Thermodynamics of noncommutative quantum Kerr Black Holes

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Thermodynamic formalism for rotating black holes, characterized by noncommutative and quantum corrections, is constructed. From a fundamental thermodynamic relation, equations of state and thermodynamic response functions are explicitly given and the effect of noncommutativity and quantum correction is discussed. It is shown that the well known divergence exhibited in specific heat is not removed by any of these corrections. However, regions of thermodynamic stability are affected by noncommutativity, increasing the available states for which the system is thermodynamically stable.

I. INTRODUCTION

It is known that the formalism of thermodynamics can be applied to the physical entropy of a black hole, obtained by considering semiclassical approaches to general relativity. The best known of these approximations was proposed by Bekenstein and Hawking in order to solve the so–called information problem [1–4]. They found that the area A of the event horizon of black holes in asymptotically flat spacetime, obeys a simple relation which is the mathematical analogous of the corresponding entropy for a black hole S_{BH}. In [5–7] this is postulated to be the fundamental thermodynamic relation of black holes, which contains all thermodynamic information of the system. Under this assumption, its classical thermodynamic formalism is constructed finding that for black holes, thermodynamic structure of the theory resemble magnetic systems instead fluids.

Thermodynamics of black holes has a long tradition, in particular related to the problem of thermodynamic stability. Since the seminal work of Hawking and Page [8], it is known that this problem can be transferred to de Sitter or Anti–de Sitter (AdS) black hole. They found that thermodynamic information of de Sitter or AdS black holes exhibit important differences with respect to black holes on asymptotically flat space time, which was latter corroborated in different works [9, 10]. Since we are considering black holes in asymptotically flat spacetime, it seems legitimate though to ask if corrections like noncommutativity or semiclassical ones are able to modify the thermodynamics of black holes in order to have thermodynamic stable systems.

In entropic representation, assumed thermodynamic fundamental relation for Kerr black holes have the form $S = S(U, J)$, where $U = Mc^2$ is the internal energy of the system and J gives its angular momentum. This relation can be written as [5],

$$S_{BH}(U, J) = \frac{2\pi k_B}{\hbar c}\left(\frac{GU^2}{c^4} + \sqrt{\frac{G^2U^4}{c^8} - c^2J^2}\right); \tag{1}$$

where G is the universal gravitational constant, $\hbar$ is the reduced Planck constant, c is the speed of light in vacuum and k_B is the Boltzmann constant. We are interested in thermodynamic implications of quantum correction to Bekenstein–Hawking (BH) entropy that have arisen in recent years, in the search for suitable candidates of quantum gravity, namely, the quest for understanding microscopic states of black holes [11, 12]; and the inclusion to black hole entropy of noncommutativity. This is given considering that coordinates of minisuperspace are noncommutative [13]. Different corrections to BH entropy have emerged from a variety of approaches in recent years, logarithmic ones are a popular choice among those; arising from quantum corrections to the string theory partition function [14]. They

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are related to the low energy or infrared properties of gravity, and are also independent of high energy or ultraviolet properties of the theory [11, 14–16]. In this work, the selected expression for quantum and noncommutative corrected entropy to work with, which will be denoted as $S^\star$, is obtained according to the ideas presented in [13]. Starting from a diffeomorphism between the Kantowski–Sachs cosmological model [17] and the Schwarzschild solution, whose line element inside the event horizon $r < 2M$ is given by:

$$ds^2 = -\left(\frac{2M}{t} - 1\right)^{-1} dt^2 + \left(\frac{2M}{t} - 1\right) dr^2 + t^2(d\theta^2 + \sin^2\theta d\phi^2); \tag{2}$$

where the temporal t and the spatial r coordinates swap their role, i.e., the transformation $t \leftrightarrow r$ is performed and considering the Misner parametrization of the Kantowski–Sachs metric it follows,

$$ds^2 = -N^2 dt^2 + e^{(2\sqrt{3}\gamma)}dr^2 + e^{(-2\sqrt{3}\gamma)}e^{(-2\sqrt{3}\Omega)}(d\theta^2 + \sin^2\theta d\phi^2). \tag{3}$$

Comparing eqs. (2) and (3) it is straightforward to note the correspondence between components of the metric tensor, in order to identify the functions N, γ and Ω. The following step is to obtain the Wheeler DeWitt (WDW) equation for Kantowski–Sachs metric given in (3), whose parametrization is related with the Schwarzschild solution given in eq. (2), finding the corresponding Hamiltonian of the system H through the Arnowitt–Deser–Misner (ADM) formalism, to introduce it into the quantum wave equation $H\Psi = 0$, where $\Psi(\gamma, \Omega)$ is the wave function. This process leads to the WDW equation whose solution can be found by separation of variables. We are interested in the solution that can be obtained when the symplectic structure of minisuperspace is modified, making the coordinates Ω and γ obey the commutation relation $[\Omega, \gamma] = i\theta$, where θ is the noncommutative parameter; this relation strongly resembles noncommutative quantum mechanics.

The above modification is introduced in terms of the Moyal product, modifying the original phase space, similarly to noncommutative quantum mechanics [18]. These modifications allow to redefine the coordinates of minisuperspace in order to obtain a noncommutative version of the WDW equation,

$$\left[\frac{\partial^2}{\partial\gamma^2} - \frac{\partial^2}{\partial\Omega^2} + 48e^{(-2\sqrt{3}\Omega + \sqrt{3}\theta P_\gamma)}\right]\Psi(\Omega, \gamma) = 0. \tag{4}$$

where P_γ is the momentum on coordinate γ. The above equation can be solved by separation of variables to obtain the corresponding wave equation [19]:

$$\Psi(\Omega, \gamma) = e^{i\sqrt{3}\nu\gamma}K_{i\nu}\left[4e^{(-\sqrt{3}(\Omega + \sqrt{3}\nu\theta/2))}\right]; \tag{5}$$

where ν is the separation constant and $K_{i\nu}$ are the modified Bessel functions. It can be noticed that the dependence of γ in the above relation is a plane wave; it is worth to mention that this contribution vanishes when thermodynamic observables are calculated.

With the above wave equation for the noncommutative Kantowski–Sachs cosmological model, we are in position to derive the corresponding modified entropy. To that purpose, the Feynman–Hibbs procedure [20] is considered to calculate the partition function of the system. In this approach the separated differential equation for Ω is expanded and with a change of variables, it can be compared with a quantum harmonic oscillator in order to obtain a natural frequency, which can be related to a potential $U(x)$ which includes a quantum correction, and it is used to calculate the partition function of the system,

$$Z = A \int_{-\infty}^{\infty} e^{-\beta U(x)}dx; \tag{6}$$

where β^{-1} is proportional to the Bekenstein–Hawking temperature and A is a constant. Through this partition function it is possible calculate any thermodynamic observable. With this considerations the following expression for noncommutative $S^\star$ is found to be,

$$S^\star = S_{BH}e^{-3\nu\theta} - \frac{1}{2}k_B \ln\left[\frac{S_{BH}}{k_B}e^{-3\nu\theta}\right] + \mathcal{O}(S_{BH}^{-1}e^{-3\nu\theta}); \tag{7}$$

The functional form of noncommutative quantum black hole entropy $S^\star$ is basically the same than quantum corrected one, besides the addition of multiplicative factor $e^{-3\nu\theta}$ to Bekenstein–Hawking entropy. For the sake of simplicity, we denote the noncommutative term in this expression as:

$$\Gamma = \exp[-3\nu\theta].$$

Through the rest of this paper, natural units: $G = \hbar = k_B = c = 1$ will be considered. In this work, the Bekenstein–Hawking entropy for Kerr black holes, presented in eq. (1) will be used to study the thermodynamic properties of the modified entropy presented above. Substituting this relation in eq. (7), the corrected entropy can be written as:

$$S^\star = 2\pi\Gamma\left(U^2 + \sqrt{U^4 - J^2}\right) - \frac{1}{2}\ln\left[2\pi\Gamma\left(U^2 + \sqrt{U^4 - J^2}\right)\right] \tag{8}$$

In the following, all the thermodynamic expressions with superindex $\star$ will stand up for the noncommutative quantum corrected quantities derived from corresponding $S^\star$ entropy, and quantities without subindexes or superindexes will represent their noncommutative Bekenstein–Hawking counterparts. It is known from observational data that noncommutative parameter in spacetime is small [22, 23]; however, for entropy $S^\star$, noncommutativity on the coordinates of minisuperspace is considered instead. It is expected such parameter to be small as well [24], nonetheless, actual bounds of θ are not well known yet. In this work parameter Γ will be considered to be bounded in the interval given by $0 < \Gamma \leq 1$.

As mentioned above, eq. (8) will be assumed as a fundamental thermodynamic relation for Kerr black holes when noncommutative and quantum corrections are considered. It is well known from classical thermodynamics that fundamental thermodynamic relations, contain all thermodynamic information of the system under study [21], as a consequence, modifications on thermodynamic information originated by the introduced corrections to entropy, are carried through all thermodynamic quantities.

In Fig. 1 curves for Bekenstein–Hawking and quantum corrected entropy are presented considering only commutative relations ($\Gamma = 1$). Fig. 1(a) considers plots for $S = S(U)$ and $S^\star = S^\star(U)$. BH entropy is above of quantum correction one, including the region of small energy where entropy is thermodynamically stable [7]. Fig. 1(b) shows the same entropy as functions of angular momentum instead, for fixed values of U; it can be noticed that Bekenstein–Hawking entropy is above $S^\star$ in all of considered dominion as well. A similar analysis can be performed over noncommutativity, finding that for small values of θ, variations over S and $S^\star$ are negligible.

The goal of this manuscript is to explore how these considerations introduced into Bekenstein–Hawking entropy, change thermodynamic information contained in this new fundamental relation, in particular, thermodynamic stability and the existence of thermodynamic phase transition for these systems.

In the following an outline of this work is presented. Section II examines the different thermodynamic equations of state and their behavior when considering aforementioned modifications to entropy. The same analysis is carried out in section III, considering thermodynamic response functions instead. In section IV, a discussion of thermodynamic stability and phase transitions for Kerr black holes is presented. Some conclusions of this work are given in section V.

II.   EQUATIONS OF STATE

Fundamental Bekenstein–Hawking thermodynamic relation in entropic representation for Kerr black holes has the form $S_{BH} = S_{BH}(U, J)$. The role of thermodynamic equations of state for Kerr black holes is played by partial derivatives of entropy, $T \equiv (\partial_S U)_J$ and $\Omega \equiv (\partial_J U)_S$ where Ω is the angular velocity of the black hole and T is its temperature, the following relations in entropic representation are defined:

$$\frac{1}{T} \equiv \left(\frac{\partial S_{BH}}{\partial U}\right)_J; \quad \frac{\Omega}{T} \equiv -\left(\frac{\partial S_{BH}}{\partial J}\right)_U. \tag{9}$$

these definitions are also valid for quantum corrected entropy $S^\star$.

Explicit equations of state in entropic representation, $T^\star = T^\star(U, J)$ and $\Omega^\star = \Omega^\star(U, J)$ for noncommutative quantum corrected entropy can be written as

$$\frac{1}{T^\star} = \frac{U\left(4\pi\Gamma\sqrt{U^4 - J^2} + 4\pi\Gamma U^2 - 1\right)}{\sqrt{U^4 - J^2}}; \tag{10a}$$

$$\frac{\Omega^\star}{T^\star} = \frac{1}{2}\frac{J\left(4\pi\Gamma\sqrt{U^4 - J^2} + 4\pi\Gamma U^2 - 1\right)}{\sqrt{U^4 - J^2}\left(U^2 + \sqrt{U^4 - J^2}\right)}. \tag{10b}$$

In addition, for nonconmutative Bekenstein–Hawking entropy the corresponding equations of state are given by,

$$\frac{1}{T} = \frac{4\pi\Gamma U\left(U^2 + \sqrt{U^4 - J^2}\right)}{\sqrt{U^4 - J^2}}, \tag{11a}$$

$$\frac{\Omega}{T} = \frac{2\pi\Gamma J}{\sqrt{U^4 - J^2}}; \tag{11b}$$
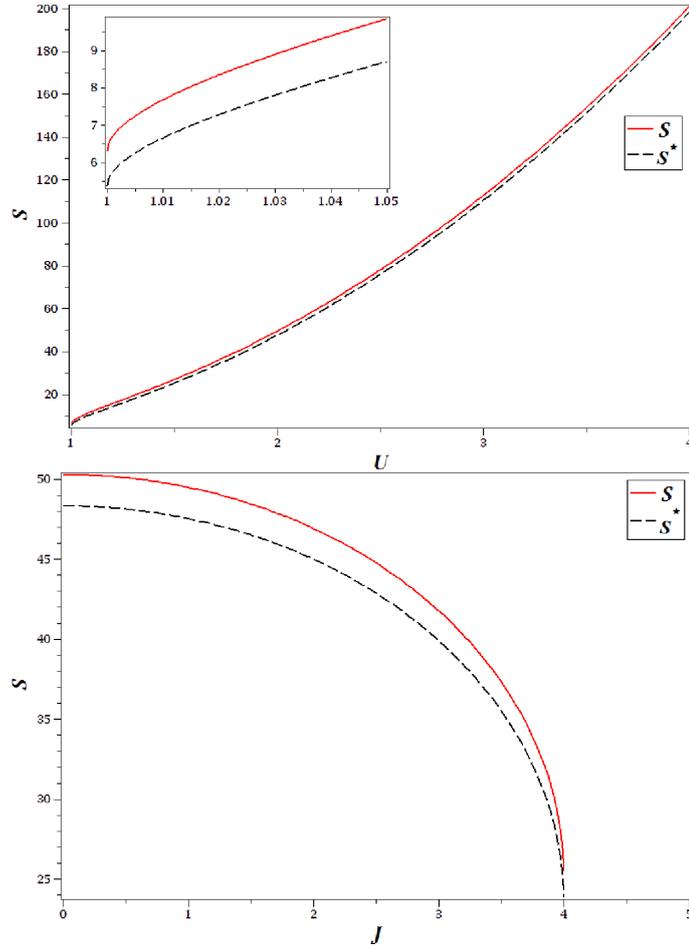
Figure 1: Comparison between Bekenstein–Hawking (solid line) and quantum corrected (dashed line) entropies.(a) Entropy as a function of internal energy for $J = 1$. (b) Entropy as a function of angular momentum for $U = 10$.

The overall effect of noncommutativity over T and $T^\star$ was analyzed, considering different values of Γ, including the commutative case ($\Gamma = 1$). A noticeable effect of this parameter over these curves can be found; nonetheless, it does not change functional behavior either of T or $T^\star$. In order to illustrate how the introduced quantum correction affect thermodynamic properties of black holes when compared with Bekenstein–Hawking ones, a graphical comparison between T and $T^\star$ is performed in Fig. 2 for $\Gamma = 1$. As expected, BH curves are very similar to those obtained through corrected entropy. Nevertheless, for temperature it is possible to remark that $T^\star(U, J)$ is slightly higher than $T(U, J)$, which is the opposite result that the one obtained for entropy, indicating that variations of entropy for a given change in its internal energy are greater for quantum corrected entropy when compared to Bekenstein–Hawking one.

It was mentioned above that when considering values in the vicinity of $\Gamma = 1$, temperature is minimally affected by noncommutativity. Smaller values of Γ were also tested, as a consequence of this consideration, maximum values capable to reach by T and T^* are noticeable increased. It must be remarked that changing this parameter does not alter the shape of the curves.

Regarding the angular velocity, it is an interesting result to remark that this property is independent of both quantum and nonconmutative corrections to entropy, namely,

$$\Omega = \Omega^\star = \frac{J}{2U\left(U^2 + \sqrt{U^4 - J^2}\right)}. \tag{12}$$

In Fig. 3 angular velocity in entropic representation is presented. Fig. 3(a) shows Ω as a function of energy for $J = 1$; in this case Ω increases until it reaches a maximum value from which it becomes complex, and it is determined by square root that appears in the denominator of eq. (12).
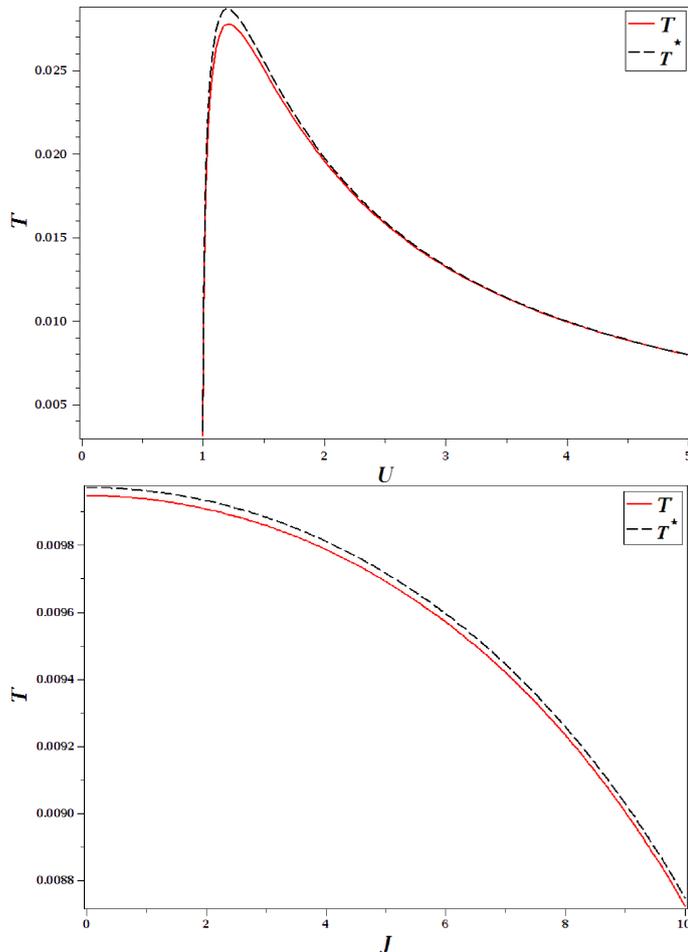
Figure 2: Plots of Bekenstein–Hawking and quantum corrected temperatures for $\Gamma = 1$. (a) T (solid) vs $T^\star$ (dash) as a function of internal energy considering $J = 1$. (b) The same plots of temperature for variations in angular momentum at $U = 10$.

III.   RESPONSE FUNCTIONS

Response functions contain valuable information about the thermodynamic behavior of systems; therefore, this topic must be addressed in order to study the changes if any, introduced to black hole thermodynamic properties by noncommutativity and quantum correction to its entropy. It is possible to define thermodynamic response functions for a Kerr black hole considering Bekenstein–Hawking entropy, following the structure exhibited by magnetic systems in such a way that resulting TdS equations [25] are completely analogous of their magnetic counterparts [7]. Subsequently and following this resemble with magnetic systems, thermodynamic response functions are defined in this work without any weight factor except for heat capacities, defined with such factor given by inverse of temperature. Even with the above considerations, thermodynamic response functions for black holes can also be constructed following the structure commonly associated with fluids, as made in Refs. [5, 26].

The first response to be analyzed is the heat capacity at constant angular momentum, defined as:

$$C_J \equiv \left(\frac{dQ}{dT}\right)_J = T\left(\frac{\partial S}{\partial T}\right)_J = \left(\frac{\partial U}{\partial T}\right)_J; \tag{13}$$

in entropic representation $T = T(U, J)$, which makes convenient to write,

$$C_J = \left(\frac{\partial T}{\partial U}\right)_J^{-1}. \tag{14}$$

Corresponding heat capacity at constant angular momentum for noncommutative Bekenstein–Hawking and quantum
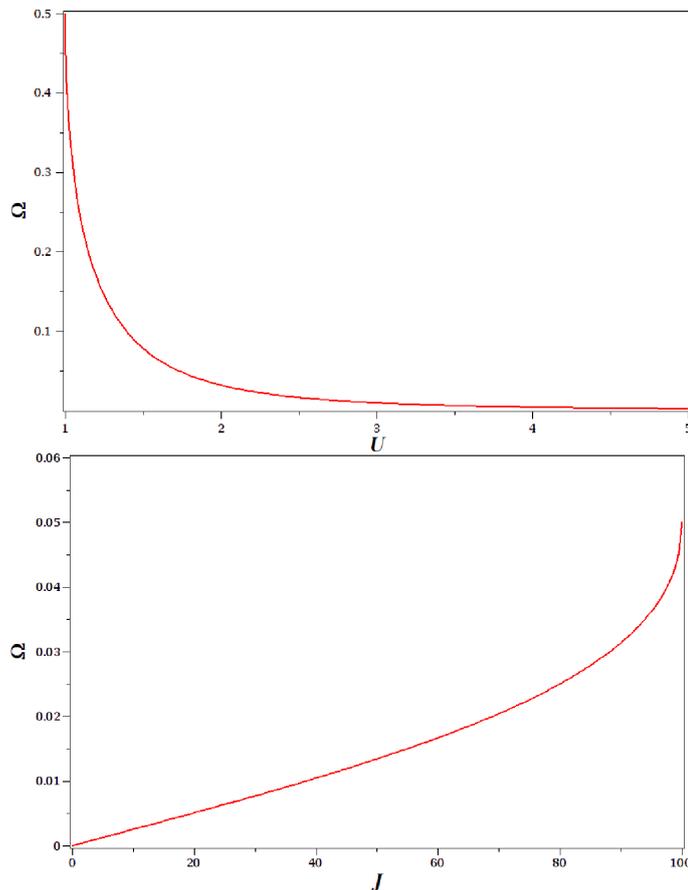
Figure 3: Plots of angular velocity for both Bekenstein–Hawking and quantum corrected entropies. (a) Ω as function of internal energy for $J = 1$. (b) Angular velocity as a function of angular momentum considering $U = 10$.

corrected entropies are respectively given by:

$$C_J = \frac{4\pi\Gamma U^2\sqrt{U^4 - J^2}\left(U^2 + \sqrt{U^4 - J^2}\right)}{\left(U^4 + J^2 - 2U^2\sqrt{U^4 - J^2}\right)}; \tag{15a}$$

$$C_J^\star = \frac{U^2\sqrt{U^4 - J^2}\left[4\pi\Gamma\sqrt{U^4 - J^2} + 4\pi\Gamma U^2 - 1\right]^2}{\left[4\pi\Gamma U^6 + U^4 - 12\pi\Gamma U^2 J^2 + 4\pi\Gamma\sqrt{U^4 - J^2}\left(U^4 - J^2\right) + J^2\right]}. \tag{15b}$$

In Fig. 4, C_J is plotted as a function of internal energy for a given angular momentum in Fig. 4(a), and as a function of J for $U = 10$ in Fig. 4(b). The most relevant feature exhibited in those plots is the divergence that appears in both curves, which divides heat capacity in two regions, one where C_J is positive and another where specific heat becomes negative. It is well known for some time, that black holes display divergences in response functions, specifically in heat capacity [5]. This feature also has been found to appear in high dimensional black hole models [26–29], and often has been related with phase transitions in black holes, this topic will be discussed and presented later.

For response functions, lower values of Γ were also tested. Specifically in Fig. 4, it can be observed that varying this parameter do not change the appearance of discontinuity of C_J. Although this divergence is not removed by noncommutativity, negative part of specific heat is reduced as Γ is reduced. Divergence in C_J can be traced to the roots of denominator in eq. (15.a):

$$U^4 + J^2 - 2U^2\sqrt{U^4 - J^2} = 0,$$

this function has one real positive root for J,

$$J_{\text{sing}} = \sqrt{-3 + 2\sqrt{3}} \cdot U^2 \approx 0.68U^2. \tag{16}$$
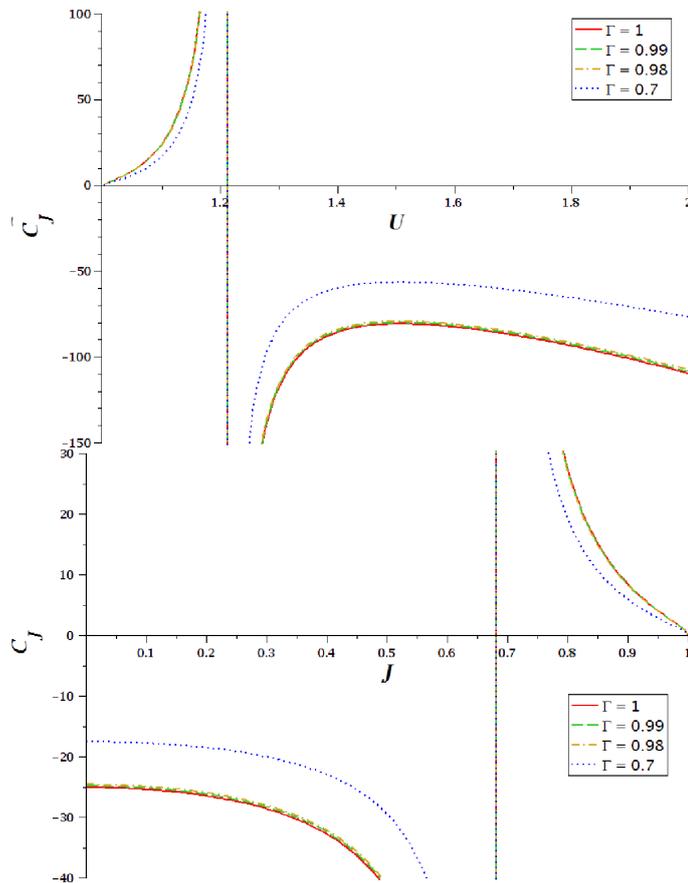
Figure 4: Specific heat capacity at constant J for Kerr black hole considering different values of Γ exhibiting a discontinuity. (a) C_J is presented as a function of energy at $J = 1$. (b) Plots of specific heat as a function of angular momentum for $U = 1$.

Which is the same value found in Ref. [5], for a Kerr black hole with Bekenstein–Hawking entropy. It must be remarked that the above root it is not affected by noncommutativity corrections.

To locate this divergence across different equations of state in entropic representation, it is necessary to substitute the above value J_{sing} in each equation of state. For example, for the angular velocity equation of state given in eq. (11b), if eq. (16) is substituted into this expression, it leads to:

$$\frac{\Omega}{T} \approx 5.83\Gamma, \tag{17}$$

or $\Omega \approx 5.8271\Gamma T$. Therefore, in the plane Ω–T there is a straight line which divides this plane in two regions where $C_J > 0$ before the divergence, and the second one where $C_J < 0$. Analogously, for Bekenstein–Hawking entropy and temperature, the corresponding function which divides the S–U plane is given by the parabola determined by:

$$S \approx 10.88\Gamma U^2; \tag{18}$$

additionally, plane T–U is divided by the following straight line,

$$T \approx 29.73\Gamma U. \tag{19}$$

One highlight from eqs. (17)–(19) is that all of them are linear functions of noncommutativity parameter Γ (and exponentially on θ). In Fig. 5 all the above corresponding thermodynamic planes are plotted, changes introduced by noncommutativity in each of those planes increase the area of the region where specific heat C_J is positive, therefore reducing the possible values for which this response function can become negative. These changes near $\Gamma = 1$ are subtle, but if Γ is taken out this neighborhood, the area where $C_J < 0$ becomes considerably smaller. This is an important result that can be related to thermodynamic stability, and it will be discussed in the next section. It is enough to indicate that noncommutativity modify the region of available thermodynamically stable states for the system.
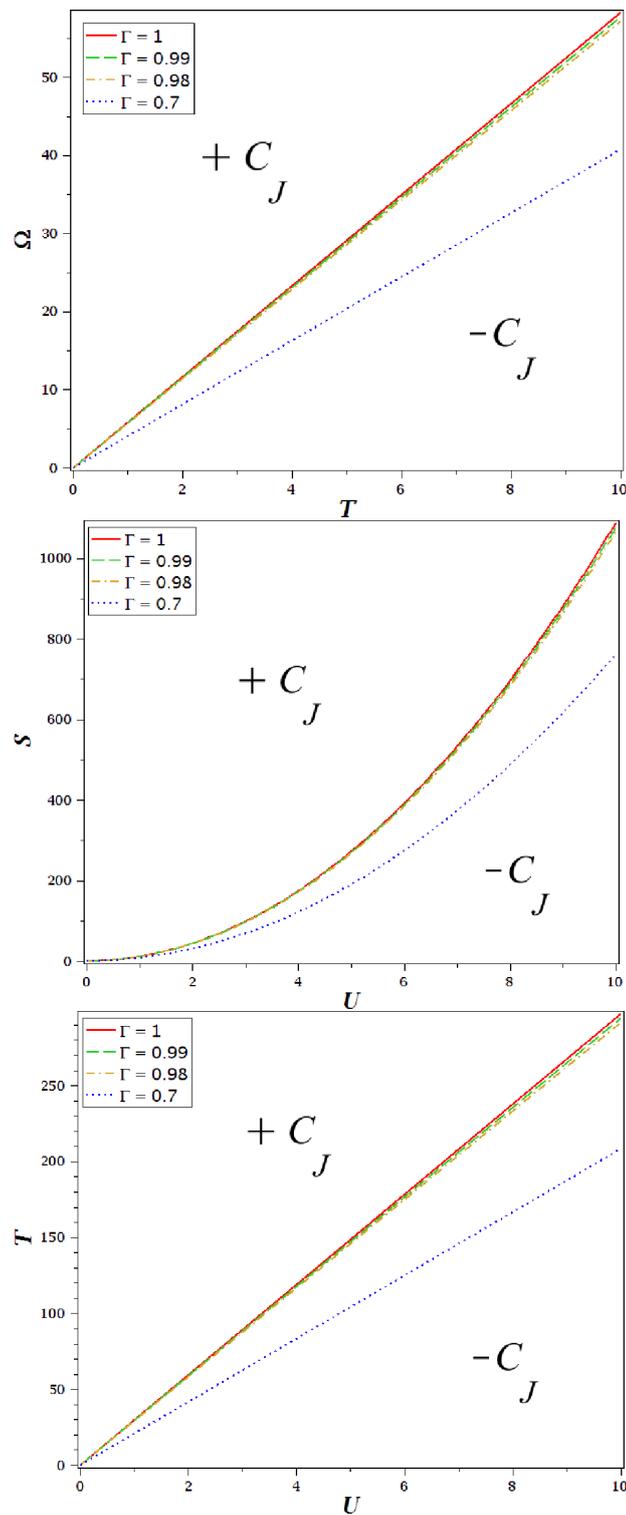
Figure 5: Thermodynamic planes considering different values of Γ showing location of divergence found in specific heat, $\pm C_J$ indicates the sign of this response function in each region. (a) Plane Ω–T divided by a line representing discontinuity given in eq. (16). (b) Plane S–U shows a parabola dividing regions where C_J is positive or negative. (c) Plane T–U depicts another straight line separating both regions.

Regarding the noncommutative quantum corrected specific heat capacity at constant angular momentum $C_J^\star$, results are fairly similar to its Bekenstein–Hawking counterpart, quantum correction do not removes the discontinuity in this

response function. Similarly as C_J, it can be located by finding the roots in denominator of eq. (15b),

$$4\pi\Gamma U^6 + U^4 - 12\pi\Gamma U^2 J^2 + 4\pi\Gamma U^4 \sqrt{U^4 - J^2} - 4\pi\Gamma J^2 \sqrt{U^4 - J^2} + J^2 = 0; \tag{20}$$

To simplify the above expression a set of manageable functions of J are obtained, by substituting a sequence of values for U into (20). It is possible to solve each of these function assuming $J > 0$, constructing a set of coordinate pairs (U, J). This process is repeated several times to obtain a relevant sample in order to accurately represent the root of eq. (20). Straightforward, a plot can be constructed, using the set (U, J) to obtain an expression for the root by least–squares method. When plotted, points clearly exhibit a quadratic behavior, and a fitting process lead us to:

$$J_{\mathrm{sing}} \approx 0.02176 + 0.68126 U^2. \tag{21}$$

When compared to eq. (16), this result show that discontinuity in $C_J^\star$ is almost the same than the one found for Bekenstein–Hawking specific heat. This result is another indication, as shown above for first–order derivatives, that thermodynamic properties obtained from noncommutative quantum corrected entropy are very close with their noncommutative BH counterparts.

Unlike results presented in eq. (16), there is not a simple function that can be used to describe behavior of neither equation of state through its corresponding phase plane for eq. (21). The only plane that can be plotted directly is $S^\star$–U one, although its functional behavior is not simple. Correspondingly to C_J, exhibited behavior in this plane for quantum corrected expression is very similar to the one obtained for noncommutative BH one, in Fig. 5(b); including the role played parameter Γ. If C_J and $C_J^\star$ are compared in the commutative case, it is found that $C_J(U, J) > C_J^\star(U, J)$ by a slight margin in all their dominion.

Another response function that can be defined for Kerr black holes is the isothermic rotational susceptibility [7],

$$\chi_T \equiv \left(\frac{\partial J}{\partial \Omega}\right)_T. \tag{22}$$

Alternative functional forms for this response function can be obtained in entropic representation. With some algebraic manipulation it is possible to write isothermic rotational susceptibility as:

$$\chi_T = \left[\left(\frac{\partial \Omega}{\partial J}\right)_U - \frac{(\partial \Omega/\partial U)_J}{(\partial T/\partial U)_J}\left(\frac{\partial T}{\partial J}\right)_U\right]^{-1}; \tag{23}$$

It is possible to work in entropic representation with the above result since both equations of state $T = T(U, J)$ and $\Omega = \Omega(U, J)$ are available.

For noncommutative Bekenstein–Hawking entropy, isothermic rotational susceptibility can be written as

$$\chi_T = -\frac{2}{U^3}\left[\left(U^2 + \sqrt{U^4 - J^2}\right)\left(U^4 + J^2 - 2U^2\sqrt{U^4 - J^2}\right)\right]; \tag{24}$$

a remarkable feature of this material property is that it is independent of the noncommutative parameter Γ, in analogy of angular velocity presented in eq. (12). Furthermore, χ_T is well defined in all its domain, on the opposite to C_J. Plots for χ_T are presented in Fig. 6, from these curves it can be noted that $\chi_T \to 0$ when $J \approx 0.68 U^2$ or equivalently, $U \approx 1.21\sqrt{J}$. $\chi_T(U)$ have a region of negative values, which is also related to thermodynamic stability.
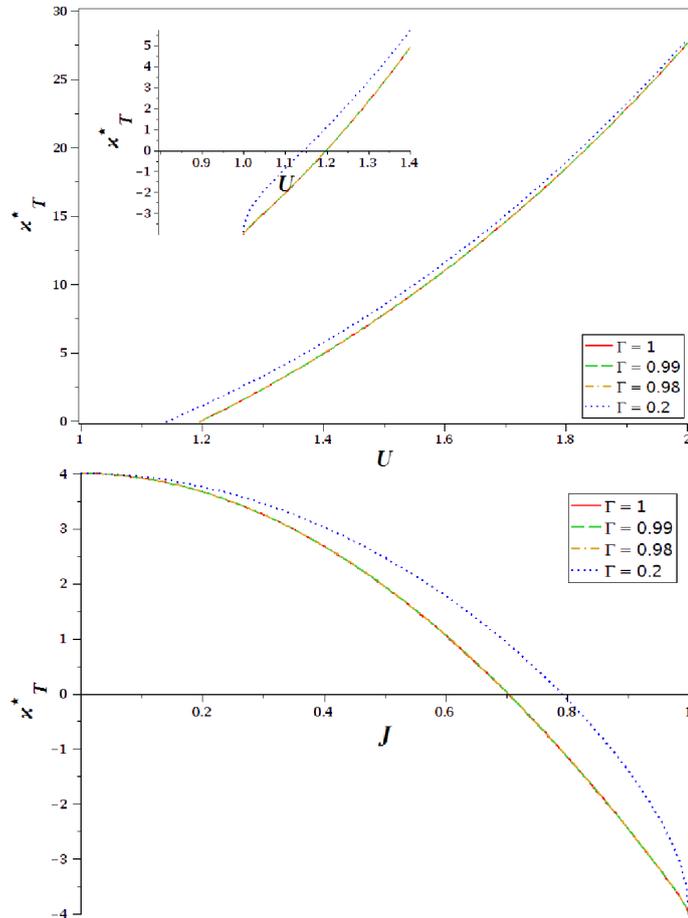
Figure 6: Noncommutative quantum corrected isothermal rotational susceptibility. (a) $\chi_T^\star$ as function of energy for $J = 1$, exhibiting a monotonically growing function. (b) Curves for isothermal susceptibility as a function of J considering $U = 1$.

With respect to noncommutative quantum corrected entropy, its corresponding isothermal rotational susceptibility is obtained by application of eq. (23),

$$\chi_T^\star = \frac{-4\pi\Gamma U^6 - U^4 + 12\pi\Gamma U^2 J^2 - J^2 + 4\pi\Gamma(U^4 - J^2)^{3/2}}{-8\pi\Gamma U^8 - U^6 + 4\pi\Gamma U^4 J^2 + 2U^2 J^2 + \sqrt{U^4 - J^2}(-8\pi\Gamma U^6 - U^4 + J^2)}$$

$$\times 2U\left(U^2 + \sqrt{U^4 - J^2}\right)^2 \circ \tag{25}$$

this relation is indeed a function of Γ, and it is presented in Fig. 6. Effect of noncommutativity in $\chi_T^\star$ is almost nonexistent, and if different values of this parameter near $\Gamma = 1$ are plotted together for $\chi_T^\star$ all resulting curves overlap. Only when the vicinity near $\Gamma \to 0$ is considered, noncommutativity effect on $\chi_T^\star$ is perceivable. Changes produced by Γ in numerator of eq. (25) are countered by its role in denominator of the same expression.

$\chi_T^\star$ goes to zero around $J \approx 0.7$ for $U = 1$, which corresponds to the location of divergence for $C_J^\star$. Negative region still appears in $\chi_T^\star$ as observed in BH isothermal rotational susceptibility. Additionally, if both χ_T and $\chi_T^\star$ are compared, the latter is always slightly above to the former, namely, $\chi_T(U, J) < \chi_T^\star(U, J)$.

Another thermodynamic response function to be analyzed is the isentropic rotational, defined as [7]:

$$\chi_S \equiv \left(\frac{\partial J}{\partial \Omega}\right)_S; \tag{26}$$

similarly to χ_T, it is straightforward to obtain by rewriting in terms of U and J,

$$\chi_S = \left[\left(\frac{\partial \Omega}{\partial J}\right)_U + \Omega\left(\frac{\partial \Omega}{\partial U}\right)_J\right]^{-1}. \tag{27}$$

This relation leads to the same result for both noncommutative Bekenstein–Hawking and quantum corrected entropies since $\Omega = \Omega^\star$, as showed in eq. (12). Therefore,

$$\chi_S = \chi_S^\star = 4U^3; \tag{28}$$

which is well defined in all its dominion. Independence of Γ in the above result is a consequence that for both S and $S^\star$, angular velocity is also independent of this parameter.

For Kerr black holes, specific heat can also be defined maintaining constant angular velocity,

$$C_\Omega \equiv \left(\frac{dQ}{dT}\right)_\Omega = T\left(\frac{\partial S}{\partial T}\right)_\Omega; \tag{29}$$

TdS equations provide a set of algebraic relations between response functions that can be applied in order to find analytical expressions for C_Ω and $C_\Omega^\star$. The following relations between material properties arise [7]:

$$\chi_T(C_\Omega - C_J) = T\alpha_\Omega^2; \tag{30a}$$

$$C_\Omega(\chi_T - \chi_S) = T\alpha_\Omega^2; \tag{30b}$$

$$\chi_S C_\Omega = \chi_T C_J. \tag{30c}$$

Where α_Ω is the coefficient of thermal induced rotation. Heat capacity at constant angular velocity can be obtained directly from eq. (30c).

Therefore, specific heat for noncommutative Bekenstein–Hawking entropy is given by:

$$C_\Omega = \frac{2\pi\Gamma\sqrt{U^4 - J^2}}{U^4}\left(-2U^2\sqrt{U^4 - J^2} - 2U^4 + J^2\right), \tag{31}$$

this expression is well defined in all the domain of its variables, and has one discontinuity in the trivial case where $U = 0$ (or $M = 0$). In Fig. 7 this response function is presented for different values of Γ, it can be noticed that C_Ω is negative in all its domain. Noncommutativity reduces the negativity of this heat capacity, finding $C_\Omega \to 0$ as $\Gamma \to 0$. Nevertheless, this response function remains always negative.

Considering noncommutative quantum corrected entropy, specific heat capacity at constant angular velocity can be expressed as:

$$C_\Omega^\star = -\frac{1}{2}\frac{\sqrt{U^4 - J^2}(4\pi\Gamma U^2 + 4\pi\Gamma\sqrt{U^4 - J^2} - 1)^2(-2U^4 - 2U^2\sqrt{U^4 - J^2} + J^2)}{-8\pi\Gamma U^8 - U^6 + 4\pi\Gamma U^4 J^2 + 2U^2 J^2 - (8\pi\Gamma U^6 + U^4 - J^2)\sqrt{U^4 - J^2}}; \tag{32}$$

its graphical representation is very similar to the one exhibited by C_Ω. Although parameter Γ plays a more complicated role in eq. (30), the overall effect of noncommutativity in $C_\Omega^\star$ leads to a very close behavior to the one observed in C_Ω. A direct comparison between both response functions shows that $C_\Omega^\star(U, J) > C_\Omega(U, J)$ in all their domain.

The last response function of Kerr black holes studied in this work is the coefficient of thermal induced rotation α_Ω [7],

$$\alpha_\Omega \equiv \left(\frac{\partial J}{\partial T}\right)_\Omega; \tag{33}$$

This material property is also calculated indirectly via relations between response functions, either eq. (30a) or eq. (30b). For noncommutative Bekenstein–Hawking entropy,

$$\alpha_\Omega = \frac{4\pi\Gamma J}{U^3}\left(U^2 + \sqrt{U^4 - J^2}\right)\sqrt{5U^4 + 4U^2\sqrt{U^4 - J^2} - J^2}; \tag{34}$$

it can be remarked that α_Ω is well behaved and has no discontinuities, excluding $U = 0$. This function is presented in Fig. 8. It can be noticed that α_Ω is reduced when smaller values of Γ are considered.
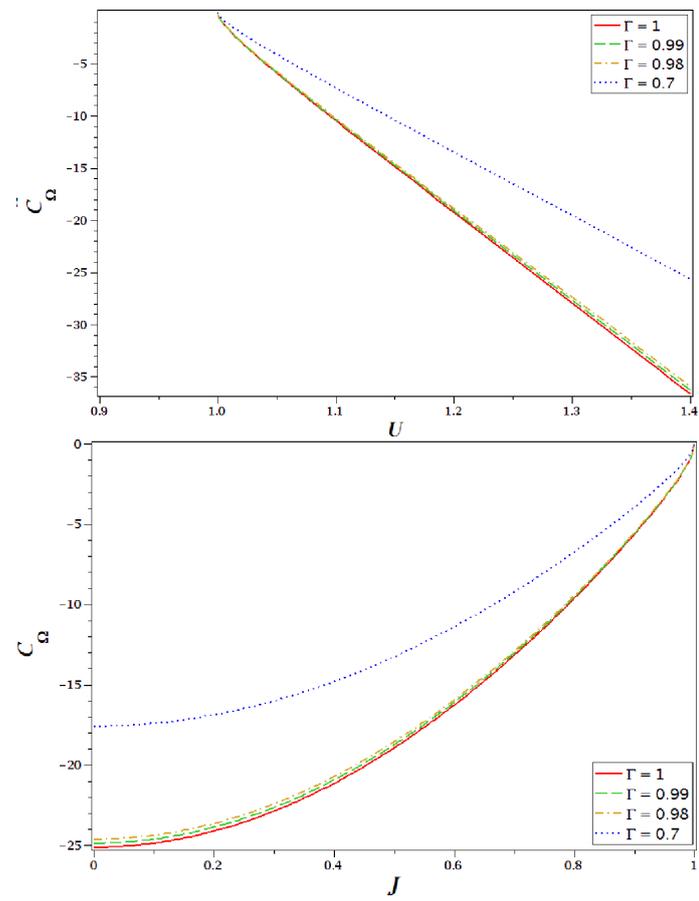
Figure 7: Specific heat capacity at constant angular velocity for different values of Γ. (a) C_Ω is plotted as a function of internal energy for $J = 1$. (b) Curves of C_Ω for $U = 1$ as a function of angular momentum.
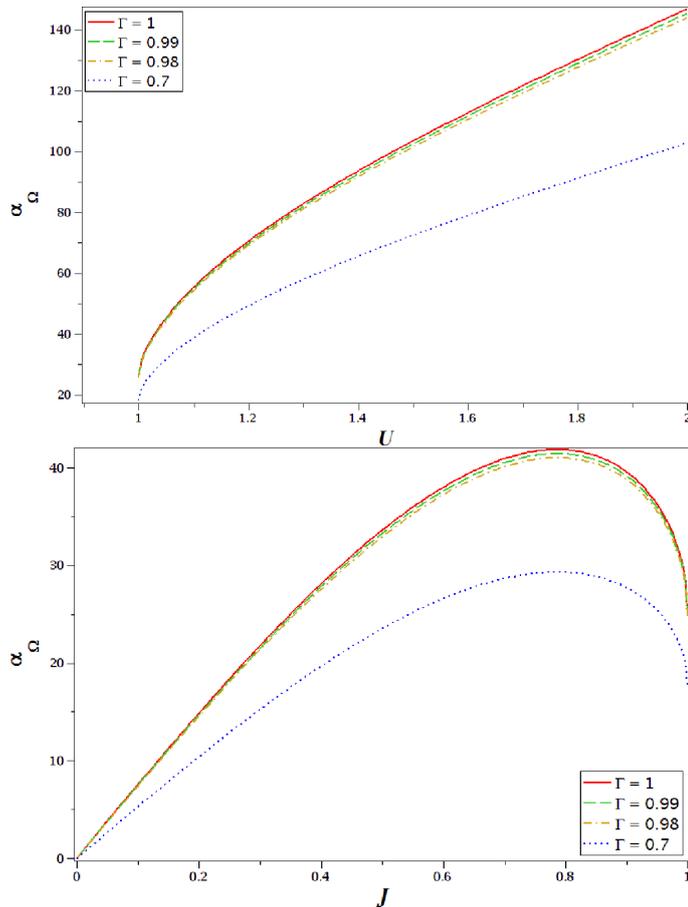
Figure 8: Coefficient of thermal induced rotation for noncommutative Bekenstein–Hawking entropy. (a) α_Ω as a function of internal energy, for $J = 1$. (b) Plots of the same coefficient varying angular momentum, for internal energy at $U = 1$.

Noncommutative quantum corrected coefficient, $\alpha_\Omega^\star$ is given by,

$$\alpha_\Omega^\star = \frac{UJ\left(U^2 + \sqrt{U^4 - J^2}\right)\left(4\pi\Gamma U^2 + 4\pi\Gamma\sqrt{U^4 - J^2} - 1\right)^{3/2}}{-8\pi\Gamma U^8 - U^6 + 4\pi\Gamma U^4 J^2 + 2U^2 J^2 + \sqrt{U^4 - J^2}(-8\pi\Gamma U^6 - U^4 + J^2)}$$

$$\times\left[36\pi\Gamma U^6 - 5U^4 - 20\pi\Gamma U^2 J^2 + J^2 - \sqrt{U^4 - J^2}(-36\pi U^4 + 4\pi J^2 + 4U^2)\right]^{1/2}. \tag{35}$$

Analogously to other response functions, quantum corrected coefficient $\alpha_\Omega^\star$ have almost the same behavior than its BH counterpart. If both curves are plotted together it is found that $\alpha_\Omega(U, J) > \alpha_\Omega^\star(U, J)$. As a summary, a comparison of thermodynamic properties between noncommutative Bekenstein–Hawking and quantum corrected entropies is presented in Table I.

Table I: Comparison between thermodynamic properties of noncommutative Bekenstein–Hawking and noncommutative quantum corrected entropies.

Response functions	Equations of state	Fundamental relation
$C_J > C_J^\star$	$T < T^\star$	$S > S^\star$
$\chi_T < \chi_T^\star$	$\Omega = \Omega^\star$	
$\chi_S = \chi_S^\star$		
$C_\Omega < C_\Omega^\star$		
$\alpha_\Omega > \alpha_\Omega^\star$		

In the following section, information provided by response functions will be used to determine whether Kerr black holes are thermodynamically stable or not.

IV.   THERMODYNAMIC STABILITY AND PHASE TRANSITION

From the analysis performed on thermodynamic response functions for both, noncommutative S and $S^\star$, an interesting result arises, specific heat capacity at constant angular momentum exhibits a singularity as showed in eqs. (16) and (21). Discontinuity in C_J for Bekenstein–Hawking entropy has been known for some time [5, 9], and it is associated with a second–order (or continuous) phase transition. Expressions calculated from Bekenstein–Hawking entropy are more manageable and, for the sake of simplicity the following analysis is performed considering only BH properties. It is expected that results obtained with these considerations are prominently similar to the ones expected for quantum corrected properties.

Thermodynamic systems passing through a fist–order phase transition have physical states for which parts of the system are in different phases, or a phase coexistence, constituting a series of not homogeneous states appearing below critical point, where phase boundaries vanish. Often these states can be identified with the aid of thermodynamic diagrams, as P–V diagrams for fluids. During phase transition equation of state remain constant, therefore a mechanical and thermal equilibrium exists [30, 31]. Maxwell construction, is a correction to violation in van der Waals equation of the requirement to have a constant pressure with volume in isotherms of phase diagram during a first–order phase transition [32]. It is helpful to find the critical point of a system in a first–order phase transition, if exists.

For a Kerr black holes isotherms in Ω– J plane must be analyzed. Criteria to find critical point is based on the pair of conjugate variables angular velocity and angular momentum, for which, the following requirements must be satisfied

$$\left(\frac{\partial\Omega}{\partial J}\right)_{T_C} = 0, \qquad \left(\frac{\partial^2\Omega}{\partial J^2}\right)_{T_C} = 0; \tag{36}$$

recalling eq. (22), it implies that isothermic rotational susceptibility must be singular at this critical point $\chi_T \to \infty$. As showed, χ_T do not have any divergence and is well behaved. Therefore, there is not critical point for Kerr Black holes. More evidence of this result can be found when constructing the isotherms in phase diagram in the plane Ω–J. Changes in concavity of the curves are expected if the system pass through a first–order phase transition, this is the region of inhomogeneous states and it is commonly named van der Waals loop, since were first observed for van der Waals equation.

In order to construct the corresponding isotherm for noncommutative Bekenstein–Hawking Kerr black holes $\Omega = \Omega(J, T)$, it is easier to proceed from thermodynamic fundamental relation in energy representation $U(S, J)$ [7],

$$U = \frac{1}{2}\sqrt{\frac{S}{\pi\Gamma} + \frac{4\pi\Gamma J^2}{S}}, \qquad \Omega = \frac{2\pi^{3/2}\Gamma J}{S\sqrt{\frac{S^2+4\pi^2\Gamma^2 J^2}{\Gamma S}}}; \tag{37}$$

using eq. (11a) for temperature, it is straightforward to obtain:

$$J = \left(4\Omega^2\left\{\left[\left(\frac{2\pi\Gamma T}{\Omega}\right)^2 + 1\right]^{3/4} + \left(\frac{2\pi\Gamma T}{\Omega}\right)\left[\left(\frac{2\pi\Gamma T}{\Omega}\right)^2 + 1\right]^{1/4}\right\}\right)^{-1}. \tag{38}$$

Inverse function $\Omega(T, J)$ can be estimated, and it is presented in Fig. 9 for different isotherms, showing commutative case. It was not possible to find an analytical expression for this relation. As noticed in this figure, for a given value of J there are two corresponding values of Ω, which can be interpreted as two possible cases for Kerr black holes, one of small mass and another with a larger one. When small temperatures are considered, it can be noticed that angular momentum have greater values available. There is no evidence of changes in concavity of the isotherms in plane Ω–J, which implies that there is no van der Waals loop. The last piece of evidence is that if the analogous of the Maxwell construction for noncommutative Kerr black holes is tried to be performed, this procedure is not satisfied by any value in their domain, indicating once again, that there is not critical point.

Therefore, continuity of first derivatives along with the lack of critical point, indicates that Kerr black holes do not pass through a first–order phase transition.
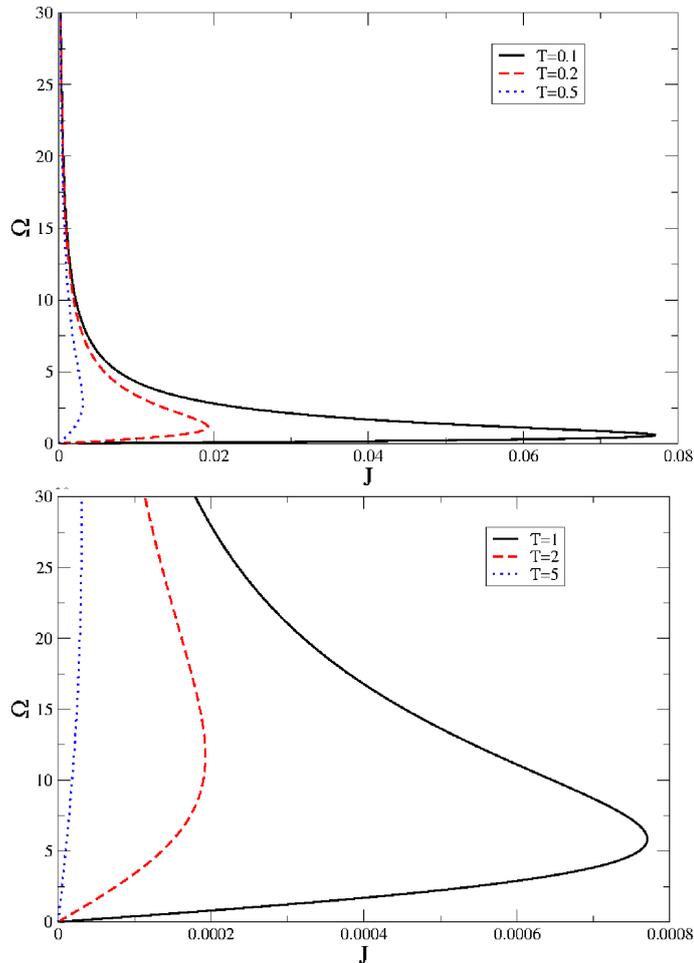
Figure 9: Isotherms in plane Ω–J for a Kerr black hole. Different temperatures were tested, exterior isotherm corresponds to the lower temperatures. Van der Waals loop do not appears in any of these isotherms.

Negative values exhibited by material properties are directly linked with the thermodynamic stability of the system. Thermodynamic equilibrium states are characterized by an extremal principle, either maximal entropy or equivalently a minimum in other thermodynamic potentials. In order to ensure that the curves of those potentials are stable, they must be concave functions of their variables. In particular, for the Kerr black holes, Gibbs potential $G(T, \Omega)$ and Helmholtz free energy $F(T, J)$ must be concave functions of both temperature and angular velocity, temperature and angular momentum, respectively. Using Legendre transformations for Kerr black holes [7],

$$S = -\left(\frac{\partial G}{\partial T}\right)_\Omega = -\left(\frac{\partial F}{\partial T}\right)_J;$$

concavity criteria requires that second derivatives satisfy the following relations [25, 32]:

$$\left(\frac{\partial^2 F}{\partial T^2}\right)_J = -\left(\frac{\partial S}{\partial T}\right)_J = -\frac{1}{T}C_J \leq 0 \qquad \text{(For: } \Delta U \to 0\text{)}, \tag{39}$$

$$\left(\frac{\partial^2 G}{\partial \Omega^2}\right)_T = -\left(\frac{\partial J}{\partial \Omega}\right)_T = -\chi_T \leq 0 \qquad \text{(For: } \Delta J \to 0\text{)}, \tag{40}$$

$$\left(\frac{\partial^2 G}{\partial T^2}\right)_\Omega = -\left(\frac{\partial S}{\partial T}\right)_\Omega = -\frac{1}{T}C_\Omega \leq 0 \qquad \text{(For: } \Delta U \to 0, \Delta J \to 0\text{)}. \tag{41}$$

From the above relations and results found in section four, particularly eqs. (15), (24) and (31), it is evident that Kerr black holes have regions where thermodynamic states do not meet these requirements, since C_J, χ_T and C_Ω are negative in those regions.

Geometric interpretation on three dimensional space S–U–J of relations given in eqs. (39)–(41) can be found in the corresponding figures of each response functions. For variations in internal energy, change of sign in C_J implicates that noncommutative quantum corrected Kerr black holes are stable for low masses, becoming unstable at greater ones, as noticed in Fig. 4(a). For variations in angular momentum showed in Fig. 6, isothermic rotational susceptibility becomes negative in the region above $J = 0.68U^2$, namely, greater values of J, the system is thermodynamically stable for low values of J. When variations in both U and J are considered, noncommutative quantum corrected Kerr black holes are always unstable, since $C_\Omega \leq 0$ in all its dominion, as presented in Fig. 7.

Noncommutativity have an important effect on thermodynamic stability of Kerr black holes, as mentioned when response functions were analyzed. Varying the value of Γ on those quantities have a direct consequence on accessible states where the system is thermodynamically stable. As showed in Fig. 5, smaller values of noncommutativity parameter, force the system to exist in a larger set of stable states.

Existence of thermodynamic instability and the divergence in C_J, reveals that the system goes through a series of metastable states from a low mass black hole to a higher mass black hole, in analogy to other metastable phenomena as superheating or supercooling. Nevertheless, the lack of a microscopic description for the system makes not possible to assure with certainty that Kerr black holes pass through a continuous phase transition [33]. However, violation of stability criteria is a strong thermodynamic argument to support this hypothesis [21].

V. CONCLUSIONS

An analysis of thermodynamic properties of noncommutative quantum corrected Kerr black holes have been given. Although resulting properties are mathematically more complicated, thermodynamic properties still retain the same functional behavior with respect to those calculated via Bekenstein–Hawking entropy. It was explicitly proven that Kerr black holes do not pass through a first–order phase transition, since the local criteria to find the critical point is not fulfilled for any value in the domain, corresponding isotherms do not exhibit van der Waals loops and the Maxwell construction cannot be obtained, all of which are characteristic of this kind of transition. Nonetheless, some second derivatives exhibit a change of sign which is an indication that those states are thermodynamically unstable. This instability and the nonexistence of a critical point suggest that the system goes through metastable states, from a low mass black hole to a high mass one, in a continuous phase transition. Regarding noncommutativity in coordinates of minisuperspace, outside vicinity where $\Gamma \approx 1$, changes introduced by this parameter over thermodynamic information of the system are relevant. In particular, it have a powerful impact on thermodynamic stability of Kerr black holes, allowing the system to be thermodynamically stable for a wider set of states. It will be interesting to explore the effect of noncommutativity over thermodynamics of black holes in (Anti) de Sitter spacetime and how noncommutativity could be considered in the Statistical Mechanics formalism of long–range interactions. These works will be reported in somewhere else.

Acknowledgments

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