# Pi Formulas

# Part 4: Ten notes related with the constant Pi

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abstract

In this note we give some formulas related to the constant Pi

# $\pi$ - SERIES

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de series en las que aparece la constante $\pi = 3.1415926...$

## 1. INTRODUCCIÓN.

En esta nota se muestra una colección de series en las que aparece la constante $\pi$, y que corresponden a casos particulares de la siguiente fórmula general:

$$ln\prod_{k=1}^{m}\left(1+x_k\right)^{\alpha_k}+\sum_{k=1}^{m}\alpha_k\tan^{-1}x_k=\sum_{n=1}^{\infty}a_n\left(\sum_{k=1}^{m}\alpha_k x_k^n\right)$$

donde $m\in\mathbb{N}$, $\alpha_k\in\mathbb{R}$, $|x_k|<1$, y los coeficientes $a_n$ estan definidos como sigue:

$$a_n=\frac{2}{n} \qquad n=4j-3 \quad , \quad j\in\mathbb{N}$$

$$a_n=0 \qquad n=4j-1 \quad , \quad j\in\mathbb{N}$$

$$a_n=-\frac{1}{2n} \qquad n=2j \quad , \quad j\in\mathbb{N}$$

$$a_n=\frac{2}{n} \qquad n=1,5,9,13,...$$

$$a_n=0 \qquad n=3,7,11,15,...$$

$$a_n=-\frac{1}{2n} \qquad n=2,4,6,8,...$$

La expresión $\sum_{k=1}^{m}\alpha_k\tan^{-1}x_k$ , es la forma general de las fórmulas del tipo Machin, por lo tanto, para valores adecuados de $\alpha_k$ y $x_k$ podemos obtener una infinidad de series que contengan la constante $\pi$.

## 2. SERIES.

2.1. $$ln\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)+\frac{\pi}{4}=\sum_{n=1}^{\infty}a_n\left(\frac{1}{2^n}+\frac{1}{3^n}\right)$$

2.2. $$ln\left(1+\frac{29}{278}\right)^{5}\left(1+\frac{3}{79}\right)^{7}+\frac{\pi}{4}=\sum_{n=1}^{\infty}a_n\left(5\left(\frac{29}{278}\right)^{n}+7\left(\frac{3}{79}\right)^{n}\right)$$

2.3. $$ln\left(1+\frac{1}{7}\right)^{5}\left(1+\frac{1}{53}\right)^{4}\left(1+\frac{1}{4443}\right)^{2}+\frac{\pi}{4}=\sum_{n=1}^{\infty}a_n\left(5\left(\frac{1}{7}\right)^{n}+4\left(\frac{1}{53}\right)^{n}+2\left(\frac{1}{4443}\right)^{n}\right)$$

2.4. $$ln\left(1+\frac{1}{18}\right)^{12}\left(1+\frac{1}{57}\right)^{8}\left(1+\frac{1}{239}\right)^{-5}+\frac{\pi}{4}=\sum_{n=1}^{\infty}a_n\left(12\left(\frac{1}{18}\right)^{n}+8\left(\frac{1}{57}\right)^{n}-5\left(\frac{1}{239}\right)^{n}\right)$$

2.5. $$ln\left(1+\frac{1}{2}\right)^{1/2}\left(1+\frac{1}{3}\right)^{-1/6}\left(1+\frac{1}{7}\right)^{-1/3}+\frac{\pi}{24}=\sum_{n=1}^{\infty}a_n\left(\left(\frac{1}{2}\right)^{n+1}-\frac{1}{2}\left(\frac{1}{3}\right)^{n+1}-\frac{1}{3}\left(\frac{1}{7}\right)^{n}\right)$$

2.6. $$ln\left(1+\frac{1}{2}\right)^{1/2}\left(1+\frac{1}{3}\right)^{7/6}\left(1+\frac{1}{7}\right)^{1/3}+\frac{5\pi}{24}=\sum_{n=1}^{\infty}a_n\left(\left(\frac{1}{2}\right)^{n+1}+\frac{7}{2}\left(\frac{1}{3}\right)^{n+1}+\frac{1}{3}\left(\frac{1}{7}\right)^{n}\right)$$

2.7. $$ln\left(1+\frac{1}{\sqrt{2}}\right)^{2\sqrt{2}}\left(1+\frac{1}{2\sqrt{2}}\right)^{\sqrt{2}}+\frac{\pi}{\sqrt{2}}=\sum_{n=1}^{\infty}a_n\left(4\left(\frac{1}{\sqrt{2}}\right)^{n+1}+4\left(\frac{1}{2\sqrt{2}}\right)^{n+1}\right)$$

2.8. $$ln\left(1+\frac{5}{19}\right)\left(1+\frac{7}{12}\right)+\frac{\pi}{4}=\sum_{n=1}^{\infty}a_n\left(\left(\frac{5}{19}\right)^{n}+\left(\frac{7}{12}\right)^{n}\right)$$

2.9. $$ln\left(1+\frac{1}{2}\right)^{\sqrt{2}}\left(1+\frac{1}{3}\right)^{\sqrt{2}+2\sqrt{3}}\left(1+\frac{1}{7}\right)^{\sqrt{3}}+\frac{\pi\left(\sqrt{2}+\sqrt{3}\right)}{4}=$$

$$=\sum_{n=1}^{\infty}a_n\left(\sqrt{2}\left(\frac{1}{2}\right)^{n}+\left(\sqrt{2}+2\sqrt{3}\right)\left(\frac{1}{3}\right)^{n}+\sqrt{3}\left(\frac{1}{7}\right)^{n}\right)$$

2.10. $$ln\left(1+\frac{1}{57}\right)^{44}\left(1+\frac{1}{239}\right)^{7}\left(1+\frac{1}{682}\right)^{-12}\left(1+\frac{1}{12943}\right)^{24}+\frac{\pi}{4}=$$

$$=\sum_{n=1}^{\infty}a_n\left(44\left(\frac{1}{57}\right)^{n}+7\left(\frac{1}{239}\right)^{n}-12\left(\frac{1}{682}\right)^{n}+24\left(\frac{1}{12943}\right)^{n}\right)$$

2.11. $$ln\left(1+\frac{1}{49}\right)^{48}\left(1+\frac{1}{57}\right)^{128}\left(1+\frac{1}{239}\right)^{-20}\left(1+\frac{1}{110443}\right)^{48}+\pi=$$

$$=\sum_{n=1}^{\infty}a_n\left(48\left(\frac{1}{49}\right)^{n}+128\left(\frac{1}{57}\right)^{n}-20\left(\frac{1}{239}\right)^{n}+48\left(\frac{1}{110443}\right)^{n}\right)$$

2.12.Para $m \in \mathbb{N}$ se tiene:

$$\ln\left(1+\frac{1}{m+1}\right)\prod_{k=1}^{m}\left(1+\frac{1}{k^2+k+1}\right)+\frac{\pi}{4}=$$

$$=\sum_{n=1}^{\infty}a_n\left(\left(\frac{1}{m+1}\right)^n+\sum_{k=1}^{m}\left(\frac{1}{k^2+k+1}\right)^n\right)$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

3. M. R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York, 1968.

4. E. Valdebenito, Pi Handbook, manuscript, unpublished, 1989 , ( 20000 fórmulas).

# COLECCIÓN DE FÓRMULAS QUE CONTIENEN LA CONSTANTE $\pi$

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de fórmulas que contienen la constante $\pi = 3.14159265...$

## 1. INTRODUCCIÓN.

El número $\pi$ se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota se muestra una colección de fórmulas que contienen la constante $\pi$ .

## 2. FÓRMULAS.

2.1.Para $m,k \in \mathbb{N}$, $m < k$ se tiene:

$$\frac{\pi}{4} = \tan^{-1}\frac{\sqrt{k(k-m)}}{k} + \sum_{n=1}^{\infty}\tan^{-1}\frac{k(\phi(n+1,k,m)-\phi(n,k,m))}{k^2+\phi(n+1,k,m)\phi(n,k,m)}$$

$$\phi(n,k,m) = \underbrace{\sqrt{k(k-m)+m\sqrt{k(k-m)+...+m\sqrt{k(k-m)}}}}_{n-radicales}$$

Ejemplo 1: $m = 1, k = 2$ :

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{\sqrt{2}} + \tan^{-1}\frac{2\left(\sqrt{2+\sqrt{2}}-\sqrt{2}\right)}{4+\sqrt{2+\sqrt{2}}\sqrt{2}} + \tan^{-1}\frac{2\left(\sqrt{2+\sqrt{2+\sqrt{2}}}-\sqrt{2+\sqrt{2}}\right)}{4+\sqrt{2+\sqrt{2+\sqrt{2}}}\sqrt{2+\sqrt{2}}} + ...$$

Ejemplo 2: $m = 1, k = 3$ :

$$\frac{\pi}{4} = \tan^{-1}\sqrt{\frac{2}{3}} + \tan^{-1}\frac{3\left(\sqrt{6+\sqrt{6}}-\sqrt{6}\right)}{9+\sqrt{6+\sqrt{6}}\sqrt{6}} + \tan^{-1}\frac{3\left(\sqrt{6+\sqrt{6+\sqrt{6}}}-\sqrt{6+\sqrt{6}}\right)}{9+\sqrt{6+\sqrt{6+\sqrt{6}}}\sqrt{6+\sqrt{6}}} + ...$$

2.2.Para $0 < a < 1$ se tiene:

$$\frac{\pi}{4} = tan^{-1}\sqrt{1-a} + \sum_{n=1}^{\infty} tan^{-1}\frac{\phi(n+1,a)-\phi(n,a)}{1+\phi(n+1,a)\phi(n,a)}$$

$$\phi(n,a) = \underbrace{\sqrt{1-a+a\sqrt{1-a+...+a\sqrt{1-a}}}}_{n-radicales}$$

2.3. $$\frac{\pi^2}{8} = \left(2\left(2-\sqrt{2}\right)\right)\times\left(2\left(2+\sqrt{2}\right)\left(2-\sqrt{2+\sqrt{2}}\right)\right)\times$$

$$\times\left(2\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\right)\times....$$

2.4. $$\frac{\pi^2}{2} = 2^2 + \sum_{n=1}^{\infty} 2^{2n}\left(3+r(n-1)-4r(n)\right)$$

$$= 2^2 + 2^2\left(3-4\sqrt{\frac{1}{2}}\right) + 2^4\left(3+\sqrt{\frac{1}{2}}-4\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\right)+....$$

$$r(n) = \underbrace{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+....+\frac{1}{2}\sqrt{\frac{1}{2}}}}}_{n-radicales} \quad , \quad r(0)=0$$

2.5.Para $0 < a < \frac{1}{4}$ se tiene:

$$\pi^3\frac{(1-2a)^4}{8(1-4a)}\psi(a) = \prod_{n=1}^{\infty}\left(1+\left(\phi(a)\right)^{2^{n+1}}\right)$$

$$\phi(a) = \prod_{n=1}^{\infty}\frac{(2n-1)^2-(1-2a)^2}{(2n-1)^2-(2a)^2}$$

$$\psi(a) = \prod_{n=1}^{\infty}\frac{\left(4n^2-(1-2a)^2\right)^4}{\left(4n^2\right)^3\left(4n^2-(1-4a)^2\right)}$$

2.6.Para $|a| < 1$ se tiene:

$$\pi\left[\frac{1-3a}{2-2a}\phi\left(\frac{1-3a}{2-2a}\right)+i\frac{a}{1-a}\phi\left(\frac{a}{1-a}\right)\right] = \prod_{n=1}^{\infty}\left(\psi\left(a^n\right)+i\psi\left(\frac{1}{2}-a^n\right)\right)$$

$$\phi(x) = \prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n}\right)^2\right) \quad , \quad \psi(x) = \prod_{n=1}^{\infty}\left(1-\left(\frac{2x}{2n-1}\right)^2\right)$$

2.7.Sean $s$, $u$, $v$ definidos como sigue:

$$s=\sqrt[3]{1+3\sqrt[3]{1+3\sqrt[3]{1+\ldots}}} \quad , \quad u=\frac{1}{2}\left(s+\sqrt{3-\frac{3}{s}}\right) \quad , \quad v=\frac{1}{2}\left(s-\sqrt{3-\frac{3}{s}}\right)$$

se tiene:

$$\pi=6\tan^{-1}\left(\frac{1}{s}\right)+2\tan^{-1}\left(\frac{s}{3+8s}\right)$$
$$\pi=6\tan^{-1}\left(\frac{1}{u}\right)-2\tan^{-1}\left(\frac{u}{8u-3}\right)$$
$$\pi=6\tan^{-1}(v)+2\tan^{-1}\left(\frac{3-8v}{v}\right)$$

2.8.Para $x=\sqrt[3]{\frac{\sqrt{5}+1}{2}}+\sqrt[3]{\frac{\sqrt{5}-1}{2}}$ , se tiene:

$$\pi=6\tan^{-1}\left(\frac{1}{x}\right)+2\tan^{-1}\left(\frac{x\sqrt{5}}{8x+3\sqrt{5}}\right)$$

2.9.

$$\frac{\pi}{2}=sen^{-1}\left(\frac{2}{3}\right)+\sum_{n=1}^{\infty}sen^{-1}\left(\frac{(3n-1)\sqrt{6n-1}}{(3n)^2}-\frac{(3n+2)\sqrt{6n+5}}{(3n+3)^2}\right)$$

2.10.

$$\pi+12\sum_{n=1}^{\infty}\tan^{-1}\left(\left(2-\sqrt{3}\right)^{n+1}\right)=12\sum_{n=0}^{\infty}\frac{(-1)^n\left(2-\sqrt{3}\right)^{2n+1}}{(2n+1)\left(1-\left(2-\sqrt{3}\right)^{2n+1}\right)}$$

$$\pi+6\sum_{n=1}^{\infty}\tan^{-1}\left(\left(\frac{1}{\sqrt{3}}\right)^{n+1}\right)=6\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left(3^n-\sqrt{3}\right)}$$

$$\pi+6\sum_{n=1}^{\infty}\tan^{-1}\left(\left(\frac{1}{3}\right)^{n}\right)+6\sum_{n=1}^{\infty}\tan^{-1}\left(\left(\frac{1}{3}\right)^{n+\frac{1}{2}}\right)=6\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\left(3^n-\sqrt{3}\right)}$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

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4. E. Valdebenito, Pi Handbook, manuscript, unpublished, 1989 , ( 20000 fórmulas).

# PRODUCTOS INFINITOS QUE CONTIENEN LAS CONSTANTES $\pi$ Y $\phi=\frac{1+\sqrt{5}}{2}$

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**

Se muestra una colección de fórmulas (Productos infinitos),que contienen las constantes $\pi = 3.141592...$, y $\phi = \frac{1+\sqrt{5}}{2}$.

## 1. INTRODUCCIÓN.

En esta nota se muestran algunos Productos infinitos que involucran las constantes

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265..., \text{ y } \phi = \frac{1+\sqrt{5}}{2}$$

## 2. FÓRMULAS.

2.1. $$\pi\phi\prod_{n=1}^{\infty}\left(1-\left(\frac{\phi}{2n}\right)^2\right) = \sqrt{2}\prod_{n=1}^{\infty}\left(1-\frac{(-1)^n(2\phi-1)}{2n-1}\right)$$

2.2. $$\frac{2\phi+3}{\pi}\prod_{n=1}^{\infty}\left(1-\left(\frac{2\phi}{2n+3}\right)^2\right) = \frac{9}{8}\prod_{n=1}^{\infty}\left(1-\frac{5}{4(n+1)^2}\right)$$

2.3. $$\pi\phi^2\prod_{n=1}^{\infty}\left(1-\left(\frac{\phi}{n+1}\right)^2\right) = 4\prod_{n=1}^{\infty}\left(1-\frac{5}{(2n+1)^2}\right) = \frac{\pi}{\phi}\prod_{n=1}^{\infty}\left(1-\frac{1}{\phi^2 n^2}\right)$$

2.4. $$2\sqrt{2}\phi\prod_{n=1}^{\infty}\left(1-\left(\frac{\phi}{2n}\right)^2\right) = \frac{4}{\pi}\prod_{n=1}^{\infty}\left(1-\frac{5}{4(2n-1)^2}\right)+(2\phi-1)\prod_{n=1}^{\infty}\left(1-\frac{5}{16n^2}\right)$$

2.5. $$\frac{2\phi-3}{\pi} = \prod_{n=1}^{\infty}\left(1-\left(\frac{2}{(2n+1)\phi}\right)^2\right) = \frac{1}{8}\prod_{n=1}^{\infty}\left(1-\frac{5}{4(n+1)^2}\right)$$

2.6. $$\pi(2\phi+3)\prod_{n=1}^{\infty}\left(1-\frac{4\phi+4}{(n+3)^2}\right) = \frac{2^8}{25}\prod_{n=1}^{\infty}\left(1-\frac{8\phi+17}{(2n+5)^2}\right)$$

2.7. $$4\sqrt{2}\phi\prod_{n=1}^{\infty}\left(1-\left(\frac{\phi}{2n+1}\right)^{2}\right)=\pi(2\phi-1)\prod_{n=1}^{\infty}\left(1-\frac{5}{16n^{2}}\right)-4\prod_{n=1}^{\infty}\left(1-\frac{5}{4(2n-1)^{2}}\right)$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

3. M. R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York, 1968.

4. E. Valdebenito, Pi Handbook, manuscript, unpublished, 1989 , ( 20000 fórmulas).

# COLECCION DE SERIES QUE INVOLUCRAN LA CONSTANTE $\pi$

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de series que contienen la constante $\pi = 3.14159265...$

## 1. INTRODUCCIÓN.

EL número Pi se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota mostramos una colección de series que involucran la clásica constante $\pi$, algunas series son generales y otras particulares.

## 2. SERIES QUE CONTIENEN LA CONSTANTE $\pi$.

2.1. $$\sum_{n=1}^{\infty}\frac{\left((n-1)!\right)^2\left(n^2+1\right)^2}{(2n)!\,n^2} = \frac{1}{3}+\frac{2\sqrt{3}\pi}{27}+\frac{\pi^2}{9}+\frac{17\pi^4}{3240}$$

2.2. $$\sum_{n=1}^{\infty}\frac{\left((n-1)!\right)^2\left(n^2-1\right)^2}{(2n)!\,n^2} = \frac{1}{3}+\frac{2\sqrt{3}\pi}{27}-\frac{\pi^2}{9}+\frac{17\pi^4}{3240}$$

2.3. $$\sum_{n=1}^{\infty}\frac{\left((n+1)!\right)^2}{(2n)!\,n^2} = \frac{1}{3}+\frac{8\sqrt{3}\pi}{27}+\frac{\pi^2}{18}$$

2.4. $$\sum_{n=1}^{\infty}\frac{(n!)^2\,n^2}{(2n+2)!} = \frac{1}{3}-\frac{4\sqrt{3}\pi}{27}+\frac{\pi^2}{18}$$

2.5. $$\sum_{n=1}^{\infty}\frac{\left((2n)!\right)^2(10n-3)}{(4n)!\,n(2n-1)} = \frac{2\sqrt{3}}{9}\pi$$

2.6. $$\sum_{n=1}^{\infty}\frac{\left((2n-1)!\right)^2(10n-3)}{(4n-1)!(2n-1)} = \frac{2\sqrt{3}}{9}\pi$$

2.7. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(24n^3+12n+1\right)}{n(n+1)(2n+1)\left(36n^2-1\right)} = \pi-3$$

2.8. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n+1-m)(2n+1+m)}{n(n+1)(2n+1)^3}=7m^2-3-\left(m^2-1\right)\pi-\frac{m^2}{8}\pi^3 \quad , \quad m>0$$

2.9. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)(2n+3)}{n(n+1)(2n+1)^3}=25-3\pi-\frac{\pi^3}{2}$$

2.10. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n(n+3)}{(n+1)(n+2)(2n+3)^3}=\frac{9}{32}\pi^3+2\pi-15$$

2.11. $$\sum_{n=1}^{\infty}\frac{(2n-m)(2n+m)}{\left(4n^2-1\right)^3}=\left(\frac{3m^2+1}{64}\right)\pi^2-\frac{m^2}{2} \quad , \quad m>0$$

2.12. $$\sum_{n=1}^{\infty}\frac{n(n+2)}{(2n+1)^3(2n+3)^3}=\frac{13}{256}\pi^2-\frac{1}{2}$$

2.13. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2sn+s-m)(2sn+s+m)}{n(n+1)(2n+1)^3}=7m^2-3s^2+\left(s^2-m^2\right)\pi-\frac{m^2}{8}\pi^3 \quad , \quad s,m\in\mathbb{R}$$

2.14. $$\sum_{n=1}^{\infty}\frac{(2sn-m)(2sn+m)}{(2n-1)^3(2n+1)^3}=\left(\frac{3m^2+s^2}{64}\right)\pi^2-\frac{m^2}{2} \quad , \quad s,m\in\mathbb{R}$$

2.15. $$\sum_{n=1}^{\infty}\frac{(2n-m)(2n+m)}{(2n-1)^4(2n+1)^4}=\frac{m^2}{2}-\left(\frac{5m^2+1}{128}\right)\pi^2-\left(\frac{m^2-1}{768}\right)\pi^4 \quad , \quad m>0$$

2.16. $$\sum_{n=1}^{\infty}\frac{n(n+2)}{(2n+1)^4(2n+3)^4}=\frac{1}{2}-\frac{21}{512}\pi^2-\frac{3}{3072}\pi^4$$

2.17. $$\sum_{n=1}^{\infty}\frac{(2sn-m)(2sn+m)}{(2n-1)^4(2n+1)^4}=\left(\frac{s^2-m^2}{768}\right)\pi^4-\left(\frac{5m^2+s^2}{128}\right)\pi^2+\frac{m^2}{2} \quad , \quad s,m\in\mathbb{R}$$

2.18. $$\sum_{n=1}^{\infty}\frac{\left(4sn^2-s-m\right)\left(4sn^2-s+m\right)}{(2n-1)^4(2n+1)^4}=\frac{m^2-s^2}{2}+\left(\frac{8s^2-5m^2}{128}\right)\pi^2-\frac{m^2}{768}\pi^4 \quad , \quad s,m\in\mathbb{R}$$

2.19. $$\sum_{n=1}^{\infty}\frac{(2rn-t)(2rn+t)\left(4r^2n^2+t^2-2r^2\right)}{(2n-1)^4(2n+1)^4}=\frac{t^2\left(t^2-2r^2\right)}{2}+\left(\frac{3r^4+10r^2t^2-5t^4}{128}\right)\pi^2-$$

$$-\frac{\left(t^2-r^2\right)^2}{768}\pi^4 \quad , \quad r,t\in\mathbb{R}$$

2.20. $$\sum_{n=1}^{\infty}\frac{(10n-7)(10n+7)(10n-1)(10n+1)}{(2n-1)^4(2n+1)^4}=-\frac{49}{2}+\frac{265}{16}\pi^2-\frac{3}{4}\pi^4$$

2.21. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\left(4\left(8+m^2\right)n^2+4\left(8+m^2\right)n+m^2\right)}{n(n+1)(2n+1)^3}=32-3m^2-\pi(\pi-m)(\pi+m)\quad,\quad m\in\mathbb{R}$$

2.22. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n(18n-1)}{(2n-1)(6n-1)(6n+1)}=\frac{1}{4}+\frac{\pi}{24}$$

2.23. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(3n+1)(3n+2)}{n(n+1)(2n+1)^3}=\frac{\pi(8+\pi)(8-\pi)-160}{32}$$

2.24. $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n(18n+1)}{(2n+1)(6n+5)(6n+7)}=\frac{67}{24}\pi-\frac{35}{4}$$

2.25. $$\sum_{n=1}^{\infty}\frac{(n!)^2(4bn+2a+2b+c)}{(2n+1)!}=\frac{2\sqrt{3}}{27}(6a+2b+3c)\pi+\frac{8b}{3}\quad,\quad a,b,c\in\mathbb{R}$$

2.26. $$\sum_{n=1}^{\infty}\frac{n!(n+1)!}{(2n+1)!}=\frac{4\sqrt{3}}{27}\pi+\frac{2}{3}$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

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# $\pi$ - FORMULAS
# Colección de fórmulas que contienen la constante $\pi$

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de fórmulas que contienen la constante $\pi = 3.14159265...$

## 1. INTRODUCCIÓN.

EL número Pi se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota se muestra una colección de fórmulas que involucran la constante $\pi$ .

## 2. FÓRMULAS.

2.1.Sea $u$ el número definido como sigue:

$$u = e^{-2e^{-2e^{-2...}}}$$
$$u = 0.426302751...$$

El número $u$ satisface la ecuación $u = e^{-2u}$ , y esta relacionado con la constante $\pi$ por la serie:

$$\sqrt{\pi} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}u^{n-1/2}\Gamma\left(-n+\frac{1}{2},u\right)$$

donde $\Gamma\left(-n+\frac{1}{2},u\right)$ es la función Gamma incompleta.

Otra forma de escribir la serie es la siguiente:

$$\sqrt{\pi} = \sqrt{u}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}A(n,u) = e^{-u}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{n!}A(n,u)$$

donde

$$A(n,u) = \cfrac{1}{u+\cfrac{n+1/2}{1+\cfrac{1}{u+\cfrac{n+3/2}{1+\cfrac{2}{u+\cfrac{n+5/2}{1+\cfrac{3}{u+...}}}}}}}$$

2.2.Para $n \in \mathbb{N}_0 = \{0,1,2,3,...\}$ se tiene:

$$\sqrt{\pi} = \frac{2^{2n} n!}{(2n)!}\left( \Gamma\left( n+\frac{1}{2},1 \right) + 2\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+2n+1)} \right)$$

donde

$$\Gamma\left( n+\frac{1}{2},1 \right) = \cfrac{e^{-1}}{1-\cfrac{2n-1}{2+\cfrac{2}{1-\cfrac{2n-3}{2+\cfrac{4}{1-\cfrac{2n-5}{2+\cfrac{6}{1-\ldots}}}}}}}$$

2.3. $$\frac{\pi}{4} e^{-\sqrt{2}} = a\left( \cos\frac{1}{\sqrt{2}} \right)\left( ch\frac{1}{\sqrt{2}} \right) - b\left( sen\frac{1}{\sqrt{2}} \right)\left( sh\frac{1}{\sqrt{2}} \right) +$$

$$+\sum_{n=1}^{\infty} \frac{(-1)^n}{(4n+2)!} \sum_{k=0}^{n-1} (-1)^k (4k+3)! - \sum_{n=1}^{\infty} \frac{(-1)^n}{(4n)!} \sum_{k=0}^{n-1} (-1)^k (4k+1)!$$

donde

$$a = \int_0^{\infty} \frac{xe^{-x}}{1+x^4} dx \quad , \quad b = \int_0^{\infty} \frac{x^3 e^{-x}}{1+x^4} dx$$

2.4.Para $a,b \in \mathbb{Z} - \{0\}$ , $c,d \in \mathbb{N}$ , $c \neq m^2$ , $m \in \mathbb{N}$ , $A = -a^2 + b^2 c - d^2$ , $B = -2bd$ , $D = (a+d)^2 - b^2 c$
se tiene:

$$\pi = 4\left( \frac{a+b\sqrt{c}}{d} \right) \sum_{n=0}^{\infty} \frac{(-1)^n \left( a_n + b_n\sqrt{c} \right)^2}{(2n+1) d^{2n}} + 4\left( \frac{A+B\sqrt{c}}{D} \right) \sum_{n=0}^{\infty} \frac{(-1)^n \left( A_n + B_n\sqrt{c} \right)^2}{(2n+1) D^{2n}}$$

donde

$$\begin{cases} a_{n+1} = aa_n + bcb_n \\ b_{n+1} = ba_n + ab_n \end{cases} \qquad a_0 = 1 \ , \ b_0 = 0$$

$$\begin{cases} A_{n+1} = AA_n + BcB_n \\ B_{n+1} = BA_n + AB_n \end{cases} \qquad A_0 = 1 \ , \ B_0 = 0$$

$$a_{n+2} = 2aa_{n+1} - \left( a^2 - b^2 c \right) a_n \qquad a_0 = 1 \ , \ a_1 = a$$

$$b_{n+2} = 2ab_{n+1} - \left( a^2 - b^2 c \right) b_n \qquad b_0 = 0 \ , \ b_1 = b$$

$$A_{n+2} = 2AA_{n+1} - \left( A^2 - B^2 c \right) A_n \qquad A_0 = 1 \ , \ A_1 = A$$

$$B_{n+2} = 2AB_{n+1} - \left( A^2 - B^2 c \right) B_n \qquad B_0 = 0 \ , \ B_1 = B$$

2.5.

$$\frac{\pi}{3\sqrt{3}}=\sum_{n=0}^{\infty}(-1)^n\, a_n$$

$$\frac{\pi}{18\left(2-\sqrt{3}\right)}=\sum_{n=0}^{\infty}(-1)^n\left(16-9\sqrt{3}\right)^n a_n$$

$$\frac{\pi}{12\left(\sqrt{2}-1\right)}=\sum_{n=0}^{\infty}(-1)^n\left(\frac{14-9\sqrt{2}}{2}\right)^n a_n$$

donde $a_0=ln\,2$, y para $n\in\mathbb{N}$, $a_n$ esta dado por la siguiente fórmula:

$$a_n=(-1)^n\binom{2n}{n}ln\,2+\sum_{\substack{k=0\\k\neq n}}^{2n}\frac{(-1)^k\binom{2n}{k}\left(2^{n-k}-1\right)}{n-k}$$

$$a_n=(-1)^n\binom{2n}{n}ln\,2+\sum_{k=0}^{n-1}\frac{(-1)^k\binom{2n}{k}\left(2^{n-k}-1\right)}{n-k}+\sum_{k=n+1}^{2n}\frac{(-1)^k\binom{2n}{k}\left(2^{n-k}-1\right)}{n-k}$$

2.6.

$$2^{\pi}=\prod_{n=0}^{\infty}\left(4\prod_{k=1}^{2n+1}exp\left(\frac{(-1)^k}{k}\right)\right)^{\frac{8(-1)^n}{2n+1}}$$

2.7.Para $m>1$ se tiene:

$$m^{\pi}=\prod_{k=0}^{\infty}\left[\prod_{n=0}^{k}exp\left(\frac{\left((m-1)/(m+1)\right)^{2n+1}}{(2n+1)(4k-4n+1)(4k-4n+3)}\right)\right]^{16}$$

Ejemplo $m=2$:

$$2^{\pi}=\prod_{k=0}^{\infty}\left[\prod_{n=0}^{k}exp\left(\frac{3^{-(2n+1)}}{(2n+1)(4k-4n+1)(4k-4n+3)}\right)\right]^{16}$$

$$2^{\pi}=\left(exp\left(\frac{3^{-1}}{1\cdot1\cdot3}\right)\right)^{16}\times\left(exp\left(\frac{3^{-1}}{1\cdot5\cdot7}\right)\cdot exp\left(\frac{3^{-3}}{3\cdot1\cdot3}\right)\right)^{16}\times$$

$$\times\left(\exp\left(\frac{3^{-1}}{1\cdot 9\cdot 11}\right)\cdot\exp\left(\frac{3^{-3}}{3\cdot 5\cdot 7}\right)\cdot\exp\left(\frac{3^{-5}}{5\cdot 1\cdot 3}\right)\right)^{16}\times....$$

2.8.

$$\pi^4+\pi^3=128\left(1+\sum_{n=1}^{\infty}\left(\frac{n+1}{(4n+1)^4}-\frac{n}{(4n+3)^4}\right)\right)$$

$$=128\left(1+\frac{2}{5^4}-\frac{1}{7^4}+\frac{3}{9^4}-\frac{2}{11^4}+...\right)$$

$$=a$$

$$\pi=\sqrt[3]{\frac{a}{1+\sqrt[3]{\frac{a}{1+\sqrt[3]{\frac{a}{1+...}}}}}}$$

$$\frac{a}{\pi}=\sqrt[4]{a^3+a^2\cdot\sqrt[4]{a^3+a^2\cdot\sqrt[4]{a^3+...}}}$$

2.9.Para $n\in\mathbb{N}$ se tiene:

$$\pi\cdot 2^{2^{-n-1}}\left(\frac{2^{n+1}+1}{2^{n+2}}\right)\prod_{k=1}^{\infty}\left(1-\left(\frac{2^{n+1}+1}{2^{n+2}k}\right)^2\right)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2^n}\right)_{2k}}{(2k)!}$$

$$\pi\cdot 2^{2^{-n-1}}\left(\frac{2^{n+2}+1}{2^{n+2}}\right)\prod_{k=1}^{\infty}\left(1-\left(\frac{2^{n+2}+1}{2^{n+2}k}\right)^2\right)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(-\frac{1}{2^n}\right)_{2k+1}}{(2k+1)!}$$

2.10.Para $0<x<\pi$ se tiene:

$$\frac{\pi^2}{8}-\frac{\pi}{2}sen(x)=\sum_{n=1}^{\infty}\left(\frac{\cos(2nx)}{2n-1}+\frac{1}{2n+1}\right)^2-\sum_{n=1}^{\infty}\left(\frac{\cos(2nx)}{2n-1}\right)^2$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

3. M. R. Spiegel, Mathematical Handbook, McGraw-Hill Book Company, New York, 1968.

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# $\pi$ - SERIES
# Colección de series que contienen el número Pi

**EDGAR VALDEBENIO V.**
**(1994)**

**Resumen**
Se muestra una colección de series para la constante $\pi = 3.1415926535...$

## 1. INTRODUCCIÓN.

El número Pi ( $\pi$ ) , se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota se muestra una colección de series para la constante $\pi$ .

## 2. SERIES.

2.1.Para $n,m \in \mathbb{N} = \{1,2,3,...\}$ , con $m < n$ se tiene:

$$\pi = 4\sum_{k=0}^{\infty}\left[\left(\frac{m}{n}\right)^{8k+1}\left(\frac{1}{8k+1}-\frac{m^2}{n^2(8k+3)}+\frac{m^4}{n^4(8k+5)}-\frac{m^6}{n^6(8k+7)}\right)+\left(\frac{n-m}{n+m}\right)^{8k+1}\left(\frac{1}{8k+1}-\frac{(n-m)^2}{(n+m)^2(8k+3)}+\frac{(n-m)^4}{(n+m)^4(8k+5)}-\frac{(n-m)^6}{(n+m)^6(8k+7)}\right)\right]$$

2.2.Para $n \in \mathbb{N}_0 = \{0,1,2,3,...\}$ se tiene:

$$\pi = 2\frac{n!}{\left(\frac{1}{2}\right)_n}\sum_{m=0}^{\infty}\frac{\binom{6m}{3m}}{2^{6m}}\left(\frac{1}{6m+2n+1}+\frac{6m+1}{2(3m+1)(6m+2n+3)}+\frac{(6m+1)(6m+3)}{4(3m+1)(3m+2)(6m+2n+5)}\right)$$

2.3.Para $0 \le x < \frac{1}{\sqrt{2}}$ se tiene:

$$\frac{\pi}{2\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{(2n)!\left(1-x^2\right)^n\left(1+2^{2n}x^{2n+1}\right)}{2^{2n}(n!)^2(2n+1)}$$

2.4.Para $n,m \in \mathbb{N}_0 = \{0,1,2,...\}$ , se tiene:

$$\pi = \frac{2^{2m+1} n!\, m!}{\left(\frac{1}{2}\right)_{m+n+1}} \sum_{k=0}^{\infty} \frac{(-1)^k (2m+1)_k \left(m-n+\frac{1}{2}\right)_k}{k! \left(m+n+\frac{3}{2}\right)_k}$$

2.5.Para $0 < a < b$ , se tiene:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{a}{b}\right)^{2n+1} + \left(\frac{b-a}{b+a}\right)^{2n+1}\right)$$

2.6.Para $0 < a < b$ , $M = a(a+b)$, $P = b(b-a)$, $Q = b(a+b)$, se tiene:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{M^{2n+1} + P^{2n+1}}{Q^{2n+1}}\right) = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\frac{C(n)}{D(n)}\right) = 4\sum_{n=0}^{\infty} (-1)^n \left(\frac{C(n)}{E(n)}\right)$$

donde

$$C(n+2) = (M+P)C(n+1) - MP\, C(n)$$

$$D(n+1) = Q^2 D(n)$$

$$C(0) = M + P \qquad C(1) = M^3 + P^3 \qquad D(0) = Q$$

$$E(n+2) = 2Q^2 E(n+1) - Q^4 E(n)$$

$$E(0) = Q \qquad E(1) = 3Q^3$$

2.7.Para $k \in \mathbb{N} = \{1,2,3,...\}$, se tiene:

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{2k}{2k+1}\right)^{2n+1} + \left(\frac{1}{4k+1}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{2k-1}{2k}\right)^{2n+1} + \left(\frac{1}{4k-1}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{2k-1}{2k+1}\right)^{2n+1} + \left(\frac{1}{2k}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{2k}{4k+1}\right)^{2n+1} + \left(\frac{2k+1}{6k+1}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{k}{k+1}\right)^{2n+1} + \left(\frac{1}{2k+1}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{2k}{3k+1}\right)^{2n+1} + \left(\frac{k+1}{5k+1}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{1}{2k}\right)^{2n+1} + \left(\frac{1}{2k+1}\right)^{2n+1} + \left(\frac{(k-1)(2k+1)}{k(2k+3)}\right)^{2n+1}\right)$$

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\left(\left(\frac{1}{2k+1}\right)^{2n+1} + \left(\frac{1}{2k+3}\right)^{2n+1} + \left(\frac{(2k+1)^2}{4k^2+12k+3}\right)^{2n+1}\right)$$

2.8.Para $m \in \mathbb{N}_0 = \{0,1,2,3,...\}$ , $a_k \in \mathbb{R}$ , $k = 0,..,m$ , se tiene:

$$\frac{\pi}{4}\sum_{k=0}^{m}(-1)^k a_k - \sum_{k=0}^{m}(-1)^k a_k \sum_{r=1}^{k}\frac{(-1)^{r-1}}{2r-1} = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{m}\frac{a_k}{2n+2k+1}$$

Ejemplos:

$$m = 2 \quad a_0 = 1 \quad a_1 = 2 \quad a_2 = 3$$

$$\frac{\pi}{2} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2n+1} + \frac{2}{2n+3} + \frac{3}{2n+5}\right)$$

2.9.

$$\pi = 8\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\left(2^{k+3}n + 2^{k+2} - 3\right)\left(2^{k+3}n + 2^{k+2} - 1\right)}$$

$$\pi = 8\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{\left(2^{n-k+3}k + 2^{n-k+2} - 3\right)\left(2^{n-k+3}k + 2^{n-k+2} - 1\right)}$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

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# COLECCIÓN DE SERIES QUE CONTIENEN LA CONSTANTE $\pi$

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de series que contienen la constante $\pi = 3.1415926535...$

## 1. INTRODUCCIÓN.

En esta nota se muestra una colección de series que contienen la constante $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , y que corresponden a casos particulares de las siguientes fórmulas generales:

$$\frac{\pi}{6\sqrt{3}} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) + \frac{1}{6}\ln\left(\frac{(1+v)^2}{1-v+v^2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}v^{3n+1}$$

$$0 \le v \le 1 \qquad (1)$$

$$\frac{\pi}{9\sqrt{3}} + \frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2v-1}{\sqrt{3}}\right) + \frac{1}{9}\ln\left(\frac{(1+v)^2}{1-v+v^2}\right) + \frac{v}{3(1+v^3)} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}v^{3n+1}$$

$$0 \le v < 1 \qquad (2)$$

## 2. EJEMPLOS DE SERIES TIPO (1)

2.1. Caso $v = \frac{1}{m}$, $m \in \mathbb{N}$ :

$$\frac{\pi}{6\sqrt{3}} + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2-m}{m\sqrt{3}}\right) + \frac{1}{6}\ln\left(\frac{(m+1)^2}{m^2-m+1}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{m}\right)^{3n+1}$$

$$\frac{\pi}{3\sqrt{3}} + \frac{\ln 2}{3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$$

$$\frac{\pi}{6\sqrt{3}} + \frac{\ln 3}{6} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{2}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{16}{7}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{3}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{25}{13}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{4}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{3}}{5}\right)+\frac{1}{6}ln\left(\frac{12}{7}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{5}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2}{3\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{49}{31}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{6}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}-\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{5}{7\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{64}{43}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{1}{7}\right)^{3n+1}$$

2.2 Caso $v=\frac{m}{m+1}$, $m\in\mathbb{N}$ :

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{m-1}{(m+1)\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{(2m+1)^2}{m^2+m+1}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{m}{m+1}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{25}{7}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{2}{3}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{49}{13}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{3}{4}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{3}}{5}\right)+\frac{1}{6}ln\left(\frac{27}{7}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{4}{5}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2}{3\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{121}{31}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{5}{6}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{5}{7\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{169}{43}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{6}{7}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{\sqrt{3}}{4}\right)+\frac{1}{6}ln\left(\frac{75}{19}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{7}{8}\right)^{3n+1}$$

2.3. Caso $v=\frac{2m-1}{2m+1}$, $m\in\mathbb{N}$ :

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{2m-3}{(2m+1)\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{(4m)^2}{4m^2+3}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{2m-1}{2m+1}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{1}{5\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{64}{19}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{3}{5}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{3}{7\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{48}{13}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{5}{7}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{5}{9\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{256}{67}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{7}{9}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{7}{11\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{400}{103}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{9}{11}\right)^{3n+1}$$

$$\frac{\pi}{6\sqrt{3}}+\frac{1}{\sqrt{3}}tan^{-1}\left(\frac{9}{13\sqrt{3}}\right)+\frac{1}{6}ln\left(\frac{192}{49}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}\left(\frac{11}{13}\right)^{3n+1}$$

## 3. EJEMPLOS DE SERIES TIPO (2)

3.1. Caso $v=\frac{1}{m}$, $m=2,3,4,\ldots$:

$$\frac{\pi}{9\sqrt{3}}+\frac{2}{3\sqrt{3}}tan^{-1}\left(\frac{2-m}{m\sqrt{3}}\right)+\frac{1}{9}ln\left(\frac{(m+1)^2}{m^2-m+1}\right)+\frac{m^2}{3\left(1+m^3\right)}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{1}{m}\right)^{3n+1}$$

$$\frac{\pi}{9\sqrt{3}}+\frac{ln3}{9}+\frac{4}{27}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{1}{2}\right)^{3n+1}$$

$$\frac{\pi}{9\sqrt{3}}-\frac{2}{3\sqrt{3}}tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)+\frac{1}{9}ln\left(\frac{16}{7}\right)+\frac{3}{28}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{1}{3}\right)^{3n+1}$$

3.2. Caso $v=\frac{m}{m+1}$, $m\in\mathbb{N}$:

$$\frac{\pi}{9\sqrt{3}}+\frac{2}{3\sqrt{3}}tan^{-1}\left(\frac{m-1}{(m+1)\sqrt{3}}\right)+\frac{1}{9}ln\left(\frac{(2m+1)^2}{m^2+m+1}\right)+\frac{m(m+1)^2}{3\left((m+1)^3+m^3\right)}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{m}{m+1}\right)^{3n+1}$$

$$\frac{\pi}{9\sqrt{3}}+\frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)+\frac{1}{9}\ln\left(\frac{25}{7}\right)+\frac{6}{35}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{2}{3}\right)^{3n+1}$$

3.3. Caso $v=\dfrac{2m-1}{2m+1}$, $m\in\mathbb{N}$ :

$$\frac{\pi}{9\sqrt{3}}+\frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{2m-3}{(2m+1)\sqrt{3}}\right)+\frac{1}{9}\ln\left(\frac{(4m)^2}{4m^2+3}\right)+\frac{(2m+1)^2(2m-1)}{3\left((2m+1)^3+(2m-1)^3\right)}=$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{2m-1}{2m+1}\right)^{3n+1}$$

$$\frac{\pi}{9\sqrt{3}}+\frac{2}{3\sqrt{3}}\tan^{-1}\left(\frac{1}{5\sqrt{3}}\right)+\frac{1}{9}\ln\left(\frac{64}{19}\right)+\frac{25}{152}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{3n+1}\left(\frac{3}{5}\right)^{3n+1}$$

## 4. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

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# $\pi$ - FÓRMULAS

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen**
Se muestra una colección de fórmulas que contienen la constante $\pi = 3.14159265...$

## 1. INTRODUCCIÓN.

El número Pi ( $\pi$ ) , se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota se muestra una colección de fórmulas que involucran la constante $\pi$ .

## 2. FÓRMULAS.

2.1.Para $0 < a < x < b$ :

$$\pi = 2\tan^{-1}\left(\frac{x-a}{x+a}\right) + 2\tan^{-1}\left(\frac{b-x}{b+x}\right) + 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{2n+1}A_n x^{2n+1}$$

$$A_n = \begin{cases} a^{-(2n+1)} & n \le -1 \\ b^{-(2n+1)} & n \ge 0 \end{cases}$$

2.2.Para $N \in \mathbb{N}$ , $N \ge 3$ , sean $y_n \in [0,1]$ , $1 \le n \le N$ , tales que: $y_1 = 0$ , $y_N = 1$ , $y_n < y_{n+1}$ $n = 1,..,N-1$ . Se tiene:

$$\frac{\pi}{2}(N-2) = sen^{-1}\left(\sqrt{1-y_2^2}\right) + sen^{-1}(y_{N-1}) + \sum_{n=2}^{N-2} sen^{-1}\left(\sqrt{1-y_n^2}\sqrt{1-y_{n+1}^2} + y_n y_{n+1}\right)$$

2.3.Para $n \in \mathbb{N} - \{1\}$ , sean $a_k > 0$ $k = 1,..,n$ , tales que: $\sqrt{2} < \sum_{k=1}^{n} a_k < \frac{\pi}{2}$ , se tiene:

$$\frac{\pi}{2}(n-1) = \sum_{k=1}^{n} sen^{-1}\left(1 - \frac{1}{2}a_k^2\right) = \sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m^2}{\left(\frac{3}{2}\right)_m}\sum_{k=1}^{n}\left(1-\frac{1}{2}a_k^2\right)^{2m+1}$$

2.4.Para $m \in \mathbb{N}$, se tiene:

$$\frac{\pi}{\underbrace{\sqrt{2-\sqrt{2+...+\sqrt{2}}}}_{m-radicales}} = 2^m + 2^m \sum_{n=1}^{\infty} \frac{\left(3 \cdot 2^{2m+2} n^2 + 1\right) P(n,m)}{(2n)!\left(2^{2m+2} n^2 - 1\right) 2^{(2m+2)n}}$$

$$\frac{\pi}{\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m-radicales}} = \frac{2^m}{2^m - 1} + 2^m \left(2^m - 1\right) \sum_{n=1}^{\infty} \frac{\left(3 \cdot 2^{2m+2} n^2 + \left(2^m - 1\right)^2\right) Q(n,m)}{(2n)!\left(2^{2m+2} n^2 - \left(2^m - 1\right)^2\right) 2^{(2m+2)n}}$$

donde

$$P(n,m) = \prod_{k=1}^{n-1} \left(2^{2m+2} k^2 - 1\right)$$

$$P(n+1,m) = P(n,m)\left(2^{2m+2} n^2 - 1\right) \qquad P(1,m) = 1$$

$$Q(n,m) = \prod_{k=1}^{n-1} \left(2^{2m+2} k^2 - \left(2^m - 1\right)^2\right)$$

$$Q(n+1,m) = Q(n,m)\left(2^{2m+2} n^2 - \left(2^m - 1\right)^2\right) \qquad Q(1,m) = 1$$

2.5.Para $a,b \in \mathbb{N}$, $a \geq b\sqrt{2}$, se tiene:

$$\pi\sqrt{a-b}\sqrt{a+b} = \frac{2\left(a^2 - b^2\right)}{a} \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2^{2n}(2n+1)} \left(1 - \frac{b^2}{a^2}\right) + 2b \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{b^2}{a^2 - b^2}\right)^n$$

Ejemplos:

$$\pi\sqrt{3} = 3 \sum_{n=0}^{\infty} \frac{\binom{2n}{n} 3^n}{2^{2n}(2n+1)} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) 3^n}$$

$$\pi = \frac{8}{5} \sum_{n=0}^{\infty} \frac{\binom{2n}{n} (4/5)^{2n}}{2^{2n}(2n+1)} + \frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{3}{4}\right)^{2n}$$

2.6.Sea $p_k$ el k-ésimo número primo, $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$,sea $H = p_2 p_3 \cdots p_k$, para $k \geq 2, k \in \mathbb{N}$,sean $R$ y $S$ números impares que satisfacen las siguientes condiciones: (1) $H = RS$ , (2) $R > S$, (3) $R^2 + S^2 \leq 6RS$, se tiene:

$$\pi\sqrt{H} = 4(R+S) \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{2n+1} \left(\frac{H}{(R+S)^2}\right)^{n+1} + (R-S) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{(R-S)^2}{4H}\right)^n$$

Ejemplos:

$$H=3\cdot 5\cdot 7\cdot 11=1155 \quad R=77 \quad S=15$$
$$H=3\cdot 5\cdot 7\cdot 11\cdot 13=15015 \quad R=143 \quad S=105$$

$$H=3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29=3234846615$$
$$R=57057 \qquad S=56695$$

$$\pi\sqrt{3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29}=$$

$$=455008\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{2n+1}\left(\frac{3234846615}{113752^2}\right)^{n+1}+362\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{181^2}{3234846615}\right)^n$$

2.7.

$$\frac{\pi}{2\sqrt{3}}=\sum_{n=0}^{\infty}\left(\frac{v^{3n+1}-u^{3n+1}}{3n+1}-\frac{v^{3n+2}-u^{3n+2}}{3n+2}\right)$$

$$-1<u<-\frac{3-\sqrt{3}}{3+\sqrt{3}} \qquad v=\frac{2+u\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)-2u}$$

2.8.

$$\frac{\pi^3}{12}=\sum_{n=1}^{\infty}\frac{\tan^{-1}(n)}{n^2}+\sum_{n=0}^{\infty}\frac{(-1)^n\zeta(2n+3)}{2n+1}$$

$\zeta(n)$ es la función zeta de Riemann

2.9.Para $k\in\mathbb{N}=\{1,2,3,...\}$, se tiene:

$$\pi\sqrt{\frac{2-r(k)}{2+r(k)}}=2\sum_{n=1}^{\infty}\frac{\left(2^{2n}-1\right)\zeta(2n)}{2^{(k+2)(2n-1)}}$$

$$\frac{1}{\pi}\sqrt{\frac{2-r(k)}{2+r(k)}}=\frac{1}{2^{k+2}}+8\sum_{n=2}^{\infty}\frac{\left(2^{2n}-1\right)\zeta(2n-2)B_n}{(2n-1)(2n)2^{(k+2)(2n-1)}B_{n-1}}$$

$$r(k)=\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k-radicales}$$

$\zeta(n)$ función zeta de Riemann

$B_n$ números de Bernoulli

2.10.Para $n\in\mathbb{N}=\{1,2,3,...\}$, se tiene:

$$\pi^n=\frac{\sqrt{2n+1}}{2^n}\prod_{k=1}^{2n}\Gamma\left(\frac{k}{2n+1}\right)$$

$$\pi^{n-1}\sqrt{\pi} = \frac{\sqrt{n}}{2^{n-1}}\prod_{k=1}^{2n-1}\Gamma\left(\frac{k}{2n}\right)$$

$$\pi^{n} = \frac{\sqrt{2n+1}\left(2n+1\right)^{2n}}{2^{n}\left(2n\right)!}\prod_{m=1}^{\infty}\frac{\left(1+\frac{1}{m}\right)^{n}}{\prod_{k=1}^{2n}\left(1+\frac{k}{\left(2n+1\right)m}\right)}$$

$$\pi^{n-1}\sqrt{\pi} = \frac{2^{n}n^{2n-\frac{1}{2}}}{\left(2n-1\right)!}\prod_{m=1}^{\infty}\frac{\left(1+\frac{1}{m}\right)^{n-\frac{1}{2}}}{\prod_{k=1}^{2n-1}\left(1+\frac{k}{2nm}\right)}$$

$$\pi^{n} = \frac{\sqrt{2n+1}\left(2n+1\right)^{2n}}{2^{n}\left(2n\right)!}\prod_{m=1}^{\infty}\frac{\left(\left(2n+1\right)_{m}\right)^{2n+1}\left(1+\frac{1}{m}\right)^{n}}{\left(\left(2n+1\right)m\right)_{2n+1}}$$

$$\pi^{n-1}\sqrt{\pi} = \frac{2^{n}n^{2n-\frac{1}{2}}}{\left(2n-1\right)!}\prod_{m=1}^{\infty}\frac{\left(2nm\right)^{2n}\left(1+\frac{1}{m}\right)^{n-\frac{1}{2}}}{\left(2nm\right)_{2n}}$$

2.11.

$$\pi = 3\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n}\left(-\sum_{k=0}^{n}\frac{\binom{2k}{k}}{2^{2k}\left(2n-2k+1\right)\left(2k-1\right)}\right)$$

$$\pi = 4\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n}\left(-\sum_{k=0}^{n}\frac{\binom{2k}{k}\left(-\frac{3}{4}\right)^{k}}{2^{2k}\left(2n-2k+1\right)\left(2k-1\right)}\right)$$

2.12.Para $n \in \mathbb{N} = \{1,2,3,...\}$ , se tiene:

$$\pi^{2} = 2^{2n+2}\left(a\left(n+1\right)\right)^{2}\sum_{k=1}^{\infty}\left[\frac{1}{\left(2^{n}\left(2k-1\right)-1\right)^{2}}+\frac{1}{\left(2^{n}\left(2k-1\right)+1\right)^{2}}\right]$$

$$a\left(n+1\right) = \sqrt{\frac{1}{2}+\frac{1}{2}a\left(n\right)} \qquad a\left(1\right) = 0$$

2.13.Para $m \in \mathbb{N}_{0} = \{0,1,2,3,...\}$ , se tiene:

$$\frac{\pi}{2}=(-1)^m\left(\ln 2+\sum_{k=1}^{m}\frac{(-1)^k}{k}\right)+(2m+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)(m+k+1)}$$

2.14.Sean $a$ , $b$ , $c$ , $d$ números reales que satisfacen las siguientes condiciones:

$$-\frac{\pi}{2}<a<\ln\left(\frac{2e^{\pi/2}-1}{2+e^{\pi/2}}\right) \qquad b=\ln\left(\frac{1+2e^a}{2-e^a}\right)$$

$$-\frac{\pi}{2}<c<\ln\left(\frac{3e^{\pi/2}-1}{3+e^{\pi/2}}\right) \qquad d=\ln\left(\frac{1+3e^c}{3-e^c}\right)$$

se tiene:

$$\frac{\pi}{2}=(b+d)-(a+c)+\sum_{n=1}^{\infty}\frac{(-1)^n E_n}{(2n)!(2n+1)}\left(b^{2n+1}-a^{2n+1}+d^{2n+1}-c^{2n+1}\right)$$

$E_n$ son los números de Euler

$$E_n=\{1,5,61,1385,...\}$$

## 3. REFERENCIAS.


1. Abramowitz, M. e I.A. Stegun, Handbook of Mathematical Functions. Nueva York: Dover, 1965.

2. I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (A. Jeffrey), Academic Press, New York, London, and Toronto, 1980.

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# ALGUNAS FÓRMULAS CLÁSICAS PARA $\pi$

**EDGAR VALDEBENITO V.**
**(2004)**

**Resumen**
Se muestran algunas fórmulas clásicas para la constante $\pi = 3.14159...$

## 1. INTRODUCCIÓN.

El número $\pi$ se define por la serie: $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ , en esta nota se muestra una colección de fórmulas que contienen la constante $\pi$ .

## 2. FÓRMULAS.

2.1.

$$\frac{2}{\pi} = \prod_{k=1}^{\infty}\prod_{m=1}^{k}\left(1-\frac{1}{\left(k(k-1)+2m\right)^2}\right)$$

2.2.

$$\frac{\pi}{4} = \prod_{k=1}^{\infty}\prod_{m=1}^{k}\left(1-\frac{1}{\left(k(k-1)+2m+1\right)^2}\right)$$

2.3.

$$\pi = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(n+1)(25n-6)\left(25n^2+24n+5\right)}{2^n\binom{3n}{n}(3n+1)(3n+2)}$$

2.4.

$$\pi = \sum_{n=0}^{\infty}\frac{25n-28}{2^n}\sum_{k=0}^{n}\frac{1}{\binom{3k}{k}}$$

2.5.

$$\pi = 4\sum_{n=0}^{\infty}\left(\frac{1}{\binom{3n}{n}}-\frac{1}{\binom{3n+3}{n+1}}\right)\left(22-(25n+47)2^{-(n+1)}\right)$$

2.6. Para $a>0$, se tiene:

$$\pi=\frac{3\sqrt{3}}{4}+6\sqrt{1+a}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!\left(1+a\right)^n}\sum_{k=0}^{n}\frac{\binom{n}{k}a^k}{\left(2n-2k+3\right)2^{2n-2k}}$$

2.7. Sean $N,M\in\mathbb{N}=\{1,2,3,..\}$, $P\in\mathbb{Z}=\{...,-3,-2,-1,0,1,2,3,...\}$, tales que: $3N^2=M^2+P$, $|P|<M^2$, $\left|P/\left(M^2+P\right)\right|<1$, se tiene:

$$\pi=\frac{6M}{N}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{3^{n-k}P^{k-1}\binom{2k-2}{k-1}}{2^{2k}\left(M^2+P\right)^{k-1}\left(n-k+1\right)\binom{2n-2k+2}{n-k+1}}$$

$$\pi=1881\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{3^{n-k}\binom{2k-2}{k-1}}{\left(2\cdot 3\cdot 11^2\right)^k\left(n-k+1\right)\binom{2n-2k+2}{n-k+1}}$$

$$\pi=7020\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\left(-1\right)^{k-1}3^{n-k}\binom{2k-2}{k-1}}{\left(2^2\cdot 3^3\cdot 5^2\right)^k\left(n-k+1\right)\binom{2n-2k+2}{n-k+1}}$$

Algunos valores para $\left(N,M,P\right)$:

$(1,1,2)$ , $(1,2,-1)$
$(3,5,2)$ , $(4,7,-1)$
$(11,19,2)$ , $(15,26,-1)$
$(2,3,3)$ , $(5,9,-6)$
$(7,12,3)$ , $(6,11,-13)$
$(5,8,11)$ , $(8,14,-4)$
$(10,17,11)$ , $(19,33,-6)$

2.8. Sean $N,M\in\mathbb{N}=\{1,2,3,..\}$, $P\in\mathbb{Z}=\{...,-3,-2,-1,0,1,2,3,...\}$, tales que: $3N^2=M^2+P$, $|P|<M^2$, $\left|P/\left(M^2+P\right)\right|<1$, se tiene:

$$\pi = 18MN \sum_{n=1}^{\infty} \frac{c_n}{(2n)!\left(12N^2\right)^n}$$

donde

$$c_n = \sum_{k=1}^{n} P^{k-1} (6N)^{2n-2k} \binom{2k-2}{k-1} (n-k)!(n-k+1)!(2n-2k+3)_{2k-2}$$

$$c_n \in \mathbb{N}$$

$$\pi = 2^5 \cdot 3^3 \cdot 7 \cdot \sum_{n=1}^{\infty} \frac{c_n}{(2n)!\, 2^{4n}\, 3^n\, 7^{2n}}$$

$$c_n = \sum_{k=1}^{n} 12^{k-1} 84^{2n-2k} \binom{2k-2}{k-1} (n-k)!(n-k+1)!(2n-2k+3)_{2k-2}$$

$$\pi = 2^3 \cdot 3^4 \cdot 7 \cdot \sum_{n=1}^{\infty} \frac{c_n}{(2n)!\, 2^{6n}\, 3^{3n}}$$

$$c_n = \sum_{k=1}^{n} (-9)^{k-1} 72^{2n-2k} \binom{2k-2}{k-1} (n-k)!(n-k+1)!(2n-2k+3)_{2k-2}$$

2.9.

$$\pi = 2 \sum_{n=1}^{\infty} \frac{2^{2n} - \binom{2n}{n}}{2^n \binom{2n}{n}}$$

$$\pi = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \left( \frac{2 \cdot 3^n - \binom{2n}{n}}{\binom{2n}{n}} \right)$$

$$\pi + 2 = \sum_{n=1}^{\infty} \frac{2^{2n}(6n-1)}{n\binom{4n}{2n}}$$

$$\pi + 2 = \frac{2}{3}\sum_{n=1}^{\infty}\frac{2^{3n}\left(63n^2-36n+4\right)}{\binom{6n}{3n}n\left(3n-1\right)}$$

2.10. Para $m\in\mathbb{N}=\{1,2,3,...\}$, se tiene:

$$\frac{2}{\pi}=\prod_{k=0}^{\infty}\left[\left(\prod_{n=1}^{m}\frac{\left(2^{n+1}k+2^n\right)^2-1}{\left(2^{n+1}k+2^n\right)^2}\right)\frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m+k+1-radicales}\right]$$

Ejemplo: $m=2$:

$$\frac{2}{\pi}=\left(\frac{1\cdot 3}{2^2}\cdot\frac{3\cdot 5}{4^2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)\left(\frac{5\cdot 7}{6^2}\cdot\frac{11\cdot 13}{12^2}\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)....$$

2.11. Para $m\in\mathbb{N}=\{1,2,3,...\}$, se tiene:

$$\frac{2}{\pi^{m+1}}\prod_{n=1}^{m}\frac{2^{n+1}}{2^n-1}=\prod_{k=1}^{\infty}\left[\left(\prod_{n=1}^{m}\left(1-\left(\frac{2^n-1}{2^{n+1}k}\right)^2\right)\right)\frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{m+k-radicales}\right]$$

Ejemplos:

$$\frac{8}{\pi^2}=\left(\frac{3\cdot 5}{4^2}\frac{\sqrt{2+\sqrt{2}}}{2}\right)\left(\frac{7\cdot 9}{8^2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)....$$

$$\frac{64}{3\pi^3}=\left(\frac{3\cdot 5}{4^2}\frac{5\cdot 11}{8^2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\right)\left(\frac{7\cdot 9}{8^2}\frac{13\cdot 19}{16^2}\frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2}\right)....$$

2.12. Para $m\in\mathbb{N}=\{1,2,3,...\}$, se tiene:

$$\frac{\pi}{4}=\sum_{n=0}^{m}\frac{(-1)^n}{2n+1}+(-1)^{m+1}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}}{2k+2m+3}$$

2.13. Sean $N,M\in\mathbb{N}=\{1,2,3,..\}$, $P\in\mathbb{Z}=\{...,-3,-2,-1,0,1,2,3,...\}$, tales que:
$2N^2=M^2+P$, $|P|<M^2$, $\left|P/\left(M^2+P\right)\right|<1$, se tiene:

$$\frac{4}{\pi}=\frac{M}{N}\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{2^{9n}\left(M^2+P\right)^n}$$

$$c_n=\sum_{k=0}^{n}\binom{2k}{k}^3\binom{2n-2k}{n-k}(6k+1)(-P)^{n-k}\,2^{7n-7k}\left(M^2+P\right)^k$$

$$c_n\in\mathbb{N}$$

Algunos valores para $(N,M,P)$:

$$(2,3,-1)$$
$$(3,4,2)$$
$$(4,5,7)$$
$$(4,6,-4)$$
$$(5,7,1)$$
$$(6,8,8)$$
$$(7,10,-2)$$

2.14.

$$\pi=\frac{6}{\sqrt{3}}-\frac{3}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{n!(2n+3)2^{2n}}$$

$$\pi=\frac{11\sqrt{3}}{6}-\frac{1}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{n!(2n+5)2^{2n}}$$

$$\pi=\frac{109\sqrt{3}}{60}-\frac{1}{40}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{n!(2n+7)2^{2n}}$$

$$\pi=\frac{16}{3\sqrt{3}}+\frac{3}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n}{n!(2n+5)2^{2n}}$$

$$\pi = \frac{653621\sqrt{3}}{360360} - \frac{1}{68640}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{n!\left(2n+17\right)2^{2n}}$$

$$\pi = \frac{980431\sqrt{3}}{540540} + \frac{1}{320320}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n}{n!\left(2n+17\right)2^{2n}}$$

$$\pi = \frac{5882587\sqrt{3}}{3243240} - \frac{1}{768768}\sum_{n=0}^{\infty}\frac{\left(\frac{7}{2}\right)_n}{n!\left(2n+17\right)2^{2n}}$$

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# NÚMERO $\pi$ , FÓRMULA

**EDGAR VALDEBENITO V.**
**(1994)**

**Resumen.** Se muestra una fórmula que involucra la constante $\pi$ .

**1. INTRODUCCIÓN**. En esta nota se muestra una fórmula que involucra la clásica constante $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$.

**2. FÓRMULA**. Para $0<x<1$, se tiene:

$$\pi^2 = U(x)+\pi V(x)$$

donde

$$U(x)=6\sum_{n=1}^{\infty}\frac{x^n}{n^2}cos\left(\frac{n\pi}{4}\right)+6\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}\left(1-\frac{x}{\sqrt{2}}\right)^{n-2k}\left(\frac{x^2}{2}\right)^k+$$
$$+3\,ln(x)\,ln\left(1+x^2-x\sqrt{2}\right)$$

$$V(x)=\frac{3}{2}tan^{-1}\left(\frac{x}{\sqrt{2}-x}\right)$$

**3. FÓRMULAS ALTERNATIVAS.**

$$\pi = V(x)+\cfrac{U(x)}{V(x)+\cfrac{U(x)}{V(x)+....}}$$

$$\pi=\sqrt{U(x)+V(x)\sqrt{U(x)+V(x)\sqrt{U(x)+....}}}$$

**4. EJEMPLO:** $x=\frac{\sqrt{2}}{3}$

$$\pi^2 = U\left(\frac{\sqrt{2}}{3}\right)+\pi V\left(\frac{\sqrt{2}}{3}\right)$$

$$V\left(\frac{\sqrt{2}}{3}\right)=\frac{3}{2}tan^{-1}\left(\frac{1}{2}\right)$$

$$U\left(\frac{\sqrt{2}}{3}\right)=2+\frac{3}{8}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\frac{2}{9}\right)^{2n}+$$

$$+9\sum_{n=1}^{\infty}(-1)^n\left(\frac{2}{9}\right)^{2n}\left(\frac{1}{(4n-1)^2}+\frac{2}{9(4n+1)^2}\right)+$$

$$+6\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}\frac{2^{n-2k}}{3^n}+3\,ln\left(\frac{5}{9}\right)ln\left(\frac{\sqrt{2}}{3}\right)$$

## 5. REFERENCIAS.


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