}\right)^2}{n^2-6n+1}\right) \quad ,n \in \mathbb{N} \tag{338}$$

Ejemplos:

$$\pi = 8\tan^{-1}\left(\frac{1}{\sqrt{6}}\right) + 4\tan^{-1}\left(\left(5-2\sqrt{6}\right)^2\right) \tag{339}$$

$$\pi = 8\tan^{-1}\left(\frac{1}{\sqrt{7}}\right) + 4\tan^{-1}\left(\frac{\left(3-\sqrt{7}\right)^2}{2}\right) \tag{340}$$

2.57.

$$\pi = 12\tan^{-1}\left(\frac{1}{\sqrt{n}}\right) + 4\tan^{-1}\left(\frac{\left(n(n-3)-(3n-1)\sqrt{n}\right)^2}{n(n-1)\left(n^2-14n+1\right)}\right) \quad ,n\in\mathbb{N} \tag{341}$$

Ejemplo:

$$\pi = 12\tan^{-1}\left(\frac{1}{\sqrt{14}}\right) + 4\tan^{-1}\left(\frac{\left(154-41\sqrt{14}\right)^2}{182}\right) \tag{342}$$

2.58.

$$\pi = 4\tan^{-1}\left(\frac{2\sqrt{3}+4\sqrt{2}}{7\sqrt{3}+7\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{5\sqrt{3}+3\sqrt{2}}{9\sqrt{3}+11\sqrt{2}}\right) \tag{343}$$

$$\pi = 4\tan^{-1}\left(\frac{5\sqrt{3}+\sqrt{2}}{10\sqrt{3}+5\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{5\sqrt{3}+4\sqrt{2}}{15\sqrt{3}+6\sqrt{2}}\right) \tag{344}$$

$$\pi = 4\tan^{-1}\left(\frac{4\sqrt{5}+\sqrt{2}}{8\sqrt{5}+5\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{2\sqrt{5}+2\sqrt{2}}{6\sqrt{5}+3\sqrt{2}}\right) \tag{345}$$

$$\pi = 4\tan^{-1}\left(\frac{3\sqrt{5}+3\sqrt{3}}{9\sqrt{5}+5\sqrt{3}}\right) + 4\tan^{-1}\left(\frac{3\sqrt{5}+\sqrt{3}}{6\sqrt{5}+4\sqrt{3}}\right) \tag{346}$$

2.59.

$$\pi = 16\tan^{-1}\left(\frac{1}{\sqrt{26}}\right) + 4\tan^{-1}\left(\frac{521-100\sqrt{26}}{521+100\sqrt{26}}\right) \tag{347}$$

2.60.

$$\pi = 20\tan^{-1}\left(\sqrt{10}-3\right) - 4\tan^{-1}\left(\frac{25\sqrt{10}-79}{3}\right) \tag{348}$$

$$\pi = 20\tan^{-1}\left(\frac{1}{3+\sqrt{11}}\right) + 4\tan^{-1}\left(\frac{7}{9963+3002\sqrt{11}}\right) \tag{349}$$

$$\pi = 24\tan^{-1}\left(\frac{1}{2\sqrt{3}+\sqrt{17}}\right) + 4\tan^{-1}\left(\frac{8870-5130\sqrt{3}-2155\sqrt{17}+1242\sqrt{51}}{8870+5130\sqrt{3}+2155\sqrt{17}+1242\sqrt{51}}\right) \tag{350}$$

$$\pi = 28\tan^{-1}\left(\sqrt{17}-4\right) - 4\tan^{-1}\left(\frac{4913\sqrt{17}-20047}{2908}\right) \tag{351}$$

$$\pi = 4\tan^{-1}\left(6\sqrt{5}-13\right) + 4\tan^{-1}\left(\frac{\sqrt{5}-1}{3}\right) \tag{352}$$

$$\pi = 4\tan^{-1}\left(\frac{1}{4\sqrt{2}}\right) + 4\tan^{-1}\left(\frac{1}{\sqrt{2}}\right) - 4\tan^{-1}\left(\frac{1}{\left(7+5\sqrt{2}\right)^2}\right) \tag{353}$$

$$\pi = 12\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right) - 12\tan^{-1}\left(\frac{2}{\left(5+3\sqrt{3}\right)^2}\right) \tag{354}$$

2.61.

$$\pi = 6\tan^{-1}\left(\left(\sqrt{3}-1\right)^2\right) + 6\tan^{-1}\left(\frac{1}{16+9\sqrt{3}}\right) \tag{355}$$

$$\pi = 6\tan^{-1}\left(\left(\sqrt{3}-1\right)^3\right) + 6\tan^{-1}\left(\frac{11}{40+19\sqrt{3}}\right) \tag{356}$$

$$\pi = 6\tan^{-1}\left(\left(\sqrt{3}-1\right)^4\right) + 6\tan^{-1}\left(\frac{7}{16+7\sqrt{3}}\right) \tag{357}$$

2.62.

$$\pi = 4\tan^{-1}\left(\left(\sqrt{3}-1\right)^3\right) + 4\tan^{-1}\left(\frac{4\sqrt{3}-3}{9}\right) \tag{358}$$

$$\pi = 4\tan^{-1}\left(\left(\sqrt{7}-2\right)^2\right) + 4\tan^{-1}\left(\frac{\sqrt{7}-1}{4}\right) \tag{359}$$

$$\pi = 4\tan^{-1}\left(\left(\sqrt{23}-4\right)^4\right) + 4\tan^{-1}\left(\frac{8\left(13\sqrt{23}-25\right)}{699}\right) \tag{360}$$

2.63.

$$\pi = 8\tan^{-1}\left(\left(\sqrt{3}-1\right)^3\right) + 8\tan^{-1}\left(\frac{23}{319+203\sqrt{2}+168\sqrt{3}+132\sqrt{6}}\right) \tag{361}$$

$$\pi = 8\tan^{-1}\left(\left(\sqrt{7}-2\right)^2\right) - 8\tan^{-1}\left(\frac{1}{105+75\sqrt{2}+40\sqrt{7}+28\sqrt{14}}\right) \tag{362}$$

2.64.

$$\pi = 8\tan^{-1}\left(\frac{1}{2^2}\right) + 8\tan^{-1}\left(\frac{1}{3^2}\right) + 4\tan^{-1}\left(\frac{1}{2^4}\right) + 4\tan^{-1}\left(\frac{1}{3^4}\right) - 4\tan^{-1}\left(\frac{24}{37649}\right) \tag{363}$$

2.65.

$$\pi = 84\tan^{-1}\left(\frac{1}{3^3}\right) + 12\tan^{-1}\left(\frac{466647286\sqrt{3}-803317073}{803317073\sqrt{3}+466647286}\right) \tag{364}$$

$$\pi = 48\tan^{-1}\left(\frac{1}{2^4}\right) + 8\tan^{-1}\left(\frac{15798015\sqrt{2}-22007647}{6209632\sqrt{2}+9588383}\right) \tag{365}$$

2.66.

$$\begin{aligned} \pi &= 12\tan^{-1}\left(\frac{1}{2^2}\right) + 4\tan^{-1}\left(\frac{5}{99}\right) = \\ &= 12\tan^{-1}\left(\frac{1}{2^2}\right) + 4\tan^{-1}\left(\frac{1}{5^2}\right) + 4\tan^{-1}\left(\frac{13}{1240}\right) = \\ &= 12\tan^{-1}\left(\frac{1}{2^2}\right) + 4\tan^{-1}\left(\frac{1}{5^2}\right) + 4\tan^{-1}\left(\frac{1}{10^2}\right) + 4\tan^{-1}\left(\frac{60}{124013}\right) \end{aligned} \tag{366}$$

2.67.

$$\pi = 24\tan^{-1}\left(5-4\sqrt{2}-\sqrt{3}+\sqrt{6}\right) + 24\tan^{-1}\left(\frac{1}{14+9\sqrt{2}-3\sqrt{3}-3\sqrt{6}}\right) \tag{367}$$

$$\pi = 24\tan^{-1}\left(4-3\sqrt{2}+3\sqrt{3}-2\sqrt{6}\right) + 24\tan^{-1}\left(\frac{1}{4+\sqrt{2}+3\sqrt{3}+\sqrt{6}}\right) \tag{368}$$

$$\pi = 24\tan^{-1}\left(3+7\sqrt{2}-6\sqrt{3}-\sqrt{6}\right) + 24\tan^{-1}\left(\frac{1}{6+\sqrt{2}+5\sqrt{3}-\sqrt{6}}\right) \tag{369}$$

$$\pi = 24\tan^{-1}\left(2+7\sqrt{2}-4\sqrt{3}-2\sqrt{6}\right) + 24\tan^{-1}\left(\frac{23}{290-57\sqrt{2}+163\sqrt{3}-41\sqrt{6}}\right) \tag{370}$$

$$\pi = 24\tan^{-1}\left(9+5\sqrt{2}-5\sqrt{3}-3\sqrt{6}\right)+24\tan^{-1}\left(\frac{47}{174+139\sqrt{2}+103\sqrt{3}+55\sqrt{6}}\right) \quad (371)$$

$$\pi = 24\tan^{-1}\left(6-5\sqrt{2}-5\sqrt{3}+4\sqrt{6}\right)+24\tan^{-1}\left(\frac{1}{7\sqrt{2}-\sqrt{3}+3\sqrt{6}}\right) \quad (372)$$

2.68.

$$\pi = 6\tan^{-1}\left(\frac{n\sqrt{3}}{3n+1}\right)+6\tan^{-1}\left(\frac{\sqrt{3}}{12n+3}\right) \quad ,n\in\mathbb{N}\cup\{0\} \quad (373)$$

$$\pi = 6\tan^{-1}\left(\frac{n\sqrt{3}}{3n+2}\right)+6\tan^{-1}\left(\frac{\sqrt{3}}{6n+3}\right) \quad ,n\in\mathbb{N}\cup\{0\} \quad (374)$$

2.69.

$$\pi = 16\tan^{-1}\left(\left(\sqrt{2}-1\right)^2\right)+8\tan^{-1}\left(\frac{7\sqrt{2}-9}{17}\right) \quad (375)$$

$$\begin{aligned}\pi &= 24\tan^{-1}\left(\left(\sqrt{2}-1\right)^3\right)+8\tan^{-1}\left(\frac{2}{11}\right)= \\ &= 24\tan^{-1}\left(\left(\sqrt{2}-1\right)^3\right)+8\tan^{-1}\left(\frac{1}{6}\right)+8\tan^{-1}\left(\frac{1}{68}\right)\end{aligned} \quad (376)$$

2.70.

$$\begin{aligned}\pi = 6\tan^{-1}&\left(\frac{\sqrt{(n+3)(3n+1)}-(n+1)\sqrt{3}}{2}\right) \\ &+6\tan^{-1}\left(\frac{\sqrt{(n+3)(3n+1)}-(n+1)\sqrt{3}}{2n}\right) \quad ,n\in\mathbb{N}\end{aligned} \quad (377)$$

Ejemplos:

$$\pi = 6\tan^{-1}\left(\frac{\sqrt{35}-3\sqrt{3}}{2}\right)+6\tan^{-1}\left(\frac{\sqrt{35}-3\sqrt{3}}{4}\right) \quad (378)$$

$$\pi = 6\tan^{-1}\left(\sqrt{15}-2\sqrt{3}\right)+6\tan^{-1}\left(\frac{\sqrt{15}-2\sqrt{3}}{3}\right) \quad (379)$$

$$\pi = 6\tan^{-1}\left(\frac{\sqrt{91}-5\sqrt{3}}{2}\right)+6\tan^{-1}\left(\frac{\sqrt{91}-5\sqrt{3}}{8}\right) \tag{380}$$

2.71.

$$\pi = 3\tan^{-1}\left(\frac{\sqrt{n^2+14n+1}-n-1}{2\sqrt{3}}\right)+3\tan^{-1}\left(\frac{\sqrt{n^2+14n+1}-n-1}{2n\sqrt{3}}\right) \quad , n\in\mathbb{N} \tag{381}$$

Ejemplos:

$$\pi = 3\tan^{-1}\left(\frac{\sqrt{33}-3}{2\sqrt{3}}\right)+3\tan^{-1}\left(\frac{\sqrt{33}-3}{4\sqrt{3}}\right) \tag{382}$$

$$\pi = 3\tan^{-1}\left(\frac{\sqrt{13}-2}{\sqrt{3}}\right)+3\tan^{-1}\left(\frac{\sqrt{13}-2}{3\sqrt{3}}\right) \tag{383}$$

2.72.

$$\begin{aligned}\pi &= 8\tan^{-1}\left(\frac{\sqrt{2(n+1)^2\sqrt{2}+3(n+1)^2+4n}-(n+1)\sqrt{2}-n-1}{2}\right)\\ &+8\tan^{-1}\left(\frac{\sqrt{2(n+1)^2\sqrt{2}+3(n+1)^2+4n}-(n+1)\sqrt{2}-n-1}{2n}\right) \quad , n\in\mathbb{N}\end{aligned} \tag{384}$$

Ejemplo:

$$\pi = 8\tan^{-1}\left(\frac{\sqrt{18\sqrt{2}+35}-3\sqrt{2}-3}{2}\right)+8\tan^{-1}\left(\frac{\sqrt{18\sqrt{2}+35}-3\sqrt{2}-3}{4}\right) \tag{385}$$

Comentario final: mostraremos mas fórmulas en un próximo preprint.

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