Double Conformal Space-Time Algebra

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Abstract

This paper introduces the $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA). $\mathcal{G}_{4,8}$ DCSTA is a straightforward extension of the $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA), which is a different form of the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA). $\mathcal{G}_{4,8}$ DCSTA extends $\mathcal{G}_{2,8}$ DCSA with spacetime boost operations and differential operators for differentiation with respect to the pseudospatial time $w = ct$ direction and time t. The spacetime boost operation can implement anisotropic dilation (directed non-uniform scaling) of quadric surface entities. DCSTA is a high-dimensional 12D embedding of the $\mathcal{G}_{1,3}$ Space-Time Algebra (STA) and is a doubling of the $\mathcal{G}_{2,4}$ Conformal Space-Time Algebra (CSTA). The 2-vector quadric surface entities of the DCSA subalgebra appear in DCSTA as quadric surfaces at zero velocity that can be boosted into moving surfaces with constant velocities that display the length contraction effect of special relativity. DCSTA inherits doubled forms of all CSTA entities and versors. The doubled CSTA entities (standard DCSTA entities) include points, hypercones, hyperplanes, hyperpseudospheres, and other entities formed as their intersections, such as planes, lines, spatial spheres and circles, and spacetime hyperboloids (pseudospheres) and hyperbolas (pseudocircles). The doubled CSTA versors (DCSTA versors) include rotor, hyperbolic rotor (boost), translator, dilator, and their compositions such as the translated-rotor, translated-boost, and translated-dilator. The DCSTA versors provide a complete set of spacetime transformation operators on all DCSTA entities. DCSTA inherits the DCSA 2-vector spatial entities for Darboux cyclides (incl. parabolic and Dupin cyclides, general quadrics, and ring torus) and gains Darboux pseudocyclides formed in spacetime with the pseudospatial time dimension. All DCSTA entities can be reflected in, and intersected with, the standard DCSTA entities. To demonstrate $\mathcal{G}_{4,8}$ DCSTA as concrete mathematics with possible applications, this paper includes sample code and example calculations using the symbolic computer algebra system *SymPy*.

Keywords: conformal geometric algebra; space-time algebra; Clifford algebra

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Contents

1 Introduction

This original research monograph[1.1] introduces the $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) (§7), which is a straightforward extension of the $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) (§4) into spacetime. $\mathcal{G}_{2,8}$ DCSA is a different form of the $\mathcal{G}_{8,2}$ *Double Conformal / Darboux Cyclide Geometric Algebra* (DCGA). $\mathcal{G}_{8,2}$ DCGA is introduced in the original research monograph [7] and in the published short paper [8], and is discussed further in the papers [5] and [6]. All of the results of $\mathcal{G}_{8,2}$ DCGA have a similar form in $\mathcal{G}_{2,8}$ DCSA, and also in $\mathcal{G}_{4,8}$ DCSTA at time $w = ct = 0$. $\mathcal{G}_{4,8}$ DCSTA is a high-dimensional 12D embedding of the $\mathcal{G}_{1,3}$ *Space-Time Algebra* (STA) (§5). $\mathcal{G}_{1,3}$ STA is introduced by HESTENES in [16]. $\mathcal{G}_{4,8}$ DCSTA is an application of the $\mathcal{G}_{4,8}$ Geometric Algebra. Geometric Algebra is introduced by HESTENES and SOBCZYK in [17]. Familiarity with Geometric Algebra and $\mathcal{G}_{8,2}$ DCGA [7] is assumed.

$\mathcal{G}_{4,8}$ DCSTA may offer new mathematical methods for some applications. However, the 12D high-dimensionality of DCSTA incurs high computational cost and applications may require an efficient implementation [10] using optimized hardware and software [13] for DCSTA. Other works on algebras similar to $\mathcal{G}_{4,8}$ DCSTA may exist in the mathematical physics literature, but no specific works essentially the same as $\mathcal{G}_{4,8}$ DCSTA were known by this author at the time of researching and writing this paper.

1.1. Revised version v5, *Oct 4, 2016*.

$\mathcal{G}_{4,8}$ DCSTA (§7) is a doubling extension of the $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra* (CSTA) (§6). $\mathcal{G}_{2,4}$ CSTA is introduced by C.J.L. Doran and A.N. Lasenby in [3] as the *spacetime conformal group*. $\mathcal{G}_{2,4}$ CSTA (§6) embeds $\mathcal{G}_{1,3}$ STA (§5) using stereographic embedding and homogenization as discussed by Perwass [20] in the context of $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) (chapters 1-3 in [25]) and as discussed (in French) by Anglès [1].

This paper is logically structured into two parts as follows.

Part I of this work is comprised of the three sections on spatial algebras,

- $\mathcal{G}_{0,3}$ *Space Algebra* (SA) (§2)
- $\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) (§3)
- $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) (§4),

which are adequate as alternatives to $\mathcal{G}_3$ APS, $\mathcal{G}_{4,1}$ CGA, and $\mathcal{G}_{8,2}$ DCGA [7], respectively.

Part II of this work is comprised of the three sections on spacetime algebras,

- $\mathcal{G}_{1,3}$ *Space-Time Algebra* (STA) (§5)
- $\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra* (CSTA) (§6)
- $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) (§7).

The algebras in Part I are spatial subalgebras of the corresponding spacetime algebras in Part II. The material of Part I should probably be understood before reading Part II.

The following six introductory subsections define the basis vector elements that are used throughout all six algebras. The first three subsections are a breakdown of the basis vector elements in DCSTA (§1.1) into the basis vector elements of its subalgebras CSTA (§1.2) and STA (§1.3). The last three subsections are a buildup of the basis vector elements in SA (§1.4) and CSA (§1.5) into DCSA (§1.6).

1.1 DCSTA basis vector elements

$\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) $\mathcal{D}$ (§7) has a basis of twelve orthonormal unit vector elements $\mathbf{e}_i : 1 \leq i \leq 12$ with metric matrix

$$m_{\mathcal{D}} = \operatorname{diag}(1,-1,-1,-1,1,-1,1,-1,-1,-1,1,-1) = [(m_{\mathcal{D}})_{ij}] = [\mathbf{e}_i \cdot \mathbf{e}_j]. \qquad (1.1)$$

All basis vector elements are orthonormal unit vectors. For any two different basis vector elements $\mathbf{u}$ and $\mathbf{v}$, their geometric product is $\mathbf{uv} = \mathbf{u} \wedge \mathbf{v}$.

1.2 CSTA basis vector elements

$\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra* (CSTA) $\mathcal{C}$ (§6)[3] has the six basis vector elements $\boldsymbol{\gamma}_i : 0 \leq i \leq 3$, $\mathbf{e}_+$, and $\mathbf{e}_-$,

$$\boldsymbol{\gamma}_i^2 = \begin{cases} 1 & : i \in \{0\} \\ -1 & : i \in \{1,2,3\} \end{cases} \qquad (1.2)$$

$$\mathbf{e}_+^2 = 1 \qquad (1.3)$$

$$\mathbf{e}_-^2 = -1. \qquad (1.4)$$

$\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra 1* (CSTA1) $\mathcal{C}^1$ (§6) has the six basis vector elements $\mathbf{e}_i : 1 \leq i \leq 6$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{1,5\} \\ -1 & : i \in \{2,3,4,6\}. \end{cases} \tag{1.5}$$

$\mathcal{G}_{2,4}$ *Conformal Space-Time Algebra 2* (CSTA2) $\mathcal{C}^2$ (§6) has the six basis vector elements $\mathbf{e}_i : 7 \leq i \leq 12$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{7,11\} \\ -1 & : i \in \{8,9,10,12\}. \end{cases} \tag{1.6}$$

CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$ (§6) are the two major subalgebras of DCSTA $\mathcal{D}$ (§7). Discussions of CSTA are mostly in terms of the generic CSTA $\mathcal{C}$. CSTA $\mathcal{C}$, CSTA1 $\mathcal{C}^1$, and CSTA2 $\mathcal{C}^2$ correspond to each other as indicated in (1.7).

$$\begin{array}{lllll} \text{CSTA} & \cong & \text{CSTA1} & \cong & \text{CSTA2} \\ \mathcal{C} & \cong & \mathcal{C}^1 & \cong & \mathcal{C}^2 \\ \boldsymbol{\gamma}_0 & \cong & \mathbf{e}_1 & \cong & \mathbf{e}_7 \\ \boldsymbol{\gamma}_1 & \cong & \mathbf{e}_2 & \cong & \mathbf{e}_8 \\ \boldsymbol{\gamma}_2 & \cong & \mathbf{e}_3 & \cong & \mathbf{e}_9 \\ \boldsymbol{\gamma}_3 & \cong & \mathbf{e}_4 & \cong & \mathbf{e}_{10} \\ \mathbf{e}_+ & \cong & \mathbf{e}_5 & \cong & \mathbf{e}_{11} \\ \mathbf{e}_- & \cong & \mathbf{e}_6 & \cong & \mathbf{e}_{12} \end{array} \tag{1.7}$$

1.3 STA basis vector elements

$\mathcal{G}_{1,3}$ *Space-Time Algebra* (STA) $\mathcal{M}$ (§5)[16] has the five elements, called the DIRAC gammas, $\boldsymbol{\gamma}_i : i \in \{0,1,2,3,5\}$,

$$\boldsymbol{\gamma}_i^2 = \begin{cases} 1 & : i \in \{0\} \\ -1 & : i \in \{1,2,3\} \\ (\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3)^2 = \mathbf{I}_{\mathcal{M}}^2 = -1 & : i = 5. \end{cases} \tag{1.8}$$

$\mathcal{G}_{1,3}$ *Space-Time Algebra 1* (STA1) $\mathcal{M}^1$ (§5) has four basis vector elements $\mathbf{e}_i : 1 \leq i \leq 4$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{1\} \\ -1 & : i \in \{2,3,4\}. \end{cases} \tag{1.9}$$

$\mathcal{G}_{1,3}$ *Space-Time Algebra 2* (STA2) $\mathcal{M}^2$ (§5) has four basis vector elements $\mathbf{e}_i : 7 \leq i \leq 10$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{7\} \\ -1 & : i \in \{8,9,10\}. \end{cases} \tag{1.10}$$

CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$ (§6) embed the subalgebras STA1 $\mathcal{M}^1$ and STA2 $\mathcal{M}^2$ (§5), respectively. The element $\boldsymbol{\gamma}_5 = \mathbf{I}_{\mathcal{M}}$ is the STA $\mathcal{M}$ unit pseudoscalar (§5.1.2). STA $\mathcal{M}$ also defines the PAULI sigmas $\boldsymbol{\sigma}_i$,

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 \tag{1.11}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 \tag{1.12}$$

$$\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_z = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0. \tag{1.13}$$

The $\mathcal{G}_{1,3}$ STA $\mathcal{M}$ (§5) elements $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ that are introduced in [16] are used in all general discussions of STA. The STA $\mathcal{M}$ elements can be identified isomorphically with either the STA1 $\mathcal{M}^1$ or STA2 $\mathcal{M}^2$ elements, per (1.7). The elements $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, and $\boldsymbol{\gamma}_3$ can also be denoted $\boldsymbol{\gamma}_x$, $\boldsymbol{\gamma}_y$, and $\boldsymbol{\gamma}_z$ when emphasizing their usage as the conventional x, y, and z spatial directions. The element $\boldsymbol{\gamma}_0$ is the pseudospatial time $w=ct$ direction.

1.4 SA basis vector elements

$\mathcal{G}_{0,3}$ *Space Algebra* (SA) $\mathcal{S}$ (§2) has three basis vector elements $\boldsymbol{\gamma}_i : 1 \leq i \leq 3$,

$$\boldsymbol{\gamma}_i^2 = -1 : i \in \{1,2,3\}. \tag{1.14}$$

$\mathcal{G}_{0,3}$ *Space Algebra 1* (SA1) $\mathcal{S}^1$ (§2) has three basis vector elements $\mathbf{e}_i : 2 \leq i \leq 4$,

$$\mathbf{e}_i^2 = -1 : i \in \{2,3,4\}. \tag{1.15}$$

$\mathcal{G}_{0,3}$ *Space Algebra 2* (SA2) $\mathcal{S}^2$ (§2) has three basis vector elements $\mathbf{e}_i : 8 \leq i \leq 10$,

$$\mathbf{e}_i^2 = -1 : i \in \{8,9,10\}. \tag{1.16}$$

SA $\mathcal{S}$, SA1 $\mathcal{S}^1$, and SA2 $\mathcal{S}^2$ correspond with each other according to (1.7).

1.5 CSA basis vector elements

$\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) $\mathcal{CS}$ (§3) has five basis vector elements $\boldsymbol{\gamma}_i : 1 \leq i \leq 3$, $\mathbf{e}_+$, and $\mathbf{e}_-$,

$$\boldsymbol{\gamma}_i^2 = -1 : i \in \{1,2,3\} \tag{1.17}$$

$$\mathbf{e}_+^2 = 1 \tag{1.18}$$

$$\mathbf{e}_-^2 = -1. \tag{1.19}$$

$\mathcal{G}_{1,4}$ *Conformal Space Algebra 1* (CSA1) $\mathcal{CS}^1$ (§3) has five basis vector elements $\mathbf{e}_i$: $2 \leq i \leq 6$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{5\} \\ -1 & : i \in \{2,3,4,6\}. \end{cases} \tag{1.20}$$

$\mathcal{G}_{1,4}$ *Conformal Space Algebra 2* (CSA2) $\mathcal{CS}^2$ (§3) has five basis vector elements $\mathbf{e}_i$: $8 \leq i \leq 12$,

$$\mathbf{e}_i^2 = \begin{cases} 1 & : i \in \{11\} \\ -1 & : i \in \{8,9,10,12\}. \end{cases} \tag{1.21}$$

CSA $\mathcal{CS}$, CSA1 $\mathcal{CS}^1$, and CSA2 $\mathcal{CS}^2$ correspond with each other according to (1.7). $\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) $\mathcal{CS}$ (§3) is very similar to the well-known $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) with differences only in the signs of some expressions.

1.6 DCSA basis vector elements

$\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) $\mathcal{DS}$ (§4) has the ten basis vector elements of $\mathcal{CS}^1$ and $\mathcal{CS}^2$ (§1.5).

$\mathcal{G}_{2,8}$ DCSA (§4) is very similar to $\mathcal{G}_{8,2}$ DCGA [7][8][5][6], except for differences in the signs of some expressions. $\mathcal{G}_{4,8}$ DCSTA (§7) becomes $\mathcal{G}_{2,8}$ DCSA (§4) when all times are zero, $w=ct=0$.

2 Space Algebra (SA)

$\mathcal{G}_{0,3}$ *Space Algebra* (SA) $\mathcal{S}$ is very similar to $\mathcal{G}_3$ *Algebra of Physical Space* (APS) [15]. Vectors in SA square negative $(\mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}) \leq 0$, which causes sign flips in many formulas adapted from $\mathcal{G}_3$ APS, $\mathcal{G}_{4,1}$ CGA (Conformal APS), or $\mathcal{G}_{8,2}$ DCGA. While $\mathcal{G}_{8,2}$ DCGA embeds two Euclidean $\mathcal{G}_3$ APS algebras, $\mathcal{G}_{2,8}$ DCSA $\mathcal{DS}$ and $\mathcal{G}_{4,8}$ DCSTA $\mathcal{D}$ embed two anti-Euclidean $\mathcal{G}_{0,3}$ SA algebras $\mathcal{S}^1$ and $\mathcal{S}^2$.

The subscript $\mathcal{S}$ denotes an element or operation in generic SA. The subscript $\mathcal{S}^1$ denotes an element or operation in SA1. The subscript $\mathcal{S}^2$ denotes an element or operation in SA2. In most formulas, these subscripts can simply be substituted to write formulas in SA, SA1, and SA2. Such duplication of similar formulas in each representation is avoided unless it adds clarity to the discussion. In similar fashion, in STA (§5), STA1, and STA2, the subscripts $\mathcal{M}$, $\mathcal{M}^1$, and $\mathcal{M}^2$ indicate elments in STA, STA1, and STA2.

The 3D spatial vectors in $\mathcal{G}_{0,3}^1$ SA are generally **bold** lowercase letters, such as $\mathbf{p} = \mathbf{p}_{\mathcal{S}}$. The 4D spacetime vectors in $\mathcal{G}_{1,3}^1$ STA (§5) are generally ***bold italic*** lowercase letters, such as $\boldsymbol{p} = \boldsymbol{p}_{\mathcal{M}}$.

2.1 SA unit pseudoscalar

The SA 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \tag{2.1}$$

$$\mathbf{I}_{\mathcal{S}}^{\sim} = (-1)^{3(3-1)/2}\mathbf{I}_{\mathcal{S}} = -\mathbf{I}_{\mathcal{S}} \tag{2.2}$$

$$\mathbf{I}_{\mathcal{S}}^2 = -\mathbf{I}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{\sim} = 1 \tag{2.3}$$

$$\mathbf{I}_{\mathcal{S}}^{-1} = \mathbf{I}_{\mathcal{S}}. \tag{2.4}$$

The SA1 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}^1}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}^1} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4. \tag{2.5}$$

The SA2 3-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}^2}$ with signature $(---)$ is

$$\mathbf{I}_{\mathcal{S}^2} = \mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}. \tag{2.6}$$

The notation $A^{\sim}$ is the *reverse* of A [4][20]. The $\mathcal{G}_{0,3}$ SA unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ is its own inverse and squares to 1 as a *hyperbolic unit*. In $\mathcal{G}_3$ APS, the unit pseudoscalar squares to -1 and is an *imaginary unit*. This difference affects how the SA dualization operation is defined.

The distinction between a *k-blade* and a *k-vector* is rarely made in this paper, and the more-general term *k-vector* is used in most cases. A *k-blade* is any element that can be factored into the *outer product of k vectors*. Blades include all 1-vectors, all pseudoscalars, all CSA and CSTA entities, and all of the doubled "standard" entities in DCSA and DCSTA. A k-vector is the sum of one or more k-blades.

2.2 SA dualization

The SA $\mathcal{S}$ *dual* $A_{\mathcal{S}}^{*\mathcal{S}}$ of an SA multivector $A_{\mathcal{S}}$ is

$$A_{\mathcal{S}}^{*} = A_{\mathcal{S}}^{*\mathcal{S}} = A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{\sim} = -A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}. \tag{2.7}$$

The SA $\mathcal{S}$ *undual* $A_{\mathcal{S}}$ of a dual SA multivector $A_{\mathcal{S}}^{*\mathcal{S}}$ is

$$A_{\mathcal{S}} = A_{\mathcal{S}}^{*}\widetilde{\mathbf{I}_{\mathcal{S}}} = -A_{\mathcal{S}}^{*}\mathbf{I}_{\mathcal{S}} \tag{2.8}$$

The SA dual and undual operations are the same, and the SA dualization is an *involution*.

The notation $A_{\mathcal{S}}$ denotes an element of the algebra denoted by $\mathcal{S}$, which is SA. In later sections, we will encounter the algebras denoted by $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{D}$, which are STA, CSTA, and DCSTA, respectively. For the subalgebras $\mathcal{S}$, $\mathcal{M}$, and $\mathcal{C}$ of $\mathcal{D}$, there are two copies of them in $\mathcal{D}$, which are denoted by $\mathcal{S}^1$ and $\mathcal{S}^2$ and similarly for other subalgebras that have a double in DCSTA.

The explicit dualization notation $A_{\mathcal{S}}^{*\mathcal{S}}$ denotes the dual of $A_{\mathcal{S}}$ in algebra $\mathcal{S}$ using the unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ of the algebra $\mathcal{S}$. The implicit dualization notation $A_{\mathcal{S}}^{*}$ denotes the same dualization as indicated by the subscript $\mathcal{S}$.

To introduce the notation further, the dualizations are

$$A^{*} = \begin{cases} A_{\mathcal{S}}^{*\mathcal{S}} = A_{\mathcal{S}}^{*} = -A_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1} & : \text{ SA } \mathcal{S} \text{ dualization (§2.2)} \\ A_{\mathcal{M}}^{*\mathcal{M}} = A_{\mathcal{M}}^{*} = A_{\mathcal{M}}\mathbf{I}_{\mathcal{M}}^{-1} & : \text{ STA } \mathcal{M} \text{ dualization (§5.2.1)} \\ A_{\mathcal{C}}^{*\mathcal{C}} = A_{\mathcal{C}}^{*} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} & : \text{ CSTA } \mathcal{C} \text{ dualization (§6.6.1)} \\ A_{\mathcal{D}}^{*\mathcal{D}} = A_{\mathcal{D}}^{*} = A_{\mathcal{D}}\mathbf{I}_{\mathcal{D}}^{-1} & : \text{ DCSTA } \mathcal{D} \text{ dualization (§7.7.1).} \end{cases} \tag{2.9}$$

Duals are typically the result of division by the unit pseudoscalar. The SA dualization is defined as division by the negative unit pseudoscalar, and the reason is explained in (§2.6) on the SA rotor. These dualizations are discussed further in later sections.

2.3 SA test vector

The symbolic SA $\mathcal{S}$ *test vector* $\mathbf{t}_{\mathcal{S}}$ is defined on the basis of the Dirac gammas [16] as

$$\mathbf{t} = \mathbf{t}_{\mathcal{S}} = x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3. \tag{2.10}$$

The symbolic SA1 $\mathcal{S}^1$ *test vector* $\mathbf{t}_{\mathcal{S}^1}$ is defined as

$$\mathbf{t}_{\mathcal{S}^1} = x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_4. \tag{2.11}$$

The symbolic SA2 $\mathcal{S}^2$ *test vector* $\mathbf{t}_{\mathcal{S}^2}$ is defined as

$$\mathbf{t}_{\mathcal{S}^2} = x\mathbf{e}_8 + y\mathbf{e}_9 + z\mathbf{e}_{10}. \tag{2.12}$$

The *symbolic* scalars x, y, and z are the conventional coordinates in space. Numerical scalars are denoted p_x, p_y, and p_z for a vector

$$\mathbf{p} = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3. \tag{2.13}$$

This distinction between symbolic values and numerical values is helpful in symbolic computations. Symbolic computations using a symbolic computer algebra software, such as *SymPy* [24] with the $\mathcal{GA}$lgebra [2] module, can assist in the study of DCSTA and other high-dimensional Geometric Algebras.

A test vector, or other test entity, holds symbolic coordinates and parameters. A non-test vector, or other non-test entity, holds numeric coordinates and parameters. In symbolic calculations, a non-test entity, or simply an actual *entity*, can be evaluated against a symbolic test entity to obtain the symbolic algebraic expression, or implicit surface function, that is represented by the entity.

2.4 SA spatial velocity vector

An SA $\mathcal{S}$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}}$ has the form

$$\mathbf{v} = \mathbf{v}_{\mathcal{S}} = v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3 = \beta c\hat{\mathbf{v}}. \quad (2.14)$$

An SA1 $\mathcal{S}^1$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}^1}$ has the form

$$\mathbf{v}_{\mathcal{S}^1} = v_x\mathbf{e}_2 + v_y\mathbf{e}_3 + v_z\mathbf{e}_4 = \beta c\hat{\mathbf{v}}_{\mathcal{S}^1}. \quad (2.15)$$

An SA2 $\mathcal{S}^2$ *spatial velocity vector* $\mathbf{v}_{\mathcal{S}^2}$ has the form

$$\mathbf{v}_{\mathcal{S}^2} = v_x\mathbf{e}_8 + v_y\mathbf{e}_9 + v_z\mathbf{e}_{10} = \beta c\hat{\mathbf{v}}_{\mathcal{S}^2}. \quad (2.16)$$

The scalars v_x, v_y, and v_z are coordinate speeds in the conventional x, y, and z spatial directions. Natural speed is $\beta = v/c = \|\mathbf{v}\|/c$, $|\beta| \le 1$. Light speed is c.

The vector units $\mathbf{e}_1$ and $\mathbf{e}_7$ are in STA1 and STA2 (§5.1.7), respectively, where they serve as the unit directions for pseudospatial time $w = ct$. Pseudospatial time coordinate w is measured in distance that light travels in time t. Clock time (coordinate time) is $t = w/c$. In standard units of meters and seconds, $c = 299792458$ m/s, exactly. In natural units, $c = 1$, which is convenient for testing calculations and graphing implicit surfaces $F(x, y, z) = 0$.

In special relativity, the constant non-negative *norm* of an SA velocity

$$\|\mathbf{v}_{\mathcal{S}}\| = \sqrt{\mathbf{v}_{\mathcal{S}} \cdot \mathbf{v}_{\mathcal{S}}^{\dagger}} = \sqrt{-\mathbf{v}_{\mathcal{S}}^2} = \sqrt{v_x^2 + v_y^2 + v_z^2} = |\beta c| \quad (2.17)$$

must not exceed light speed c

$$0 \le \|\mathbf{v}_{\mathcal{S}}\| \le c. \quad (2.18)$$

The unit direction of velocity $\mathbf{v}$ is

$$\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|. \quad (2.19)$$

The *conjugate* $\mathbf{v}^{\dagger}$ is discussed by PERWASS in [20]. The conjugate of any STA multivector $A_{\mathcal{M}}$, including any SA multivector, is

$$A_{\mathcal{M}}^{\dagger} = \boldsymbol{\gamma}_0 \widetilde{A_{\mathcal{M}}} \boldsymbol{\gamma}_0. \quad (2.20)$$

The conjugation formula (Eq. 2.20) is valid for any $\mathcal{G}_{1,q}$ Geometric Algebra, where $\boldsymbol{\gamma}_0 = \mathbf{e}_1$ and $\boldsymbol{\gamma}_0^2 = 1$. The conjugate takes the reverse $\widetilde{A_{\mathcal{M}}}$ and reflects it in $\boldsymbol{\gamma}_0$ (as a versor sandwich product). The reflection in $\boldsymbol{\gamma}_0$ has the effect of inverting the sign on any anti-Euclidean vector $\mathbf{v} \in \mathcal{G}_{0,q}^1$. The conjugate of an SA vector is simply its negative, $\mathbf{v}^{\dagger} = -\mathbf{v}$. The conjugate has the positive-definite property

$$\begin{aligned} A_{\mathcal{M}} \cdot A_{\mathcal{M}}^{\dagger} &= A_{\mathcal{M}}^{\dagger} \cdot A_{\mathcal{M}} > 0 && (2.21)\\ &= \|A_{\mathcal{M}}\|^2 && (2.22)\end{aligned}$$

that produces the squared *norm* (or squared magnitude) of $A_{\mathcal{M}}$. If $A \ne 0$ is *null* $A^2 = 0$, then its conjugate $A^{\dagger}$ can still be used to obtain its norm $\|A\| = \sqrt{A \cdot A^{\dagger}}$ and its *pseudoinverse*

$$A^{+} = A^{\dagger}/\|A\|^2 \quad (2.23)$$

such that $A\cdot A^{+}=A^{+}\cdot A=1$. The division $A/\|A\|$ is called the *normalization* (or norm-unit) of A and is especially useful since it is valid for all k-vectors, including null k-vectors. The normalization is different than the "modulus-unit" $A/|A|$, for $|A|\neq 0$, where $|A|^2=A^2$ is the indefinite *squared modulus* (or squared interval) of A that can be positive, negative or zero. Both the norm-unit $A/\|A\|$ and modulus-unit $A/|A|$ have uses in this paper.

The notations as used by HESTENES in [16] are *not* adopted here and conflict with the notations as they are adopted here. In [16], the notation $\mathbf{v}^{\dagger}$ is called *hermitian conjugation* and is the *reverse* of an element in STA. The *reverse* $\mathbf{v}^{\sim}$ is the notation that is adopted here. In [16], the notation $\mathbf{v}^{*}$ is called *space conjugation* and is anti-Euclidean conjugation in STA. The notation $\mathbf{v}^{*}$ is adopted here as the *dual* of $\mathbf{v}$. The *conjugate* $\mathbf{v}^{\dagger}$ is adopted here, following PERWASS in [20], and is discussed by this author in [9].

In general, a conjugation is an operation that selectively changes the signs on only certain elements and there are many kinds of conjugations and notations. It is thought that the notations that have been adopted here are the ones most commonly adopted in the current literature on Geometric Algebra. The notations of HESTENES in [16] may be found in physics literature.

2.5 SA spatial position vector

An SA $\mathcal{S}$ *spatial position vector* $\mathbf{p}_{\mathcal{S}}(t)$, as a function of time t, has the form

$$\mathbf{p}(t)=\mathbf{p}_{\mathcal{S}}(t) \;=\; p_x\gamma_1+p_y\gamma_2+p_z\gamma_3=\mathbf{p}_0+\mathbf{v}t. \tag{2.24}$$

An SA1 $\mathcal{S}^1$ *spatial position vector* $\mathbf{p}_{\mathcal{S}^1}$ has the form

$$\mathbf{p}_{\mathcal{S}^1}(t) \;=\; p_x\mathbf{e}_2+p_y\mathbf{e}_3+p_z\mathbf{e}_4. \tag{2.25}$$

An SA2 $\mathcal{S}^2$ *spatial position vector* $\mathbf{p}_{\mathcal{S}^2}$ has the form

$$\mathbf{p}_{\mathcal{S}^2} \;=\; p_x\mathbf{e}_8+p_y\mathbf{e}_9+p_z\mathbf{e}_{10}. \tag{2.26}$$

The scalars p_x, p_y, and p_z are coordinate positions in the conventional x, y, and z spatial directions. The vector $\mathbf{p}_0$ is the *initial position* at time $t=0$, and $\mathbf{v}$ is the velocity. A spatial vector $\mathbf{p}$ is the spatial position of a particle, observer, or other *observable* object. For all time $\forall t$, $\mathbf{p}(t)$ is the spatial *path* of the spacetime *worldline* of an observable moving at contant velocity $\mathbf{v}$. This paper only considers constant velocities in special relativity.

2.6 SA rotor

A *rot*ation operat*or* R, called a *rotor*, can be understood in terms of ratios (or products) of unit vectors, which are called *versors*. The term *versor*, which seems to mean *vers*ion operat*or*, was coined by WILLIAM ROWAN HAMILTON in [14]. A rotor is isomorphic to a quaternion versor as discussed at length by this author in [9]. The concept of versors is generalized to k-versors in [17]. A k-versor is the product of k unit vectors. In DCSTA, we will encounter 4-versors for rotation, translation, dilation, boost, and planar reflection, and 2-versors for inversions in hyperpseudospheres.

In SA, the unit bivector rotor elements are the ratios

$$\mathbf{i}=\mathbf{k}/\mathbf{j} \;\cong\; \gamma_3/\gamma_2=-\gamma_3\gamma_2 \tag{2.27}$$
$$\mathbf{j}=\mathbf{i}/\mathbf{k} \;\cong\; \gamma_1/\gamma_3=-\gamma_1\gamma_3 \tag{2.28}$$
$$\mathbf{k}=\mathbf{j}/\mathbf{i} \;\cong\; \gamma_2/\gamma_1=-\gamma_2\gamma_1. \tag{2.29}$$

The SA duals of the SA unit vector elements are

$$\gamma_1^* = \gamma_1 \mathbf{I}_{\mathcal{S}}^{\sim} = -\gamma_1 \mathbf{I}_{\mathcal{S}} = -\gamma_1 \mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_1(\gamma_1\gamma_2\gamma_3) = -\gamma_3\gamma_2 \cong \mathbf{i} \quad (2.30)$$

$$\gamma_2^* = \gamma_2 \mathbf{I}_{\mathcal{S}}^{\sim} = -\gamma_2 \mathbf{I}_{\mathcal{S}} = -\gamma_2 \mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_2(\gamma_1\gamma_2\gamma_3) = -\gamma_1\gamma_3 \cong \mathbf{j} \quad (2.31)$$

$$\gamma_3^* = \gamma_3 \mathbf{I}_{\mathcal{S}}^{\sim} = -\gamma_3 \mathbf{I}_{\mathcal{S}} = -\gamma_3 \mathbf{I}_{\mathcal{S}}^{-1} = -\gamma_3(\gamma_1\gamma_2\gamma_3) = -\gamma_2\gamma_1 \cong \mathbf{k}. \quad (2.32)$$

The SA dualization is defined such that the isomorphism ($\cong$) to quaternion units is via duals.

The dual of an SA vector $\mathbf{x}_{\mathcal{S}}$ is

$$\mathbf{x}^* = \mathbf{x}_{\mathcal{S}}^* = -\mathbf{x}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}^{-1}. \quad (2.33)$$

The dual SA vector $\mathbf{x}^*$ is the rotor element or logarithm of a rotor $R = e^{\mathbf{x}^*}$, where

$$e^A = \exp(A) \quad (2.34)$$

$$= \sum_{n=0}^{\infty} \frac{A^n}{n!} \quad (2.35)$$

$$= \sum_{n=0}^{\infty} \frac{A^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{A^{2n+1}}{(2n+1)!} \quad (2.36)$$

$$= \cosh(A) + \sinh(A) \quad (2.37)$$

for any multivector A. For a scalar x, imaginary unit $i = \sqrt{-1}$, hyperbolic unit $j^2 = 1$, and null (nilpotent) unit $\varepsilon^2 = 0$, we also have the standard formulas

$$\cosh(ix) = \cos(x) \quad (2.38)$$

$$\sinh(ix) = i\sin(x) \quad (2.39)$$

$$\cosh(jx) = \cosh(x) \quad (2.40)$$

$$\sinh(jx) = j\sinh(x) \quad (2.41)$$

$$\cosh(\varepsilon x) = 1 \quad (2.42)$$

$$\sinh(\varepsilon x) = \varepsilon x. \quad (2.43)$$

Given the SA unit vector

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\mathcal{S}} = \frac{\mathbf{x}_{\mathcal{S}}}{\|\mathbf{x}_{\mathcal{S}}\|} = \frac{\mathbf{x}_{\mathcal{S}}}{\sqrt{-\mathbf{x}_{\mathcal{S}}^2}} \quad (2.44)$$

as the *axis* of rotation, and $\|\mathbf{x}_{\mathcal{S}}\| = \frac{1}{2}\theta$ as *half* the non-negative *angle* θ of rotation, then $(\hat{\mathbf{x}}^*)^2 = -1$ and $\hat{\mathbf{x}}^* \cong \sqrt{-1}$, and the *rotor* $R_{\mathcal{S}}$ for the rotation is

$$R = R_{\mathcal{S}} = e^{\mathbf{x}^*} = \exp(\mathbf{x}^*) = \cosh(\mathbf{x}^*) + \sinh(\mathbf{x}^*) \quad (2.45)$$

$$= \cosh\left(\frac{1}{2}\theta\hat{\mathbf{x}}^*\right) + \sinh\left(\frac{1}{2}\theta\hat{\mathbf{x}}^*\right) \quad (2.46)$$

$$= \cos\left(\frac{1}{2}\theta\right) + \hat{\mathbf{x}}^* \sin\left(\frac{1}{2}\theta\right) \quad (2.47)$$

$$= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{x}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}^{\sim} \quad (2.48)$$

$$= \exp\left(\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}^*\right) = e^{\frac{1}{2}\theta\hat{\mathbf{x}}_{\mathcal{S}}^*}. \quad (2.49)$$

The unit vector axis $\hat{\mathbf{x}}_{\mathcal{S}}$ and unit pseudoscalar $\mathbf{I}_{\mathcal{S}}$ may be in SA1 or SA2 by changing subscript $\mathcal{S}$ to $\mathcal{S}^1$ or $\mathcal{S}^2$ for rotors $R_{\mathcal{S}^1}$ and $R_{\mathcal{S}^2}$, respectively.

In $\mathcal{G}_{4,1}$ CGA, $\mathcal{G}_{1,4}$ CSA, and $\mathcal{G}_{2,4}$ CSTA, the isotropic dilator D (§6.6.6) has a logarithm unit that is isomorphic ($\cong$) to a hyperbolic unit j, and the translator T (§6.6.4) has a logarithm unit that is isomorphic to a null (nilpotent) unit ε. The CSTA boost B (§6.6.8), like a dilator D, has a logarithm unit that is isomorphic to a hyperbolic unit j and can be used as a directed non-uniform (anisotropic) dilation operator (§7.7.9) in DCSTA.

The *rotor operation*, or *versor "sandwich" operation*, that rotates any SA multivector $A_{\mathcal{S}}$ around the axis $\hat{\mathbf{x}}_{\mathcal{S}}$ by angle θ is

$$A'_{\mathcal{S}} = R_{\mathcal{S}} A_{\mathcal{S}} R_{\mathcal{S}}^{-1} = R_{\mathcal{S}} A_{\mathcal{S}} \tilde{R}_{\mathcal{S}}. \tag{2.50}$$

The SA multivector $A_{\mathcal{S}}$ is typically a vector $\mathbf{a}_{\mathcal{S}}$, but it can be any multivector in SA.

The sense of positive angle θ rotation around an axis $\hat{\mathbf{x}}$ usually follows the *right-hand rule* on a right-handed axes model. The sense of positive rotation around an axis follows the similar *left-hand rule* on a left-handed axes model. The choice of axes model does *not* affect the rotation mathematics, but it affects the orientation, or handedness, of the axes and the *interpretation* of rotation results on the chosen axes model.

In general, the rotor operation is an example of a *versor operation*, and each ith vector $\mathbf{a}_{ijk}$, of the jth k-blade $\mathbf{A}_{jk}$ of the k-vector $A_k = \langle A\rangle_k$ of a multivector A in a $\mathcal{G}_{p,q}$ Geometric Algebra $n = p + q$, is transformed by a $(1 \leq m \leq n)$-versor R as the versor operation

$$A' = RAR^{-1} \tag{2.51}$$

$$= R\left(\sum_{k=0}^{n} \langle A\rangle_k\right)R^{-1} \tag{2.52}$$

$$= R\left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}} \mathbf{A}_{jk}\right)\right)R^{-1} \tag{2.53}$$

$$= R\left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}}\left(\bigwedge_{i=1}^{k} \mathbf{a}_{ijk}\right)\right)\right)R^{-1} \tag{2.54}$$

$$= \left(\sum_{k=0}^{n}\left(\sum_{j=1}^{\binom{n}{k}}\left(\bigwedge_{i=1}^{k} R\mathbf{a}_{ijk}R^{-1}\right)\right)\right). \tag{2.55}$$

For grade $k = 0$, the 0-vector $\langle A\rangle_0$ is defined as the scalar part of A, which is not transformed by any versor operation since scalars commute with all multivectors. The general versor R is called an m-versor [17], which is the product of m vectors with inverses. This result is called *versor outermorphism*, which is discussed by Perwass in [20] and by this author in [9].

If R is a rotor, then the whole multivector A is rotated, as a rigid body of blades or as a surface entity, vector by vector in the multivector and preserves the angles between all vectors as a conformal transformation. Other angle-preserving (i.e. conformal) transformations of surface points or geometric surface entities are uniform surface scaling (isotropic dilation) and surface translation, which preserve the angles between surface features. The conformal transformations, as versor operations, for dilation and translation of surface points or entities require the embedding of surface representations or surface points into the $\mathcal{G}_{p+1,q+1}$ Conformal Geometric Algebra (CGA) or conformal space model. The same conformal rotation versor R of the base algebra, the $\mathcal{G}_{p,q}$ Geometric Algebra (GA), is also valid in the embedding of $\mathcal{G}_{p,q}$ GA into its higher-dimensional conformal algebra $\mathcal{G}_{p+1,q+1}$ CGA. In a CGA, some types of surfaces have a full representation as a multivector-valued surface entity A, but general surfaces are represented point-wise as embedded surface points that can be transformed by versor operations. This should be familiar from books on CGA such as [4], where surface entities for flats and rounds are the types of surface entities available in $\mathcal{G}_{4,1}$ CGA. In $\mathcal{G}_{8,2}$ DCGA [7][5][6], we gain 2-vector surface entities for quadrics, Dupin cyclides, and Darboux cyclides. In $\mathcal{G}_{4,8}$ DCSTA, we gain 2-vector surface entities for quadrics that can be boosted and anisotropically dilated. In $\mathcal{G}_{2,4}$ CSTA, we have entities for spacetime flats and hyperbolics (pseudorounds). Still, other general surfaces that do not have any multivector-valued entity A representation in the algebra must be represented point-wise as meshes or clouds of surface points. When they are available in a CGA, multivector-valued surface entities are powerful representations of complete surfaces that have advantages over point-wise representations of surfaces, and they can be transformed by versor operations (e.g., $A' = RAR^{-1}$) as versor outermorphism, which preserves blades as transformed blades of the same grade that are composed of transformed vectors.

The axis $\hat{\mathbf{x}}$ of a rotor R is a directional unit vector that represents a line through the origin around which to rotate. A general line is represented by any point $\mathbf{q}$ on the line and the unit direction $\hat{\mathbf{x}}$ of the line. The rotation of a point $\mathbf{p}$ around a general line requires a composition of translations and rotation $TRT^{\sim}\mathbf{p}TR^{\sim}T^{\sim}$. The translation $T^{\sim}$ translates by $-\mathbf{q}$, which translates any point $\mathbf{q}$ on the line to the origin and translates point $\mathbf{p}$ relatively as $\mathbf{p}-\mathbf{q}$. The rotation R rotates around the unit direction $\hat{\mathbf{x}}$ of the line, which is translated to the origin by $T^{\sim}$. The translation T translates by $\mathbf{q}$, which translates the rotated point back to a position relative to the general line. The versor $TRT^{\sim}$ is a translated rotor. See the CSTA 2-versor *translated-rotor* (§6.6.5).

3 Conformal Space Algebra (CSA)

The $\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) is very similar to $\mathcal{G}_{4,1}$ *Conformal Geometric Algebra* (CGA) [7], with only some changes in signs. This section is a very straightforward expanded adaptation of §2 *"CGA1 and CGA2"* in [7], with some additional notes on differences between similar CSA and CGA entities.

All of the $\mathcal{G}_{1,4}$ CSA entities and versors can be doubled into the corresponding $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) (§4) entities and versors that are similar to those in $\mathcal{G}_{8,2}$ *Double Conformal Geometric Algebra* (DCGA) [7].

3.1 CSA point

3.1.1 Introduction

In the following discussion about the *conformal embedding* of $\mathcal{G}_{0,3}$ SA point $\mathbf{p}=\mathbf{p}_{\mathcal{S}}$ (§2.5) into $\mathcal{G}_{1,4}$ CSA point $\mathbf{P}_{\mathcal{CS}}=\mathcal{C}(\mathbf{p})$, the generic $\mathcal{G}_{1,4}$ CSA $\mathcal{CS}$ algebra is used with vector units $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3, \mathbf{e}_+, \mathbf{e}_-$ (§6.2.2). The discussion is similar in CSA1 $\mathcal{CS}^1$ with vector units $\mathbf{e}_2, \mathbf{e}_3$, $\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6$, and in CSA2 $\mathcal{CS}^2$ with vector units $\mathbf{e}_8, \mathbf{e}_9, \mathbf{e}_{10}, \mathbf{e}_{11}, \mathbf{e}_{12}$. Perwass [20] discusses the conformal embedding, and some of his notation has been adopted here.

3.1.2 Stereographic embedding

In $\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA), the *conformal embedding* of a SA *point* (§2.5)

$$\mathbf{p}_{\mathcal{S}}=\mathbf{p} \;=\; x\boldsymbol{\gamma}_1+y\boldsymbol{\gamma}_2+z\boldsymbol{\gamma}_3 \tag{3.1}$$

starts with a *stereographic embedding* of $\mathbf{p}$ onto an anti-Euclidean 3-hypersphere or 3-sphere $\mathbb{S}^{0,3}$ using the *stereographic 3-sphere south pole* $-\mathbf{e}_-$ (§6.2.2) as the origin of projecting rays. In $\mathcal{G}_{4,1}$ CGA [7], $\mathbf{e}_+$ is the stereographic 3-sphere $\mathbb{S}^3$ north pole for projecting rays.

The signature of an SA unit vector $\hat{\mathbf{p}}^2=-1$ is negative (anti-Euclidean), and to form a spherical geometry or metric, $\hat{\mathbf{p}}$ and the stereographic pole $\mathbf{e}_-$ must have the same signature. The choice is arbitrary to use either the north pole $+\mathbf{e}_-$ or south pole $-\mathbf{e}_-$ as the origin for projecting rays. But as shown below, if the south pole $-\mathbf{e}_-$ is used, then it leads to familiar definitions for the point at the origin $\mathbf{e}_{o\gamma}$ (§6.2.2) and at infinity $\mathbf{e}_{\infty\gamma}$ (§6.2.3).

The stereographic embedding of $\mathbf{p}$ on the 3-sphere is the intersection of the ray (line) from $-\mathbf{e}_-$ to $\mathbf{p}$ with the 3-sphere (see [20], or Figure 1 in [7]). The vectors $-\mathbf{e}_-$ and $\mathbf{p}$ are perpendicular, and we can treat the embedding of $\mathbf{p}$ as similar to a 1D axis embedding onto a stereographic 1-sphere or circle.

The identities

$$\begin{aligned}
\|\mathbf{p}\| &= \sqrt{\mathbf{p}\cdot\mathbf{p}^{\dagger}}=\sqrt{-\mathbf{p}^2}=\sqrt{x^2+y^2+z^2} && (3.2)\\
\hat{\mathbf{p}} &= \frac{\mathbf{p}}{\|\mathbf{p}\|} && (3.3)\\
\mathbf{p} &= \|\mathbf{p}\|\hat{\mathbf{p}} && (3.4)\\
\mathbf{p}^2 &= -\|\mathbf{p}\|^2=-(x^2+y^2+z^2) && (3.5)
\end{aligned}$$

are used in the following.

The *stereographic embedding* $\mathcal{S}$ of $\mathbf{p}=\|\mathbf{p}\|\hat{\mathbf{p}}$ is the intersection point (α,β)

$$\mathcal{S}(\mathbf{p})=\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}}) \;=\; \alpha\hat{\mathbf{p}}-\beta\mathbf{e}_- \tag{3.6}$$

where the line through $-\mathbf{e}_-$ and $\|\mathbf{p}\|\hat{\mathbf{p}}$ intersect the unit 3-circle on the $\hat{\mathbf{p}}\mathbf{e}_-$-hyperplane. The Minkowski *homogenization* $\mathcal{H}_{\mathcal{M}}$ of $\mathcal{S}(\mathbf{p})$ is

$$\mathcal{H}_{\mathcal{M}}(\mathcal{S}(\mathbf{p})) \;=\; \alpha\hat{\mathbf{p}}-\beta\mathbf{e}_--\mathbf{e}_+. \tag{3.7}$$

The vectors $\mathbf{e}_-$ and $\mathbf{e}_+$ are the basis of a Minkowski plane. It is an arbitrary choice to add or subtract the homogeneous unit $\mathbf{e}_+$, but subtracting leads to familiar formulas for $\mathbf{e}_{o\gamma}$ and $\mathbf{e}_{\infty\gamma}$. The anti-Euclidean signature of vectors in $\mathcal{G}^1_{0,3}$ SA leads to making many opposite choices about the CSA embedding as compared to CGA that has Euclidean vectors in $\mathcal{G}^1_3$ APS. By following a ray from $-\mathbf{e}_-$ to a point $\mathbf{p}$, it can be seen how the ray intersects the 3-sphere, and the origin embeds to $\mathbf{e}_{o\gamma}=\mathbf{e}_--\mathbf{e}_+$ and the point at infinity embeds to $\mathbf{e}_{\infty\gamma}=-\mathbf{e}_--\mathbf{e}_+$.

For other points $\mathbf{p}$ that are not at the origin or infinity, the values for α and β on a unit circle are solved as follows. The initial relations are the unit circle

$$\alpha^2+\beta^2 = 1 \tag{3.8}$$

and, by similar triangles, the line

$$\frac{1-\beta}{\alpha} = \frac{1}{\|\mathbf{p}\|}. \tag{3.9}$$

Then, β is

$$\begin{aligned} \alpha^2 &= 1-\beta^2=(1+\beta)(1-\beta)=((1-\beta)\|\mathbf{p}\|)^2 && (3.10)\\ (1+\beta) &= (1-\beta)\|\mathbf{p}\|^2 && (3.11)\\ \beta\|\mathbf{p}\|^2+\beta &= \|\mathbf{p}\|^2-1 && (3.12)\\ \beta &= \frac{\|\mathbf{p}\|^2-1}{\|\mathbf{p}\|^2+1} && (3.13) \end{aligned}$$

and α is

$$\begin{aligned} \alpha &= (1-\beta)\|\mathbf{p}\| && (3.14)\\ &= \left(1-\frac{\|\mathbf{p}\|^2-1}{\|\mathbf{p}\|^2+1}\right)\|\mathbf{p}\| && (3.15)\\ &= \left(\frac{\|\mathbf{p}\|^2+1}{\|\mathbf{p}\|^2+1}-\frac{\|\mathbf{p}\|^2-1}{\|\mathbf{p}\|^2+1}\right)\|\mathbf{p}\| && (3.16)\\ &= \frac{2\|\mathbf{p}\|}{\|\mathbf{p}\|^2+1}. && (3.17) \end{aligned}$$

The stereographic embedding of $\|\mathbf{p}\|\hat{\mathbf{p}}$, denoted $\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}})$, can now be written as

$$\begin{aligned} \mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}}) &= \alpha\hat{\mathbf{p}}-\beta\mathbf{e}_- && (3.18)\\ &= \left(\frac{2\|\mathbf{p}\|}{\|\mathbf{p}\|^2+1}\right)\hat{\mathbf{p}}-\left(\frac{\|\mathbf{p}\|^2-1}{\|\mathbf{p}\|^2+1}\right)\mathbf{e}_-. && (3.19) \end{aligned}$$

3.1.3 Homogenization

The homogenization of $\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}})$, denoted $\mathcal{H}_{\mathcal{M}}(\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}}))$, can be written as

$$\mathbf{P}=\mathcal{H}_{\mathcal{M}}(\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}})) = \left(\frac{2\|\mathbf{p}\|}{\|\mathbf{p}\|^2+1}\right)\hat{\mathbf{p}}-\left(\frac{\|\mathbf{p}\|^2-1}{\|\mathbf{p}\|^2+1}\right)\mathbf{e}_--\mathbf{e}_+. \tag{3.20}$$

3.1.4 Conformal embedding

The point entity $\mathbf{P}$ is the *conformal embedding* of $\mathbf{p}$. Since the point entity $\mathbf{P}$ is homogeneous, and $\|\mathbf{p}\|^2+1$ is never zero, it can be scaled by $\frac{\|\mathbf{p}\|^2+1}{2}$ to define $\mathbf{P}$ as

$$\begin{aligned} \mathbf{P} &\simeq \mathcal{H}_{\mathcal{M}}(\mathcal{S}(\|\mathbf{p}\|\hat{\mathbf{p}})) && (3.21)\\ &= \mathcal{C}(\mathbf{p})=\mathcal{C}_{1,4}(\mathbf{p}) && (3.22)\\ &= \|\mathbf{p}\|\hat{\mathbf{p}}-\frac{\|\mathbf{p}\|^2-1}{2}\mathbf{e}_--\frac{\|\mathbf{p}\|^2+1}{2}\mathbf{e}_+ && (3.23)\\ &= \|\mathbf{p}\|\hat{\mathbf{p}}-\frac{\|\mathbf{p}\|^2}{2}(\mathbf{e}_-+\mathbf{e}_+)+\frac{1}{2}(\mathbf{e}_--\mathbf{e}_+) && (3.24)\\ &= \mathbf{p}+\frac{1}{2}\mathbf{p}^2(\mathbf{e}_++\mathbf{e}_-)+\frac{1}{2}(-\mathbf{e}_++\mathbf{e}_-). && (3.25) \end{aligned}$$

3.1.5 Origin point

When $\|\mathbf{p}\| = 0$,

$$\mathbf{P}_{\|\mathbf{p}\|=0} = \frac{1}{2}(-\mathbf{e}_+ + \mathbf{e}_-) = \mathbf{e}_{o\gamma} \tag{3.26}$$

representing the point at the origin $\mathbf{e}_{o\gamma}$ (§6.2.2).

3.1.6 Infinity point

Dividing $\mathbf{P}$ by $\frac{1}{2}\mathbf{p}^2$ and taking the limit as $\|\mathbf{p}\| \to \infty$, we find that

$$\mathbf{P}_{\|\mathbf{p}\|\to\infty} = \mathbf{e}_+ + \mathbf{e}_- = \mathbf{e}_{\infty\gamma} \tag{3.27}$$

represents the point at infinity $\mathbf{e}_{\infty\gamma}$ (§6.2.3). The vector $\mathbf{e}_+$ can be reinterpreted as the stereographic north pole, and the vector $\mathbf{e}_-$ can be reinterpreted as the homogeneous unit.

By taking inner products, it can be shown that conformal embedded points are null vectors $\mathbf{P}^2 = 0$ on the null 4-cone [4][20] in $\mathcal{G}^1_{1,4}$.

A frequently used inner product, worth remembering, is

$$\mathbf{e}_{o\gamma} \cdot \mathbf{e}_{\infty\gamma} = -1. \tag{3.28}$$

3.1.7 CSA point definition

The CSA embedding of SA vector $\mathbf{p}$ as CSA point $\mathbf{P}$ can now be defined as

$$\mathbf{P} = \mathbf{P}_{\mathcal{CS}} = \mathbf{P}_{\mathcal{C}_{1,4}} = \mathcal{C}(\mathbf{p}) = \mathcal{C}_{1,4}(\mathbf{p}) = \mathbf{p} + \frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}. \tag{3.29}$$

The CSA point embedding $\mathcal{C}(\mathbf{p})$ is the same as the CSTA point embedding $\mathcal{C}(\mathbf{p})$ (§6.2.4) of a spatial SA point $\mathbf{p}$. $\mathcal{G}_{1,4}$ CSA is a subalgebra of $\mathcal{G}_{2,4}$ CSTA.

3.1.8 CSA point normalization

A *normalized point* $\hat{\mathbf{P}}$ has unit scale on the homogeneous component $\mathbf{e}_{o\gamma}$ as

$$\hat{\mathbf{P}} = \frac{\mathbf{P}}{-\mathbf{P} \cdot \mathbf{e}_{\infty\gamma}}. \tag{3.30}$$

Points are assumed to be initially normalized as the embedding $\mathbf{P} = \mathcal{C}(\mathbf{p})$. After performing operations (§3.4) on a point (or other entity), a point (or other entity) may, in general, no longer be normalized (have unit scale). Some operations do not preserve scale.

3.1.9 CSA point projection

The *projection* (inverse embedding) of a CSA *point* $\mathbf{P}_{\mathcal{CS}}$ back to an SA *vector* $\mathbf{p}_{\mathcal{S}}$ is

$$\mathbf{p}_{\mathcal{S}} = (\hat{\mathbf{P}}_{\mathcal{CS}} \cdot \mathbf{I}_{\mathcal{S}})\mathbf{I}_{\mathcal{S}}^{-1} \tag{3.31}$$

where $\mathbf{I}_{\mathcal{S}}$ is the SA *unit pseudoscalar* (§2.1). $\hat{\mathbf{P}}_{\mathcal{CS}}$ must be a *normalized point* (§3.1.8).

3.1.10 Distance between two points

The *distance* $d(\mathbf{p}, \mathbf{q})$ between two CSA points $\mathbf{P} = \mathcal{C}(\mathbf{p})$ and $\mathbf{Q} = \mathcal{C}(\mathbf{q})$ is

$$d(\mathbf{p}, \mathbf{q}) = \sqrt{2\hat{\mathbf{P}} \cdot \hat{\mathbf{Q}}} \tag{3.32}$$

$$= \sqrt{-(\mathbf{p} - \mathbf{q})^2}. \tag{3.33}$$

In $\mathcal{G}_{4,1}$ CGA, distance is $d=\sqrt{-2\hat{\mathbf{P}}\cdot\hat{\mathbf{Q}}}$, with a difference in sign. The points $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ must be *normalized points* (§3.1.8).

3.2 CSA GIPNS entities

A CSA point $\mathbf{T}_{\mathcal{CS}}=\mathcal{C}(\mathbf{t})=\mathcal{C}_{1,4}(\mathbf{t})$ (§3.1) is on a spatial CSA *Geometric Inner Product Null Space* (GIPNS) surface *entity* $\mathbf{X}_{\mathcal{CS}}$ if $\mathbf{T}_{\mathcal{CS}}\cdot\mathbf{X}_{\mathcal{CS}}=0$ [20].

Using a CSTA spacetime point $\boldsymbol{T}_{\mathcal{C}}$ (§6.2.4) instead of a CSA spatial point $\mathbf{T}_{\mathcal{CS}}$ (§3.1) causes a change in the meaning of the test $\boldsymbol{T}_{\mathcal{C}}\cdot\mathbf{X}_{\mathcal{CS}}$, where $\mathbf{X}_{\mathcal{CS}}$ is then interpreted as a CSTA spacetime entity $\boldsymbol{X}_{\mathcal{C}}=\mathbf{X}_{\mathcal{CS}}$. A round CSA *surface* entity $\mathbf{X}_{\mathcal{CS}}$ is a CSTA hyperbolic *hypersurface* entity $\boldsymbol{X}_{\mathcal{C}}$ that grows in radius in space with time. A flat CSA surface entity $\mathbf{X}_{\mathcal{CS}}$ is a flat CSTA hypersurface entity $\boldsymbol{X}_{\mathcal{C}}$ that gains the span of the pseudospatial time dimension $w\boldsymbol{\gamma}_0$.

3.2.1 CSA GIPNS sphere

The CSA GIPNS 1-vector *sphere* $\mathbf{S}_{\mathcal{CS}}$, centered at CSA point $\mathbf{P}_{\mathcal{CS}}$ (§3.1) with radius r or with surface point $\mathbf{Q}_{\mathcal{CS}}$, is defined as

$$\begin{aligned}\mathbf{S}_{\mathcal{CS}} &= \mathbf{P}_{\mathcal{CS}}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma} &(3.34)\\ &= \mathbf{P}_{\mathcal{CS}}+(\mathbf{P}_{\mathcal{CS}}\cdot\mathbf{Q}_{\mathcal{CS}})\mathbf{e}_{\infty\gamma} &(3.35)\\ &= \mathbf{P}_{\mathcal{CS}}-\frac{1}{2}(\mathbf{p}-\mathbf{q})^2\mathbf{e}_{\infty\gamma}. &(3.36)\end{aligned}$$

In $\mathcal{G}_{4,1}$ CGA [7], the sign on r^2 in Eq. 3.34 is negative. However, Eqs. 3.35 and 3.36, for a sphere $\mathbf{S}_{\mathcal{CS}}$ defined by center point $\mathbf{P}_{\mathcal{CS}}$ and surface point $\mathbf{Q}_{\mathcal{CS}}$, are the same in both CGA and CSA.

For any point $\mathbf{T}=\mathcal{C}(\mathbf{t})$ on sphere $\mathbf{S}$ at center $\mathbf{P}=\mathcal{C}(\mathbf{p})$ with radius r,

$$d^2(\mathbf{t},\mathbf{p})=2\mathbf{T}\cdot\mathbf{P} \;=\; r^2 \qquad (3.37)$$

and

$$\begin{aligned}\mathbf{T}\cdot\mathbf{P}-\frac{1}{2}r^2 &= &(3.38)\\ \mathbf{T}\cdot\left(\mathbf{P}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma}\right) &= &(3.39)\\ \mathbf{T}\cdot\mathbf{S} &= 0. &(3.40)\end{aligned}$$

A sphere with unit scale $\hat{\mathbf{S}}$ has a unit scale center point $\hat{\mathbf{P}}$, and

$$\hat{\mathbf{S}}^2 \;=\; -r^2. \qquad (3.41)$$

As a CSTA spacetime entity $\boldsymbol{\Sigma}_{\mathcal{C}}=\mathbf{S}_{\mathcal{CS}}$, the sphere gains the span of $w\boldsymbol{\gamma}_0$ and is a spacetime *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{C}}$ (§6.4.5) centered at $\mathbf{P}_{\mathcal{CS}}$ with initial radius $r_0=r$. If $r=0$, then it is a *hypercone* (§6.4.2) centered on $\mathbf{P}_{\mathcal{CS}}$. The *imaginary sphere* with negative sign on r^2 (squared imaginary radius) becomes the *imaginary hyperpseudosphere* $\boldsymbol{\Xi}_{\mathcal{C}}$ (§6.4.6) in CSTA.

3.2.2 CSA GIPNS plane

The CSA GIPNS 1-vector *plane* $\mathbf{\Pi}_{\mathcal{CS}}$, normal to unit vector $\hat{\mathbf{n}}$ at distance d from the origin, or through point $\mathbf{p}$, is defined as

$$\begin{aligned}\mathbf{\Pi}_{\mathcal{CS}} &= \hat{\mathbf{n}} - d\mathbf{e}_{\infty\gamma} & (3.42)\\ &= \hat{\mathbf{n}} + (\mathbf{p}\cdot\hat{\mathbf{n}})\mathbf{e}_{\infty\gamma}. & (3.43)\end{aligned}$$

In $\mathcal{G}_{4,1}$ CGA [7], the plane is $\hat{\mathbf{n}} + d\mathbf{e}_\infty$, which is a difference in sign compared to Eq. 3.42. However, Eq. 3.43 is the same in CGA and CSA.

A *unit plane* $\hat{\mathbf{\Pi}}$ has a unit scale, normalized component $\hat{\mathbf{n}}$, and for a spatial plane

$$\hat{\mathbf{\Pi}}^2 = \hat{\mathbf{\Pi}}\cdot\hat{\mathbf{\Pi}} = -1. \quad (3.44)$$

As a CSTA spacetime entity $\boldsymbol{E}_{\mathcal{C}} = \mathbf{\Pi}_{\mathcal{CS}}$, the plane gains the span of $w\boldsymbol{\gamma}_0$ and is a spacetime *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ (§6.4.3) through point $d\hat{\mathbf{n}}$ or $\mathbf{p}$ in the 3D spacetime unit direction $\hat{\mathbf{n}}^{*\mathcal{M}}$ (§5.2.1) perpendicular to $\hat{\mathbf{n}}$ in spacetime.

Equation 3.43 for $\mathbf{\Pi}_{\mathcal{CS}}$ can be obtained as the translation (§3.4.2) by $\mathbf{p}$ of the plane $\hat{\mathbf{n}}$, which is through the origin, as

$$\begin{aligned}T\hat{\mathbf{n}}T^{\sim} &= \left(1-\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}\right)\hat{\mathbf{n}}\left(1-\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}\right)^{\sim} & (3.45)\\ &= \left(\hat{\mathbf{n}}+\frac{1}{2}\mathbf{p}\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}\right)\left(1+\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}\right) & (3.46)\\ &= \hat{\mathbf{n}}+\frac{1}{2}\hat{\mathbf{n}}\mathbf{p}\mathbf{e}_{\infty\gamma}+\frac{1}{2}\mathbf{p}\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma} & (3.47)\\ &= \hat{\mathbf{n}}+\frac{1}{2}(\hat{\mathbf{n}}\mathbf{p}+\mathbf{p}\hat{\mathbf{n}})\mathbf{e}_{\infty\gamma} & (3.48)\\ &= \hat{\mathbf{n}}+(\mathbf{p}\cdot\hat{\mathbf{n}})\mathbf{e}_{\infty\gamma}. & (3.49)\end{aligned}$$

The plane $\mathbf{\Pi}_{\mathcal{CS}}$ is a sphere $\mathbf{S}_{\mathcal{CS}}$ (§3.2.1) through point $d\hat{\mathbf{n}}$ with center $d\hat{\mathbf{n}}+r\hat{\mathbf{n}}$ and radius $r\rightarrow\infty$ as

$$\begin{aligned}\mathbf{S}_{\mathcal{CS}} &= \mathcal{C}(d\hat{\mathbf{n}}+r\hat{\mathbf{n}})+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma} & (3.50)\\ &= (d+r)\hat{\mathbf{n}}-\frac{1}{2}(d+r)^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma} & (3.51)\\ &= (d+r)\hat{\mathbf{n}}-\frac{1}{2}(d^2+2dr)\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} & (3.52)\\ &\simeq \hat{\mathbf{n}}-\frac{1}{2}\frac{(d^2+2dr)}{(d+r)}\mathbf{e}_{\infty\gamma}+\frac{\mathbf{e}_{o\gamma}}{(d+r)}, & (3.53)\end{aligned}$$

and using L'Hôpital's rule as

$$\begin{aligned}\mathbf{\Pi}_{\mathcal{CS}} &= \lim_{r\rightarrow\infty}\mathbf{S}_{\mathcal{CS}} & (3.54)\\ &= \lim_{r\rightarrow\infty}\left(\hat{\mathbf{n}}-\frac{1}{2}\frac{\partial_r(d^2+2dr)}{\partial_r(d+r)}\mathbf{e}_{\infty\gamma}+\frac{\mathbf{e}_{o\gamma}}{(d+r)}\right) & (3.55)\\ &= \lim_{r\rightarrow\infty}\left(\hat{\mathbf{n}}-\frac{1}{2}\frac{2d}{1}\mathbf{e}_{\infty\gamma}+\frac{\mathbf{e}_{o\gamma}}{(d+r)}\right) & (3.56)\\ &= \hat{\mathbf{n}}-d\mathbf{e}_{\infty\gamma}. & (3.57)\end{aligned}$$

Reflection of a point $\mathbf{P}$ in a plane $\mathbf{\Pi}$ is

$$\begin{aligned}\mathbf{P}' &= \mathbf{\Pi}\mathbf{P}\mathbf{\Pi}^{\sim} &(3.58)\\ &= \mathbf{\Pi}\mathbf{P}\mathbf{\Pi} &(3.59)\\ &= T\hat{\mathbf{n}}T^{\sim}\mathbf{P}T\hat{\mathbf{n}}T^{\sim}, &(3.60)\end{aligned}$$

which is a translated reflection in $\hat{\mathbf{n}}$. If there is no translation, then

$$\begin{aligned}\mathbf{P}' &= \hat{\mathbf{n}}\left(\mathbf{p}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}\right)\hat{\mathbf{n}} &(3.61)\\ &= \hat{\mathbf{n}}(\mathbf{p}^{||\hat{\mathbf{n}}}+\mathbf{p}^{\perp\hat{\mathbf{n}}})\hat{\mathbf{n}}+\frac{1}{2}\mathbf{p}^2\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}\hat{\mathbf{n}}+\hat{\mathbf{n}}\mathbf{e}_{o\gamma}\hat{\mathbf{n}} &(3.62)\\ &= (-\mathbf{p}^{||\hat{\mathbf{n}}}+\mathbf{p}^{\perp\hat{\mathbf{n}}})+\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}. &(3.63)\end{aligned}$$

In $\mathcal{G}_{4,1}$ CGA, this same reflection in $\hat{\mathbf{n}}$ is slightly different, and results in

$$\mathbf{P}'_{4,1} = (\mathbf{p}^{||\hat{\mathbf{n}}}-\mathbf{p}^{\perp\hat{\mathbf{n}}})-\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty}-\mathbf{e}_{o}, \quad (3.64)$$

such that the result is scaled by -1 and is not normalized. Normalizing $\mathbf{P}'_{4,1}$ (like §3.1.8) and projecting (like §3.1.9) still gives the correct result. In $\mathcal{G}_{1,4}$ CSA, $\hat{\mathbf{n}}$ is naturally a plane entity for the plane through the origin perpendicular to $\hat{\mathbf{n}}$ since the reflection in $\hat{\mathbf{n}}$ is the planar reflection. In $\mathcal{G}_{4,1}$ CGA, $\hat{\mathbf{n}}$ is not naturally a plane entity, but is just a vector, since the reflection in $\hat{\mathbf{n}}$ is vector reflection. It could be argued that $\mathcal{G}_{1,4}$ CSA is the more-natural Conformal Geometric Algebra $\mathcal{G}_{p+1,q+1}$ for conformal 3-space than is $\mathcal{G}_{4,1}$ CGA.

3.2.3 CSA GIPNS line

The CSA GIPNS 2-vector *line* $\mathbf{L}_{\mathcal{CS}}$, in the direction of the unit vector $\hat{\mathbf{d}}$, perpendicular to the SA unit bivector $\hat{\mathbf{D}}=\hat{\mathbf{d}}^{*\mathcal{S}}=-\hat{\mathbf{d}}\mathbf{I}_{\mathcal{S}}^{-1}$ formed by SA dualization (§2.2), and through SA point $\mathbf{p}$, is defined as

$$\begin{aligned}\mathbf{L}_{\mathcal{CS}} &= \hat{\mathbf{d}}^{*\mathcal{S}}-(\mathbf{p}\cdot\hat{\mathbf{d}}^{*\mathcal{S}})\mathbf{e}_{\infty\gamma} &(3.65)\\ &= \hat{\mathbf{D}}-(\mathbf{p}\cdot\hat{\mathbf{D}})\mathbf{e}_{\infty\gamma}. &(3.66)\end{aligned}$$

The $\mathcal{G}_{4,1}$ CGA line entity $\mathbf{L}$ [7] has the same form as the CSA line $\mathbf{L}_{\mathcal{CS}}$, with no differences in signs. The SA unit pseudoscalar is $\mathbf{I}_{\mathcal{S}}=\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$ (§2.1). The SA unit bivector $\hat{\mathbf{D}}$ is the planar direction passing through $\mathbf{p}$. $\mathbf{L}_{\mathcal{CS}}$ can be obtained as a translation (§3.4.2) by $\mathbf{p}$ of the line $\hat{\mathbf{D}}$, which is the line through the origin perpendicular to $\hat{\mathbf{D}}$, and is similar to the translation of a plane through the origin (§3.2.2). The line $\mathbf{L}_{\mathcal{CS}}$ can also be obtained as the intersection of two non-parallel CSA planes (§3.2.2) as

$$\mathbf{L}_{\mathcal{CS}} = \mathbf{\Pi}_{\mathcal{CS}_1}\wedge\mathbf{\Pi}_{\mathcal{CS}_2}. \quad (3.67)$$

As a CSTA spacetime entity $\mathbf{\Pi}_{\mathcal{C}}=\mathbf{L}_{\mathcal{CS}}$, the line gains the span of $w\boldsymbol{\gamma}_0$ and is a spacetime *plane* $\mathbf{\Pi}_{\mathcal{C}}$ (§6.4.10) through $\mathbf{p}$ in the direction $\boldsymbol{D}=\hat{\mathbf{d}}\boldsymbol{\gamma}_0=\hat{\mathbf{D}}\mathbf{I}_{\mathcal{M}}$.

3.2.4 CSA GIPNS circle

The CSA GIPNS 2-vector *circle* $\mathbf{C}_{\mathcal{CS}}$ is defined as

$$\mathbf{C}_{\mathcal{CS}} = \mathbf{S}_{\mathcal{CS}}\wedge\mathbf{\Pi}_{\mathcal{CS}} \quad (3.68)$$

which is the intersection of a CSA sphere $\mathbf{S}_{\mathcal{CS}}$ and CSA plane $\mathbf{\Pi}_{\mathcal{CS}}$. An intuitive construction uses a sphere centered at a point $\mathbf{p}$ with radius r_0 and a plane that intersects the sphere through the center $\mathbf{p}$ to form a circle of radius r_0 in the plane.

As a CSTA spacetime entity $\boldsymbol{S}_{\mathcal{C}} = \boldsymbol{C}_{\mathcal{CS}}$, the circle gains the span of $w\boldsymbol{\gamma}_0$ and is a spacetime *hyperboloid of one sheet (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}} = \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}}$ (§6.4.8) centered on $\mathbf{p}$ with initial radius r_0 that grows with time in the spatial plane. The radius grows as $r = \sqrt{(w-p_w)^2 + r_0^2}$. In the spacetime of $\boldsymbol{E}_{\mathcal{C}}$, the hyperpseudosphere $\boldsymbol{S}_{\mathcal{C}}$ is a spacetime hyperboloid aligned around the time axis $w\boldsymbol{\gamma}_0$. If $\mathbf{S}_{\mathcal{CS}}$ is an imaginary sphere, then $\boldsymbol{S}_{\mathcal{C}} = \boldsymbol{C}_{\mathcal{CS}}$ is a *hyperboloid of two sheets (imaginary pseudosphere)* (§6.4.9) in spacetime.

3.3 CSA GOPNS entities

A CSA point $\mathbf{T}_{\mathcal{CS}} = \mathcal{C}(\mathbf{t}) = \mathcal{C}_{1,4}(\mathbf{t})$ (§3.1) is on a CSA *Geometric Outer Product Null Space* (GOPNS) surface entity $\mathbf{X}^{*\mathcal{CS}}$ if $\mathbf{T}_{\mathcal{CS}} \wedge \mathbf{X}^{*\mathcal{CS}} = 0$ [20], where

$$\mathbf{X}^{*\mathcal{CS}} = \mathbf{X}^{*}_{\mathcal{CS}} \;=\; \mathbf{X}_{\mathcal{CS}}\mathbf{I}^{-1}_{\mathcal{CS}} = \mathbf{X}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}} \tag{3.69}$$

is the CSA dual (§3.4.1) of the CSA GIPNS entity $\mathbf{X}_{\mathcal{CS}}$ (§3.2). An entity and its dual entity represent the same geometric surface.

A CSA GOPNS entity can be directly formed as the wedge of up to five CSA points

$$\mathbf{X}^{*\mathcal{CS}} \;=\; \bigwedge \mathbf{P}_{\mathcal{CS}_i}, \qquad \text{for } 1 \le i \le 5. \tag{3.70}$$

3.3.1 CSA GOPNS sphere

The CSA GOPNS 4-vector *sphere* $\mathbf{S}^{*\mathcal{CS}}$ is the wedge of four non-coplanar CSA points $\mathbf{P}_{\mathcal{CS}_i}$ (§3.1) on the sphere

$$\mathbf{S}^{*\mathcal{CS}} \;=\; \mathbf{P}_{\mathcal{CS}_1} \wedge \mathbf{P}_{\mathcal{CS}_2} \wedge \mathbf{P}_{\mathcal{CS}_3} \wedge \mathbf{P}_{\mathcal{CS}_4} \tag{3.71}$$

$$\;=\; \mathbf{S}_{\mathcal{CS}}\mathbf{I}^{-1}_{\mathcal{C}} = \mathbf{S}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}} \tag{3.72}$$

and is the CSA dual (§3.4.1) of the CSA GIPNS 1-vector *sphere* $\mathbf{S}_{\mathcal{CS}}$ (§3.2.1).

The CSA unit pseudoscalar is $\mathbf{I}_{\mathcal{CS}} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_-$ (§3.4.1), and the CSA dualization and undualization are both multiplication with $\mathbf{I}_{\mathcal{CS}} = \mathbf{I}^{-1}_{\mathcal{CS}}$ as an involution. In $\mathcal{G}_{4,1}$ CGA, the dualization is an anti-involution. The SA dualization (§2.2) is also an involution by multiplying with $\mathbf{I}^{\sim}_{\mathcal{S}} = -\mathbf{I}_{\mathcal{S}}$.

3.3.2 CSA GOPNS plane

The CSA GOPNS 4-vector *plane* $\mathbf{\Pi}^{*\mathcal{CS}}$ is the wedge of three non-collinear CSA points $\mathbf{P}_{\mathcal{CS}_i}$ (§3.1) on the plane and the point $\mathbf{e}_{\infty\gamma}$

$$\mathbf{\Pi}^{*\mathcal{CS}} \;=\; \mathbf{P}_{\mathcal{CS}_1} \wedge \mathbf{P}_{\mathcal{CS}_2} \wedge \mathbf{P}_{\mathcal{CS}_3} \wedge \mathbf{e}_{\infty\gamma} \tag{3.73}$$

$$\;=\; \mathbf{\Pi}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}} \tag{3.74}$$

and is the CSA dual (§3.4.1) of CSA GIPNS 1-vector *plane* $\mathbf{\Pi}_{\mathcal{CS}}$ (§3.2.2).

3.3.3 CSA GOPNS line

The CSA GOPNS 3-vector *line* $\mathbf{L}^{*\mathcal{CS}}$ is the wedge of two CSA points $\mathbf{P}_{\mathcal{C}_i}$ (§3.1) on the line and the point $\mathbf{e}_{\infty\gamma}$

$$\mathbf{L}^{*\mathcal{CS}} \;=\; \mathbf{P}_{\mathcal{CS}_1} \wedge \mathbf{P}_{\mathcal{CS}_2} \wedge \mathbf{e}_{\infty\gamma} \tag{3.75}$$

$$\;=\; \mathbf{L}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}} \tag{3.76}$$

and is the CSA dual (§3.4.1) of the CSA GIPNS 2-vector *line* $\mathbf{L}_{\mathcal{CS}}$ (§3.2.3).

3.3.4 CSA GOPNS circle

The CSA GOPNS 3-vector *circle* $\mathbf{C}^{*\mathcal{CS}}$ is the wedge of three CSA points $\mathbf{P}_{\mathcal{C}_i}$ (§3.1) on the circle

$$\begin{aligned}\mathbf{C}^{*\mathcal{CS}} &= \mathbf{P}_{\mathcal{CS}_1}\wedge\mathbf{P}_{\mathcal{CS}_2}\wedge\mathbf{P}_{\mathcal{CS}_3} && (3.77)\\ &= \mathbf{C}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}} && (3.78)\end{aligned}$$

and is the CSA dual (§3.4.1) of the CSA GIPNS 2-vector *circle* $\mathbf{C}_{\mathcal{CS}}$ (§3.2.4).

3.3.5 CSA GOPNS point pair

The CSA GOPNS 2-vector *point pair* $\mathbf{2}^{*\mathcal{CS}}$ is the wedge of two *finite* CSA points $\mathbf{P}_{\mathcal{C}_i}$ (§3.1)

$$\begin{aligned}\mathbf{2}^{*\mathcal{CS}}=\mathbf{2}^{*}_{\mathcal{CS}} &= \mathbf{P}_{\mathcal{CS}_1}\wedge\mathbf{P}_{\mathcal{CS}_2} && (3.79)\\ &= \mathbf{2}_{\mathcal{CS}}\mathbf{I}_{\mathcal{C}} && (3.80)\end{aligned}$$

and is the CSA dual (§3.4.1) of the CSA GIPNS 3-vector *point pair* $\mathbf{2}_{\mathcal{CS}}$. If one of the points is $\mathbf{e}_{\infty\gamma}$, then it is a CSA GOPNS 2-vector *flat point* $\mathbb{P}^{*\mathcal{CS}}$ (§3.3.6).

The *point pair decomposition* [4]

$$\hat{\mathbf{P}}_{\mathcal{CS}_\pm} = \frac{\mathbf{2}^{*}_{\mathcal{CS}}\mp\sqrt{(\mathbf{2}^{*}_{\mathcal{CS}})^2}}{-\mathbf{e}_{\infty\gamma}\cdot\mathbf{2}^{*}_{\mathcal{CS}}} = (\mathbf{2}^{*}_{\mathcal{CS}}\mp\sqrt{\mathbf{2}^{*}_{\mathcal{CS}}\cdot\mathbf{2}^{*}_{\mathcal{CS}}})(-\mathbf{e}_{\infty\gamma}\cdot\mathbf{2}^{*}_{\mathcal{CS}})^{-1} \quad (3.81)$$

gives the two normalized (unit scale) finite *points* $\hat{\mathbf{P}}_{\mathcal{CS}_+}$ and $\hat{\mathbf{P}}_{\mathcal{CS}_-}$ of the point pair $\mathbf{2}^{*\mathcal{CS}}$. The CSA point pair decomposition is the same form as the CSTA point pair decomposition (Eq. 6.217).

3.3.6 CSA GOPNS flat point

The CSA GOPNS 2-vector *flat point* $\mathbb{P}^{*\mathcal{CS}}$ is the wedge of one *finite* CSA point $\mathbf{P}_{\mathcal{CS}}$ (§3.1) and the point $\mathbf{e}_{\infty\gamma}$

$$\begin{aligned}\mathbb{P}^{*\mathcal{CS}} &= \mathbf{P}_{\mathcal{CS}}\wedge\mathbf{e}_{\infty\gamma} && (3.82)\\ &= \mathbb{P}_{\mathcal{CS}}\mathbf{I}_{\mathcal{C}} && (3.83)\end{aligned}$$

and is the CSA dual (§3.4.1) of the CSA GIPNS 3-vector *flat point* $\mathbb{P}_{\mathcal{CS}}$.

As explained in [4] in the context of $\mathcal{G}_{4,1}$ CGA, a CSA GIPNS 3-vector *flat point* $\mathbb{P}_{\mathcal{CS}}$ can represent the intersection of a CSA GIPNS 1-vector *plane* $\mathbf{\Pi}_{\mathcal{CS}}$ (§3.2.2) and CSA GIPNS 2-vector *line* $\mathbf{L}_{\mathcal{CS}}$ (§3.2.3) as

$$\mathbb{P}_{\mathcal{CS}} = \mathbf{\Pi}_{\mathcal{CS}}\wedge\mathbf{L}_{\mathcal{CS}}. \quad (3.84)$$

The CSA *point* $\mathbf{P}_{\mathcal{CS}}$ of CSA GOPNS 2-vector *flat point* $\mathbb{P}^{*\mathcal{CS}}$ is *projected* [4] as

$$\mathbf{p}_{\mathcal{S}}=\mathcal{C}^{-1}_{1,4}(\mathbf{P}_{\mathcal{CS}}) = \frac{(\mathbf{e}_{o\gamma}\wedge\mathbf{e}_{\infty\gamma})\cdot(\mathbf{e}_{o\gamma}\wedge\mathbb{P}^{*\mathcal{CS}})}{-(\mathbf{e}_{o\gamma}\wedge\mathbf{e}_{\infty\gamma})\cdot\mathbb{P}^{*\mathcal{CS}}} = \frac{-\mathbb{P}^{*\mathcal{CS}}}{(\mathbf{e}_{o\gamma}\wedge\mathbf{e}_{\infty\gamma})\cdot\mathbb{P}^{*\mathcal{CS}}}\cdot\mathbf{e}_{o\gamma}-\mathbf{e}_{o\gamma}, \quad (3.85)$$

which is the same form as the CSTA flat point projection (Eq. 6.239).

3.3.7 CSA GOPNS point

The CSA GOPNS null 1-vector *point* $\mathbf{P}_{\mathcal{CS}}$ is the CSA null *point* (embedding) $\mathbf{P}_{\mathcal{CS}}=\mathcal{C}(\mathbf{p})$ (§3.1), which is also the CSA GIPNS null 1-vector *point* $\mathbf{P}_{\mathcal{CS}}$.

The CSA GOPNS null 4-vector *point* $\mathbf{P}^{*\mathcal{CS}}$ is

$$\mathbf{P}^{*\mathcal{CS}} = \mathbf{P}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}}, \tag{3.86}$$

which is also the CSA GIPNS null 4-vector *point* $\mathbf{P}^{*\mathcal{CS}}$. The CSA null 4-vector *point* $\mathbf{P}^{*\mathcal{CS}}$ is usually taken into its undual as $\mathbf{P}_{\mathcal{CS}} = \mathbf{P}^{*\mathcal{CS}}\mathbf{I}_{\mathcal{CS}}$.

The CSTA null point embedding $\mathbf{P}_{\mathcal{CS}} = \mathcal{C}(\mathbf{p})$ (§3.1) represents the implicit surface

$$(x - p_x)^2 + (y - p_y)^2 + (z - p_z)^2 = 0 \tag{3.87}$$

of a sphere with radius $r = 0$ centered at $\mathbf{p} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3$.

The CSTA *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p})$ (§6.2.4) is the CSTA GIPNS null 1-vector *hypercone* (§6.4.2) and is the CSTA GOPNS null 1-vector *point* (§6.5.2). The CSTA GIPNS null 1-vector hypercone (point embedding) represents the implicit surface

$$(w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0 \tag{3.88}$$

of a hypercone in spacetime that is centered at $\boldsymbol{p} = w\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3$. The different metrics of $\mathcal{G}_{0,3}$ SA (§2) and $\mathcal{G}_{1,3}$ STA (§5) make a difference in the implicit surfaces represented by their conformal point embeddings.

3.4 CSA operations

The rotor R (§3.4.3), dilator D (§3.4.5), and translator T (§3.4.2) are called *versors* $V \in \{R, D, T\}$. Their operation on a CSA entity $\mathbf{X}$ has the form $\mathbf{X}' = V\mathbf{X}V^{-1}$, called a *versor operation*. The versor V of the operation often has an exponential form which can be expanded by Taylor series into circular trigonometric, hyperbolic trigonometric, or dual number form.

3.4.1 CSA dualization

The $\mathcal{G}_{0,3}$ SA *unit pseudoscalar* $\mathbf{I}_{\mathcal{S}}$ (§2.1) is defined as

$$\mathbf{I}_{\mathcal{S}} = \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_1\gamma_2\gamma_3 \tag{3.89}$$

$$\mathbf{I}_{\tilde{\mathcal{S}}} = (-1)^{3(3-1)/2}\mathbf{I}_{\mathcal{S}} = -\mathbf{I}_{\mathcal{S}} \tag{3.90}$$

$$\mathbf{I}_{\mathcal{S}}^2 = -\mathbf{I}_{\mathcal{S}}\mathbf{I}_{\tilde{\mathcal{S}}} = 1 \tag{3.91}$$

$$\mathbf{I}_{\mathcal{S}}^{-1} = -\mathbf{I}_{\tilde{\mathcal{S}}} = \mathbf{I}_{\mathcal{S}} \tag{3.92}$$

and its *negative* is defined as the SA dualization operator (§2.2) on multivectors in $\mathcal{G}_{0,3}$ SA. A k-blade $\mathbf{B} \in \mathcal{G}_{0,3}^k$ is taken to its $\mathcal{G}_{0,3}$ SA dual $\mathbf{B}^{*\mathcal{S}} \in \mathcal{G}_{0,3}^{3-k}$ of grade $3 - k$ as

$$\mathbf{B}^{*\mathcal{S}} = -\mathbf{B}\mathbf{I}_{\mathcal{S}}^{-1} = -\mathbf{B} \cdot \mathbf{I}_{\mathcal{S}}. \tag{3.93}$$

Duals represent the same objects from two converse spatial spans, and duals have different behavior as operators or versors.

As a 1-versor, an SA unit vector $\hat{\mathbf{x}}$ acts as a reflector in the plane through the origin perpendicular to $\hat{\mathbf{x}}$. As a 2-versor, the SA dual bivector $\hat{\mathbf{X}} = \hat{\mathbf{x}}^{*\mathcal{S}}$ acts as a versor for successive reflections in two perpendicular planes for a rotation by π around $\hat{\mathbf{x}}$. The unit bivector $\hat{\mathbf{X}}$ is the generator of rotations around $\hat{\mathbf{x}}$, and its exponential $R = \exp\left(\frac{1}{2}\theta\hat{\mathbf{X}}\right)$ is the rotor R for rotations around $\hat{\mathbf{x}}$ by angle θ anticlockwise on right-handed axes.

The $\mathcal{G}_{1,4}$ CSA *unit pseudoscalar* $\mathbf{I}_{\mathcal{CS}}$ is defined as

$$\begin{aligned}
\mathbf{I}_{\mathcal{CS}} &= \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_- && (3.94)\\
\mathbf{I}_{\mathcal{CS}}^{\sim} &= (-1)^{5(5-1)/2}\mathbf{I}_{\mathcal{CS}} = \mathbf{I}_{\mathcal{CS}} && (3.95)\\
\mathbf{I}_{\mathcal{CS}}^{2} &= \mathbf{I}_{\mathcal{CS}}\mathbf{I}_{\mathcal{CS}}^{\sim} = 1 && (3.96)\\
\mathbf{I}_{\mathcal{CS}}^{-1} &= \mathbf{I}_{\mathcal{CS}}^{\sim} = \mathbf{I}_{\mathcal{CS}} && (3.97)
\end{aligned}$$

and is the dualization operator on CSA entities that takes CSA GIPNS entities to or from CSA GOPNS entities. A CSA entity $\mathbf{X} \in \mathcal{G}_{1,4}^{k}$ of grade k is taken to its CSA dual entity $\mathbf{X}^{*\mathcal{CS}} \in \mathcal{G}_{1,4}^{5-k}$ of grade $5-k$ as

$$\mathbf{X}^{*\mathcal{CS}} = \mathbf{X}\mathbf{I}_{\mathcal{CS}}^{-1} = \mathbf{X}\cdot\mathbf{I}_{\mathcal{CS}}. \qquad (3.98)$$

The CSA 5-vector unit pseudoscalar $\mathbf{I}_{\mathcal{CS}}$ can be interpreted as the CSA GOPNS 5-vector *space* entity that represents the entire 3D space of the vector point space $\mathcal{G}_{0,3}^{1}$.

CSA dualization and undualization are both multiplication of any CSA entity with $\mathbf{I}_{\mathcal{CS}}$.

3.4.2 CSA translator

A *translator* is a *translat*ion operat*or* or vers*or*.

The CSA 2-versor *translator* T, by an SA vector $\mathbf{d} = d_x\boldsymbol{\gamma}_1 + d_y\boldsymbol{\gamma}_2 + d_z\boldsymbol{\gamma}_3$, is defined as

$$\begin{aligned}
T &= 1 - \frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma} && (3.99)\\
&= e^{-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}}. && (3.100)
\end{aligned}$$

The $\mathcal{G}_{1,4}$ CSA translator T has the same exact form as the $\mathcal{G}_{4,1}$ CGA translator, and is the same as the CSTA translator (§6.6.4) for translation by an STA spacetime displacement $\boldsymbol{d}$. Any CSA entity $\mathbf{X}$ is translated by the SA vector $\mathbf{d}$ as

$$\mathbf{X} = T\mathbf{X}T^{\sim}. \qquad (3.101)$$

Spatial translation is also defined by reflection in two parallel CSA planes (§3.2.2) as

$$\mathbf{X}' = \boldsymbol{\Pi}_2\boldsymbol{\Pi}_1\mathbf{X}\boldsymbol{\Pi}_1\boldsymbol{\Pi}_2 \qquad (3.102)$$

which translates by *twice* the vector $\mathbf{d} = (d_2 - d_1)\hat{\mathbf{n}}$, with common normal unit vector $\hat{\mathbf{n}}$ of each plane (they are parallel) and plane distances from origin d_1 and d_2 of planes $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$, respectively. The translator T for translation by displacement $2\mathbf{d}$ is

$$\begin{aligned}
T &= -\boldsymbol{\Pi}_2\boldsymbol{\Pi}_1 && (3.103)\\
&= -(\hat{\mathbf{n}} - d_2\mathbf{e}_{\infty\gamma})(\hat{\mathbf{n}} - d_1\mathbf{e}_{\infty\gamma}) && (3.104)\\
&= -(-1 - d_1\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma} + d_2\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}) && (3.105)\\
&= 1 - (d_2 - d_1)\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma} && (3.106)\\
&= 1 - \mathbf{d}\mathbf{e}_{\infty\gamma}. && (3.107)
\end{aligned}$$

A CSA point $\mathbf{P} = \mathbf{P}_{\mathcal{CS}} = \mathcal{C}(\mathbf{p})$ (§3.1) is translated by $\mathbf{d}$ as

$$\begin{aligned}
\mathcal{C}(\mathbf{p}+\mathbf{d}) &= T\mathbf{P}T^{\sim} && (3.108)\\
&= T\left(\mathbf{p} + \frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}\right)T^{\sim}, && (3.109)
\end{aligned}$$

where the vector $\mathbf{p}$ transforms as a plane (§3.2.2)

$$\begin{aligned} T\mathbf{p}T^{\sim} &= \left(1-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right)\mathbf{p}\left(1-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right)^{\sim} && (3.110)\\ &= \mathbf{p}+\frac{1}{2}\mathbf{d}\mathbf{p}\mathbf{e}_{\infty\gamma}+\frac{1}{2}\mathbf{p}\mathbf{d}\mathbf{e}_{\infty\gamma} && (3.111)\\ &= \mathbf{p}+(\mathbf{p}\cdot\mathbf{d})\mathbf{e}_{\infty\gamma}, && (3.112)\end{aligned}$$

and $\mathbf{e}_{\infty\gamma}$ is translation invariant

$$\begin{aligned} T\left(\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}\right)T^{\sim} &= \left(1-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right)\left(\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}\right)\left(1+\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right) && (3.113)\\ &= \frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}, && (3.114)\end{aligned}$$

and $\mathbf{e}_{o\gamma}$ is translated as

$$\begin{aligned} T\mathbf{e}_{o\gamma}T^{\sim} &= \left(1-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right)\mathbf{e}_{o\gamma}\left(1+\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right) && (3.115)\\ &= \left(\mathbf{e}_{o\gamma}-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\mathbf{e}_{o\gamma}\right)\left(1+\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\right) && (3.116)\\ &= \mathbf{e}_{o\gamma}-\frac{1}{2}\mathbf{d}\mathbf{e}_{\infty\gamma}\mathbf{e}_{o\gamma}+\frac{1}{2}\mathbf{e}_{o\gamma}\mathbf{d}\mathbf{e}_{\infty\gamma}-\frac{1}{4}\mathbf{d}^2\mathbf{e}_{\infty\gamma}\mathbf{e}_{o\gamma}\mathbf{e}_{\infty\gamma} && (3.117)\\ &= \mathbf{e}_{o\gamma}-\mathbf{d}(\mathbf{e}_{\infty\gamma}\cdot\mathbf{e}_{o\gamma})-\frac{1}{4}\mathbf{d}^2(-2\mathbf{e}_{\infty\gamma}) && (3.118)\\ &= \mathbf{d}+\frac{1}{2}\mathbf{d}^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.119)\\ &= \mathcal{C}(\mathbf{d}), && (3.120)\end{aligned}$$

and finally, adding gives

$$\begin{aligned} T\mathbf{P}T^{\sim} &= \mathbf{p}+(\mathbf{p}\cdot\mathbf{d})\mathbf{e}_{\infty\gamma}+\frac{1}{2}\mathbf{p}^2\mathbf{e}_{\infty\gamma}+\mathbf{d}+\frac{1}{2}\mathbf{d}^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.121)\\ &= (\mathbf{p}+\mathbf{d})+\frac{1}{2}(\mathbf{p}\mathbf{d}+\mathbf{d}\mathbf{p}+\mathbf{p}^2+\mathbf{d}^2)\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.122)\\ &= (\mathbf{p}+\mathbf{d})+\frac{1}{2}(\mathbf{p}+\mathbf{d})^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.123)\\ &= \mathcal{C}(\mathbf{p}+\mathbf{d}). && (3.124)\end{aligned}$$

A surface entity represents a set of points, and translating an entity represents translating the set of surface points. A similar argument holds for rotation (§3.4.3) and dilation (§3.4.5) of entities, and for the boost (§6.6.8) of entities in CSTA.

3.4.3 CSA rotor

A *rotor* is a *rot*ation operat*or* or vers*or*. The CSA rotor R is the same as the SA rotor (§2.6).

The CSA 2-versor *rotor* R, for rotation around unit vector axis $\hat{\mathbf{n}}$ by θ radians, is defined as

$$\begin{aligned} R &= \cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{n}}^{*\mathcal{S}} && (3.125)\\ &= e^{\frac{1}{2}\theta\hat{\mathbf{n}}^{*\mathcal{S}}}=e^{\frac{1}{2}\theta\hat{\mathbf{N}}}. && (3.126)\end{aligned}$$

The SA unit bivector $\hat{\mathbf{N}} = \hat{\mathbf{n}}^{*\mathcal{S}}$ represents the plane of rotation. Any multivector in $\mathcal{G}_{0,3}$ SA or any $\mathcal{G}_{1,4}$ CSA entity $\mathbf{X}$ is rotated as

$$\mathbf{X}' = R\mathbf{X}R^{\sim} \quad (3.127)$$
$$= R\mathbf{X}R^{-1} \quad (3.128)$$

where $R^{\sim}$, the *reverse* of R, is equal to the inverse R^{-1}. Rotation is also defined by successive reflections in two non-parallel CSA planes (§3.2.2) as

$$\mathbf{X}' = \mathbf{\Pi}_2\mathbf{\Pi}_1\mathbf{X}\mathbf{\Pi}_1\mathbf{\Pi}_2 \quad (3.129)$$

which rotates $\mathbf{X}$ by *twice* the angle between the planes from $\mathbf{\Pi}_1$ to $\mathbf{\Pi}_2$. The axis of rotation is the line of intersection $\mathbf{L}_{\mathcal{CS}} = \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2$ (§3.2.3) of the two non-parallel planes, which can be a line through the origin or a general line. The orientation of the rotation is by right-hand rule on right-handed axes, or by left-hand rule on left-handed axes, around the line from $\mathbf{\Pi}_1$ toward $\mathbf{\Pi}_2$.

3.4.4 CSA translated-rotor

The rotation of an entity $\mathbf{E}$ by angle θ around an arbitrary line $\mathbf{L}$ (§3.2.3) through point $\mathbf{p}$ with direction $\hat{\mathbf{d}}$ is a composition of translations (§3.4.2) and rotations (§3.4.3)

$$\mathbf{E}' = TRT^{\sim}\mathbf{E}TR^{\sim}T^{\sim} \quad (3.130)$$
$$= R_{\mathbf{d}}^{\mathbf{p}}\mathbf{E}R_{\mathbf{d}}^{\mathbf{p}\sim}, \quad (3.131)$$

where

$$\mathbf{d} = \frac{1}{2}\theta\hat{\mathbf{d}} \quad (3.132)$$
$$R = \exp\left(\frac{\theta}{2}\hat{\mathbf{d}}^{*\mathcal{S}}\right) \quad (3.133)$$
$$= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(-\hat{\mathbf{d}}\mathbf{I}_{\mathcal{S}}^{-1})^{(\S 2.6)} \quad (3.134)$$
$$T = \exp\left(-\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}\right) \quad (3.135)$$
$$= 1 - \frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}, \quad (3.136)$$

and

$$R_{\mathbf{d}}^{\mathbf{p}} = TRT^{\sim} \quad (3.137)$$
$$= \exp\left(\frac{\theta}{2}T\hat{\mathbf{d}}^{*\mathcal{S}}T^{\sim}\right) \quad (3.138)$$
$$= \exp\left(\frac{\theta}{2}(\hat{\mathbf{d}}^{*\mathcal{S}} - (\mathbf{p}\cdot\hat{\mathbf{d}}^{*\mathcal{S}})\mathbf{e}_{\infty\gamma})^{(\S 3.2.3)}\right) \quad (3.139)$$
$$= \exp\left(\frac{\theta}{2}\hat{\mathbf{L}}\right). \quad (3.140)$$

The CSA GIPNS 2-versor *translated-rotor* $R_{\mathbf{d}}^{\mathbf{p}}$, for rotation by angle θ around the *unit line* $\hat{\mathbf{L}}$ (§3.2.3), is defined as

$$R_{\mathbf{d}}^{\mathbf{p}} = \exp\left(\frac{\theta}{2}\hat{\mathbf{L}}\right) \quad (3.141)$$

$$= \cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\hat{\mathbf{L}}, \quad (3.142)$$

where

$$\hat{\mathbf{L}} = \hat{\mathbf{d}}^{*\mathcal{S}} - \left(\mathbf{p}\cdot\hat{\mathbf{d}}^{*\mathcal{S}}\right)\mathbf{e}_{\infty\gamma}. \quad (3.143)$$

A unit line $\hat{\mathbf{L}}$ is a line with unit scale, when the line direction is a unit vector $\hat{\mathbf{d}}$. Then,

$$\hat{\mathbf{L}}^2 = -1. \quad (3.144)$$

Alternatively, the line can be the axis-angle line

$$\mathbf{L} = \frac{\theta}{2}\hat{\mathbf{L}}, \quad (3.145)$$

and the translated-rotor is

$$R_{\mathbf{L}} = \exp(\mathbf{L}). \quad (3.146)$$

3.4.5 CSA dilator

A *dilator* is a *dilat*ion operat*or* or vers*or*.

The CSA 2-versor *isotropic dilator* D, by dilation factor d, is defined as

$$D = \frac{1}{2}(1+d)+\frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma} \quad (3.147)$$

$$\simeq 1+\frac{(1-d)}{(1+d)}\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma} \quad (3.148)$$

$$\simeq e^{\operatorname{atanh}\left(\frac{(1-d)}{(1+d)}\right)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}} = e^{-\frac{1}{2}\ln(d)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}}. \quad (3.149)$$

Any $\mathcal{G}_{1,4}$ CSA entity $\mathbf{X}$ is dilated by the factor d as

$$\mathbf{X}' = D\mathbf{X}D^{\sim}. \quad (3.150)$$

The first form of D (Eq. 3.147) is the most applicable since it allows $d<0$ and usually gives the expected results in that case. The forms of D using atanh or ln cannot accept $d\leq 0$ since those functions would return an infinite result for $d=0$ or complex numbers which are not valid in a geometric algebra over real numbers.

It should be noted that a zero dilation factor $d=0$ is *generally not valid*. Finite surface entities having $\mathbf{e}_{o\gamma}$ or its dual $\mathbf{e}_{o\gamma}^{*\mathcal{CS}}$ as a term will dilate by factor 0 into $\mathbf{e}_{o\gamma}$ or its dual (up to scale), which is a *valid* result. All other entities dilate by factor 0 into the scalar 0, which is an *invalid* result. Dilation by factor 0 is valid on the CSA GIPNS *sphere* and CGA *point* and their duals. Using $d=0$ in the first form of D gives

$$D = \frac{1}{2}+\frac{1}{2}\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma} \quad (3.151)$$

which, as can be checked, dilates any normalized CSA GIPNS sphere $\mathbf{S}$ (§3.2.1) or CSA point $\mathbf{P}_{\mathcal{CS}}$ (§3.1) into

$$D\mathbf{S}D^{\sim} \;=\; D\mathbf{P}_{\mathcal{CS}}D^{\sim} = D\mathbf{e}_{o\gamma}D^{\sim} = \mathbf{e}_{o\gamma}. \tag{3.152}$$

A DCSA entity must have the DCSA origin point $\mathbf{e}_o$ or its dual $\mathbf{e}_o^{*\mathcal{DS}}$ as a term for DCSA dilation by factor $d=0$ to be valid. This is explained further in the section on the DCSA GIPNS cyclide.

Dilation is also defined by inversions in two concentric spheres (§3.2.1) as

$$\mathbf{X}' \;=\; \mathbf{S}_2\mathbf{S}_1\mathbf{X}\mathbf{S}_1\mathbf{S}_2 \tag{3.153}$$

which dilates by $d=\frac{r_2^2}{r_1^2}$, with radius r_1 of $\mathbf{S}_1$ and radius r_2 of $\mathbf{S}_2$. The dilator D is derived from successive inversions in two spheres centered at the origin, but it is also possible for the spheres to be centered at any point and to dilate relative to that point.

3.4.6 CSA spherical inversion

The inversion of point $\mathbf{P}=\hat{\mathbf{P}}=\mathcal{C}(\mathbf{p})=\mathcal{C}(\mathbf{c}+\mathbf{d})$ (§3.1) in sphere $\mathbf{S}$ (§3.2.1), at center $\mathbf{C}=\mathcal{C}(\mathbf{c})$ with radius r, where $\mathbf{p}$ is at displacement $\mathbf{d}$ from $\mathbf{c}$, is

$$\begin{aligned}
\mathbf{P}' &= \mathbf{S}\mathbf{P}\mathbf{S}^{\sim} = \mathbf{S}\mathbf{P}\mathbf{S} && (3.154)\\
&= \left(\mathbf{C}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma}\right)\mathbf{P}\left(\mathbf{C}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma}\right) && (3.155)\\
&= \mathbf{C}\mathbf{P}\mathbf{C}+\frac{1}{2}r^2\mathbf{e}_{\infty\gamma}\mathbf{P}\mathbf{C}+\frac{1}{2}r^2\mathbf{C}\mathbf{P}\mathbf{e}_{\infty\gamma}+\frac{1}{4}r^4\mathbf{e}_{\infty\gamma}\mathbf{P}\mathbf{e}_{\infty\gamma} && (3.156)\\
&= 2(\mathbf{C}\cdot\mathbf{P})\mathbf{C}+\frac{1}{2}r^2(\mathbf{e}_{\infty\gamma}\mathbf{P}\mathbf{C}+\mathbf{C}\mathbf{P}\mathbf{e}_{\infty\gamma})+\frac{1}{4}r^4 2(\mathbf{e}_{\infty\gamma}\cdot\mathbf{P})\mathbf{e}_{\infty\gamma} && (3.157)\\
&= d^2\mathbf{C}+\frac{1}{2}r^2(\mathbf{e}_{\infty\gamma}(\mathbf{P}\cdot\mathbf{C}+\mathbf{P}\wedge\mathbf{C})+(\mathbf{C}\cdot\mathbf{P}+\mathbf{C}\wedge\mathbf{P})\mathbf{e}_{\infty\gamma})-\frac{1}{2}r^4\mathbf{e}_{\infty\gamma} && (3.158)\\
&= -\mathbf{d}^2\mathbf{C}+\left(r^2\mathbf{C}\cdot\mathbf{P}-\frac{1}{2}r^4\right)\mathbf{e}_{\infty\gamma}+\frac{1}{2}r^2(\mathbf{e}_{\infty\gamma}\cdot(\mathbf{P}\wedge\mathbf{C})+(\mathbf{C}\wedge\mathbf{P})\cdot\mathbf{e}_{\infty\gamma}) && (3.159)\\
&= -\mathbf{d}^2\mathbf{C}+\left(-\frac{1}{2}r^2\mathbf{d}^2-\frac{1}{2}r^4\right)\mathbf{e}_{\infty\gamma}+\frac{1}{2}r^2((\mathbf{C}\wedge\mathbf{P})\cdot\mathbf{e}_{\infty\gamma}+(\mathbf{C}\wedge\mathbf{P})\cdot\mathbf{e}_{\infty\gamma}) && (3.160)\\
&= -\mathbf{d}^2\mathbf{C}-\frac{1}{2}(r^2\mathbf{d}^2+r^4)\mathbf{e}_{\infty\gamma}+r^2(\mathbf{C}\wedge\mathbf{P})\cdot\mathbf{e}_{\infty\gamma} && (3.161)\\
&= -\mathbf{d}^2\mathbf{C}-\frac{1}{2}(r^2\mathbf{d}^2+r^4)\mathbf{e}_{\infty\gamma}+r^2\mathbf{P}-r^2\mathbf{C} && (3.162)\\
&= r^2\mathbf{P}-(\mathbf{d}^2+r^2)\mathbf{C}-\frac{1}{2}(r^2\mathbf{d}^2+r^4)\mathbf{e}_{\infty\gamma} && (3.163)\\
&= -\mathbf{d}^2\mathbf{c}+r^2\mathbf{d}+\frac{1}{2}(r^2(\mathbf{c}+\mathbf{d})^2-(\mathbf{d}^2+r^2)\mathbf{c}^2-r^2\mathbf{d}^2-r^4)\mathbf{e}_{\infty\gamma}-\mathbf{d}^2\mathbf{e}_{o\gamma} && (3.164)\\
&\simeq \mathbf{c}+\frac{r^2\mathbf{d}}{-\mathbf{d}^2}+\frac{1}{2}\frac{(r^2\mathbf{c}\mathbf{d}+r^2\mathbf{d}\mathbf{c}-\mathbf{d}^2\mathbf{c}^2-r^4)}{-\mathbf{d}^2}\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}, \quad \text{for } \|\mathbf{d}\|\neq 0, && (3.165)\\
&\simeq \mathbf{c}-r^2\mathbf{d}^{-1}+\frac{1}{2}(-r^2\mathbf{c}\mathbf{d}^{-1}-r^2\mathbf{d}^{-1}\mathbf{c}+\mathbf{c}^2+r^4\mathbf{d}^{-2})\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.166)\\
&\simeq \mathbf{c}-r^2\mathbf{d}^{-1}+\frac{1}{2}(\mathbf{c}-r^2\mathbf{d}^{-1})^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} && (3.167)\\
&\simeq \mathbf{c}+r^2\|\mathbf{d}\|^{-1}\hat{\mathbf{d}}+\frac{1}{2}(\mathbf{c}+r^2\|\mathbf{d}\|^{-1}\hat{\mathbf{d}})^2\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma}. && (3.168)
\end{aligned}$$

The algebra looks involved, but the result is important. If $\mathbf{p}$ is on sphere $\mathbf{S}$, then $\mathbf{p}=\mathbf{c}+r\hat{\mathbf{d}}$, $\mathbf{d}=r\hat{\mathbf{d}}$, and

$$r^2\|\mathbf{d}\|^{-1}\hat{\mathbf{d}} \;=\; r\hat{\mathbf{d}}, \tag{3.169}$$

such that points on the sphere are invariant, up to scale by r^2 (Eq. 3.163), by inversion. When $r=1$, $\mathbf{d}$ transforms to

$$r^2\|\mathbf{d}\|^{-1}\hat{\mathbf{d}} \;=\; \|\mathbf{d}\|^{-1}\hat{\mathbf{d}}, \tag{3.170}$$

which is a displacement of inverse magnitude from the center $\mathbf{c}$. This accounts for the name *inversion*. Equation 3.162 shows that when $\mathbf{P}=\mathbf{C}$, then

$$\mathbf{SCS} \;=\; -\frac{1}{2}r^4\mathbf{e}_{\infty\gamma}. \tag{3.171}$$

These results confirm that inversions of points in spheres are correct. Now, consider a test point $\mathbf{T}$, GOPNS entity $\mathbf{X}^*$, and the outermorphism of their test by a sphere $\mathbf{S}$

$$\begin{aligned}
\mathbf{S}(\mathbf{T}\wedge\mathbf{X}^*)\mathbf{S} &= (\mathbf{STS})\wedge(\mathbf{SX}^*\mathbf{S}) && (3.172)\\
&= \mathbf{T}'\wedge\bigwedge \mathbf{SP}_i\mathbf{S} && (3.173)\\
&= \mathbf{T}'\wedge\mathbf{X}^{*\prime}. && (3.174)
\end{aligned}$$

Since inversions of points are correct, the inversion of $\mathbf{X}^*$ into the entity $\mathbf{X}^{*\prime}$ is also correct. Considering the GIPNS entity

$$\begin{aligned}
\mathbf{X} &= \mathbf{X}^*\mathbf{I}_{\mathcal{CS}} && (3.175)\\
&= \mathbf{X}^*\cdot\mathbf{I}_{\mathcal{CS}}, && (3.176)
\end{aligned}$$

then the inversion of the test is

$$\begin{aligned}
\mathbf{S}(\mathbf{T}\cdot(\mathbf{X}^*\cdot\mathbf{I}_{\mathcal{CS}}))\mathbf{S} &= \mathbf{S}((\mathbf{T}\wedge\mathbf{X}^*)\cdot\mathbf{I}_{\mathcal{CS}})\mathbf{S} && (3.177)\\
&= \mathbf{S}(\mathbf{T}\wedge\mathbf{X}^*)\mathbf{I}_{\mathcal{CS}}\mathbf{S} && (3.178)\\
&= \mathbf{S}(\mathbf{T}\wedge\mathbf{X}^*)\mathbf{S}\mathbf{I}_{\mathcal{CS}}. && (3.179)
\end{aligned}$$

With these results, the CSA GIPNS *sphere* $\mathbf{S}$ (§3.2.1) can be defined as the CSA 1-versor *invers*ion operat*or* (*inversor*) $\mathbf{S}$ for spherical inversion in the sphere $\mathbf{S}$ of any CSA entity $\mathbf{X}$. The inversion (inversor) operation is the versor sandwich product

$$\mathbf{X}' \;=\; \mathbf{SXS}. \tag{3.180}$$

In general, the round or hyperbolic entities defined in a Conformal Geometric Algebra (CGA) $\mathcal{G}_{p+1,q+1}$ are inversors, and the flat entities are *reflect*ion operat*ors* (*reflectors*). In a Double Conformal Geometric Algebra (DCGA) $\mathcal{G}_{2p+2,2q+2}$, the doubled inversors and reflectors are DCGA 2-versor inversors and reflectors.

The CSTA 1-vector *hyperpseudosphere* $\mathbf{\Sigma}_{\mathcal{C}}$ (§6.4.5) is an inversor for both spherical inversion in space, and pseudospherical (hyperboloidal) inversion in a 3D spacetime. Inversion in the CSTA 2-vector *spacetime hyperboloid (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}$ (§6.4.8) is also valid.

In DCSTA, the 2-versors $\mathbf{\Sigma}_{\mathcal{D}}$ (§7.3.3) and $\mathbf{\Xi}_{\mathcal{D}}$ (§7.3.4) are valid for the inversion of doubled entities and all DCSTA 2-vector entities (7.5), including the DCSTA 2-vector *quadric* and *pseudoquadric* surfaces, with different results for the inversion of doubled entities, quadrics, and pseudoquadrics.

3.4.7 CSA translated-dilator

The CSA 2-versor *translated-dilator* $D_d^{\mathbf{p}}$, for isotropic dilation by factor $d \geq 0$ centered around the unit-scale point $\hat{\mathbf{P}}_{\mathcal{CS}} = \mathcal{C}(\mathbf{p})$ (§3.1.8), is defined as

$$D_d^{\mathbf{p}} = \exp\left(\frac{1}{2}\ln(d)\hat{\mathbb{P}}^{*\mathcal{CS}}\right), \tag{3.181}$$

where the unit CSA GOPNS 2-vector *flat point* (§3.3.6) is

$$\hat{\mathbb{P}}^{*\mathcal{CS}} = \hat{\mathbf{P}}_{\mathcal{CS}} \wedge \mathbf{e}_{\infty\gamma}. \tag{3.182}$$

The translated-dilator is derived as $D_d^{\mathbf{p}} = T^{\mathbf{p}} D_d T^{\mathbf{p}\sim}$, by translating (§3.4.2) the dilator (§3.4.5) by $\mathbf{p}$.

3.4.8 CSA motor

A *motor* is a *mot*ion operat*or*. A rotation around an SA unit vector axis $\hat{\mathbf{n}}$, followed by a translation parallel to $\hat{\mathbf{n}}$ are commutative operations. Either the translation or the rotation can be done first, and the other second, to reach the same final position. This commutative operation, being a screw or helical motion, can be seen physically without mathematics. The motor is a special case where the commutative rotor and translator can be composed into a single versor M with an *exponential form* as

$$\begin{aligned} M &= RT = TR && (3.183)\\ &= e^{\frac{1}{2}\theta\hat{\mathbf{n}}^{*\mathcal{S}}} e^{-\frac{1}{2}d\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}} = e^{-\frac{1}{2}d\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}} e^{\frac{1}{2}\theta\hat{\mathbf{n}}^{*\mathcal{S}}} && (3.184)\\ &= e^{\frac{1}{2}\theta\hat{\mathbf{n}}^{*\mathcal{S}} - \frac{1}{2}d\hat{\mathbf{n}}\mathbf{e}_{\infty\gamma}} && (3.185)\\ &= e^{-\frac{1}{2}\hat{\mathbf{n}}(\theta\mathbf{I}_{\mathcal{S}} + d\mathbf{e}_{\infty\gamma})}. && (3.186) \end{aligned}$$

The exponents or *logarithms* of commutative exponentials can be added. A motor can be used to model smoothly-interpolated *screw*, *twistor*, or helical motions, performed in n steps using the nth root of M

$$M^{\frac{1}{n}} = e^{-\frac{1}{2n}\hat{\mathbf{n}}(\theta\mathbf{I}_{\mathcal{S}} + d\mathbf{e}_{\infty\gamma})} \tag{3.187}$$

applied at each step.

3.4.9 CSA intersection

CSA GIPNS *intersection* entities, which represent the surface intersections of two or more CSA GIPNS entities, are formed by the wedge of the CSA GIPNS entities. The CSA GIPNS circle is defined as a CSA GIPNS intersection entity $\mathbf{C}_{\mathcal{CS}} = \mathbf{S}_{\mathcal{CS}} \wedge \mathbf{\Pi}_{\mathcal{CS}}$.

Almost any combination of CSA GIPNS entities may be wedged to form a CSA GIPNS intersection entity up to grade 4, except that the CSA GIPNS 2-vector line and circle entities that are coplanar cannot be intersected unless their common plane is first contracted out of each of them, then the common plane is wedged back onto their intersection entity.

Like any CSA GIPNS entity, a CSA GIPNS intersection entity $\mathbf{X}$ can be taken dual as $\mathbf{X}^{*\mathcal{CS}} = \mathbf{X}/\mathbf{I}_{\mathcal{CS}}$ (§3.4.1) into its CSA GOPNS intersection entity $\mathbf{X}^{*\mathcal{CS}}$.

De Morgan's law for the intersection $\mathbf{X}$ of two objects $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{X} = \text{not}((\text{not}\,\mathbf{A})\,\text{and}\,(\text{not}\,\mathbf{B})) \tag{3.188}$$

and translates into the CSA intersection

$$\mathbf{X}^{*\mathcal{CS}} = (\mathbf{A}^{*\mathcal{CS}} \wedge \mathbf{B}^{*\mathcal{CS}})^{*\mathcal{CS}}. \tag{3.189}$$

This is just the creation of the CSA GOPNS intersection entity $\mathbf{X}^{*\mathcal{CS}}$ of two *CGA GOPNS entities* $\mathbf{A}$ and $\mathbf{B}$. In this case, $\mathbf{A}^{*\mathcal{CS}}$ and $\mathbf{B}^{*\mathcal{CS}}$ are the *undual* CSA GIPNS entities, which can then be intersected by wedge product. The CSA GIPNS intersection $\mathbf{X}$ is then dualized as the CGA GOPNS entity $\mathbf{X}^{*\mathcal{CS}}$. The classical view of intersections is by working with spanning objects, which are the CSA GOPNS entities.

3.5 CSA1 and CSA2 notations

The CSA1 and CSA2 spaces are used as exact copies of CSA. All that is needed is a little notation to separate the two spaces.

Multivectors in the $\mathcal{G}_{0,3}$ SA1 subspace of $\mathcal{G}_{1,4}$ CSA1 use the subscript $\mathcal{S}^1$. For example, an SA1 vector $\mathbf{p}$ in CSA1 is denoted

$$\mathbf{p}_{\mathcal{S}^1} = p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_4. \tag{3.190}$$

CSA1 entities use the subscript $\mathcal{CS}^1$. For example, the embedding of $\mathbf{p}_{\mathcal{S}^1}$ as a CSA1 point $\mathbf{P}_{\mathcal{CS}^1}$ is denoted

$$\mathbf{P}_{\mathcal{CS}^1} = \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1}) \tag{3.191}$$

$$= \mathbf{p}_{\mathcal{S}^1} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^1}^2\mathbf{e}_{\infty 1} + \mathbf{e}_{o1} \tag{3.192}$$

where

$$\mathbf{e}_{\infty 1} = (\mathbf{e}_5 + \mathbf{e}_6) \tag{3.193}$$

$$\mathbf{e}_{o1} = \frac{1}{2}(-\mathbf{e}_5 + \mathbf{e}_6). \tag{3.194}$$

The CSA1 point embedding function has been named $\mathcal{C}_1$. Likewise, a CSA1 surface entity is named $\mathbf{X}_{\mathcal{CS}^1}$. The CSA1 point at the origin $\mathbf{e}_{o1}$ and point at infinity $\mathbf{e}_{\infty 1}$ are named with suffix 1 to indicate their version as being the CSA1 and CSTA1 versions.

Multivectors in the $\mathcal{G}_{0,3}$ SA2 subspace of CSA2 space use the subscript $\mathcal{S}^2$ (e.g., $\mathbf{p}_{\mathcal{S}^2}$). CSA2 entities use the subscript $\mathcal{CS}^2$ (e.g., $\mathbf{X}_{\mathcal{CS}^2}$).

With this notation, the CSA1 unit pseudoscalars are

$$\mathbf{I}_{\mathcal{S}^1} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 \tag{3.195}$$

$$\mathbf{I}_{\mathcal{CS}^1} = \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6 \tag{3.196}$$

and the CSA2 unit pseudoscalars are

$$\mathbf{I}_{\mathcal{S}^2} = \mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10} \tag{3.197}$$

$$\mathbf{I}_{\mathcal{CS}^2} = \mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12}. \tag{3.198}$$

An SA2 vector $\mathbf{p}$ in CSA2 is denoted

$$\mathbf{p}_{\mathcal{S}^2} = p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. \tag{3.199}$$

The CSA2 point embedding is

$$\begin{aligned}\mathbf{P}_{\mathcal{CS}^2} &= \mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) & (3.200)\\ &= \mathbf{p}_{\mathcal{S}^2} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^2}^2\mathbf{e}_{\infty 2} + \mathbf{e}_{o2} & (3.201)\end{aligned}$$

where

$$\begin{aligned}\mathbf{e}_{\infty 2} &= (\mathbf{e}_{11} + \mathbf{e}_{12}) & (3.202)\\ \mathbf{e}_{o2} &= \frac{1}{2}(-\mathbf{e}_{11} + \mathbf{e}_{12}). & (3.203)\end{aligned}$$

The CSA2 point at the origin $\mathbf{e}_{o2}$ and point at infinity $\mathbf{e}_{\infty 2}$ are named with suffix 2 to indicate their version as being the CSA2 and CSTA2 versions.

A versor O (rotor, dilator, translator, or motor) in CSA1 is denoted $O_{\mathcal{CS}^1}$, and in CGA2 is denoted $O_{\mathcal{CS}^2}$.

4 Double Conformal Space Algebra (DCSA)

The $\mathcal{G}_{2,8}$ *Double Conformal Space Algebra* (DCSA) is a doubling of the $\mathcal{G}_{1,4}$ *Conformal Space Algebra* (CSA) (§3) and is the spatial subalgebra of $\mathcal{G}_{4,8}$ *Double Conformal Space-Time Algebra* (DCSTA) (§7).

$\mathcal{G}_{2,8}$ DCSA is very similar to the $\mathcal{G}_{8,2}$ *Double Conformal Geometric Algebra* (DCGA) that is introduced in [7], and this section is directly based on [7]. Compared to DCGA, there are some changes in notation and some changes in signs to account for the changes in signatures. The discussions on the DCSA entities include new details on how the entities can be used within DCSTA. In DCSTA, the spatial DCSA entities gain the ability to be boosted using the DCSTA boost operator (§7.7.3), and their inversions in hyperpseudospheres (§7.3.3) and reflections in hyperplanes (§7.3.2) are also possible.

4.1 DCSA point

4.1.1 DCSA point embedding

The standard DCSA *null* 2-vector *point* entity $\mathbf{P}_{\mathcal{DS}}$ is the embedding of a vector

$$\mathbf{p} = \mathbf{p}_{\mathcal{S}} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \tag{4.1}$$

as

$$\begin{aligned}\mathbf{P}_{\mathcal{DS}} &= \mathcal{D}(\mathbf{p}) & (4.2)\\ &= \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1}) \wedge \mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) & (4.3)\\ &= \mathbf{P}_{\mathcal{CS}^1} \wedge \mathbf{P}_{\mathcal{CS}^2} & (4.4)\end{aligned}$$

where

$$\mathbf{p}_{\mathcal{S}^1} = p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_4 \tag{4.5}$$
$$\mathbf{p}_{\mathcal{S}^2} = p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. \tag{4.6}$$

The DCSA point $\mathbf{P}_{\mathcal{DS}}$ is the wedge (or geometric product) of a CSA1 point $\mathbf{P}_{\mathcal{CS}^1}$ with a CSA2 point $\mathbf{P}_{\mathcal{CS}^2}$ (§3.5). All DCSA points are null 2-vectors, $\mathbf{P}_{\mathcal{DS}}^2 = 0$.

CSA1 and CSA2 points and surface entities can be rotated, translated, and dilated using CSA1 and CSA2 versors for these operations. The wedge of a CSA1 versor with its copy CSA2 versor (rotor, translator, dilator, or motor) creates the DCSA versor on DCSA points and surface entities.

4.1.2 DCSA origin point

The DCSA point at the origin $\mathbf{e}_o$ is defined as

$$\mathbf{e}_o = \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}. \tag{4.7}$$

This is also the DCSTA origin point (§7.2.1.1). The CSA origin point $\mathbf{e}_{o\gamma}$ (§3.1.5) in CSA1 $\mathbf{e}_{o1}$ and CSA2 $\mathbf{e}_{o2}$ (§3.5) are multiplied to form the DCSA origin point $\mathbf{e}_o$.

4.1.3 DCSA infinity point

The DCSA point at infinity $\mathbf{e}_\infty$ is defined as

$$\mathbf{e}_\infty = \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \tag{4.8}$$

This is also the DCSTA infinity point (§7.2.1.2). The CSA infinity point $\mathbf{e}_{\infty\gamma}$ (§3.1.6) in CSA1 $\mathbf{e}_{\infty 1}$ and CSA2 $\mathbf{e}_{\infty 2}$ (§3.5) are multiplied to form the DCSA infinity point $\mathbf{e}_\infty$.

As in CSA, the DCSA origin and infinity points have the inner product

$$\mathbf{e}_\infty \cdot \mathbf{e}_o = -1. \tag{4.9}$$

4.1.4 Distance between two DCSA points

The squared-squared distance d^4 between two DCSA points $\mathbf{P}_{\mathcal{DS}_1}$ and $\mathbf{P}_{\mathcal{DS}_2}$ is

$$\begin{aligned}
d^4 &= -4\mathbf{P}_{\mathcal{DS}_1} \cdot \mathbf{P}_{\mathcal{DS}_2} && (4.10)\\
&= -4(\mathbf{P}_{\mathcal{CS}_1^1} \wedge \mathbf{P}_{\mathcal{CS}_1^2}) \cdot (\mathbf{P}_{\mathcal{CS}_2^1} \wedge \mathbf{P}_{\mathcal{CS}_2^2}) && (4.11)\\
&= -4(\mathbf{P}_{\mathcal{CS}_1^1} \cdot ((\mathbf{P}_{\mathcal{CS}_1^2} \cdot \mathbf{P}_{\mathcal{CS}_2^1})\mathbf{P}_{\mathcal{CS}_2^2} - \mathbf{P}_{\mathcal{CS}_2^1}(\mathbf{P}_{\mathcal{CS}_1^2} \cdot \mathbf{P}_{\mathcal{CS}_2^2}))) && (4.12)\\
&= -4((\mathbf{P}_{\mathcal{CS}_1^2} \cdot \mathbf{P}_{\mathcal{CS}_2^1})(\mathbf{P}_{\mathcal{CS}_1^1} \cdot \mathbf{P}_{\mathcal{CS}_2^2}) - (\mathbf{P}_{\mathcal{CS}_1^1} \cdot \mathbf{P}_{\mathcal{CS}_2^1})(\mathbf{P}_{\mathcal{CS}_1^2} \cdot \mathbf{P}_{\mathcal{CS}_2^2})) && (4.13)\\
&= -4\left((0)(0) - \left(\frac{d^2}{2}\right)\left(\frac{d^2}{2}\right)\right). && (4.14)
\end{aligned}$$

The squared distance d^2 between points is also

$$\begin{aligned}
d^2 &= 2\frac{-\mathbf{P}_{\mathcal{DS}_1} \cdot \mathbf{e}_{\infty 2}}{(\mathbf{P}_{\mathcal{DS}_1} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} \cdot \frac{-\mathbf{P}_{\mathcal{DS}_2} \cdot \mathbf{e}_{\infty 2}}{(\mathbf{P}_{\mathcal{DS}_2} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} && (4.15)\\
&= 2\mathbf{P}_{\mathcal{CS}_1^1} \cdot \mathbf{P}_{\mathcal{CS}_2^1} && (4.16)
\end{aligned}$$

where each DCSA point is contracted and renormalized into a CSA1 point.

4.1.5 DCSA point projection (inverse embedding)

The projection of a DCSA point $\mathbf{P}_{\mathcal{DS}}$ back to an SA1 vector $\mathbf{p}$ is

$$
\begin{aligned}
\mathbf{P}_{\mathcal{CS}^1} &\simeq \mathbf{P}_{\mathcal{DS}}\cdot\mathbf{e}_{\infty 2} && (4.17)\\
\mathbf{p} &= \frac{(\mathbf{P}_{\mathcal{CS}^1}\cdot\mathbf{I}_{\mathcal{S}^1})\mathbf{I}_{\mathcal{S}^1}^{-1}}{-\mathbf{P}_{\mathcal{CS}^1}\cdot\mathbf{e}_{\infty 1}} && (4.18)\\
&= \big(\hat{\mathbf{P}}_{\mathcal{CS}^1}\cdot\mathbf{I}_{\mathcal{S}^1}\big)\mathbf{I}_{\mathcal{S}^1}^{-1}. && (4.19)
\end{aligned}
$$

4.1.6 DCSA point value-extraction elements

The DCSA null 2-vector *point* $\mathbf{T}_{\mathcal{DS}}=\mathcal{D}(\mathbf{t})$ allows for the extraction of more polynomial terms than only the x, y, z, and $\mathbf{t}^2=-(x^2+y^2+z^2)$ terms that the CSA null 1-vector point $\mathbf{T}_{\mathcal{CS}}$ (§3.1) embeds. The terms that can be extracted from a point determine what polynomial equations or entities can be represented as GIPNS entities that test against the point.

When expanded, the DCSA point $\mathbf{T}_{\mathcal{DS}}=\mathcal{D}(\mathbf{t})$ is

$$
\begin{aligned}
\mathbf{T}_{\mathcal{DS}} &= \left(\mathbf{t}_{\mathcal{S}^1}+\frac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 1}+\mathbf{e}_{o1}\right)\wedge\left(\mathbf{t}_{\mathcal{S}^2}+\frac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 2}+\mathbf{e}_{o2}\right) && (4.20)\\
&= \mathbf{t}_{\mathcal{S}^1}\wedge\mathbf{t}_{\mathcal{S}^2}+\mathbf{t}_{\mathcal{S}^1}\wedge\mathbf{e}_{o2}+\mathbf{e}_{o1}\wedge\mathbf{t}_{\mathcal{S}^2}+ && (4.21)\\
&\quad \frac{1}{2}\mathbf{t}^2\mathbf{e}_{\infty 1}\wedge(\mathbf{t}_{\mathcal{S}^2}+\mathbf{e}_{o2})+\frac{1}{2}\mathbf{t}^2(\mathbf{t}_{\mathcal{S}^1}+\mathbf{e}_{o1})\wedge\mathbf{e}_{\infty 2}+\\
&\quad \frac{1}{4}\mathbf{t}^4\mathbf{e}_{\infty}+\mathbf{e}_o
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{t}=\mathbf{t}_{\mathcal{S}^1} &= x\mathbf{e}_2+y\mathbf{e}_3+z\mathbf{e}_4 && (4.22)\\
\mathbf{t}_{\mathcal{S}^2} &= x\mathbf{e}_8+y\mathbf{e}_9+z\mathbf{e}_{10} && (4.23)\\
\mathbf{t}^2 &= -(x^2+y^2+z^2) && (4.24)\\
\mathbf{t}^4 &= x^4+y^4+z^4+2x^2y^2+2y^2z^2+2z^2x^2. && (4.25)
\end{aligned}
$$

Fully expanding, $\mathbf{T}_{\mathcal{DS}}$ is

$$
\begin{aligned}
\mathbf{T}_{\mathcal{DS}} = {} & \frac{1}{2}(x\mathbf{t}^2-x)\mathbf{e}_2\mathbf{e}_{11}+\frac{1}{2}(x\mathbf{t}^2+x)\mathbf{e}_2\mathbf{e}_{12}+\frac{1}{2}(x\mathbf{t}^2-x)\mathbf{e}_5\mathbf{e}_8+\frac{1}{2}(x\mathbf{t}^2+x)\mathbf{e}_6\mathbf{e}_8+ \qquad (4.26)\\
& \frac{1}{2}(y\mathbf{t}^2-y)\mathbf{e}_3\mathbf{e}_{11}+\frac{1}{2}(y\mathbf{t}^2+y)\mathbf{e}_3\mathbf{e}_{12}+\frac{1}{2}(y\mathbf{t}^2-y)\mathbf{e}_5\mathbf{e}_9+\frac{1}{2}(y\mathbf{t}^2+y)\mathbf{e}_6\mathbf{e}_9+\\
& \frac{1}{2}(z\mathbf{t}^2-z)\mathbf{e}_4\mathbf{e}_{11}+\frac{1}{2}(z\mathbf{t}^2+z)\mathbf{e}_4\mathbf{e}_{12}+\frac{1}{2}(z\mathbf{t}^2-z)\mathbf{e}_5\mathbf{e}_{10}+\frac{1}{2}(z\mathbf{t}^2+z)\mathbf{e}_6\mathbf{e}_{10}+\\
& xy\mathbf{e}_2\mathbf{e}_9+xy\mathbf{e}_3\mathbf{e}_8+yz\mathbf{e}_3\mathbf{e}_{10}+yz\mathbf{e}_4\mathbf{e}_9+xz\mathbf{e}_2\mathbf{e}_{10}+xz\mathbf{e}_4\mathbf{e}_8+\\
& x^2\mathbf{e}_2\mathbf{e}_8+y^2\mathbf{e}_3\mathbf{e}_9+z^2\mathbf{e}_4\mathbf{e}_{10}+\\
& \frac{1}{4}(\mathbf{t}^4-1)\mathbf{e}_5\mathbf{e}_{12}+\frac{1}{4}(\mathbf{t}^4-1)\mathbf{e}_6\mathbf{e}_{11}+\\
& \frac{1}{4}(\mathbf{t}^4-2\mathbf{t}^2+1)\mathbf{e}_5\mathbf{e}_{11}+\frac{1}{4}(\mathbf{t}^4+2\mathbf{t}^2+1)\mathbf{e}_6\mathbf{e}_{12}.
\end{aligned}
$$

The vector $\mathbf{t}$, and its DCSA point embedding $\mathbf{T}_{\mathcal{DS}}=\mathcal{D}(\mathbf{t})$, will be used as a *test point* for position on surfaces.

The DCSA 2-vector *extraction elements* T_s are defined as

$$
\begin{aligned}
T_x &= \frac{1}{2}(\mathbf{e}_{\infty 2}\wedge\mathbf{e}_2+\mathbf{e}_8\wedge\mathbf{e}_{\infty 1}) &(4.27)\\
T_y &= \frac{1}{2}(\mathbf{e}_{\infty 2}\wedge\mathbf{e}_3+\mathbf{e}_9\wedge\mathbf{e}_{\infty 1}) &(4.28)\\
T_z &= \frac{1}{2}(\mathbf{e}_{\infty 2}\wedge\mathbf{e}_4+\mathbf{e}_{10}\wedge\mathbf{e}_{\infty 1}) &(4.29)
\end{aligned}
$$

$$
\begin{aligned}
T_{x^2} &= \mathbf{e}_8\wedge\mathbf{e}_2 &(4.30)\\
T_{y^2} &= \mathbf{e}_9\wedge\mathbf{e}_3 &(4.31)\\
T_{z^2} &= \mathbf{e}_{10}\wedge\mathbf{e}_4 &(4.32)
\end{aligned}
$$

$$
\begin{aligned}
T_{xy} &= \frac{1}{2}(\mathbf{e}_9\wedge\mathbf{e}_2+\mathbf{e}_8\wedge\mathbf{e}_3) &(4.33)\\
T_{yz} &= \frac{1}{2}(\mathbf{e}_{10}\wedge\mathbf{e}_3+\mathbf{e}_9\wedge\mathbf{e}_4) &(4.34)\\
T_{zx} &= \frac{1}{2}(\mathbf{e}_8\wedge\mathbf{e}_4+\mathbf{e}_{10}\wedge\mathbf{e}_2) &(4.35)
\end{aligned}
$$

$$
\begin{aligned}
T_{x\boldsymbol{t}^2} &= \mathbf{e}_{o2}\wedge\mathbf{e}_2+\mathbf{e}_8\wedge\mathbf{e}_{o1} &(4.36)\\
T_{y\boldsymbol{t}^2} &= \mathbf{e}_{o2}\wedge\mathbf{e}_3+\mathbf{e}_9\wedge\mathbf{e}_{o1} &(4.37)\\
T_{z\boldsymbol{t}^2} &= \mathbf{e}_{o2}\wedge\mathbf{e}_4+\mathbf{e}_{10}\wedge\mathbf{e}_{o1} &(4.38)
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= -\mathbf{e}_{\infty} &(4.39)\\
T_{\boldsymbol{t}^2} &= \mathbf{e}_{o2}\wedge\mathbf{e}_{\infty 1}+\mathbf{e}_{\infty 2}\wedge\mathbf{e}_{o1} &(4.40)\\
T_{\boldsymbol{t}^4} &= -4\mathbf{e}_o. &(4.41)
\end{aligned}
$$

The DCSA 2-vector *extraction elements* T_s are *inner product extraction operators* on the DCSA point $\mathbf{T}_{\mathcal{DS}}$. The value s can be extracted from point $\mathbf{T}_{\mathcal{DS}}$ as

$$s = T_s\cdot\mathbf{T}_{\mathcal{DS}}=\mathbf{T}_{\mathcal{DS}}\cdot T_s. \tag{4.42}$$

The DCSA extraction operators T_s, which are a subset of the DCSTA extraction operators (§7.2.3) with $\boldsymbol{t}=\boldsymbol{t}_{\mathcal{M}}=\mathbf{t}_{\mathcal{S}}$, are used to define most of the DCSA GIPNS 2-vector surface entities.

In the metric of $\mathcal{G}_{2,8}$ DCSA,

$$\mathbf{t}^2=\mathbf{t}_{\mathcal{S}}^2 = -(x^2+y^2+z^2), \tag{4.43}$$

which has the opposite sign of $\mathbf{t}^2=\mathbf{t}_{\mathcal{E}}^2$ in $\mathcal{G}_{8,2}$ DCGA. The $\mathcal{G}_{2,8}$ DCSA extraction operators $T_{\mathbf{t}^2}$, $T_{x\mathbf{t}^2}$, $T_{y\mathbf{t}^2}$, and $T_{z\mathbf{t}^2}$ each produce the opposite sign compared to the similar extraction operator in $\mathcal{G}_{8,2}$ DCGA. These sign differences require sign changes in the definitions of any entities that include these extraction elements.

Two properties of the extraction elements are

$$\mathbf{e}_{\infty}\cdot T_s = \begin{cases} 0 & : \; T_s\neq T_{\mathbf{t}^4}\\ 4 & : \; T_s=T_{\mathbf{t}^4}\end{cases} \tag{4.44}$$

$$\mathbf{e}_{o}\cdot T_s = \begin{cases} 0 & : \; T_s\neq T_1\\ 1 & : \; T_s=T_1.\end{cases} \tag{4.45}$$

The first property, about the point at infinity $\mathbf{e}_\infty$, has the consequence that all DCSA GIPNS 2-vector surface entities (§4.2.20) without a term in $T_{\mathbf{t}^4}$ are entities having the surface point $\mathbf{e}_\infty$. In particular, the DCSA GIPNS 2-vector ellipsoid surface entity (§4.2.2) is generally considered to be a finite closed surface, yet in DCGA it always has the surface point $\mathbf{e}_\infty$. Other surface entities can also unexpectedly have the surface point $\mathbf{e}_\infty$. This possible problem about $\mathbf{e}_\infty$ will be mentioned again in the section on the DCSA GIPNS 2-vector ellipsoid entity $\mathbf{E}$ (§4.2.2). This possible problem about $\mathbf{e}_\infty$ will also be discussed further in the section on inversions in spheres and on parabolic cyclides (§4.2.20.3). The second property, about the point at the origin $\mathbf{e}_o$, does not pose any known problems.

The spherical inverse surface entity $\mathbf{SES}^\sim$ of any surface entity $\mathbf{E}$ without a term in $T_{\mathbf{t}^4}$ will *always* have the inversion sphere $\mathbf{S}$ (§4.2.3) center point $\mathbf{P}_\mathcal{D}$ as a surface point. The point at infinity $\mathbf{e}_\infty$ always reflects into the inversion sphere center point $\mathbf{P}_{\mathcal{DS}}$, or the reverse. All open surfaces are expected to have the point $\mathbf{e}_\infty$, and their inverse surfaces are expected to have the inversion sphere center point. Unexpectedly, the inverse surface entity of the ellipsoid entity when reflected in a sphere will always have a *singular outlier surface point* at the inversion sphere center point (see Fig. 4.16). A singular outlier point may be invisible in a surface plot.

4.2 DCSA GIPNS entities

Many of the DCSA 2-vector *Geometric Inner Product Null Space* (GIPNS) surface entities are constructed using the value extractions $T_s \cdot \mathbf{T}_{\mathcal{DS}}$ (§4.1.6) from the DCSA point entity (§4.1). The DCSA GIPNS surface entities are the undual surface entities in DCSA since the direct construction of DCSA *Geometric Outer Product Null Space* (GOPNS) surface entities is limited to the wedge of up to four DCSA points, which cannot construct all of the DCSA GOPNS surface entities. The DCSA GIPNS surface entities can be rotated (§4.4.1), dilated (§4.4.2), and translated (§4.4.3) by DCSA versors, and they can be intersected (§4.4.5) with the bi-CSA GIPNS surface entities.

A DCSA test point $\mathbf{T}_{\mathcal{DS}}$ that is on a DCSA GIPNS surface entity $\mathbf{S}$ must satisfy the GIPNS condition

$$\mathbf{T}_{\mathcal{DS}} \cdot \mathbf{S} \;=\; 0. \tag{4.46}$$

The DCSA GIPNS k-vector surface entity $\mathbf{S}$ represents the set $\mathbb{NI}_G(\mathbf{S} \in \mathcal{G}^k_{2,8})$ of all 3D vector test points $\mathbf{t}$ that are surface points

$$\mathbb{NI}_G(\mathbf{S} \in \mathcal{G}^k_{2,8}) \;=\; \left\{\, \mathbf{t} \in \mathcal{G}^1_{0,3} \;:\; (\mathcal{D}(\mathbf{t}) = \mathbf{T}_{\mathcal{DS}}) \cdot \mathbf{S} = 0 \,\right\}. \tag{4.47}$$

4.2.1 DCSA GIPNS toroid

The implicit quartic equation for a circular toroid (torus), which is positioned at the origin and surrounds the z-axis, is

$$\mathbf{t}^4 - 2\mathbf{t}^2(R^2 - r^2) + (R^2 - r^2)^2 - 4R^2(x^2 + y^2) \;=\; 0 \tag{4.48}$$

where

$$\mathbf{t} = \mathbf{t}_\mathcal{S} \;=\; x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 \in \mathcal{G}^1_{0,3} \tag{4.49}$$

is a test point, R is the major radius, and r is the minor radius. The equation is true if the test point $\mathbf{t}$ is on the toroid. The radius R is that of a circle around the origin in the xy-plane. The radius r is that of circles centered on the circle of R and which span the z-axis dimension for $z = \pm r$. The toroid spans $x, y = \pm(R + r)$.

The DCSA GIPNS 2-vector *toroid* surface entity $\mathbf{O}$ is defined as

$$\mathbf{O} = T_{\mathbf{t}^4} - 2(R^2 - r^2)T_{\mathbf{t}^2} + (R^2 - r^2)^2 T_1 - 4R^2(T_{x^2} + T_{y^2}). \tag{4.50}$$

A test DCSA point $\mathbf{T}_{\mathcal{DS}} = \mathcal{D}(\mathbf{t})$ is on the toroid surface represented by $\mathbf{O}$ if $\mathbf{T}_{\mathcal{DS}} \cdot \mathbf{O} = 0$. Using symbolic mathematics software, such as the *Geometric Algebra Module* [2] for *Sympy* [24] by Alan Bromborsky et al., the inner product $\mathbf{T}_{\mathcal{DS}} \cdot \mathbf{O}$ generates the *scalar* implicit surface function of the toroid when $\mathbf{t}$ is a variable symbolic vector. When $\mathbf{t}$ is a specific vector, $\mathbf{T}_{\mathcal{DS}} \cdot \mathbf{O}$ is a test operation on the toroid for the specific point.

We can denote the *DCSA-dual* (§4.4.6) of $\mathbf{O}$ as $\mathbf{O}^{*\mathcal{DS}}$, and define it as

$$\mathbf{O}^{*\mathcal{DS}} = \mathbf{O}/\mathbf{I}_{\mathcal{DS}} = \mathbf{O}\mathbf{I}_{\mathcal{DS}}^{-1} = -\mathbf{O} \cdot \mathbf{I}_{\mathcal{DS}}. \tag{4.51}$$

The DCSA GOPNS 8-vector *toroid* surface entity is $\mathbf{O}^{*\mathcal{DS}}$, where a test point $\mathbf{t}$ on the surface must satisfy the GOPNS condition $\mathbf{T}_{\mathcal{DS}} \wedge \mathbf{O}^{*\mathcal{DS}} = 0$. Since $\mathbf{T}_{\mathcal{DS}}$ is a 2-vector and $\mathbf{O}^{*\mathcal{DS}}$ is an 8-vector, then $\mathbf{T}_{\mathcal{DS}} \wedge \mathbf{O}^{*\mathcal{DS}}$ is the DCSA 10-vector *pseudoscalar* implicit surface function of the toroid when $\mathbf{t}$ is a variable symbolic vector. The *undual* operation (§4.4.6) returns the DCSA GIPNS surface $\mathbf{O} = \mathbf{O}^{*\mathcal{DS}} \cdot \mathbf{I}_{\mathcal{DS}}$. The other DCSA GOPNS surface entities will be discussed later in this paper.

Although the toroid $\mathbf{O}$ is created at the origin and aligned around the z-axis, it can then be rotated, dilated, and translated away from the origin using DCSA versor operations. Like all DCSA GIPNS surface entities, the DCSA GIPNS toroid can be intersected with any bi-CSA GIPNS (2, 4, or 6)-vector surface, which are 2-vector spheres and planes, 4-vector circles and lines, and 6-vector point-pairs.

Since the toroid $\mathbf{O}$ is constructed with an extraction term $T_{\mathbf{t}^4} = -4\mathbf{e}_o$, it is a DCSA *closed-surface* entity that does not include $\mathbf{e}_\infty$ as a surface point, and it can be dilated (§4.4.2) by a zero dilation factor $d = 0$ into $\mathbf{e}_o$. The inverse toroid entity, when reflected in a standard DCSA GIPNS 2-vector sphere (§4.2.3), does not have a singular outlier surface point at the center point of the inversion sphere. The standard DCSA GIPNS 2-vector sphere $\mathbf{S}$ (§4.2.3) also has these closed-surface characteristics, but the ellipsoid $\mathbf{E}$ (§4.2.2) does not.

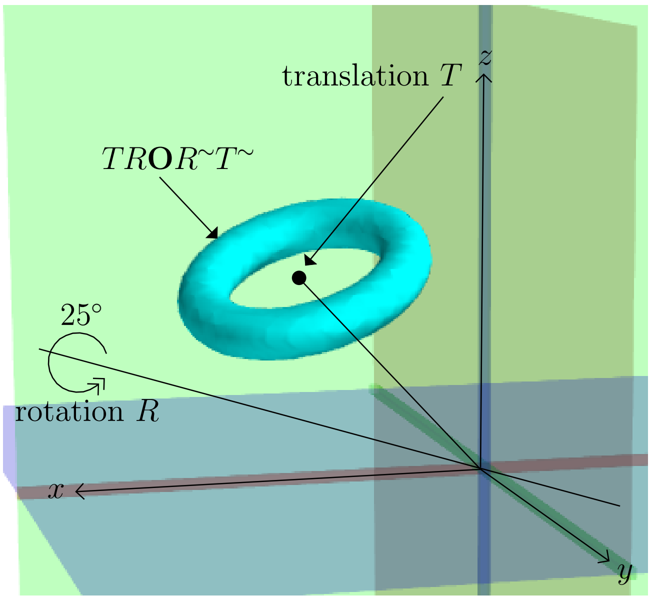

Figure 4.1. DCSA toroid rotated and translated

4.2.2 DCSA GIPNS ellipsoid

The implicit quadric equation for a principal axes-aligned ellipsoid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_x)^2}{r_z^2}-1 \;=\; 0 \tag{4.52}$$

where $\mathbf{p} = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3$ (§2.5) is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c). Expanding the squares, the equation can be written as

$$\frac{-2p_xx}{r_x^2}+\frac{-2p_yy}{r_y^2}+\frac{-2p_zz}{r_z^2}+\left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}\right)+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right) \;=\; 0. \tag{4.53}$$

Using the DCSA point value-extraction elements (§4.1.6), an ellipsoid equation can be constructed. This construction will be similar for the remaining surface entities that follow.

The DCSA GIPNS 2-vector *ellipsoid* surface entity $\mathbf{E}$ is defined as

$$\begin{aligned}\mathbf{E} \;=\; & \frac{-2p_xT_x}{r_x^2}+\frac{-2p_yT_y}{r_y^2}+\frac{-2p_zT_z}{r_z^2}+\\ & \frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}+\frac{T_{z^2}}{r_z^2}+\\ & \left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right)T_1.\end{aligned} \tag{4.54}$$

A DCSA 2-vector point $\mathbf{T}_{\mathcal{DS}} = \mathcal{D}(\mathbf{t})$ is tested against the DCSA 2-vector ellipsoid $\mathbf{E}$ as

$$\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{E}\begin{cases} <0 & : \mathbf{t} \text{ is } \textit{inside} \text{ ellipsoid}\\ =0 & : \mathbf{t} \text{ is } \textit{on} \text{ ellipsoid}\\ >0 & : \mathbf{t} \text{ is } \textit{outside} \text{ ellipsoid.}\end{cases} \tag{4.55}$$

It was first mentioned in Section 4.1.6, on the DCSA point $\mathbf{T}_{\mathcal{DS}}$ and value-extraction operators T_s, that the ellipsoid entity $\mathbf{E}$ has the **possible problem** that it includes the point at infinity $\mathbf{e}_\infty$ as a surface point according to the test just given above. We could *define* the invariant test $\mathbf{e}_\infty \cdot \mathbf{E} = 0$ as an invalid test, or we could accept that $\mathbf{e}_\infty$ is a valid surface point of the particular *DCSA ellipsoid entity* $\mathbf{E}$ but not of ellipsoids in general.

An inverse ellipsoid surface entity, which is an ellipsoid entity $\mathbf{E}$ that has been reflected in a standard DCSA 2-vector sphere $\mathbf{S}$ (§4.2.3) as $\mathbf{SES}^\sim$, will always have a *singular outlier surface point* that is exactly the center point $\mathbf{P}_{\mathcal{DS}}$ of the inversion sphere $\mathbf{S}$ and the test $\mathbf{P}_{\mathcal{DS}} \cdot (\mathbf{SES}^\sim) = 0$ will always hold true. The point $\mathbf{e}_\infty$ on the ellipsoid entity $\mathbf{E}$ is reflected into $\mathbf{P}_{\mathcal{DS}}$. The inverse ellipsoid surface entity $\mathbf{SES}^\sim$ is otherwise a correctly formed surface entity of one of the types that should be expected, which is either a quartic Darboux cyclide (§4.2.20) or a cubic parabolic cyclide (§4.2.20.3). The outlier point is often invisible in surface plots. See Figure 4.16, which shows an ellipsoid reflection that produces a Darboux cyclide.

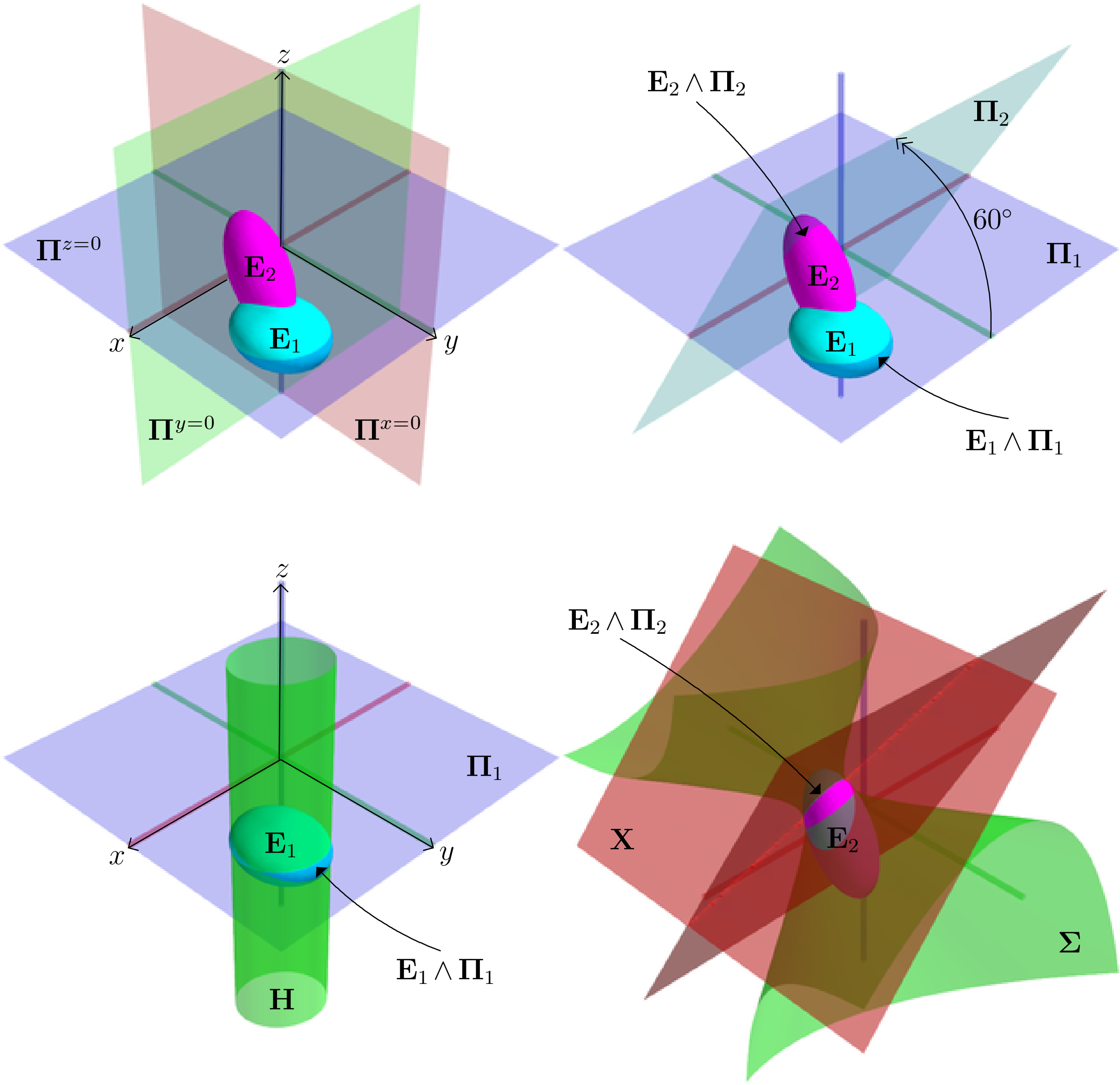

Figure 4.2. DCSA ellipsoids rotated, translated, and intersected with planes

Figure 4.2 shows two ellipsoids that have been rotated (§4.4.1) and translated (§4.4.3) into their intersecting positions using DCSA versor operations. The DCSA GIPNS ellipsoid $\mathbf{E}_1$ ($r_x=4$, $r_y=5$, $r_z=3$) is rotated 25° around the line $\mathbf{n}=\frac{1}{\sqrt{2}}(-\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)$, then rotated 45° around the z-axis, then translated by $\mathbf{d}=10\boldsymbol{\gamma}_1+10\boldsymbol{\gamma}_2$. The DCSA GIPNS ellipsoid $\mathbf{E}_2$ ($p_z=6$, $r_x=2$, $r_y=3$, $r_z=6$) is rotated -35° around the line $\mathbf{n}=\frac{1}{\sqrt{2}}(-\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)$, then rotated 35° around the z-axis, then translated by $\mathbf{d}=10\boldsymbol{\gamma}_1+10\boldsymbol{\gamma}_2$. The ellipsoids intersect in a curved ellipse which, unfortunately, could *not* be represented as an intersection entity.

Although not rigorously proved here, in tests performed by this author, the ellipsoid and all other DCSA entities can be intersected with the standard DCSA sphere (§4.2.3), plane (§4.2.5), line (§4.2.4), and circle (§4.2.6) entities (bi-CSA entities), but DCSA entities cannot be intersected in full generality.

The upper-left image in Figure 4.2 shows the ellipsoids with standard planes drawn. The upper-right image shows the ellipsoids drawn with DCSA GIPNS plane $\boldsymbol{\Pi}_1$ representing the plane $z=0$, and with the DCSA GIPNS plane $\boldsymbol{\Pi}_2$ representing the plane $z=0$

rotated 60° around the x-axis. The lower-left image shows the DCSA GIPNS intersection entity $\mathbf{E}_1 \wedge \mathbf{\Pi}_1$; the green elliptic cylinder $\mathbf{H}$ (§4.2.7) is an intersection entity component and represents the ellipse in which they intersect. The lower-right image shows the DCSA GIPNS intersection entity $\mathbf{E}_2 \wedge \mathbf{\Pi}_2$; the green *hyperboloid of one sheet* $\mathbf{\Sigma}$ (§4.2.11) and the red *non-parallel planes pair* $\mathbf{X}$ (§4.2.16) are intersection entity components which are also coincident and represent the intersection (§4.4.5).

4.2.3 DCSA GIPNS sphere

The standard DCSA GIPNS 2-vector *sphere* $\mathbf{S}$ will be defined as a bi-CSA sphere, *not* the DCSA GIPNS ellipsoid $\mathbf{E}$ (§4.2.2) with equal semi-diameters $r = r_x = r_y = r_z$.

The DCSA GIPNS ellipsoid $\mathbf{E}$ with $r = r_x = r_y = r_z$ can be *reformulated* into the DCSA GIPNS 2-vector *ellipsoid-based sphere* entity $\mathbf{\Theta}$ as

$$\begin{aligned} \mathbf{\Theta} &= -2(p_x T_x + p_y T_y + p_z T_z) + T_{x^2} + T_{y^2} + T_{z^2} + (p_x^2 + p_y^2 + p_z^2 - r^2) T_1 && (4.56) \\ &\simeq (p_x T_x + p_y T_y + p_z T_z) - \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) T_1 - \frac{1}{2}(T_{x^2} + T_{y^2} + T_{z^2}) + \frac{1}{2} r^2 T_1. && (4.57) \end{aligned}$$

Taking $r = 0$ suggests that the sphere $\mathbf{\Theta}$ degenerates into some type of point entity. With $T_1 = -\mathbf{e}_\infty$, the middle term has a familiar CSA point (§3.1.7) form. However, if this were a CSA point, the last term should reduce to $\mathbf{e}_o$, but it does not. The result here is that, the sphere entity $\mathbf{\Theta}$ with $r = 0$ degenerates into a DCSA GIPNS *non-null* 2-vector *point* entity, which is not the standard DCSA *null* 2-vector *point* that we might expect. The DCSA GIPNS ellipsoid $\mathbf{E}$ can be reformulated into a kind of sphere entity $\mathbf{\Theta}$ that degenerates into a kind of non-null point entity when $r = 0$. However, $r = 0$ is *invalid* for an ellipsoid entity $\mathbf{E}$, and only in the limit $r \to 0$ does $\mathbf{E}$ approach a point $\mathbf{\Theta}$ with $r = 0$. We can also form a sphere in another way which *does* degenerate into a standard DCSA point.

The standard DCSA GIPNS 2-vector *sphere* surface entity $\mathbf{S}$, also being called a bi-CSA GIPNS 2-vector sphere, is defined as

$$\mathbf{S} = \mathbf{S}_{\mathcal{DS}} = \mathbf{S}_{\mathcal{CS}^1} \wedge \mathbf{S}_{\mathcal{CS}^2} \tag{4.58}$$

where

$$\begin{aligned} \mathbf{S}_{\mathcal{CS}^1} &= \mathbf{P}_{\mathcal{CS}^1} + \frac{1}{2} r^2 \mathbf{e}_{\infty 1} && (4.59) \\ \mathbf{S}_{\mathcal{CS}^2} &= \mathbf{P}_{\mathcal{CS}^2} + \frac{1}{2} r^2 \mathbf{e}_{\infty 2}. && (4.60) \end{aligned}$$

The CSA1 GIPNS 1-vector sphere $\mathbf{S}_{\mathcal{CS}^1}$ and the CSA2 GIPNS 1-vector sphere $\mathbf{S}_{\mathcal{CS}^2}$ (§3.2.1), both representing the same sphere, with radius r at center position $\mathbf{p} = \mathbf{p}_{\mathcal{S}}$ in 3D anti-Euclidean SA space $\mathcal{S}$ (§2.5), are wedged to form the DCSA or bi-CSA GIPNS sphere $\mathbf{S}$. If $r = 0$, the sphere is degenerated into a DCSA point

$$\mathbf{P}_{\mathcal{DS}} = \mathbf{P}_{\mathcal{CS}^1} \wedge \mathbf{P}_{\mathcal{CS}^2} \tag{4.61}$$

that would represent the center position of the sphere. This form of sphere allows greater consistency, and it can also be intersected with any DCSA GIPNS entity. A sphere that is formed using the DCSA GIPNS ellipsoid can only be intersected with bi-CSA GIPNS entities. In general, the other bi-CSA GIPNS entities for lines (§4.2.4), circles (§4.2.6), and planes (§4.2.5) follow this same pattern, that they are the wedge of the CSA1 and CSA2 copies (§3.5) of the entity.

A DCSA 2-vector point $\mathbf{T}_{\mathcal{DS}} = \mathcal{D}(\mathbf{t})$ is tested against the standard DCSA GIPNS 2-vector sphere $\mathbf{S}$ as

$$2\left(\frac{-\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{e}_{\infty 2}}{(\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{e}_{\infty 2})\cdot\mathbf{e}_{\infty 1}}\right)\cdot\left(\frac{-\mathbf{S}\cdot\mathbf{e}_{\infty 2}}{(\mathbf{S}\cdot\mathbf{e}_{\infty 2})\cdot\mathbf{e}_{\infty 1}}\right)\begin{cases} <0 & : \ \mathbf{t} \text{ is } \textit{inside} \text{ sphere} \\ =0 & : \ \mathbf{t} \text{ is } \textit{on} \text{ sphere} \\ >0 & : \ \mathbf{t} \text{ is } \textit{outside} \text{ sphere} \\ >0 & : \ =d^2, \text{ squared tangent.} \end{cases} \quad (4.62)$$

To determine inside or outside, this incidence test requires the bi-CSA point $\mathbf{T}_{\mathcal{DS}}$ (§4.1) to be contracted into a CSA1 point (§3.1), and the bi-CSA sphere $\mathbf{S}$ to be contracted into a CSA1 sphere (§3.2.1), and both are renormalized. The entity $\mathbf{e}_{\infty 2}$ is both a CSA2 point and a CSA2 sphere of infinite radius, and it serves as the contraction operator on both the point and sphere into CSA1 entities, up to scale. The result is reduced to a CSA1 incidence test. When the test is positive, it is the squared distance d^2 from the point to the sphere along any line tangent to the sphere surface. Similarly for other bi-CSA entities, they can be contracted into CSA1 entities and then all the usual CSA tests are available on them.

4.2.4 DCSA GIPNS line

The DCSA GIPNS 4-vector *line* 1D surface entity $\mathbf{L}$ is defined as

$$\mathbf{L} = \mathbf{L}_{\mathcal{DS}} = \mathbf{L}_{\mathcal{CS}^1} \wedge \mathbf{L}_{\mathcal{CS}^2} \quad (4.63)$$

where

$$\mathbf{L}_{\mathcal{CS}^1} = \mathbf{D}_{\mathcal{S}^1} - (\mathbf{p}_{\mathcal{S}^1}\cdot\mathbf{D}_{\mathcal{S}^1})\mathbf{e}_{\infty 1} \quad (4.64)$$

$$\mathbf{L}_{\mathcal{CS}^2} = \mathbf{D}_{\mathcal{S}^2} - (\mathbf{p}_{\mathcal{S}^2}\cdot\mathbf{D}_{\mathcal{S}^2})\mathbf{e}_{\infty 2}. \quad (4.65)$$

This is the wedge of the line as represented in CSA1 with the same line as represented in CSA2 (§3.2.3). It could also be called a bi-CSA GIPNS line entity. The $\mathbf{D}$ are unit bivectors perpendicular to the line, and $\mathbf{p}$ is any sample point on the line. The SA undual (§2.2) unit vector $\mathbf{d} = -\mathbf{D}\mathbf{I}_{\mathcal{S}}$, or $\mathbf{d}_{\mathcal{S}^1} = -\mathbf{D}_{\mathcal{S}^1}\mathbf{I}_{\mathcal{S}^1}$ and $\mathbf{d}_{\mathcal{S}^2} = -\mathbf{D}_{\mathcal{S}^2}\mathbf{I}_{\mathcal{S}^2}$, is in the direction of the line.

4.2.5 DCSA GIPNS plane

The DCSA GIPNS 2-vector *plane* surface entity $\mathbf{\Pi}$ is defined as

$$\mathbf{\Pi} = \mathbf{\Pi}_{\mathcal{DS}} = \mathbf{\Pi}_{\mathcal{CS}^1} \wedge \mathbf{\Pi}_{\mathcal{CS}^2} \quad (4.66)$$

where

$$\mathbf{\Pi}_{\mathcal{CS}^1} = \mathbf{n}_{\mathcal{S}^1} - d\mathbf{e}_{\infty 1} \quad (4.67)$$

$$\mathbf{\Pi}_{\mathcal{CS}^2} = \mathbf{n}_{\mathcal{S}^2} - d\mathbf{e}_{\infty 2}. \quad (4.68)$$

This is the wedge of the plane as represented in CSA1 with the same plane as represented in CSA2 (§3.2.2). It could also be called a bi-CSA GIPNS plane entity. The vector $\mathbf{n}$ is a unit vector perpendicular (normal) to the plane, and the scalar d is the distance of the plane from the origin.

The DCSA GIPNS 4-vector line $\mathbf{L}$ (§4.2.4) can also be defined as the intersection of two DCSA GIPNS planes as

$$\begin{aligned} \mathbf{L} &= \mathbf{\Pi}_1 \wedge \mathbf{\Pi}_2 & (4.69)\\ &= (\mathbf{n}_{1\mathcal{S}^1} - d_1\mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{1\mathcal{S}^2} - d_1\mathbf{e}_{\infty 2}) \wedge (\mathbf{n}_{2\mathcal{S}^1} - d_2\mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{2\mathcal{S}^2} - d_2\mathbf{e}_{\infty 2}) & (4.70)\\ &= -((\mathbf{n}_{1\mathcal{S}^1} - d_1\mathbf{e}_{\infty 1}) \wedge (\mathbf{n}_{2\mathcal{S}^1} - d_2\mathbf{e}_{\infty 1})) \wedge ((\mathbf{n}_{1\mathcal{S}^2} - d_1\mathbf{e}_{\infty 2}) \wedge (\mathbf{n}_{2\mathcal{S}^2} - d_2\mathbf{e}_{\infty 2})) & (4.71) \end{aligned}$$

$$\begin{aligned}
&\simeq (\mathbf{n}_{1\mathcal{S}^1}\wedge\mathbf{n}_{2\mathcal{S}^1}-(d_2\mathbf{n}_{1\mathcal{S}^1}-d_1\mathbf{n}_{2\mathcal{S}^1})\mathbf{e}_{\infty1})\wedge(\mathbf{n}_{1\mathcal{S}^2}\wedge\mathbf{n}_{2\mathcal{S}^2}-(d_2\mathbf{n}_{1\mathcal{S}^2}-d_1\mathbf{n}_{2\mathcal{S}^2})\mathbf{e}_{\infty2}) &(4.72)\\
&\simeq (\mathbf{D}_{\mathcal{S}^1}-(\mathbf{p}_{\mathcal{S}^1}\cdot\mathbf{D}_{\mathcal{S}^1})\mathbf{e}_{\infty1})\wedge(\mathbf{D}_{\mathcal{S}^2}-(\mathbf{p}_{\mathcal{S}^2}\cdot\mathbf{D}_{\mathcal{S}^2})\mathbf{e}_{\infty2}) &(4.73)\\
&\simeq \mathbf{L}_{\mathcal{CS}^1}\wedge\mathbf{L}_{\mathcal{CS}^2}, &(4.74)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}_{\mathcal{S}^1} &= \mathbf{n}_{1\mathcal{S}^1}\wedge\mathbf{n}_{2\mathcal{S}^1} &(4.75)\\
\mathbf{D}_{\mathcal{S}^2} &= \mathbf{n}_{1\mathcal{S}^2}\wedge\mathbf{n}_{2\mathcal{S}^2} &(4.76)\\
\mathbf{p}_{\mathcal{S}^1}\cdot\mathbf{D}_{\mathcal{S}^1} &= (\mathbf{p}_{\mathcal{S}^1}\cdot\mathbf{n}_{1\mathcal{S}^1})\mathbf{n}_{2\mathcal{S}^1}-(\mathbf{p}_{\mathcal{S}^1}\cdot\mathbf{n}_{2\mathcal{S}^1})\mathbf{n}_{1\mathcal{S}^1} &(4.77)\\
&= -d_1\mathbf{n}_{2\mathcal{S}^1}+d_2\mathbf{n}_{1\mathcal{S}^1} &(4.78)\\
\mathbf{p}_{\mathcal{S}^2}\cdot\mathbf{D}_{\mathcal{S}^2} &= (\mathbf{p}_{\mathcal{S}^2}\cdot\mathbf{n}_{1\mathcal{S}^2})\mathbf{n}_{2\mathcal{S}^2}-(\mathbf{p}_{\mathcal{S}^2}\cdot\mathbf{n}_{2\mathcal{S}^2})\mathbf{n}_{1\mathcal{S}^2} &(4.79)\\
&= -d_1\mathbf{n}_{2\mathcal{S}^2}+d_2\mathbf{n}_{1\mathcal{S}^2} &(4.80)
\end{aligned}$$

such that $\mathbf{p}$ is any point on both planes (the line), and $\mathbf{D}=\mathbf{d}^{*\mathcal{S}}=-\mathbf{d}\mathbf{I}_{\mathcal{S}}^{-1}$ (§2.2) is the unit bivector perpendicular to the line. The unit vector $\mathbf{d}=-\mathbf{D}\mathbf{I}_{\mathcal{S}}$ points in the direction of the line. Other bi-CSA GIPNS entities are formed similarly as the wedge of the entity in CSA1 with the same entity in CSA2.

Some of the subscripting notation may seem confusing. For example, $\mathbf{n}_{1\mathcal{S}^1}$ is the first of the two anti-Euclidean 3D unit vectors in $\mathcal{G}_{0,3}$ SA1 (§2.5), and this could also be denoted as $\mathbf{n}_{\mathcal{S}^1_1}$. Recall that $\mathcal{G}_{0,3}$ SA1 $\mathcal{S}^1$ is the algebra with unit pseudoscalar $\mathbf{I}_{\mathcal{S}^1}=\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$, and it is a subalgebra of $\mathcal{G}_{1,4}$ CSA1 $\mathcal{CS}^1$ with unit pseudoscalar $\mathbf{I}_{\mathcal{CS}^1}=\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6$ (§3.5). The $\mathcal{G}_{1,4}$ CSA2 $\mathcal{CS}^2$ algebra uses notations $\mathbf{n}_{1\mathcal{S}^2}$ or $\mathbf{n}_{\mathcal{S}^2_1}$, where $\mathbf{I}_{\mathcal{S}^2}=\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}$ and $\mathbf{I}_{\mathcal{CS}^2}=\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12}$ (§3.5). The subscripting indicates the index number for multiple entities sharing the same name, and also the algebra in which the entity exists. Finally, $\mathbf{n}_{1\mathcal{S}^1}$ and $\mathbf{n}_{1\mathcal{S}^2}$ have the same index number 1, so they represent the same 3D unit vector $\mathbf{n}$ copied or doubled into the $\mathcal{S}^1$ and $\mathcal{S}^2$ anti-Euclidean subalgebras of $\mathcal{CS}^1$ CSA1 and $\mathcal{CS}^2$ CSA2, respectively.

4.2.6 DCSA GIPNS circle

A circle is the intersection of a sphere and plane. We can intersect a bi-CSA GIPNS 2-vector plane $\mathbf{\Pi}$ (§4.2.5) with either a bi-CSA GIPNS 2-vector sphere $\mathbf{S}$ (§4.2.3) or with a spherical DCSA GIPNS 2-vector ellipsoid $\mathbf{E}$ (§4.2.2) and get two different GIPNS 4-vector circle entities. The first can be intersected again with any other entity, but the latter can only be intersected again with another bi-CSA GIPNS entity.

Intersections (§4.4.5) are limited to an GIPNS intersection entity of maximum grade 8, so up to four 2-vector entities, two 4-vector entities, or a 4-vector entity and two 2-vector GIPNS entities can be intersected, but only one of the intersecting entities can be a DCSA GIPNS 2-vector *Darboux cyclide* (§4.2.20).

As the standard DCSA GIPNS 4-vector *circle* 1D surface entity $\mathbf{C}$, we will define it as the bi-CSA GIPNS circle

$$\begin{aligned}
\mathbf{C}=\mathbf{C}_{\mathcal{DS}} &= \mathbf{S}_{\mathcal{DS}}\wedge\mathbf{\Pi}_{\mathcal{DS}} &(4.81)\\
&= \mathbf{S}_{\mathcal{CS}^1}\wedge\mathbf{S}_{\mathcal{CS}^2}\wedge\mathbf{\Pi}_{\mathcal{CS}^1}\wedge\mathbf{\Pi}_{\mathcal{CS}^2} &(4.82)\\
&= -(\mathbf{S}_{\mathcal{CS}^1}\wedge\mathbf{\Pi}_{\mathcal{CS}^1})\wedge(\mathbf{S}_{\mathcal{CS}^2}\wedge\mathbf{\Pi}_{\mathcal{CS}^2}) &(4.83)\\
&\simeq \mathbf{C}_{\mathcal{CS}^1}\wedge\mathbf{C}_{\mathcal{CS}^2}. &(4.84)
\end{aligned}$$

4.2.7 DCSA GIPNS elliptic cylinder

An axes-aligned elliptic cylinder is the limit of an ellipsoid as one of the semi-diameters approaches ∞. This limit eliminates the terms of the cylinder axis from the implicit ellipsoid equation.

The x-axis aligned cylinder takes $r_x \to \infty$, reducing the ellipsoid equation to

$$\frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - 1 \;=\; 0. \tag{4.85}$$

Similarly, the y-axis and z-axis aligned cylinders are

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - 1 \;=\; 0 \tag{4.86}$$

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - 1 \;=\; 0 \tag{4.87}$$

where $\mathbf{p} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3$ is the position (or shifted origin, or center) of the ellipsoid, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c).

The DCSA GIPNS 2-vector *x,y,z-axis aligned cylinder* surface entities $\mathbf{H}^{||\{x,y,z\}}$ are defined as

$$\mathbf{H}^{||x} \;=\; \frac{-2p_yT_y}{r_y^2} + \frac{-2p_zT_z}{r_z^2} + \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - 1\right)T_1 \tag{4.88}$$

$$\mathbf{H}^{||y} \;=\; \frac{-2p_xT_x}{r_x^2} + \frac{-2p_zT_z}{r_z^2} + \frac{T_{x^2}}{r_x^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z^2}{r_z^2} - 1\right)T_1 \tag{4.89}$$

$$\mathbf{H}^{||z} \;=\; \frac{-2p_xT_x}{r_x^2} + \frac{-2p_yT_y}{r_y^2} + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - 1\right)T_1. \tag{4.90}$$

These elliptic cylinders are created as axes-aligned, but like all DCSA entities, they can be rotated, dilated, and translated using DCSA versor operations (§4.4).

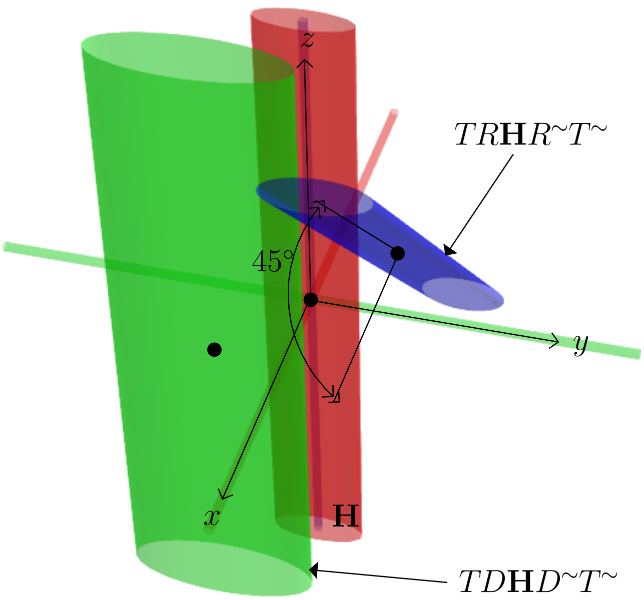

Figure 4.3. DCSA elliptic cylinders

Figure 4.3 shows a red DCSA GIPNS z-axis aligned elliptic cylinder $\mathbf{H}$ at the origin with semi-diameters $r_x = 1$ and $r_y = 3$. The green cylinder is the red cylinder dilated by factor 2 and translated $5\boldsymbol{\gamma}_1 - 5\boldsymbol{\gamma}_2$ using DCSA versors. The blue cylinder is the red cylinder rotated 45° around the y-axis and translated $-5\boldsymbol{\gamma}_1 + 5\boldsymbol{\gamma}_2$.

4.2.8 DCSA GIPNS elliptic cone

An axis-aligned elliptic cone is an axis-aligned cylinder that is linearly scaled along the axis.

The implicit quadric equation for an x-axis aligned cone is

$$\frac{(y-p_y)^2}{r_y^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(x-p_x)^2}{r_x^2} = 0. \tag{4.91}$$

where $\mathbf{p} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3$ is the position (or shifted origin, or center) of the cone apex, and r_x, r_y, r_z are the semi-diameters (often denoted a, b, c) of the ellipsoid upon which the cone is based. When

$$\frac{(x-p_x)^2}{r_x^2} = 1 \tag{4.92}$$

the cross section of the cone is the size of the similar cylinder. When $x = p_x$ the cross section of the cone is degenerated into the cone apex point.

Similarly, the implicit equations for y-axis and z-axis aligned cones are

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(z-p_z)^2}{r_z^2} - \frac{(y-p_y)^2}{r_y^2} = 0 \tag{4.93}$$

$$\frac{(x-p_x)^2}{r_x^2} + \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} = 0. \tag{4.94}$$

The GIPNS cone entities are constructed similarly to the ellipsoid and cylinder entities.

The DCSA GIPNS 2-vector *{x,y,z}-axis aligned elliptic cone* surface entities $\mathbf{K}^{||\{x,y,z\}}$ are defined as

$$\mathbf{K}^{||x} = 2\left(\frac{p_xT_x}{r_x^2} - \frac{p_yT_y}{r_y^2} - \frac{p_zT_z}{r_z^2}\right) - \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2}\right)T_1 \tag{4.95}$$

$$\mathbf{K}^{||y} = 2\left(\frac{p_yT_y}{r_y^2} - \frac{p_zT_z}{r_z^2} - \frac{p_xT_x}{r_x^2}\right) + \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} + \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z^2}{r_z^2}\right)T_1 \tag{4.96}$$

$$\mathbf{K}^{||z} = 2\left(\frac{p_zT_z}{r_z^2} - \frac{p_yT_y}{r_y^2} - \frac{p_xT_x}{r_x^2}\right) + \frac{T_{x^2}}{r_x^2} + \frac{T_{y^2}}{r_y^2} - \frac{T_{z^2}}{r_z^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2}\right)T_1. \tag{4.97}$$

These elliptic cones are created as axes-aligned, but they can be rotated, dilated, and translated using DCSA versor operations (§4.4). All the DCSA surfaces can have general position, but we initially define them in axes-aligned position for simplicity. Defining the surfaces in general position may be possible if the value-extraction operations T_{xy}, T_{yz}, and T_{zx} are employed.

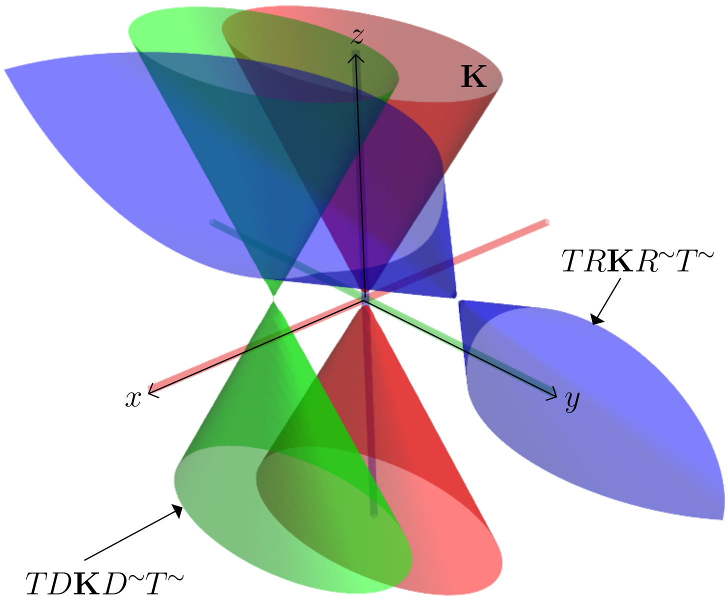

Figure 4.4. DCSA elliptic cones

Figure 4.4 shows some DCSA GIPNS cones positioned and transformed similar to the elliptic cylinders of Figure 4.3. The dilation of a cone does not change the cone shape, but it does dilate the cone center position to effectively translate a cone that is not initially at the origin to be further from the origin by the dilation factor.

4.2.9 DCSA GIPNS elliptic paraboloid

The elliptic paraboloid has a cone-like shape that opens up or down. The other paraboloid that would open the other way is imaginary with no real solution points.

The implicit quadric equation of a z-axis aligned elliptic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(z-p_z)}{r_z} \;=\; 0. \tag{4.98}$$

The surface opens up the z-axis for $r_z>0$, and opens down the z-axis for $r_z<0$. Similar equations for x-axis and y-axis aligned elliptic paraboloids are

$$\frac{(z-p_z)^2}{r_z^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(x-p_x)}{r_x} \;=\; 0 \tag{4.99}$$

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(z-p_z)^2}{r_z^2}-\frac{(y-p_y)}{r_y} \;=\; 0. \tag{4.100}$$

Expanding the squares, the z-axis aligned equation is

$$\frac{-2p_xx}{r_x^2}+\frac{-2p_yy}{r_y^2}+\frac{-z}{r_z}+\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z}{r_z}\right) \;=\; 0 \tag{4.101}$$

and the x-axis and y-axis aligned equations are

$$\frac{-2p_z z}{r_z^2}+\frac{-2p_y y}{r_y^2}+\frac{-x}{r_x}+\frac{z^2}{r_z^2}+\frac{y^2}{r_y^2}+\left(\frac{p_z^2}{r_z^2}+\frac{p_y^2}{r_y^2}+\frac{p_x}{r_x}\right) = 0 \tag{4.102}$$

$$\frac{-2p_x x}{r_x^2}+\frac{-2p_z z}{r_z^2}+\frac{-y}{r_y}+\frac{x^2}{r_x^2}+\frac{z^2}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_z^2}{r_z^2}+\frac{p_y}{r_y}\right) = 0. \tag{4.103}$$

The DCSA GIPNS 2-vector *{x,y,z}-axis aligned elliptic paraboloid* surface entities $\mathbf{V}^{||\{x,y,z\}}$ are defined as

$$\mathbf{V}^{||x} = \frac{-2p_z T_z}{r_z^2}+\frac{-2p_y T_y}{r_y^2}+\frac{-T_x}{r_x}+\frac{T_{z^2}}{r_z^2}+\frac{T_{y^2}}{r_y^2}+\left(\frac{p_z^2}{r_z^2}+\frac{p_y^2}{r_y^2}+\frac{p_x}{r_x}\right)T_1 \tag{4.104}$$

$$\mathbf{V}^{||y} = \frac{-2p_x T_x}{r_x^2}+\frac{-2p_z T_z}{r_z^2}+\frac{-T_y}{r_y}+\frac{T_{x^2}}{r_x^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_z^2}{r_z^2}+\frac{p_y}{r_y}\right)T_1 \tag{4.105}$$

$$\mathbf{V}^{||z} \quad \frac{-2p_x T_x}{r_x^2}+\frac{-2p_y T_y}{r_y^2}+\frac{-T_z}{r_z}+\frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}+\frac{p_z}{r_z}\right)T_1. \tag{4.106}$$

A DCSA 2-vector point $\mathbf{T}_{\mathcal{DS}}=\mathcal{D}(\mathbf{t})$ is tested against the DCSA 2-vector paraboloid $\mathbf{V}$ as

$$\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{V}\begin{cases} <0 & : \ \mathbf{t} \text{ is } \textit{inside} \text{ paraboloid} \\ =0 & : \ \mathbf{t} \text{ is } \textit{on} \text{ paraboloid} \\ >0 & : \ \mathbf{t} \text{ is } \textit{outside} \text{ paraboloid.} \end{cases} \tag{4.107}$$

This is similar to the ellipsoid incidence test, and this test is similar for many of the surfaces.

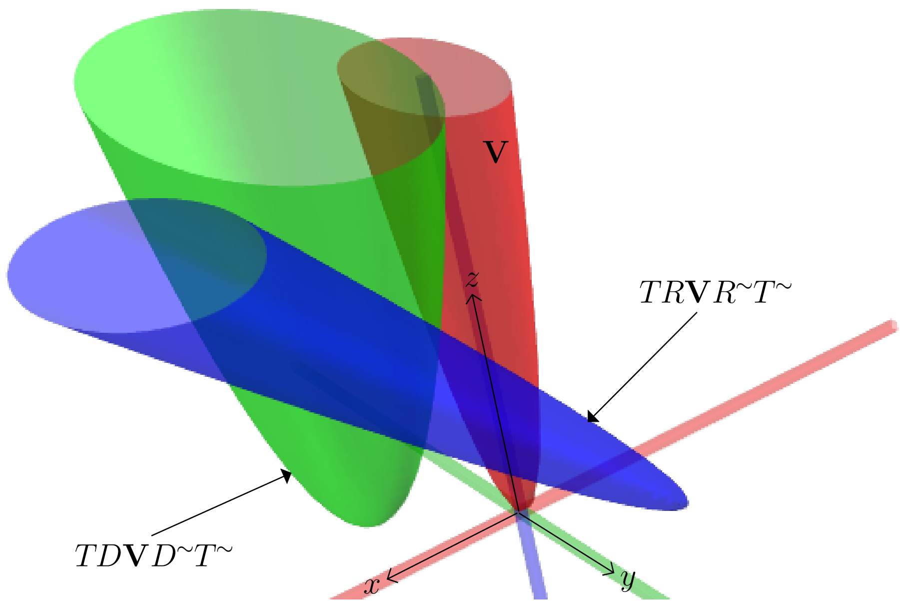

Figure 4.5. DCSA elliptic paraboloids

4.2.10 DCSA GIPNS hyperbolic paraboloid

The hyperbolic paraboloid has a saddle shape. The saddle can be mounted or aligned on a *saddle* axis with another axis chosen as the *up* axis. The third axis may be called the *straddle* axis.

The implicit quadric equation of a hyperbolic paraboloid is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} \;=\; 0. \tag{4.108}$$

This particular form of the equation has saddle x-axis, straddle y-axis, and up z-axis for $r_z > 0$ or up negative z-axis for $r_z < 0$. By its similarity to the z-axis aligned elliptic paraboloid with the elliptic y-axis inverted, this particular form can be seen as z-axis aligned. Other forms can be made by transposing axes, or by rotation around diagonal lines using DCSA rotor (§4.4.1) operations.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{-z}{r_z} + \frac{x^2}{r_x^2} + \frac{-y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) \;=\; 0. \tag{4.109}$$

The DCSA GIPNS 2-vector *z-axis aligned hyperbolic paraboloid* surface entity $\mathbf{M}$ is defined as

$$\mathbf{M} \;=\; \frac{-2p_x T_x}{r_x^2} + \frac{2p_y T_y}{r_y^2} + \frac{-T_z}{r_z} + \frac{T_{x^2}}{r_x^2} + \frac{-T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) T_1. \tag{4.110}$$

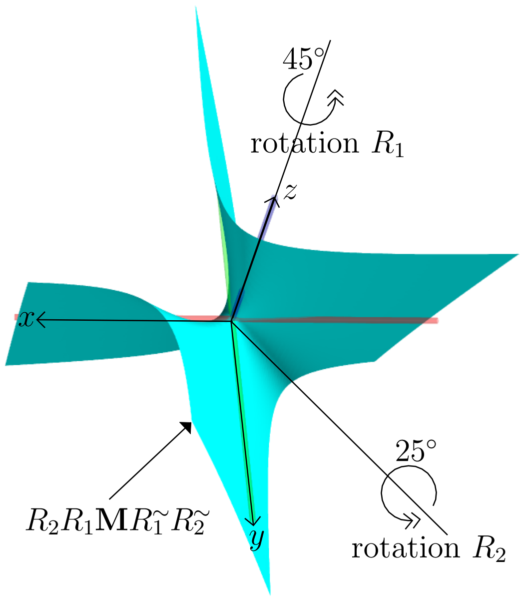

Figure 4.6. DCSA hyperbolic paraboloid rotated twice

Figure 4.6 shows the hyperbolic paraboloid entity $\mathbf{M}$, which is centered on the origin with parameters $r_x = r_y = r_z = 1$, and which was initially z-axis aligned. It was then rotated twice. The first rotation was 45° around the blue z-axis, pointing nearly out of the page. The second rotation was 25° around the line $\mathbf{n} = \frac{1}{\sqrt{2}}(-\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2)$ pointing toward the lower-right of the page. The rotations follow the right-hand rule on a right-handed axes model.

4.2.11 DCSA GIPNS hyperboloid of one sheet

The hyperboloid of one sheet has a shape that is similar to an hourglass which continues to open both upward and downward. The implicit quadric equation is

$$\frac{(x-p_x)^2}{r_x^2}+\frac{(y-p_y)^2}{r_y^2}-\frac{(z-p_z)^2}{r_z^2}-1 \;=\; 0. \tag{4.111}$$

This particular form opens up and down the z-axis. Planes parallel to the z-axis cut hyperbola sections. Planes perpendicular to the z-axis cut ellipse sections. At $z=p_z$, the ellipse section has a minimum size of the similar cylinder. Other forms can be made by transposing axes, or by rotation around diagonal lines using DCSA rotor operations (§4.4.1).

Expanding the squares, the equation is

$$\frac{-2p_xx}{r_x^2}+\frac{-2p_yy}{r_y^2}+\frac{2p_zz}{r_z^2}+\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{-z^2}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}-\frac{p_z^2}{r_z^2}-1\right) \;=\; 0. \tag{4.112}$$

The DCSA GIPNS 2-vector *z-axis aligned hyperboloid of one sheet* surface entity $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} \;=\; 2\left(\frac{p_zT_z}{r_z^2}-\frac{p_xT_x}{r_x^2}-\frac{p_yT_y}{r_y^2}\right)+\frac{T_{x^2}}{r_x^2}+\frac{T_{y^2}}{r_y^2}-\frac{T_{z^2}}{r_z^2}+\left(\frac{p_x^2}{r_x^2}+\frac{p_y^2}{r_y^2}-\frac{p_z^2}{r_z^2}-1\right)T_1. \tag{4.113}$$

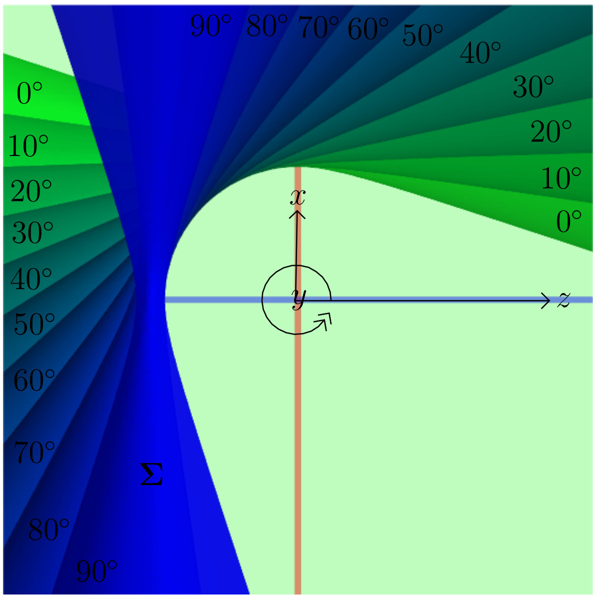

Figure 4.7. Rotation of DCSA hyperboloid of one sheet

Figure 4.7 is an orthographic (parallel projection) view from above the zx-plane that shows the hyperboloid $\mathbf{\Sigma}$ with $r_x=1$, $r_y=2$, $r_z=3$, initially with green color, positioned at $p_x=10$, and aligned up and down the z-axis. It is then rotated using a DCSA rotor R (§4.4.1) by 90° in 10° steps as its color fades to blue, with final position at $p_z=-10$ and aligned up and down the x-axis. The rotation is counter-clockwise around the y-axis coming out of the page on a right-handed system of axes. The x-axis is red and positive up, the y-axis is green (not visible), and the z-axis is blue and positive to the right. The axes are drawn by rendering thin elliptic cylinder entities (§4.2.7). The right-hand rule, holding the y-axis, provides orientation for this rotation. The hyperboloid is rotated about the origin, around the y-axis, as a rigid body of points. In the symbolic computer algebra system (CAS) *Sympy* [24], the hyperboloid equation itself, as a DCSA entity, was rotated symbolically and graphed at each step using the *MayaVi* [21] data visualization software.

4.2.12 DCSA GIPNS hyperboloid of two sheets

The hyperboloid of two sheets has the shapes of two separate hyperbolic dishes; one opens upward, and the other one opens downward. The shape is like an hourglass that is pinched closed and the two halves are also separated by some distance. The implicit quadric equation is

$$-\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2}+\frac{(z-p_z)^2}{r_z^2}-1 \;=\; 0. \tag{4.114}$$

This particular form has the two dishes opening up and down the z-axis. The dishes are separated by distance $2r_z$ centered at p_z. At $|z-p_z|=\sqrt{2}r_z$, the sections perpendicular to the z-axis are the size of the similar cylinder.

Expanding the squares, the equation is

$$\frac{2p_xx}{r_x^2}+\frac{2p_yy}{r_y^2}-\frac{2p_zz}{r_z^2}-\frac{x^2}{r_x^2}-\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2}+\left(\frac{-p_x^2}{r_x^2}+\frac{-p_y^2}{r_y^2}+\frac{p_z^2}{r_z^2}-1\right) \;=\; 0. \tag{4.115}$$

The DCSA GIPNS 2-vector *z-axis aligned hyperboloid of two sheets* surface entity $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi} \;=\; 2\left(\frac{p_xT_x}{r_x^2}+\frac{p_yT_y}{r_y^2}-\frac{p_zT_z}{r_z^2}\right)-\frac{T_{x^2}}{r_x^2}-\frac{T_{y^2}}{r_y^2}+\frac{T_{z^2}}{r_z^2}+\left(\frac{p_z^2}{r_z^2}-\frac{p_x^2}{r_x^2}-\frac{p_y^2}{r_y^2}-1\right)T_1. \tag{4.116}$$

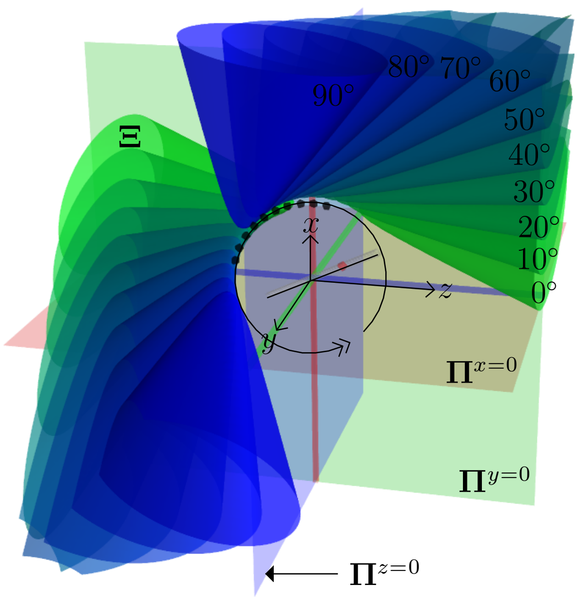

Figure 4.8. Rotation of DCSA hyperboloid of two sheets

Figure 4.8 shows a perspective view of the hyperboloid of two sheets $\mathbf{\Xi}$ initially with green color, centered at $p_x = 5$, $p_y = -5$, and with semi-diameters $r_x = 1$, $r_y = 2$, $r_z = 3$. The black dots (small sphere entities) are the center positions as the surface is rotated around the white line through the origin and the red point $5\boldsymbol{\gamma}_1 + 10\boldsymbol{\gamma}_2 + 5\boldsymbol{\gamma}_3$. The rotation is by 90° in 10° steps until it reaches the position of the blue surface. The first black dot is on the xy-plane (blue plane), and then the black dots go under the blue plane along an arc directly around the axis of rotation. The surface is carried along as a rigid body by the rotation using a DCSA rotor operation (§4.4.1). The symbolic CAS *Sympy* was used for each rotation step, where an exact symbolic equation of the hyperboloid was generated by the rotated entity and graphed using *MayaVi* data visualization software.

4.2.13 DCSA GIPNS parabolic cylinder

The implicit quadric equation for the z-axis aligned parabolic cylinder is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)}{r_y} \;=\; 0. \tag{4.117}$$

The z coordinate is free, which creates a type of cylinder with parabolic sections that open up the y-axis for $r_y > 0$, and open down the y-axis for $r_y < 0$. The similar equations for x-axis and y-axis aligned parabolic cylinders are

$$\frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)}{r_z} \;=\; 0 \tag{4.118}$$

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(z-p_z)}{r_z} \;=\; 0 \tag{4.119}$$

with parabolas that open up or down the z-axis. Other forms can be made by transpositions or by using DCSA versor operations (§4.4).

Expanding the squares, the equations are

$$\frac{-2p_x x}{r_x^2} - \frac{y}{r_y} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right) \;=\; 0 \tag{4.120}$$

$$\frac{-2p_y y}{r_y^2} - \frac{z}{r_z} + \frac{y^2}{r_y^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) \;=\; 0 \tag{4.121}$$

$$\frac{-2p_x x}{r_x^2} - \frac{z}{r_z} + \frac{x^2}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right) \;=\; 0. \tag{4.122}$$

The DCSA GIPNS 2-vector *{x,y,z}-axis aligned parabolic cylinder* surface entities $\mathbf{B}^{||\{x,y,z\}}$ are defined as

$$\mathbf{B}^{||x} \;=\; \frac{-2p_y T_y}{r_y^2} - \frac{T_z}{r_z} + \frac{T_{y^2}}{r_y^2} + \left(\frac{p_y^2}{r_y^2} + \frac{p_z}{r_z}\right) T_1 \tag{4.123}$$

$$\mathbf{B}^{||y} \;=\; \frac{-2p_x T_x}{r_x^2} - \frac{T_z}{r_z} + \frac{T_{x^2}}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_z}{r_z}\right) T_1 \tag{4.124}$$

$$\mathbf{B}^{||z} \;=\; \frac{-2p_x T_x}{r_x^2} - \frac{T_y}{r_y} + \frac{T_{x^2}}{r_x^2} + \left(\frac{p_x^2}{r_x^2} + \frac{p_y}{r_y}\right) T_1. \tag{4.125}$$

These are created as axes-aligned surfaces, but can be rotated, dilated, and translated using DCSA versor operations (§4.4).

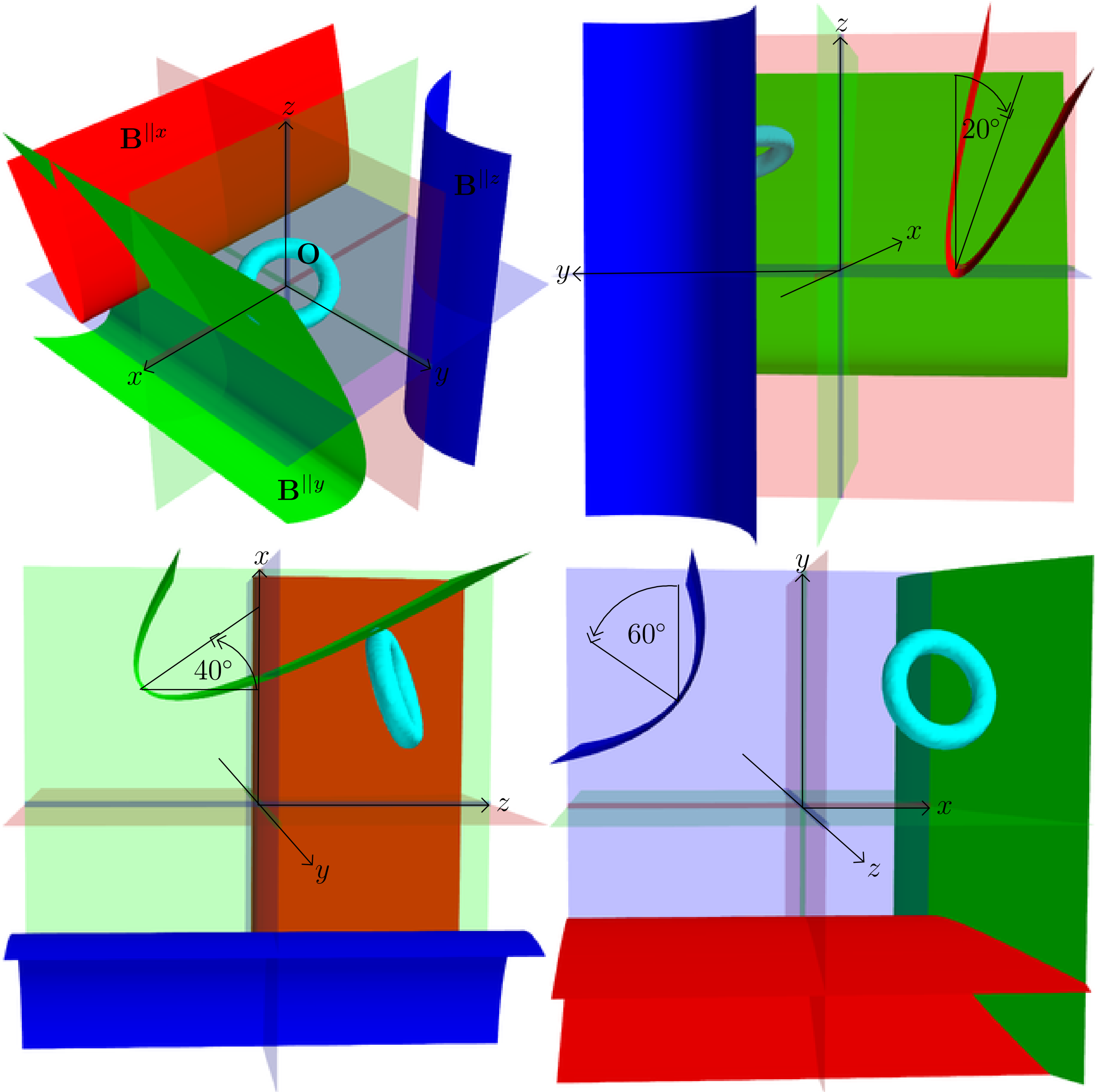

Figure 4.9. DCSA parabolic cylinders and toroid rotated and translated

Figure 4.9 shows multiple perspective views of the DCSA GIPNS 2-vector parabolic cylinders and toroid (§4.2.1) surface entities rendered together in one scene. The red cylinder is x-axis aligned, $r_y = 1$, $r_z = 1$, rotated 20° around the x-axis, and then translated by $\mathbf{d} = -10\boldsymbol{\gamma}_2$ from the origin. The green cylinder is y-axis aligned, $r_x = 2$, $r_z = 1$, rotated 40° around the y-axis, and then translated by $\mathbf{d} = 10\boldsymbol{\gamma}_1 - 10\boldsymbol{\gamma}_3$ from the origin. The blue cylinder is z-axis aligned, $r_x = 4$, $r_y = 1$, rotated 60° around the z-axis, and then translated by $\mathbf{d} = -10\boldsymbol{\gamma}_1 + 10\boldsymbol{\gamma}_2$ from the origin. The toroid, with $R = 4$ and $r = 1$, is rotated 25° around the axis $\mathbf{n} = \frac{1}{\sqrt{2}}(-\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2)$, and then translated by $\mathbf{d} = 10\boldsymbol{\gamma}_1 + 10\boldsymbol{\gamma}_2 + 10\boldsymbol{\gamma}_3$ from the origin. The rotations follow the right-hand rule on right-handed axes. The rotation-translations were performed as compositions of DCSA rotors (§4.4.1) and translators

(§4.4.3). Symbolic CAS *Sympy* was used to generate exact equations of the transformed entities, which were then graphed using the *MayaVi* data visualization software.

4.2.14 DCGA GIPNS hyperbolic cylinder

The implicit quadric equation for the z-axis aligned hyperbolic cylinder is

$$\frac{(x-p_x)^2}{r_x^2} - \frac{(y-p_y)^2}{r_y^2} - 1 = 0. \tag{4.126}$$

The z coordinate is free, which creates a type of cylinder with hyperbolic sections that open up and down the x-axis. The hyperbola branches are separated by distance $2r_x$ centered at $\mathbf{p} = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3$. The asymptotes are the lines

$$(y-p_y) = \pm\frac{r_y}{r_x}(x-p_x) \tag{4.127}$$

through (p_x, p_y), where in the limit as $x \to \pm\infty$ the -1 becomes insignificant.

The similar equations for x-axis and y-axis aligned hyperbolic cylinders are

$$\frac{(y-p_y)^2}{r_y^2} - \frac{(z-p_z)^2}{r_z^2} - 1 = 0 \tag{4.128}$$

$$\frac{(z-p_z)^2}{r_z^2} - \frac{(x-p_x)^2}{r_x^2} - 1 = 0 \tag{4.129}$$

with hyperbolas that open up and down the y-axis or z-axis. Other forms can be made by transpositions or by using DCSA versor operations (§4.4).

Expanding the squares, the equations for x, y, z-aligned hyperbolic cylinders are

$$\frac{-2p_y y}{r_y^2} + \frac{2p_z z}{r_z^2} + \frac{y^2}{r_y^2} - \frac{z^2}{r_z^2} + \left(\frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right) = 0 \tag{4.130}$$

$$\frac{-2p_z z}{r_z^2} + \frac{2p_x x}{r_x^2} + \frac{z^2}{r_z^2} - \frac{x^2}{r_x^2} + \left(\frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2} - 1\right) = 0 \tag{4.131}$$

$$\frac{-2p_x x}{r_x^2} + \frac{2p_y y}{r_y^2} + \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} - 1\right) = 0. \tag{4.132}$$

The DCSA GIPNS 2-vector *{x,y,z}-axis aligned hyperbolic cylinder* surface entities $\mathbf{J}^{||\{x,y,z\}}$ are defined as

$$\mathbf{J}^{||x} = \frac{-2p_y T_y}{r_y^2} + \frac{2p_z T_z}{r_z^2} + \frac{T_{y^2}}{r_y^2} - \frac{T_{z^2}}{r_z^2} + \left(\frac{p_y^2}{r_y^2} - \frac{p_z^2}{r_z^2} - 1\right)T_1 \tag{4.133}$$

$$\mathbf{J}^{||y} = \frac{-2p_z T_z}{r_z^2} + \frac{2p_x T_x}{r_x^2} + \frac{T_{z^2}}{r_z^2} - \frac{T_{x^2}}{r_x^2} + \left(\frac{p_z^2}{r_z^2} - \frac{p_x^2}{r_x^2} - 1\right)T_1 \tag{4.134}$$

$$\mathbf{J}^{||z} = \frac{-2p_x T_x}{r_x^2} + \frac{2p_y T_y}{r_y^2} + \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2} - 1\right)T_1. \tag{4.135}$$

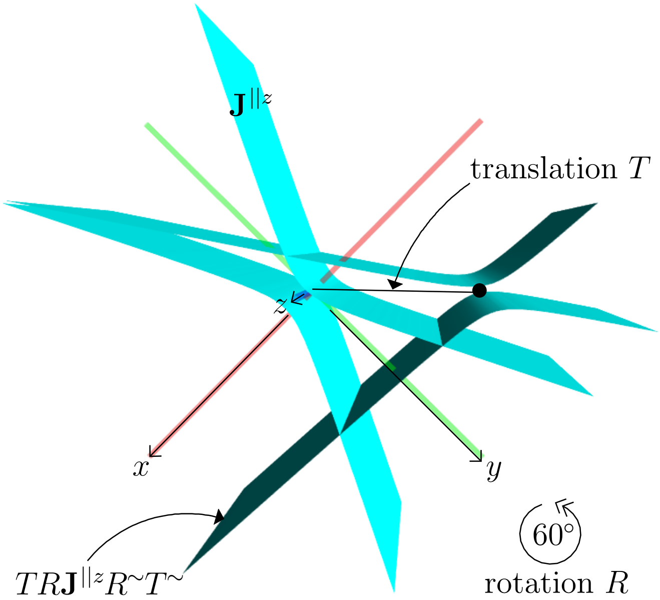

Figure 4.10. DCSA hyperbolic cylinder rotated and translated

Figure 4.10 shows the z-axis aligned hyperbolic cylinder, with initial parameters $p_x = 0$, $p_y = 0$, $r_x = 1$, and $r_y = 2$. The second rendering of it is rotated 60° around the z-axis and then translated by $\mathbf{d} = -10\boldsymbol{\gamma}_1 + 10\boldsymbol{\gamma}_2$ using a composition of DCGA rotor R (§4.4.1) and translator T (§4.4.3) operations.

4.2.15 DCSA GIPNS parallel planes pair

Parallel pairs of axes-aligned planes are represented by the simple quadratic equations in one variable

$$(x - p_{x1})(x - p_{x2}) = 0 \tag{4.136}$$
$$(y - p_{y1})(y - p_{y2}) = 0 \tag{4.137}$$
$$(z - p_{z1})(z - p_{z2}) = 0. \tag{4.138}$$

Each solution is a plane. Expanding the equations gives

$$x^2 - (p_{x1} + p_{x2})x + p_{x1}p_{x2} = 0 \tag{4.139}$$
$$y^2 - (p_{y1} + p_{y2})y + p_{y1}p_{y2} = 0 \tag{4.140}$$
$$z^2 - (p_{z1} + p_{z2})z + p_{z1}p_{z2} = 0. \tag{4.141}$$

The DCSA GIPNS 2-vector *parallel {x,y,z}-planes pair* entities $\mathbf{\Pi}^{\perp\{x,y,z\}}$ are defined as

$$\mathbf{\Pi}^{\perp x} = T_{x^2} - (p_{x1} + p_{x2})T_x + p_{x1}p_{x2}T_1 \tag{4.142}$$
$$\mathbf{\Pi}^{\perp y} = T_{y^2} - (p_{y1} + p_{y2})T_y + p_{y1}p_{y2}T_1 \tag{4.143}$$
$$\mathbf{\Pi}^{\perp z} = T_{z^2} - (p_{z1} + p_{z2})T_z + p_{z1}p_{z2}T_1. \tag{4.144}$$

These surfaces can also be described as being types of cylinders with cross sections being two parallel lines.

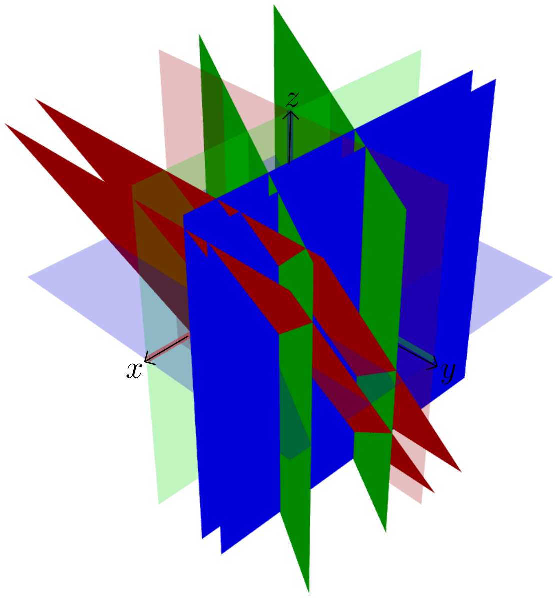

Figure 4.11. DCSA parallel planes pairs rotated

Figure 4.11 shows the DCSA GIPNS parallel planes pair entities rotated using DCSA rotor operations (§4.4.1). The red planes pair is initially perpendicular to the x-axis through points $p_{x1}=4$ and $p_{x2}=8$, then it is rotated 30° around the y-axis. The green planes pair is initially perpendicular to the y-axis through points $p_{y1}=-5$ and $p_{y2}=5$, then it is rotated 60° around the z-axis. The blue planes pair is initially perpendicular to the z-axis through points $p_{z1}=-10$ and $p_{z2}=-7$, then it is rotated 90° around the x-axis until it is perpendicular to the y-axis through the points $p_{y1}=10$ and $p_{y2}=7$.

4.2.16 DCSA GIPNS non-parallel planes pair

The implicit quadric equation for a pair of intersecting, non-parallel planes that are parallel to the z-axis is

$$\frac{(x-p_x)^2}{r_x^2}-\frac{(y-p_y)^2}{r_y^2} = 0. \tag{4.145}$$

This equation can be written as

$$(y-p_y) = \pm\frac{r_y}{r_x}(x-p_x) \tag{4.146}$$

with the z coordinate free to range. This surface can also be described as a kind of cylinder with a cross section in plane z that is two lines with slopes $\pm\frac{r_y}{r_x}$ intersecting at $\mathbf{p}=p_x\gamma_1+p_y\gamma_2+z\gamma_3$.

Expanding the squares, the equation is

$$\frac{-2p_x x}{r_x^2}+\frac{2p_y y}{r_y^2}+\frac{x^2}{r_x^2}-\frac{y^2}{r_y^2}+\left(\frac{p_x^2}{r_x^2}-\frac{p_y^2}{r_y^2}\right) = 0. \tag{4.147}$$

The DCSA GIPNS 2-vector *{x,y,z}-axis aligned non-parallel planes pair* entities $\mathbf{X}^{||\{x,y,z\}}$ are defined as

$$\mathbf{X}^{||x} = \frac{-2p_yT_y}{r_y^2}+\frac{2p_zT_z}{r_z^2}+\frac{T_{y^2}}{r_y^2}-\frac{T_{z^2}}{r_z^2}+\left(\frac{p_y^2}{r_y^2}-\frac{p_z^2}{r_z^2}\right)T_1 \tag{4.148}$$

$$\mathbf{X}^{||y} = \frac{-2p_zT_z}{r_z^2}+\frac{2p_xT_x}{r_x^2}+\frac{T_{z^2}}{r_z^2}-\frac{T_{x^2}}{r_x^2}+\left(\frac{p_z^2}{r_z^2}-\frac{p_x^2}{r_x^2}\right)T_1 \tag{4.149}$$

$$\mathbf{X}^{||z} = \frac{-2p_xT_x}{r_x^2} + \frac{2p_yT_y}{r_y^2} + \frac{T_{x^2}}{r_x^2} - \frac{T_{y^2}}{r_y^2} + \left(\frac{p_x^2}{r_x^2} - \frac{p_y^2}{r_y^2}\right)T_1. \tag{4.150}$$

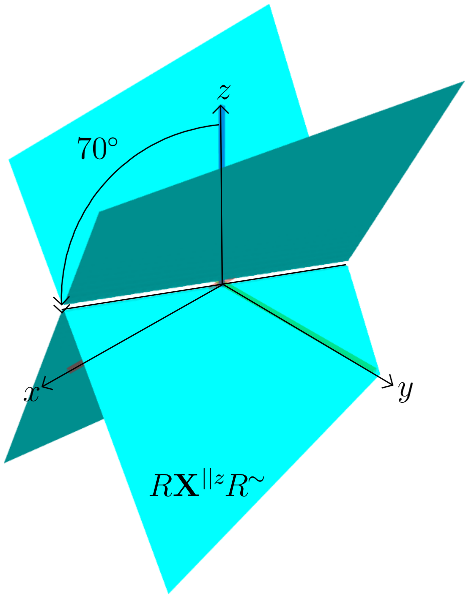

Figure 4.12. DCSA non-parallel planes pair rotated

Figure 4.12 shows the entity $\mathbf{X}^{||z}$, initially having planes with slopes $\pm\frac{r_y}{r_x} = \pm\frac{1}{2}$ that cross at the origin point $p_x = 0$, $p_y = 0$ in the xy-plane. It is then rotated using a DCSA rotor R (§4.4.1) around the y-axis by 70°. The line of crossing points was initially the z-axis, but after rotation the crossing line is at 70° off the z-axis, around the y-axis. Like the other DCSA entities, the non-parallel planes pair entities can be transformed into general positions using DCSA versor operations (§4.4).

4.2.17 DCSA GIPNS ellipse

The ellipse is a conic section, and like all conic sections it can be made as the intersection (§4.4.5) of a plane (§4.2.5) and cone (§4.2.8), but we are not limited to intersecting with cones. A simple ellipse representation is made as the intersection of a plane and elliptic cylinder (§4.2.7). The parabola (§4.2.18) and hyperbola (§4.2.19) are also conic sections, and their simple representations are as planes intersecting parabolic (§4.2.13) and hyperbolic (§4.2.14) cylinders. We can just define these conic sections as these plane and cylinder intersections, but these conic sections could be formed by a wide variety of other possible intersections.

The DCSA GIPNS 4-vector *xy-plane ellipse* 1D surface entity $\boldsymbol{\epsilon}^{||xy}$ is defined as

$$\boldsymbol{\epsilon}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{H}^{||z} \tag{4.151}$$

where the DCSA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ (§4.2.5) is the entity for the plane $z = 0$, and the DCSA GIPNS 2-vector *elliptic cylinder* $\mathbf{H}^{||z}$ (§4.2.7) is as previously defined and directly represents an ellipse in the xy-plane. Other similar ellipse entities are the wedges of other planes with other elliptic cylinders that are aligned differently.

A DCSA GIPNS ellipse entity $\boldsymbol{\epsilon}$, or its dual DCSA GOPNS ellipse entity $\boldsymbol{\epsilon}^{*\mathcal{DS}} = \boldsymbol{\epsilon}\mathbf{I}_{\mathcal{DS}}^{-1}$, can be rotated, dilated, and translated using DCSA versor operations (§4.4), where *versor outermorphism* is applied to the wedge of plane and cylinder that form the ellipse entity. In versor operations on the ellipse entity, the plane and cylinder are each transformed by the versor operations, and then the transformed plane and cylinder are intersected.

The invariant test $\mathbf{e}_\infty \cdot \boldsymbol{\epsilon}^{||xy} = 0$ seems to indicate that the ellipse reaches to infinity, but this should be considered as an *invalid* test.

4.2.18 DCSA GIPNS parabola

The DCSA GIPNS 4-vector *xy-plane parabola* 1D surface entity $\boldsymbol{\rho}^{||xy}$ is defined as

$$\boldsymbol{\rho}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{B}^{||z} \tag{4.152}$$

where the DCSA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ (§4.2.5) is the entity for the plane $z = 0$, and the DCSA GIPNS 2-vector *parabolic cylinder* $\mathbf{B}^{||z}$ (§4.2.13) is as previously defined and directly represents a parabola in the xy-plane. Other similar parabola entities are the wedges of other planes with other parabolic cylinders that are aligned differently.

4.2.19 DCSA GIPNS hyperbola

The DCSA GIPNS 4-vector *xy-plane hyperbola* 1D surface entity $\boldsymbol{\eta}^{||xy}$ is defined as

$$\boldsymbol{\eta}^{||xy} = \mathbf{\Pi}^{z=0} \wedge \mathbf{J}^{||z} \tag{4.153}$$

where the DCSA GIPNS 2-vector *plane* $\mathbf{\Pi}^{z=0}$ (§4.2.5) is the entity for the plane $z = 0$, and the DCSA GIPNS 2-vector *hyperbolic cylinder* $\mathbf{J}^{||z}$ (§4.2.14) is as previously defined and directly represents a hyperbola in the xy-plane. Other similar hyperbola entities are the wedges of other planes with other hyperbolic cylinders that are aligned differently.

4.2.20 DCSA GIPNS Darboux cyclide

The implicit quartic equation for a *Darboux cyclide* [19] surface is

$$\begin{aligned} A\mathbf{t}^4 + B\mathbf{t}^2 + & \\ Cx\mathbf{t}^2 + Dy\mathbf{t}^2 + Ez\mathbf{t}^2 + & \\ Fx^2 + Gy^2 + Hz^2 + & \\ Ixy + Jyz + Kzx + & \\ Lx + My + Nz + O &= 0 \end{aligned} \tag{4.154}$$

where $\mathbf{t} = x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3$ is a test point and the $A \ldots O$ are 15 real scalar constants. The point $\mathbf{t}$ (§2.3) is on the cyclide surface if the equation holds good. The square

$$\mathbf{t}^2 = \mathbf{t}_{\mathcal{S}}^2 = -(x^2 + y^2 + z^2) \tag{4.155}$$

has the opposite sign compared to $\mathbf{t}_{\mathcal{E}}^2$ in $\mathcal{G}_{8,2}$ DCGA [7][8].

The DCSA GIPNS 2-vector *Darboux cyclide* surface entity $\mathbf{\Omega}$ is defined as

$$\begin{aligned} \mathbf{\Omega} = {} & AT_{\mathbf{t}^4} + BT_{\mathbf{t}^2} + \\ & CT_{x\mathbf{t}^2} + DT_{y\mathbf{t}^2} + ET_{z\mathbf{t}^2} + \\ & FT_{x^2} + GT_{y^2} + HT_{z^2} + \\ & IT_{xy} + JT_{yz} + KT_{zx} + \\ & LT_x + MT_y + NT_z + OT_1. \end{aligned} \tag{4.156}$$

All DCSA versor operations (§4.4) are valid on the Darboux cyclide entity $\mathbf{\Omega}$ and its dual $\mathbf{\Omega}^{*\mathcal{DS}}$. The Darboux cyclide entity $\mathbf{\Omega}$ can be intersected with DCGA GIPNS planes (§4.2.5), spheres (§4.2.3), lines (§4.2.4), and circles (§4.2.6).

Entities with $A \neq 0$ have valid dilator operations (§4.4.2) with all dilation factors, including dilation factor $d = 0$. If $A \neq 0$, then $\mathbf{\Omega}$ dilates by factor 0 into $AT_{\mathbf{t}^4} = -4A\mathbf{e}_o$, which is a *valid* result representing the point at the origin. If $A = 0$, then $\mathbf{\Omega}$ dilates by factor 0 into scalar 0, which is an *invalid* result. Said differently, GIPNS entities that have $\mathbf{e}_o$ as a term dilate by factor 0 into $\mathbf{e}_o$ (up to scale), and other GIPNS entities dilate by factor 0 into scalar 0. The duals of such dilations are either $\mathbf{e}_o^{*\mathcal{DS}}$ or 0.

It was first discussed in Section 4.1.6, on the DCSA point $\mathbf{T}_{\mathcal{DS}}$ and extraction operators T_s, and then mentioned again in the section on the DCSA GIPNS 2-vector ellipsoid surface entity $\mathbf{E}$, that any DCSA GIPNS 2-vector surface entity without a term in $T_{\mathbf{t}^4}$ has the surface point $\mathbf{e}_\infty$. This includes some closed surfaces that would not be expected to have the point $\mathbf{e}_\infty$.

The constant B and the constants F, G, H allow alternative formulations of an entity $\mathbf{\Omega}$. If $F = G = H$, then F could be added to B to form a simpler entity having fewer terms by eliminating F, G, H. If an amount b is subtracted from each of F, G, H, then it can be added back as $(B + b)$, or the reverse. The surface represented by the entity $\mathbf{\Omega}$ is not affected by the specific choice of how to use B, F, G, H, but other metrical properties could be affected. Metrical properties include the scalar results returned by the inner products of entities, which are often distance measures between surfaces.

The Darboux cyclide entity $\mathbf{\Omega}$ is the most general form of DCSA GIPNS 2-vector surface entity that can be defined using the DCSA point $\mathbf{T}_{\mathcal{DS}}$ value-extraction operators $s = T_s \cdot \mathbf{T}_{\mathcal{DS}}$ (§4.1.6). DCSA could be described as a conformal geometric algebra on Darboux cyclide surface entities in 3D space. DCSA could also be $\mathcal{G}_{8,2}$ *Darboux Cyclide Space Algebra*. All of the DCSA GIPNS 2-vector quadric surface entities and the toroid entity, and also their inversive or cyclidic surface forms when reflected in DCSA spheres (§4.2.3), can be represented as instances of the Darboux cyclide entity $\mathbf{\Omega}$.

An instance of the DCSA GIPNS 2-vector Darboux cyclide surface entity $\mathbf{\Omega}$ can be produced by one or more *inversions in* DCSA GIPNS 2-vector *spheres* $\mathbf{S}_i$ (§4.2.3) of any DCSA GIPNS 2-vector surface entity $\mathbf{\Upsilon}$. For example, the inversion of a DCSA GIPNS 2-vector quadric or toroid surface entity $\mathbf{\Upsilon}$ in a DCSA GIPNS 2-vector sphere entity $\mathbf{S}$ is the reflection $\mathbf{\Omega} = \mathbf{S}\mathbf{\Upsilon}\mathbf{S}^\sim$, which is an instance of the Darboux cyclide surface entity $\mathbf{\Omega}$ that appears to be $\mathbf{\Upsilon}$ reflected in the sphere $\mathbf{S}$. The sphere $\mathbf{S}$ can be visualized as a spherical mirrored surface when $\mathbf{\Upsilon}$ is located entirely outside $\mathbf{S}$, and the cyclidic reflection of $\mathbf{\Upsilon}$ is seen on the surface of $\mathbf{S}$ or inside of $\mathbf{S}$. Successive inversions or reflections of $\mathbf{\Upsilon}$ in multiple spheres $\mathbf{S}_i$ transforms $\mathbf{\Upsilon}$ into a succession of different cyclide surface entities, all based on the initial shape of $\mathbf{\Upsilon}$. The distinction between inversion and reflection, which concerns whether or not the orientation of the surface remains the same or becomes inside-out, is not being made here.

Dual DCSA GOPNS 8-vector surface entities $\mathbf{\Upsilon}^{*\mathcal{DS}}$ can also be reflected in a sphere $\mathbf{S}$, or in its dual $\mathbf{S}^{*\mathcal{DS}}$, to produce an instance of the dual DCSA GOPNS 8-vector Darboux cyclide surface entity $\mathbf{\Omega}^{*\mathcal{DS}}$.

A singular outlier surface point $\mathbf{P}_{\mathcal{DS}}$ will exist on the inverse surface entity $\mathbf{S}\mathbf{\Upsilon}\mathbf{S}^\sim$ of any DCSA GIPNS 2-vector *closed surface* entity $\mathbf{\Upsilon}$ without a term in $T_{\mathbf{t}^4} = -4\mathbf{e}_o$. The singular outlier surface point $\mathbf{P}_{\mathcal{DS}}$ is always the center point of the inversion sphere $\mathbf{S}$. The inverse surface entity $\mathbf{S}\mathbf{\Upsilon}\mathbf{S}^\sim$ of an *open surface* entity $\mathbf{\Upsilon}$ that is known to reach $\mathbf{e}_\infty$ is expected to have the point $\mathbf{P}_{\mathcal{DS}}$, as it does. If $\mathbf{\Upsilon}$ is a *closed surface* entity, then it does not actually reach to infinity, and yet *any such entity* $\mathbf{\Upsilon}$ without a term in $T_{\mathbf{t}^4}$ has the surface point $\mathbf{e}_\infty$ and has an inverse surface $\mathbf{S\Upsilon S}^\sim$ that has the inversion sphere center point $\mathbf{P}_{\mathcal{DS}}$ as a singular outlier surface point. The inverse of point $\mathbf{e}_\infty$ is always the inversion sphere center point $\mathbf{P}_{\mathcal{DS}}$, or the reverse. A singular outlier surface point may be invisible on a surface plot. In particular, for any DCSA GIPNS 2-vector ellipsoid surface entity $\mathbf{E}$, its inverse surface entity $\mathbf{SES}^\sim$ has the inversion sphere $\mathbf{S}$ center point $\mathbf{P}_{\mathcal{DS}}$ as an (invisible) singular outlier surface point. An unexpected $\mathbf{e}_\infty$ or outlier point $\mathbf{P}_{\mathcal{DS}}$ is a **possible problem** for an application, but awareness of their existence may allow for a workaround to mitigate any possible problem caused by their existence.

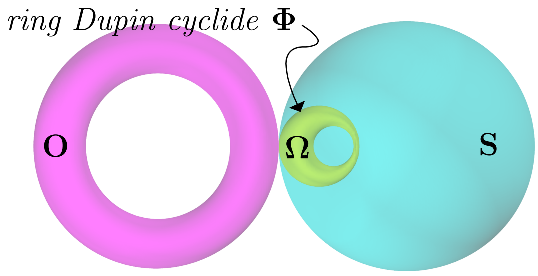

Figure 4.13. Toroid $\mathbf{O}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SOS}^\sim$

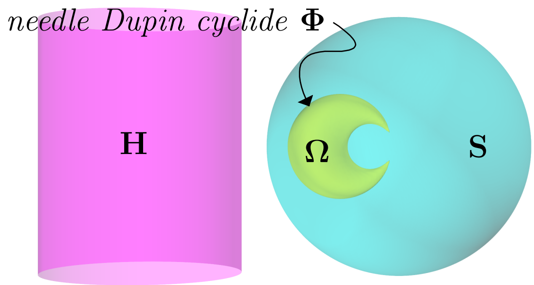

Figure 4.14. Cylinder $\mathbf{H}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SHS}^\sim$

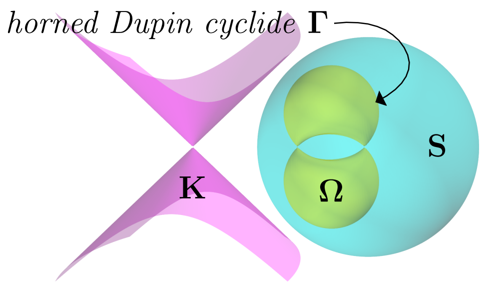

Figure 4.15. Cone $\mathbf{K}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SKS}^\sim$

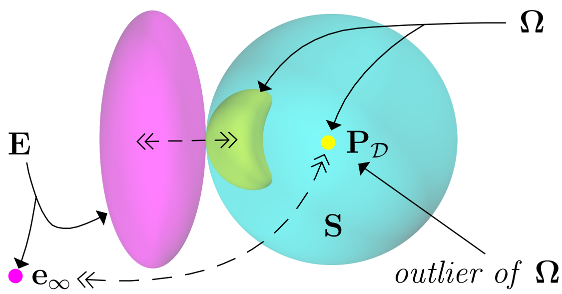

Figure 4.16. Ellipsoid $\mathbf{E}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SES}^\sim$

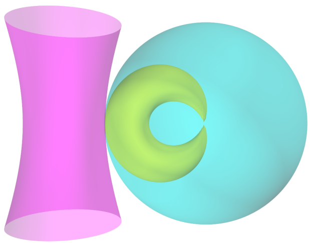

Figure 4.17. Hyperboloid of one sheet $\mathbf{\Sigma}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{S\Sigma S}^{\sim}$

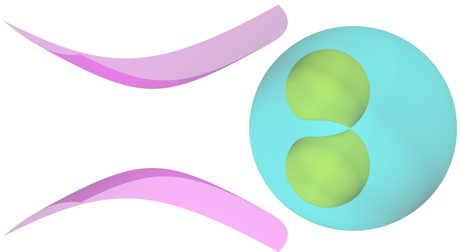

Figure 4.18. Hyperboloid of two sheets $\mathbf{\Xi}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{S\Xi S}^{\sim}$

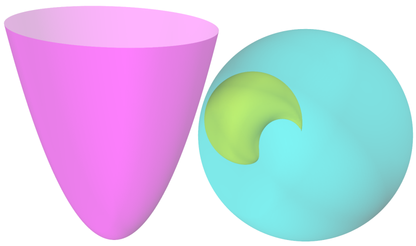

Figure 4.19. Paraboloid $\mathbf{V}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SVS}^{\sim}$

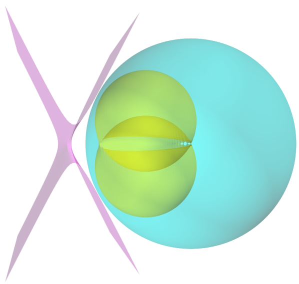

Figure 4.20. Hyperbolic paraboloid $\mathbf{M}$ reflected in sphere $\mathbf{S}$, $\mathbf{\Omega} = \mathbf{SMS}^{\sim}$

It is beyond the scope of this paper to analyze and define every possible type of cyclide that can be defined as instances of the Darboux cyclide entity $\mathbf{\Omega}$. However, as an example of what can be defined, we can consider the subsets of cyclides known as *Dupin cyclides* and *parabolic cyclides* [11][23]. The Dupin cyclides can generalize the circular toroid (torus) by creating cyclides based on torus inversion in a sphere. As shown in Figures 4.13, 4.14, 4.15, 4.16, 4.17, 4.18, 4.19, and 4.20, it is possible to define many other specific cyclide entities based on each of the quadric surfaces reflected in spheres.

4.2.20.1 DCSA GIPNS Dupin cyclide

The implicit quartic equation for a Dupin cyclide surface is

$$(-\mathbf{t}^2+(b^2-\mu^2))^2-4(ax-c\mu)^2-4b^2y^2 \;=\; 0 \tag{4.157}$$

where $\mathbf{t}=x\boldsymbol{\gamma}_1+y\boldsymbol{\gamma}_2+z\boldsymbol{\gamma}_3$ is a test point. Expanding this equation gives

$$\mathbf{t}^4-2\mathbf{t}^2(b^2-\mu^2)+(b^2-\mu^2)^2-4(a^2x^2-2ac\mu x+c^2\mu^2)-4b^2y^2 \;=\; 0. \tag{4.158}$$

The DCSA GIPNS 2-vector *Dupin cyclide* surface entity $\mathbf{\Phi}$ is defined as

$$\begin{aligned}\mathbf{\Phi} \;=\; & T_{\mathbf{t}^4}-2T_{\mathbf{t}^2}(b^2-\mu^2)+ \\ & -4a^2T_{x^2}-4b^2T_{y^2}+ \\ & 8ac\mu T_x+((b^2-\mu^2)^2-4c^2\mu^2)T_1.\end{aligned} \tag{4.159}$$

The scalar parameters of the surface are a, b, c, μ, with b always squared. The Dupin cyclide can be described as a surface that envelops a family of spheres defined by two initial spheres of *minor radii*, r_1 and r_2, centered on a circle of *major radius* R. To gain a more intuitive expression of the Dupin cyclide equation, we can define the parameters as

$$a \;=\; R \tag{4.160}$$
$$\mu \;=\; \frac{1}{2}(r_1+r_2) \tag{4.161}$$
$$c \;=\; \frac{1}{2}(r_1-r_2) \tag{4.162}$$
$$b^2 \;=\; a^2-c^2. \tag{4.163}$$

The Dupin cyclide $\mathbf{\Phi}$ is now defined by the three radii parameters, R, r_1, r_2. When $r=r_1=r_2$, the Dupin cyclide $\mathbf{\Phi}$ is exactly the same entity as the toroid $\mathbf{O}$ with parameters R and r.

The DCSA GIPNS Dupin cyclide $\mathbf{\Phi}$ has the following *related points*:

$$\begin{aligned}\text{Center of initial sphere } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1 \\ \text{Center of initial sphere } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1 \\ \text{Center of ring or spindle hole in the cyclide} &: +c\boldsymbol{\gamma}_1 \\ \text{Center of sphere enclosing entire cyclide} &: -c\boldsymbol{\gamma}_1 \\ \text{Radius around } -c\boldsymbol{\gamma}_1 \text{ enclosing entire cyclide} &: \mu+R.\end{aligned}$$

Twelve *surface points* on the Dupin cyclide $\mathbf{\Phi}$ are:

$$\begin{aligned}\text{2 surface points on } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1 \pm r_1\boldsymbol{\gamma}_1 \\ \text{2 surface points on } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1 \pm r_1\boldsymbol{\gamma}_3 \\ \text{2 surface points on } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1 \pm r_2\boldsymbol{\gamma}_1 \\ \text{2 surface points on } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1 \pm r_2\boldsymbol{\gamma}_3 \\ \text{2 surface points} &: -c\boldsymbol{\gamma}_1 \pm (\mu+R)\boldsymbol{\gamma}_2 \\ \text{2 surface points} &: +c\boldsymbol{\gamma}_1 \pm (\mu-R)\boldsymbol{\gamma}_2.\end{aligned}$$

The Dupin cyclide $\mathbf{\Phi}$ is initially created having these 12 points. All DCSA versor operations are valid on the Dupin cyclide $\mathbf{\Phi}$ and can be used to rotate, dilate, and translate it into another general position.

The type of cyclide or torus represented by $\mathbf{\Phi}$ is determined by:

$$\begin{aligned}
\textit{Ring cyclide} \text{ when } &: (r_1 + r_2) < 2R\\
\textit{Spindle cyclide} \text{ when } &: (r_1 + r_2) > 2R\\
\textit{Horn cyclide} \text{ when } &: (r_1 + r_2) = 2R\\
\textit{Ring torus} \text{ when } &: (r_1 = r_2) < R\\
\textit{Spindle torus} \text{ when } &: (r_1 = r_2) > R\\
\textit{Horn torus} \text{ when } &: (r_1 = r_2) = R.
\end{aligned}$$

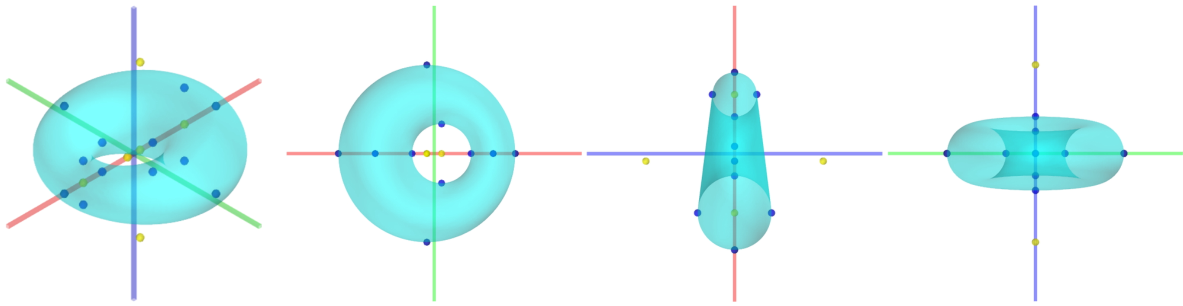

Figure 4.21. Ring cyclide $\mathbf{\Phi}$, $(r_1 + r_2) < 2R$

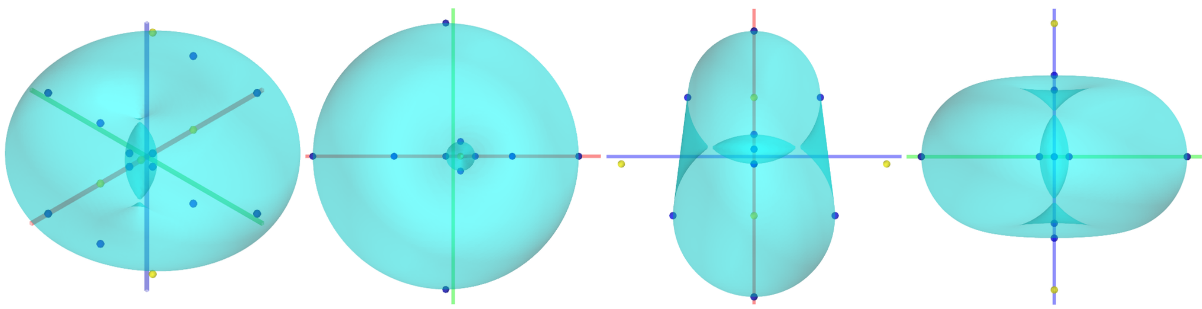

Figure 4.22. Spindle cyclide $\mathbf{\Phi}$, $(r_1 + r_2) > 2R$

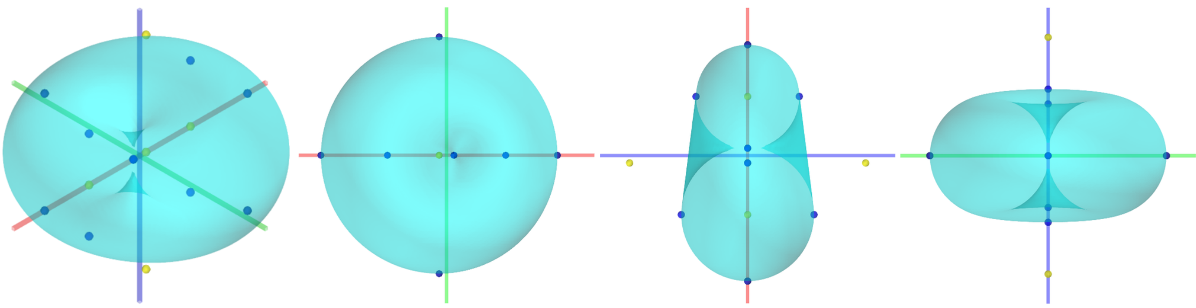

Figure 4.23. Horn cyclide $\mathbf{\Phi}$, $(r_1 + r_2) = 2R$

Figures 4.21, 4.22, and 4.23 show three types of the Dupin cyclide $\mathbf{\Phi}$. The torus types, not shown, are ordinary toroid surfaces and they are exactly the same entities as formed by the toroid entity $\mathbf{O}$. The inversion of a circular cylinder in a sphere can form a *needle cyclide* [23], which is a ring cyclide having $r_1 = 0$ or $r_2 = 0$, $r_1 \neq r_2$.

4.2.20.2 DCSA GIPNS horned Dupin cyclide

The *horned Dupin cyclide* is a modification of the *Dupin cyclide* that causes both of the initial spheres to shrink until they meet in points. The horned Dupin cyclide is formed by swapping μ and c in the implicit equation of the Dupin cyclide. The parameters a, b, c, μ are defined as

$$\begin{aligned} a &= R & (4.164)\\ \mu &= \frac{1}{2}(r_1+r_2) & (4.165)\\ c &= \frac{1}{2}(r_1-r_2) & (4.166)\\ b^2 &= a^2-\mu^2. & (4.167) \end{aligned}$$

The DCSA GIPNS 2-vector *horned Dupin cyclide* surface entity $\mathbf{\Gamma}$ is defined as

$$\begin{aligned} \mathbf{\Gamma} &= T_{\mathbf{t}^4} - 2T_{\mathbf{t}^2}(b^2-c^2) + \\ &\quad -4a^2T_{x^2} - 4b^2T_{y^2} + \\ &\quad 8ac\mu T_x + ((b^2-c^2)^2 - 4c^2\mu^2)T_1. \end{aligned} \quad (4.168)$$

The DCSA GIPNS horned Dupin cyclide $\mathbf{\Gamma}$ has the following *related points*:

$$\begin{aligned} \text{Center of initial sphere } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1\\ \text{Center of initial sphere } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1\\ \text{Center of ring or spindle hole in the cyclide} &: +c\boldsymbol{\gamma}_1\\ \text{Center of sphere enclosing entire cyclide} &: -c\boldsymbol{\gamma}_1\\ \text{Radius around } -c\boldsymbol{\gamma}_1 \text{ enclosing entire cyclide} &: \mu+R. \end{aligned}$$

Twelve *surface points* on the horned Dupin cyclide $\mathbf{\Gamma}$ are:

$$\begin{aligned} \text{2 surface points on } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1 \pm r_1\boldsymbol{\gamma}_1\\ \text{2 surface points on } \mathbf{S}_1 \text{ with radius } r_1 &: -R\boldsymbol{\gamma}_1 \pm r_1\boldsymbol{\gamma}_3\\ \text{2 surface points on } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1 \pm r_2\boldsymbol{\gamma}_1\\ \text{2 surface points on } \mathbf{S}_2 \text{ with radius } r_2 &: +R\boldsymbol{\gamma}_1 \pm r_2\boldsymbol{\gamma}_3\\ \text{2 surface points} &: -\mu\boldsymbol{\gamma}_1 \pm (c+R)\boldsymbol{\gamma}_2\\ \text{2 surface points} &: +\mu\boldsymbol{\gamma}_1 \pm (c-R)\boldsymbol{\gamma}_2. \end{aligned}$$

The horned Dupin cyclide $\mathbf{\Gamma}$ is initially created having these 12 points. All DCSA versor operations are valid on the horned Dupin cyclide $\mathbf{\Gamma}$ and can be used to rotate, dilate, and translate it into another general position.

The type of cyclide or torus represented by $\mathbf{\Gamma}$ is determined by:

$$\begin{aligned} \textit{Horned ring cyclide} \text{ when} &: (r_1+r_2) < 2R\\ \textit{Horned spindle cyclide} \text{ when} &: (r_1+r_2) > 2R\\ \textit{Horned spheres} \text{ (two } \textit{tangent spheres}\text{) when} &: (r_1+r_2) = 2R\\ \textit{Horned ring torus} \text{ when} &: (r_1=r_2) < R\\ \textit{Horned spindle torus} \text{ when} &: (r_1=r_2) > R\\ \textit{Horned spheres} \text{ when} &: (r_1=r_2) = R. \end{aligned}$$

The *horned spheres* represents the union or product of two implicit surface functions for two spheres of radius r_1 and r_2 that touch in a single tangent point, and it is an instance of a *spheres pair* cyclide entity. A different and more general spheres pair entity, the DCSA GIPNS 2-vector *spheres pair* entity $\boldsymbol{\xi}$, can be defined as the wedge of a CSA1 GIPNS sphere $\mathbf{S}_{1\mathcal{C}\mathcal{S}^1}$ and another CSA2 GIPNS sphere $\mathbf{S}_{2\mathcal{C}\mathcal{S}^2}$ as $\boldsymbol{\xi} = \mathbf{S}_{1\mathcal{C}\mathcal{S}^1} \wedge \mathbf{S}_{2\mathcal{C}\mathcal{S}^2}$ (§3.2.1). The spheres pair $\boldsymbol{\xi}$ can be transformed by the DCSA versors and intersected with standard DCSA spheres, planes, lines, and circles but not with any quadric or cyclidic surface entities.

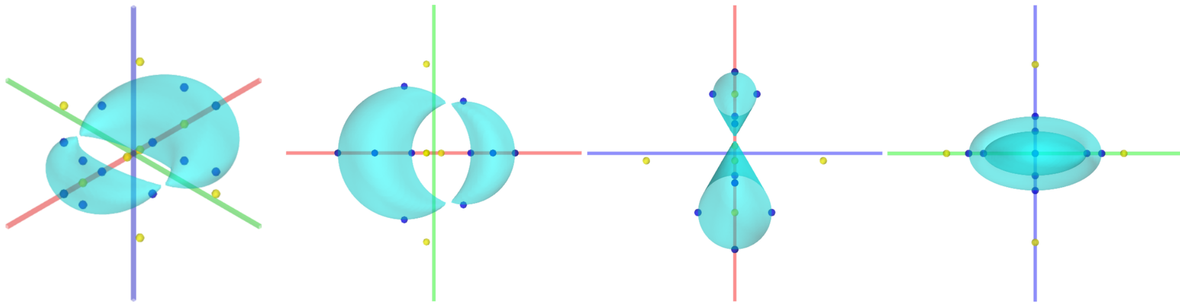

Figure 4.24. Horned ring cyclide $\mathbf{\Gamma}$, $(r_1 + r_2) < 2R$

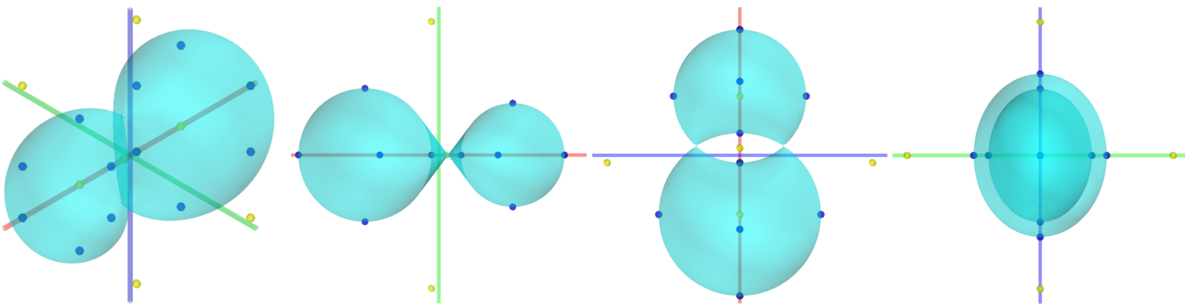

Figure 4.25. Horned spindle cyclide $\mathbf{\Gamma}$, $(r_1 + r_2) > 2R$

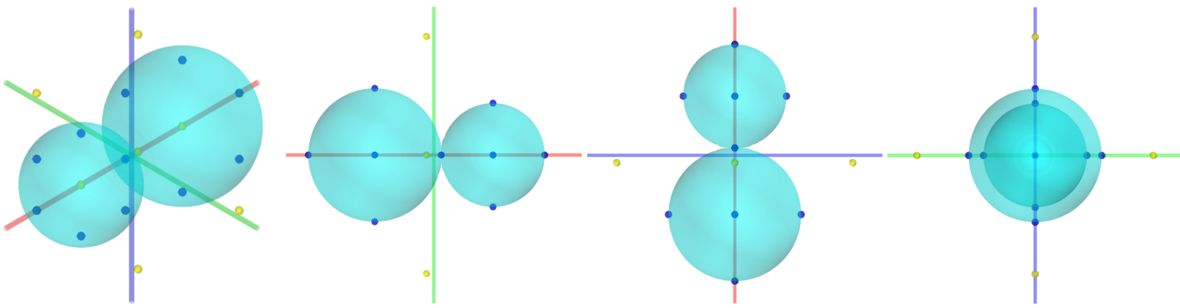

Figure 4.26. Horned spheres cyclide $\mathbf{\Gamma}$, $(r_1 + r_2) = 2R$

Figures 4.24, 4.25, and 4.26 show three types of the horned Dupin cyclide $\mathbf{\Gamma}$. The three other types with $r_1 = r_2$, not shown, are symmetrical versions of the three types shown. As defined, the Dupin cyclides are symmetrical across the planes $y = 0$ and $z = 0$, and are also symmetrical across the plane $x = 0$ only when $r_1 = r_2$. The horned Dupin cyclides can be formed as the inversions of circular cones in spheres.

All of the Dupin cyclide entities have term $T_{\mathbf{t}^4} = -4\mathbf{e}_o$ and are true closed surfaces that do not have surface point $\mathbf{e}_\infty$. Therefore, the Dupin cyclide entities $\mathbf{\Phi}$ and $\mathbf{\Gamma}$, like the standard sphere $\mathbf{S}$ and toroid $\mathbf{O}$ entities, are well-behaved entities that do not have a singular outlier point at the inversion sphere center under inversion in a sphere.

4.2.20.3 DCSA GIPNS parabolic cyclide

The DCSA GIPNS 2-vector *parabolic cyclide* surface entity $\mathbf{\Psi}$ can be defined as

$$\begin{aligned}\mathbf{\Psi} &= BT_{\mathbf{t}^2}+CT_{x\mathbf{t}^2}+DT_{y\mathbf{t}^2}+ET_{z\mathbf{t}^2}+\\ &\quad FT_{x^2}+GT_{y^2}+HT_{z^2}+\\ &\quad IT_{xy}+JT_{yz}+KT_{zx}+\\ &\quad LT_x+MT_y+NT_z+OT_1\end{aligned} \tag{4.169}$$

with C,D,E not all zero. A *degenerate parabolic cyclide* has $C=D=E=0$. All DCSA versor operations are valid on the parabolic cyclide entity $\mathbf{\Psi}$, and it can be intersected with the standard DCSA GIPNS sphere (§4.2.3), plane (§4.2.5), line (§4.2.4), and circle (§4.2.6) entities that are defined as special bi-CGA entities.

The parabolic cyclide entity $\mathbf{\Psi}$ is simply the Darboux cyclide entity $\mathbf{\Omega}$ (§4.2.20) with $A=0$, and it is a cubic surface entity. Without a term in $T_{\mathbf{t}^4}=-4\mathbf{e}_o$, the surface entity $\mathbf{\Psi}$ has surface point $\mathbf{e}_\infty$ and is generally an open-surface entity. The DCSA GIPNS 2-vector ellipsoid entity $\mathbf{E}$ (§4.2.2) is a degenerate parabolic cyclide entity that becomes a closed-surface entity with a singular outlier surface point at $\mathbf{e}_\infty$.

An instance of the DCSA 2-vector *parabolic cyclide* surface entity $\mathbf{\Psi}$ can be produced as the inversion of a DCSA GIPNS 2-vector Darboux cyclide surface entity $\mathbf{\Omega}$ (§4.2.20) in a standard DCSA GIPNS 2-vector sphere surface entity $\mathbf{S}$ (§4.2.3) that is *centered on a surface point* of $\mathbf{\Omega}$.

The inversion sphere $\mathbf{S}$, with center point $\mathbf{P}_{\mathcal{DS}}=\mathcal{D}(\mathbf{p})$ *on the surface* of an entity $\mathbf{\Omega}$, gives the inverse surface $\mathbf{\Psi}$ as

$$\mathbf{\Psi} = \mathbf{S}\mathbf{\Omega}\mathbf{S}^\sim = (\mathbf{S}_{\mathcal{C}S^1}\wedge\mathbf{S}_{\mathcal{C}S^2})\mathbf{\Omega}(\mathbf{S}_{\mathcal{C}S^2}\wedge\mathbf{S}_{\mathcal{C}S^1}) = \mathbf{S}_{\mathcal{C}S^1}\mathbf{S}_{\mathcal{C}S^2}\mathbf{\Omega}\mathbf{S}_{\mathcal{C}S^2}\mathbf{S}_{\mathcal{C}S^1} \tag{4.170}$$

$$= \left(\mathbf{P}_{\mathcal{C}S^1}+\frac{1}{2}r^2\mathbf{e}_{\infty 1}\right)\left(\mathbf{P}_{\mathcal{C}S^2}+\frac{1}{2}r^2\mathbf{e}_{\infty 2}\right)\mathbf{\Omega}\mathbf{S}_{\mathcal{C}S^2}\mathbf{S}_{\mathcal{C}S^1}. \tag{4.171}$$

If expanded further with the surface point condition $\mathbf{P}_{\mathcal{DS}}\cdot\mathbf{\Omega}=0$, then it is found that $(\mathbf{P}_{\mathcal{DS}}=\mathbf{P}_{\mathcal{C}S^1}\wedge\mathbf{P}_{\mathcal{C}S^2})\longleftrightarrow\mathbf{e}_\infty$, or that $\mathbf{P}_{\mathcal{DS}}$ goes to $\mathbf{e}_\infty$ and $\mathbf{e}_\infty$ goes to $\mathbf{P}_{\mathcal{DS}}$. Surface points of $\mathbf{\Omega}$ that are outside $\mathbf{S}$ are brought inside $\mathbf{S}$, and surface points of $\mathbf{\Omega}$ that are inside $\mathbf{S}$ are taken outside $\mathbf{S}$.

A surface point $\mathcal{D}(\mathbf{p}+\mathbf{d})$ of $\mathbf{\Omega}$ is transformed by inversion in sphere $\mathbf{S}$ centered at $\mathbf{p}$ with radius r as

$$\mathbf{S}\mathcal{D}(\mathbf{p}+\mathbf{d})\mathbf{S}^{-1} = \begin{cases}\mathcal{D}\left(\mathbf{p}+\frac{r^2}{\|\mathbf{d}\|^2}\mathbf{d}\right) & : \|\mathbf{d}\|\neq 0\\ \mathbf{e}_\infty & : \|\mathbf{d}\|\to 0\\ \mathcal{D}(\mathbf{p}) & : \|\mathbf{d}\|\to\infty\\ \mathcal{D}(\mathbf{p}+\|\mathbf{d}\|^{-1}\hat{\mathbf{d}}) & : r=1\\ \mathcal{D}(\mathbf{p}+\mathbf{d}) & : \|\mathbf{d}\|=r\end{cases} \tag{4.172}$$

$$\mathbf{S}^{-1} = \mathbf{S}^{-2}\mathbf{S}=\frac{1}{-r^4}\mathbf{S}=\frac{1}{r^4}\mathbf{S}^\sim. \tag{4.173}$$

The displacement $\mathbf{d}=\|\mathbf{d}\|\hat{\mathbf{d}}$ from the inversion sphere center $\mathbf{p}$ is transformed into the inverse magnitude displacement $\|\mathbf{d}\|^{-1}\hat{\mathbf{d}}$ when the inversion sphere has radius $r=1$. As an *inver*sion operat*or*, an inversion sphere $\mathbf{S}$ could be called an *inversor*, especially if it has radius $r=1$ where $\mathbf{S}^2=-1$ as a proper *versor* with unit magnitude.

In general, an inversion sphere $\mathbf{S}$ can have any center point $\mathbf{P}_{\mathcal{DS}}$ and any radius r. If $r=0$, then $\mathbf{S}=\mathbf{P}_{\mathcal{DS}}$, which is a finite point that could be $\mathbf{e}_o$. An infinite radius $r=\infty$ is represented by $\mathbf{S}=\mathbf{e}_\infty$. Inversion in any point $\mathbf{S}=\mathbf{P}_{\mathcal{DS}}$ or $\mathbf{S}=\mathbf{e}_\infty$ sends everything into the point and produces the point $\mathbf{S}$, *unless* the point $\mathbf{S}$ is a surface point of $\mathbf{\Omega}$ and then any point sent into itself produces the nil scalar 0 result for the entire surface inversion.

Open surfaces $\mathbf{\Psi}$ that extend out to infinity $\mathbf{e}_\infty$ are the only surfaces that reflect $\mathbf{S\Psi S}^\sim$ continuously into the center point $\mathbf{P}_{\mathcal{DS}}$ of an inversion sphere $\mathbf{S}$. Conversely, by placing the center point $\mathbf{P}_{\mathcal{DS}}$ of an inversion sphere $\mathbf{S}$ on any surface $\mathbf{\Omega}$ and then reflecting $\mathbf{S\Omega S}^\sim$ the surface outward, the resulting open surface $\mathbf{\Psi} = \mathbf{S\Omega S}^\sim$ extends out to infinity $\mathbf{e}_\infty$ and must be a plane or curved sheet that is either a parabolic cyclide $\mathbf{\Psi}$ or a degenerate parabolic cyclide that is one of the quadric surfaces. Quadric surfaces are degenerate parabolic cyclides, and all other curved-sheet cubic surface entities are instances of the parabolic cyclide entity.

Sphere, plane, line, and circle entities can be created as the *standard* DCSA GIPNS sphere, plane, line, and circle entities $\mathbf{S}$ (§4.2.3), $\mathbf{\Pi}$ (§4.2.5), $\mathbf{L}$ (§4.2.4), $\mathbf{C}$ (§4.2.6) which are defined as bi-CSA entities. Sphere, plane, line, and circle entities can also be created as *non-standard* entities that are instances of degenerate parabolic cyclides using only linear and quadratic extraction terms (§4.1.6). Only the *standard* sphere, plane, line, and circle entities operate as inversion or reflection operators. All DCSA surface entities $\mathbf{\Upsilon}$ can be reflected in the standard sphere $\mathbf{S}$, plane $\mathbf{\Pi}$, line $\mathbf{L}$, and circle $\mathbf{C}$. Reflection in a line $\mathbf{L\Upsilon L}^\sim$ rotates $\mathbf{\Upsilon}$ by 180° around the line. The results of inversion or reflection operations on standard and non-standard sphere, plane, line, and circle entities in the standard ones are not the same. Reflections and inversions of the standard entities produce another one of the standard entities. Reflections and inversions of the non-standard entities can produce cubic surfaces that represent the expected surfaces but which also have a singular outlier surface point at the inversion sphere center point. All parabolic cyclides and degenerate parabolic cyclides have the point $\mathbf{e}_\infty$ which reflects into an inversion sphere center point.

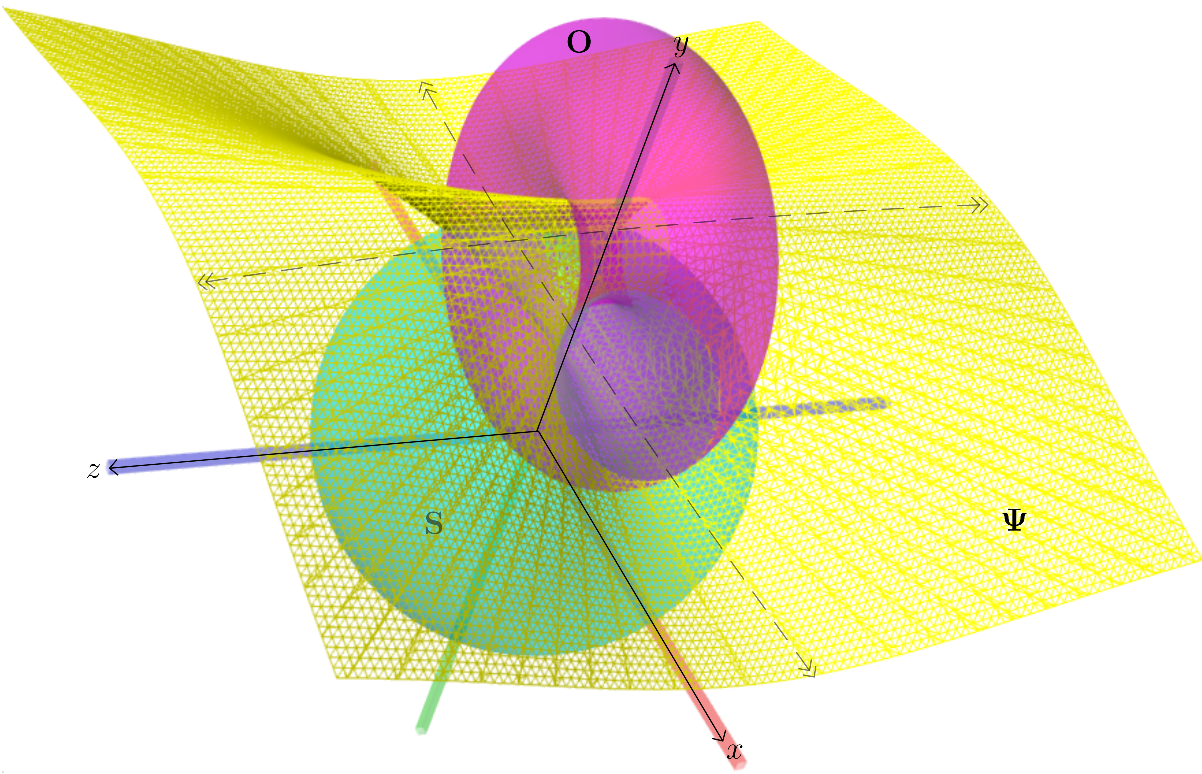

Figure 4.27. Toroid $\mathbf{O}$ on inversion sphere $\mathbf{S}$ center $\mathbf{P}_{\mathcal{DS}} = \mathbf{e}_o$, $\mathbf{\Psi} = \mathbf{SOS}^\sim$

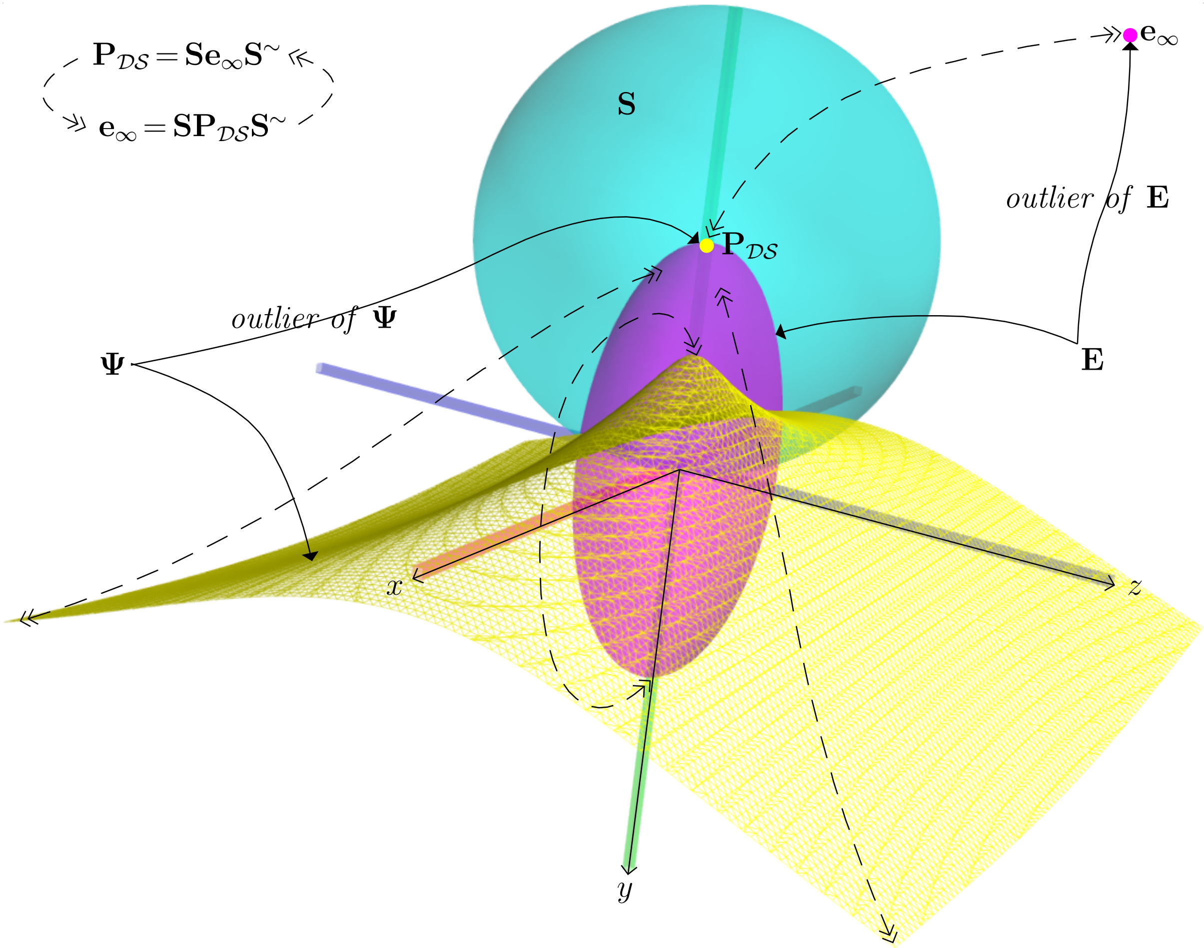

Figure 4.28. Ellipsoid $\mathbf{E}$ on inversion sphere $\mathbf{S}$ center $\mathbf{P}_{\mathcal{DS}} = \mathcal{D}(-10\gamma_2)$, $\mathbf{\Psi} = \mathbf{S}\mathbf{E}\mathbf{S}^\sim$

4.3 DCSA GOPNS entities

Up to four DCSA points (§4.1) can be wedged to form DCSA *geometric outer product null space* (GOPNS) 4,6,8-vector surface entities of the surface types available in CSA (§3.3). Unfortunately, the wedge of more than four points, as required for the quadric surfaces, does not work with DCSA points.

The DCSA GOPNS surface entities for quadric surfaces and the toroid would require more than four points to define them. For quadric surfaces in general position, it takes 5 points in 2D, and 9 points in 3D to define a quadric surface. If limited to principal axes-aligned surfaces, it still requires 6 points in 3D to define a quadric surface, as in $\mathcal{G}_{6,3}$ *Quadric Geometric Algebra* (QGA) [27][9][18]. Therefore, it seems that it is not possible in DCSA to directly represent the DCSA GOPNS quadric surfaces as the wedge of DCSA surface points. When more than four DCSA surface points are required to define a surface, then more complicated formulas are still possible but they resolve back to the GIPNS entities.

In general, we can always obtain a DCSA GOPNS surface entity $\mathbf{S}^{*\mathcal{DS}}$ by taking the DCSA dual (§4.4.6) of a DCSA GIPNS surface entity $\mathbf{S}$ as $\mathbf{S}^{*\mathcal{DS}} = \mathbf{S}\mathbf{I}_{\mathcal{DS}}^{-1}$. All DCSA versor operations (§4.4) are valid on both the DCSA GIPNS entities and their dual DCSA GOPNS entities.

The following four subsections define the four DCSA GOPNS surface entities which can be constructed as wedges of up four DCSA surface points. These four DCSA GOPNS surface entities are just the DCSA analogues of the CSA GOPNS surface entities.

A DCSA test point $\mathbf{T}_{\mathcal{DS}}$ that is on a DCSA GOPNS surface entity $\mathbf{S}^{*\mathcal{DS}}$ must satisfy the GOPNS condition

$$\mathbf{T}_{\mathcal{DS}} \wedge \mathbf{S}^{*\mathcal{DS}} = 0. \quad (4.174)$$

The DCSA GOPNS k-vector surface entity $\mathbf{S}^{*\mathcal{DS}}$ represents the set $\mathbb{NO}_G(\mathbf{S}^{*\mathcal{DS}} \in \mathcal{G}^k_{2,8})$ of all 3D vector test points $\mathbf{t}$ that are surface points

$$\mathbb{NO}_G(\mathbf{S}^{*\mathcal{DS}} \in \mathcal{G}^k_{2,8}) = \left\{ \mathbf{t} \in \mathcal{G}^1_{0,3} : (\mathcal{D}(\mathbf{t}) = \mathbf{T}_{\mathcal{DS}}) \wedge \mathbf{S}^{*\mathcal{DS}} = 0 \right\}. \quad (4.175)$$

4.3.1 DCSA GOPNS sphere

The DCSA GOPNS 8-vector *sphere* $\mathbf{S}^{*\mathcal{DS}}$ is defined as the wedge of four DCSA points $\mathbf{P}_{\mathcal{DS}_i}$ (§4.1) on the sphere as

$$\begin{aligned} \mathbf{S}^{*\mathcal{DS}} &= \mathbf{P}_{\mathcal{DS}_1} \wedge \mathbf{P}_{\mathcal{DS}_2} \wedge \mathbf{P}_{\mathcal{DS}_3} \wedge \mathbf{P}_{\mathcal{DS}_4} & (4.176) \\ &= \mathbf{S}\mathbf{I}^{-1}_{\mathcal{DS}} & (4.177) \end{aligned}$$

and is the DCSA dual of the DCSA GIPNS 2-vector sphere $\mathbf{S}$ (§4.2.3).

4.3.2 DCSA GOPNS plane

The DCSA GOPNS 8-vector *plane* $\mathbf{\Pi}^{*\mathcal{DS}}$ is defined as the wedge of three DCSA points $\mathbf{P}_{\mathcal{DS}_i}$ (§4.1) on the plane and the DCSA point at infinity $\mathbf{e}_\infty$ as

$$\begin{aligned} \mathbf{\Pi}^{*\mathcal{DS}} &= \mathbf{P}_{\mathcal{DS}_1} \wedge \mathbf{P}_{\mathcal{DS}_2} \wedge \mathbf{P}_{\mathcal{DS}_3} \wedge \mathbf{e}_\infty & (4.178) \\ &= \mathbf{\Pi}\mathbf{I}^{-1}_{\mathcal{DS}} & (4.179) \end{aligned}$$

and is the DCSA dual of the DCSA GIPNS 2-vector plane $\mathbf{\Pi}$ (§4.2.5).

4.3.3 DCSA GOPNS line

The DCSA GOPNS 6-vector *line* $\mathbf{L}^{*\mathcal{DS}}$ is defined as the wedge of two DCSA points $\mathbf{P}_{\mathcal{DS}_i}$ (§4.1) on the line and the DCSA point at infinity $\mathbf{e}_\infty$ as

$$\begin{aligned} \mathbf{L}^{*\mathcal{DS}} &= \mathbf{P}_{\mathcal{DS}_1} \wedge \mathbf{P}_{\mathcal{DS}_2} \wedge \mathbf{e}_\infty & (4.180) \\ &= \mathbf{L}\mathbf{I}^{-1}_{\mathcal{DS}} & (4.181) \end{aligned}$$

and is the DCSA dual of the DCSA GIPNS 4-vector line $\mathbf{L}$ (§4.2.4).

4.3.4 DCSA GOPNS circle

The DCSA GOPNS 6-vector *circle* $\mathbf{C}^{*\mathcal{DS}}$ is defined as the wedge of three DCSA points $\mathbf{P}_{\mathcal{DS}_i}$ (§4.1) on the circle as

$$\begin{aligned} \mathbf{C}^{*\mathcal{DS}} &= \mathbf{P}_{\mathcal{DS}_1} \wedge \mathbf{P}_{\mathcal{DS}_2} \wedge \mathbf{P}_{\mathcal{DS}_3} & (4.182) \\ &= \mathbf{C}\mathbf{I}^{-1}_{\mathcal{DS}} & (4.183) \end{aligned}$$

and is the DCSA dual of the DCSA GIPNS 4-vector circle **C** (§4.2.6).

4.4 DCSA operations

Any CSA1 versor $V_{\mathcal{CS}^1}$ and its copy CSA2 versor $V_{\mathcal{CS}^2}$ are multiplied to form the corresponding doubled DCSA versor $V_{\mathcal{DS}}$ as

$$\begin{aligned} V_{\mathcal{DS}} &= V_{\mathcal{CS}^1} \wedge V_{\mathcal{CS}^2} && (4.184)\\ &= V_{\mathcal{CS}^1} V_{\mathcal{CS}^2}. && (4.185)\end{aligned}$$

In theory, a DCSA versor $V_{\mathcal{DS}}$ can operate on any DCSA entity $\mathbf{X}$ using the versor "sandwich" operation

$$\mathbf{X}' = V_{\mathcal{DS}} \mathbf{X} V_{\mathcal{DS}}^{\sim} \quad (4.186)$$

to produce the transformed entity $\mathbf{X}'$ as expected for the operation, which can be rotation (§4.4.1), dilation (§4.4.2), translation (§4.4.3), or a composition of these operations.

In practice, the DCSA versor operations can be extremely slow, depending on the particular software and hardware that is employed for computations. However, a large speed-up may be gained by using the CSA versors for rotation (§3.4.3), dilation (§3.4.5), and translation (§3.4.2), or a composition of these operations, directly on any DCSA entity $\mathbf{X}$ as a succession of CSA operations,

$$\begin{aligned} \mathbf{X}' &= V_{\mathcal{DS}} \mathbf{X} V_{\mathcal{DS}}^{\sim} && \text{(extremely slow DCSA computations)} && (4.187)\\ &= V_{\mathcal{CS}^1} (V_{\mathcal{CS}^2} \mathbf{X} V_{\mathcal{CS}^2}^{\sim}) V_{\mathcal{CS}^1}^{\sim} && \text{(much faster CSA computations).} && (4.188)\end{aligned}$$

The only difference is to take advantage of the associativity of the geometric and outer products. The speed-up can be very large, turning an impractical DCSA versor operation into a practical succession of CSA versor operations. Taking advantage of associativity can also speed up the DCSTA versor operations (§7.7).

Practical applications may become increasingly feasible with advances in *Geometric Algebra Computing for Heterogeneous Systems* [13] and with advances in *Embedded Coprocessors for Native Execution of Geometric Algebra Operations* [12].

4.4.1 DCSA rotor

The DCSA 4-versor *rotor* R is defined as

$$R = R_{\mathcal{CS}^1} \wedge R_{\mathcal{CS}^2}. \quad (4.189)$$

The CSA rotors (§3.4.3) for the same rotation operation in CSA1 and CSA2 are wedged as the DCSA rotor R. All DCSA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be generally rotated around any axis by any angle by the DCSA rotor operation

$$\mathbf{X}' = R \mathbf{X} R^{\sim}. \quad (4.190)$$

The CSA translated-rotor (§3.4.4) can be doubled into the DCSA translated-rotor.

4.4.2 DCSA dilator

The DCSA 4-versor *dilator* D is defined as

$$D = D_{\mathcal{CS}^1} \wedge D_{\mathcal{CS}^2}. \tag{4.191}$$

The CSA dilators (§3.4.5) for the same dilation operation in CSA1 and CSA2 are wedged as the DCSA dilator D. All DCSA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be dilated by the DCSA dilator operation

$$\mathbf{X}' = D\mathbf{X}D^{\sim}. \tag{4.192}$$

Keep in mind that dilation also dilates the position of an entity, which may cause an unexpected translational movement. To *scale* an entity, it should be translated to be centered on the origin, dilated around the origin, and then translated back. The CSA translated-dilator (§3.4.7) can be doubled into the DCSA translated-dilator.

4.4.3 DCSA translator

The DCSA 4-versor *translator* T is defined as

$$T = T_{\mathcal{CS}^1} \wedge T_{\mathcal{CS}^2}. \tag{4.193}$$

The CSA translators (§3.4.2) for the same translation operation in CSA1 and CSA2 are wedged as the DCSA translator T. All DCSA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be translated by the DCSA translator operation

$$\mathbf{X}' = T\mathbf{X}T^{\sim}. \tag{4.194}$$

4.4.4 DCSA motor

The DCSA 4-versor *motor* M is defined as

$$M = M_{\mathcal{CS}^1} \wedge M_{\mathcal{CS}^2}. \tag{4.195}$$

The CSA motors (§3.4.8) for the same motion operation in CSA1 and CSA2 are wedged as the DCSA motor M. All DCSA entities $\mathbf{X}$, including both GIPNS and GOPNS, can be moved by the DCSA motor operation

$$\mathbf{X}' = M\mathbf{X}M^{\sim}. \tag{4.196}$$

All versors can be translated by the translators, including the motor.

4.4.5 DCSA intersection

Inversions in the *standard* DCSA GIPNS 2-vector sphere $\mathbf{S}$ (§4.2.3), and reflections in the standard DCSA GIPNS 2-vector plane $\mathbf{\Pi}$ (§4.2.5) are valid operations on all DCSA GIPNS entities. The dilator D (§4.4.2) operation is defined by inversions in spheres. The rotor R (§4.4.1) operation is defined by reflections in planes. These are operations that are known to work correctly, based on inversions and reflections.

If any DCSA GIPNS 2-vector entity $\mathbf{\Upsilon}$ (§4.2.20) and sphere $\mathbf{S}$ (§4.2.3) *are intersecting* in curve $\mathbf{X}$ on $\mathbf{S}$, then the inversion of $\mathbf{\Upsilon}$ in the sphere $\mathbf{\Omega}=\mathbf{S\Upsilon S}^\sim$ is also intersecting with both $\mathbf{\Upsilon}$ and $\mathbf{S}$ through the same intersection $\mathbf{X}$. If entity $\mathbf{\Upsilon}$ and plane $\mathbf{\Pi}$ (§4.2.5) are intersecting, then the reflection of $\mathbf{\Upsilon}$ in the plane $\mathbf{\Pi\Upsilon\Pi}^\sim$ is also intersecting with both $\mathbf{\Upsilon}$ and $\mathbf{\Pi}$ through the same intersection. Planar reflection is a special case of spherical inversion when the sphere radius $r\to\infty$ and $\mathbf{S}\to\mathbf{\Pi}$ through three plane points. As $\mathbf{\Upsilon}$ becomes a sphere $\mathbf{S}_2$ or plane $\mathbf{\Pi}$ denoted by $\mathbf{\Upsilon}\to\mathbf{S}_2|\mathbf{\Pi}$, then $\mathbf{\Omega}\to\mathbf{\Pi}|\mathbf{S}_2$, and $\mathbf{X}\to\mathbf{C}$ to form a circular intersection, but $\mathbf{\Omega}$ and $\mathbf{X}$ have various possible surface and curve shapes when $\mathbf{\Upsilon}\neq\mathbf{S}_2|\mathbf{\Pi}$.

For the inversion, we have

$$\begin{aligned}
\mathbf{\Omega} &= \mathbf{S\Upsilon S}^\sim = (\mathbf{S}\cdot\mathbf{\Upsilon}+\mathbf{S}\times\mathbf{\Upsilon}+\mathbf{S}\wedge\mathbf{\Upsilon})\mathbf{S}^\sim && (4.197)\\
&= (\mathbf{S}\cdot\mathbf{\Upsilon})\mathbf{S}^\sim+(\mathbf{\Upsilon}\times\mathbf{S})\mathbf{S}-\mathbf{S}^2(\mathbf{\Upsilon}\wedge\mathbf{S})\mathbf{S}\mathbf{S}^{-2} && (4.198)\\
&= (\mathbf{S}\cdot\mathbf{\Upsilon})\mathbf{S}^\sim+\frac{1}{2}(\mathbf{\Upsilon S}-\mathbf{S\Upsilon})\mathbf{S}-\mathbf{S}^2\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon}) && (4.199)\\
&= 2(\mathbf{S}\cdot\mathbf{\Upsilon})\mathbf{S}^\sim+\mathbf{\Upsilon S}^2-2\mathbf{S}^2\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon}) && (4.200)\\
\mathbf{X}=\mathbf{\Omega}\wedge\mathbf{S} &= \mathbf{S}^2(\mathbf{\Upsilon}\wedge\mathbf{S})-2\mathbf{S}^2\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon})\mathbf{S}=-\mathbf{S}^2(\mathbf{\Upsilon}\wedge\mathbf{S}). && (4.201)
\end{aligned}$$

This expression of $\mathbf{X}$ implies that $\mathbf{X}$ represents *something* in common with both $\mathbf{\Upsilon}$ and $\mathbf{\Omega}$ in relation to $\mathbf{S}$. That something is their intersection, and $\mathbf{X}$ is the entity that constructs and represents their intersection.

The product $\times$ is the commutator product on 2-vectors that produces another 2-vector. The operation $\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon})=(\mathbf{\Upsilon}\wedge\mathbf{S})\mathbf{S}^{-1}$ is the perpendicular projection or *rejection* of $\mathbf{\Upsilon}$ from $\mathbf{S}$, and $\mathcal{P}_{\mathbf{S}}(\mathbf{\Upsilon})=(\mathbf{\Upsilon}\cdot\mathbf{S})\mathbf{S}^{-1}$ is the parallel *projection* of $\mathbf{\Upsilon}$ on $\mathbf{S}$ [17]. The operation $\mathcal{P}^\times_{\mathbf{S}}(\mathbf{\Upsilon})=(\mathbf{\Upsilon}\times\mathbf{S})\mathbf{S}^{-1}$ is another projection of $\mathbf{\Upsilon}$ on $\mathbf{S}$. For 2-vectors, $\mathbf{S}\times\mathbf{\Upsilon}=-\mathbf{\Upsilon}\times\mathbf{S}$, but $\mathbf{S}\wedge\mathbf{\Upsilon}=\mathbf{\Upsilon}\wedge\mathbf{S}$. Note that $\mathbf{S}^\sim=-\mathbf{S}=-\mathbf{S}^2\mathbf{S}^{-1}$, and $\mathbf{S}^2=-r^4$ where r is the radius of sphere $\mathbf{S}$. If $r\to\infty$, then $\mathbf{S}\to\mathbf{\Pi}$, $\mathbf{S}^2\to\mathbf{\Pi}^2=-1$, and $\mathbf{X}=\mathbf{\Omega}\wedge\mathbf{S}=\mathbf{\Upsilon}\wedge\mathbf{S}$.

The pair of inverse surface entities $\mathbf{\Upsilon}$ and $\mathbf{\Omega}$ could also be defined as

$$\begin{aligned}
\mathbf{\Upsilon} &= \mathbf{\Upsilon S S}^{-1}=\mathcal{P}_{\mathbf{S}}(\mathbf{\Upsilon})+\mathcal{P}^\times_{\mathbf{S}}(\mathbf{\Upsilon})+\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon}) && (4.202)\\
\mathbf{\Omega} &= \mathbf{S\Upsilon S}^{-1}=\mathcal{P}_{\mathbf{S}}(\mathbf{\Upsilon})-\mathcal{P}^\times_{\mathbf{S}}(\mathbf{\Upsilon})+\mathcal{P}^\perp_{\mathbf{S}}(\mathbf{\Upsilon}) && (4.203)
\end{aligned}$$

and then $\mathbf{X}=\mathbf{\Omega}\wedge\mathbf{S}=\mathbf{\Upsilon}\wedge\mathbf{S}$ exactly, but $\mathbf{\Omega}=\mathbf{S\Upsilon S}^\sim$ will be assumed henceforth.

The test for a point $\mathbf{T}_{\mathcal{DS}}$ on the intersection entity $\mathbf{X}$ is

$$\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{X} = \mathbf{T}_{\mathcal{DS}}\cdot(\mathbf{\Omega}\wedge\mathbf{S})=-\mathbf{S}^2\mathbf{T}_{\mathcal{DS}}\cdot(\mathbf{\Upsilon}\wedge\mathbf{S}) \qquad (4.204)$$

where if $\mathbf{T}_{\mathcal{DS}}\cdot\mathbf{X}=0$, then the point $\mathbf{T}_{\mathcal{DS}}$ is on the intersection of all three surfaces represented by $\mathbf{S}$, $\mathbf{\Upsilon}$, and $\mathbf{\Omega}$.

The DCSA GIPNS 4-vector intersection entity $\mathbf{X} = \boldsymbol{\Upsilon} \wedge \mathbf{S}$ is derived from the inversion of $\boldsymbol{\Upsilon}$ in $\mathbf{S}$. Proving this precisely may be simple if certain algebraic steps are taken correctly. $\boldsymbol{\Upsilon}, \boldsymbol{\Omega}$, and $\mathbf{X}$ are generally *not blades*, and $\mathbf{T}_{\mathcal{DS}}$ is a null 2-blade. Most of the usual algebraic identities are valid only on blades that are the product of non-null vectors or blades. Therefore, the algebraic steps leading to a clear proof may require unusual identities or other results.

DCSA GIPNS intersection entities $\boldsymbol{\Upsilon} \wedge \boldsymbol{\Pi}$, $\boldsymbol{\Upsilon} \wedge \mathbf{L}$, and $\boldsymbol{\Upsilon} \wedge \mathbf{C}$, for intersections of any DCSA GIPNS entity $\boldsymbol{\Upsilon}$ with the standard DCSA GIPNS 2-vector plane $\boldsymbol{\Pi}$ (§4.2.5), 4-vector line $\mathbf{L}$ (§4.2.4), and 4-vector circle $\mathbf{C}$ (§4.2.6), are also derived from reflections or inversions. Line $\mathbf{L} = \boldsymbol{\Pi}_1 \wedge \boldsymbol{\Pi}_2$ and circle $\mathbf{C} = \mathbf{S} \wedge \boldsymbol{\Pi}$ are just intersections of sphere and plane entities.

In DCSA, inversions or reflections work only in the standard DCSA 2-vector sphere $\mathbf{S}$ (§4.2.3) and plane $\boldsymbol{\Pi}$ (§4.2.5). In DCSTA, inversions or reflections work only in the standard DCSTA 2-vector hyperpseudospheres $\boldsymbol{\Sigma}_{\mathcal{D}}$ (§7.3.3) and $\boldsymbol{\Xi}_{\mathcal{D}}$ (§7.3.4), and in the DCSTA 2-vector hyperplane $\boldsymbol{E}_{\mathcal{D}}$ (§7.3.2). Inversions do not work in other DCSA entities, and therefore other entities cannot form intersection entities with each other. For example, the DCSA GIPNS 2-vector quadric surface entities do not work as inversion or reflection operators, and therefore they cannot form intersection entities with each other. The intersection of two entities depends on at least one of them being a valid inversion operator on the other entity.

In symbolic calculations, the simplest intersection entities, such as $\mathbf{X} = \boldsymbol{\Upsilon} \wedge \mathbf{S}$, are 4-vectors with many 4-blade terms or components. The scalar magnitude of each 4-blade is an implicit surface function for a surface that is coincident with the intersection represented by $\boldsymbol{\Upsilon} \wedge \mathbf{S}$. Not every blade holds a unique implicit surface function, but the number of unique functions can exceed ten. Figures 4.29 and 4.30 are plots of intersections that are showing all of the unique implicit surfaces that are extracted from the blades of the intersection entities.

The foregoing discussion has not given any rigorous proof of the correctness of the intersection entities. $\mathcal{G}_{2,8}$ DCSA is a large and complicated pseudo-Euclidean algebra, and there could be unforeseen cases where intersections do not work as expected. Therefore, the following box serves as a mild warning before continuing.

> Although not rigorously proved here, the intersection tests performed by this author supported the following claims given in this subsection about DCSA intersection. Detailed examinations of ellipsoid-plane and ellipsoid-sphere intersections are shown in Figures 4.29 and 4.30. These claims should be considered preliminary, and require additional research to prove for certain what intersections are valid or invalid.

The set $\mathcal{S} = \{\mathbf{S}, \boldsymbol{\Pi}\}$ of *standard bi-CSA GIPNS entities* includes all instances of the DCSA GIPNS 2-vector sphere $\mathbf{S}$ (§4.2.3) and plane $\boldsymbol{\Pi}$ (§4.2.5). These two entities are defined in previous sections on them. The DCSA GIPNS 4-vector line $\mathbf{L} = \boldsymbol{\Pi}_1 \wedge \boldsymbol{\Pi}_2$ (§4.2.4) and circle $\mathbf{C} = \mathbf{S} \wedge \boldsymbol{\Pi}$ (§4.2.6) are extended standard bi-CSA GIPNS entities that are the intersections of spheres and planes.

The DCSA GIPNS *intersection* entity $\mathbf{X}$ is the wedge of $2 \le n \le 4$ standard bi-CSA GIPNS entities $\mathbf{B}_i \in \mathcal{S}$, or is the wedge of $1 \le n \le 3$ entities $\mathbf{B}_i \in \mathcal{S}$ and *one* DCSA GIPNS 2-vector entity $\mathbf{A}_{\langle 2 \rangle} \notin \mathcal{S}$ that is not a standard bi-CSA GIPNS entity. Only *one* DCSA GIPNS 2-vector Darboux cyclide surface entity $\mathbf{A}_{\langle 2 \rangle} = \boldsymbol{\Omega}$ (§4.2.20) (or any degenerate) can be included in a wedge that forms an intersection entity $\mathbf{X}$. Unfortunately, the Darboux cyclide entities, including the quadric surfaces, cannot be intersected directly with each other by wedge products since they are invalid inversion operators. These claims are summarized by the following definition.

The DCSA GIPNS *intersection* entity $\mathbf{X}$ of grade $4 \le k \le 8$ is defined as

$$\mathbf{X}_{\langle 4\le k\le 8\rangle} = \begin{cases} \bigwedge_{i=1}^{2\le n\le 4} \mathbf{B}_i & : \ \mathbf{B}_i \in \mathcal{S} \text{ and } \mathcal{S} = \{\mathbf{S}, \mathbf{\Pi}\} \\ \mathbf{A}_{\langle 2\rangle} \wedge \bigwedge_{i=1}^{1\le n\le 3} \mathbf{B}_i & : \ \mathbf{A}_{\langle 2\rangle} \notin \mathcal{S} \text{ and } \mathbf{B}_i \in \mathcal{S}. \end{cases} \tag{4.205}$$

The maximum grade for a valid intersection entity $\mathbf{X}$ is grade 8. The grade of the wedges is divisible by 2, making the next grade above 8 to be 10, proportional to the DCSA unit pseudoscalar $\mathbf{I}_{\mathcal{DS}}$. No valid entity is a pseudoscalar, unless the whole 3D space is considered to be a flat entity.

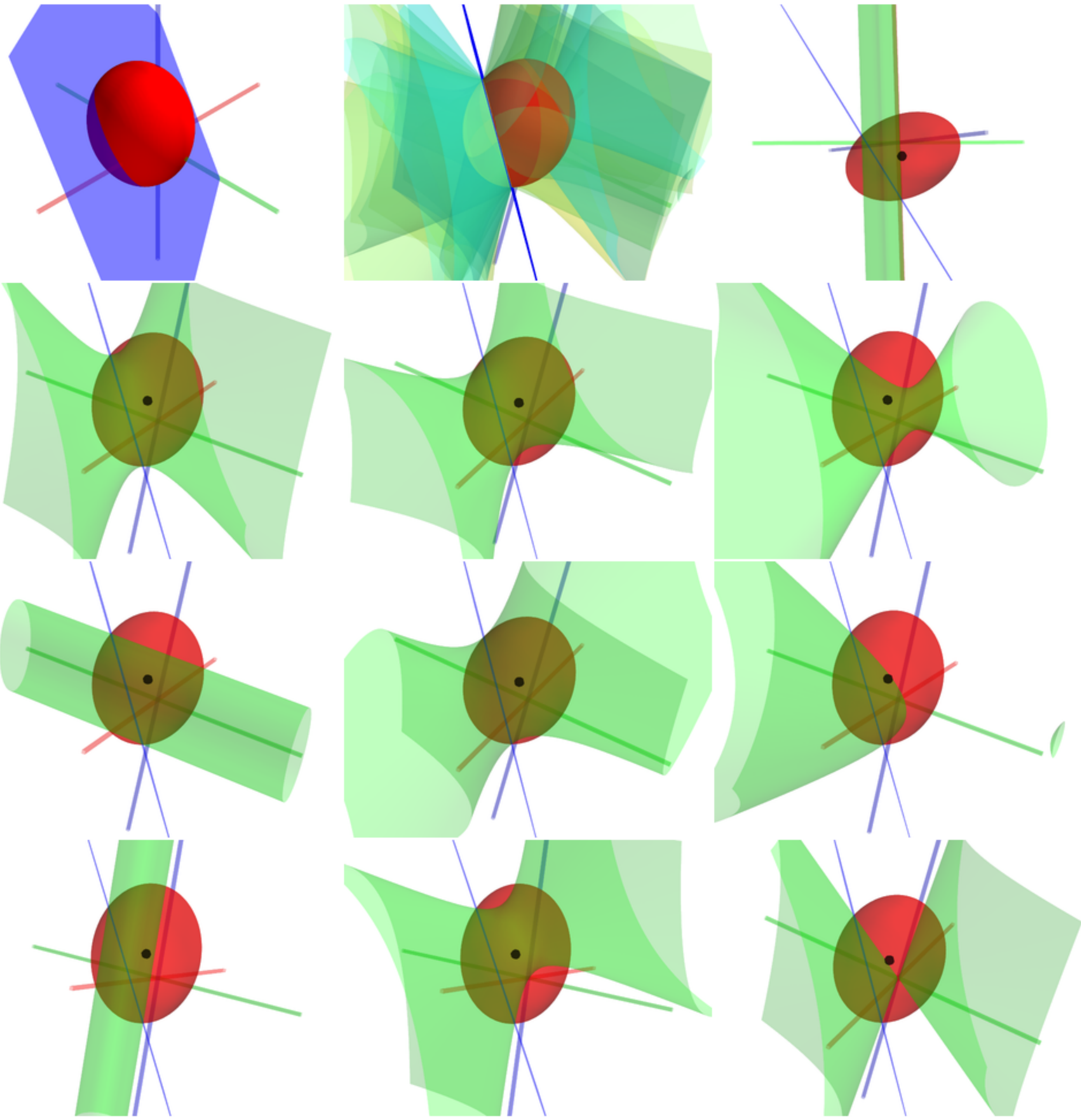

Figure 4.29. Intersection of ellipsoid and plane in general positions

Figure 4.29 shows the details of a DCSA GIPNS 4-vector intersection entity $\mathbf{E} \wedge \mathbf{\Pi}$ representing the intersection of a DCSA GIPNS 2-vector ellipsoid $\mathbf{E}$ (§4.2.2) and DCSA GIPNS 2-vector plane $\mathbf{\Pi}$ (§4.2.5), both rotated and translated differently into general positions that have an intersection. The red ellipsoid $\mathbf{E}$ has initial parameters $r_x = 5$, $r_y = 7$, $r_z = 9$, $p_x = 1$, $p_y = -2$, $p_z = 3$, and is then rotated 30° around the blue z-axis. The *Sympy* test code for the ellipsoid was:

```
Rotor(e4,30*pi*Pow(180,-1))*
GIPNS_Ellipsoid(1,-2,3,5,7,9)*
Rotor(e4,30*pi*Pow(180,-1)).rev()
```

The black dot is the ellipsoid center position. The blue plane $\mathbf{\Pi}$ is initially perpendicular to the x-axis through the origin, then transformed according to the following code:

```
Rotor(e2,30*pi*Pow(180,-1))*
Translator(-4*e3)*
Rotor(e4,-60*pi*Pow(180,-1))*
GIPNS_Plane(e2,0)*
Rotor(e4,-60*pi*Pow(180,-1)).rev()*
Translator(-4*e3).rev()*
Rotor(e2,30*pi*Pow(180,-1)).rev()
```

Their DCSA GIPNS intersection is $\mathbf{X} = \mathbf{E} \wedge \mathbf{\Pi}$. The various images in Figure 4.29 show components of $\mathbf{X}$ that represent other surfaces that are all coincident with the intersection of the ellipsoid and plane. There were ten unique components in $\mathbf{X}$. These components are cylinders, hyperboloids, and a cone. The intersection entity $\mathbf{X}$ represents the locus of points that are simultaneously located on all ten of these surfaces, which is an ellipse-shaped intersection of the ellipsoid and plane.

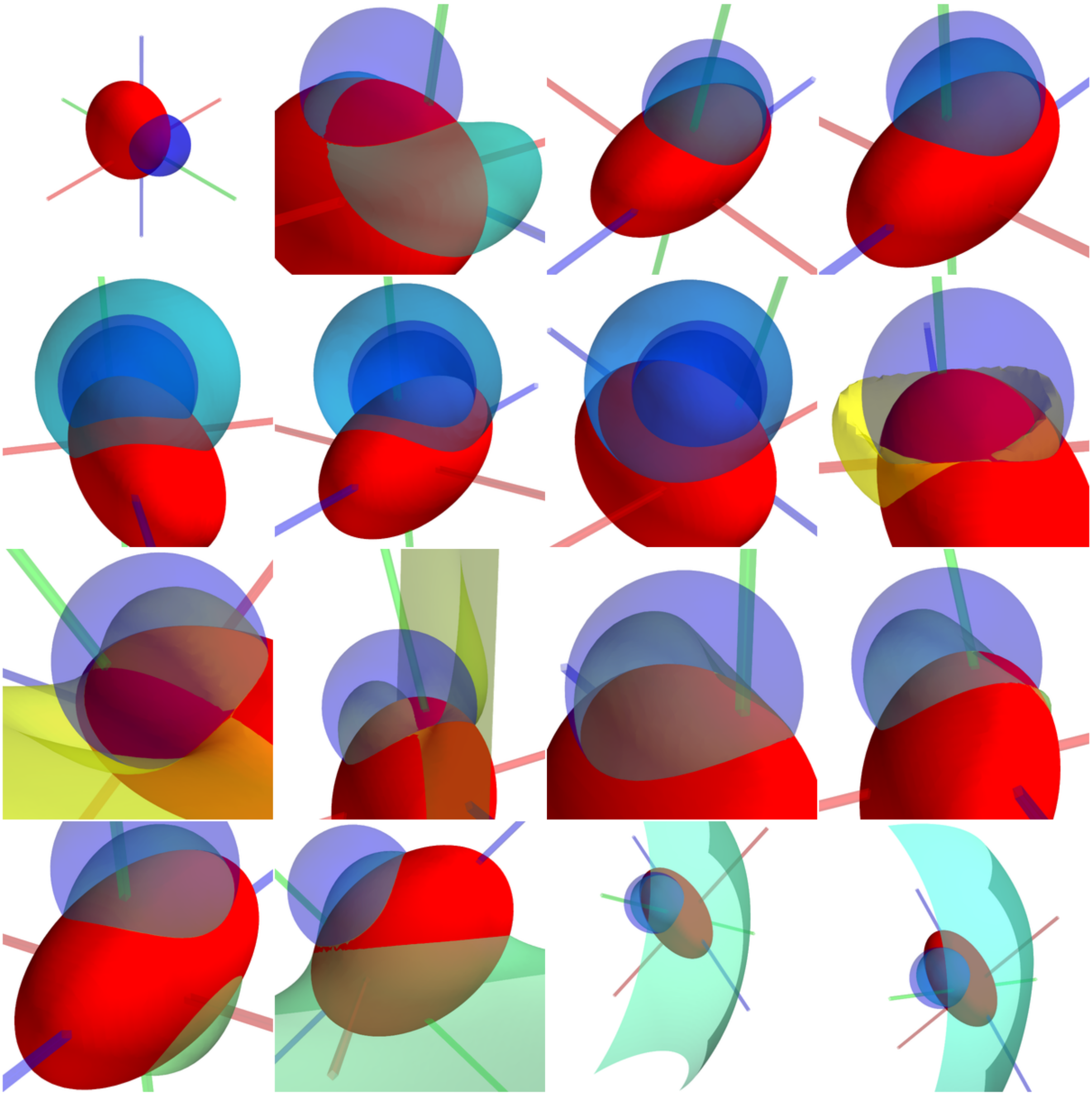

Figure 4.30. Intersection of ellipsoid and sphere in general positions

Figure 4.30 shows the same red DCSA GIPNS ellipsoid $\mathbf{E}$ as in Figure 4.29, but now intersected with a blue DCSA GIPNS sphere $\mathbf{S}$ of radius $r=5$ at position $\boldsymbol{\gamma}_1+5\boldsymbol{\gamma}_2+3\boldsymbol{\gamma}_3 \cong \mathbf{e}_2+5\mathbf{e}_3+3\mathbf{e}_4$. The DCSA GIPNS intersection entity is now $\mathbf{X}=\mathbf{E}\wedge\mathbf{S}$. The shape of the intersection appears like a curved ellipse or curved circle. The components of the entity $\mathbf{X}$ represent 15 other unique surfaces that are also coincident with the intersection of $\mathbf{E}$ and $\mathbf{S}$. The images of Figure 4.30 show how each of these 15 surfaces intersect with the intersection of $\mathbf{E}$ and $\mathbf{S}$. Some of these surfaces are unusually shaped, and some have two sheets. The DCSA GIPNS intersection entity $\mathbf{X}$ represents the simultaneous locus or intersection of all of the involved surfaces and appears to be a valid intersection entity for the ellipsoid and sphere.

The DCSA GIPNS 2-vector quadric surface entities, of the types not available in CSA, could *not* be wedged with each other to form valid intersection entities - *incorrect* or *invalid* intersection entities resulted from their wedge. More generally, the DCSA GIPNS Darboux cyclide entities $\boldsymbol{\Omega}$ cannot be intersected with each other by wedge products, but *one* can be intersected with standard bi-CSA GIPNS entities. As a curiosity, it was noticed that the sum and the difference of two intersecting DCSA GIPNS quadric surface entities represent two more coincident intersecting surfaces.

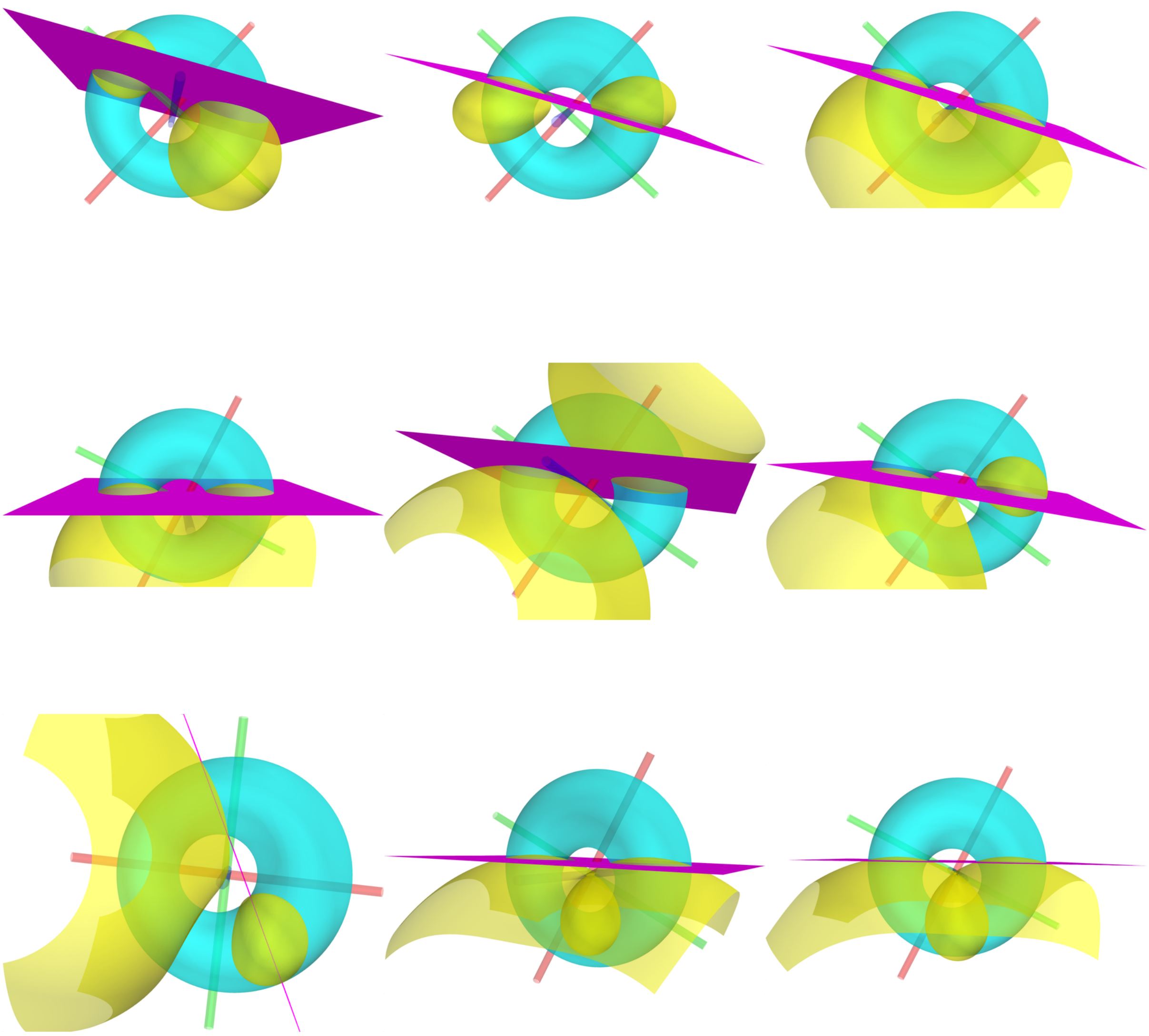

Figure 4.31. Intersection $\mathbf{\Phi}\wedge\mathbf{\Pi}$ of ring Dupin cyclide $\mathbf{\Phi}$ and plane $\mathbf{\Pi}$

4.4.6 DCSA dualization

The DCSA *unit pseudoscalar* $\mathbf{I}_{\mathcal{DS}}$ is defined as

$$\begin{aligned}\mathbf{I}_{\mathcal{DS}} &= \mathbf{I}_{\mathcal{CS}^1}\wedge\mathbf{I}_{\mathcal{CS}^2} && (4.206)\\ &= \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12} && (4.207)\end{aligned}$$

and is the DCSA dualization operator on all DCSA entities.

Properties of $\mathbf{I}_{\mathcal{DS}}$ include

$$\begin{aligned}\mathbf{I}_{\mathcal{DS}}^{\sim} &= (-1)^{10(10-1)/2}\mathbf{I}_{\mathcal{DS}} = -\mathbf{I}_{\mathcal{DS}} && (4.208)\\ \mathbf{I}_{\mathcal{DS}}^2 &= -\mathbf{I}_{\mathcal{DS}}\mathbf{I}_{\mathcal{DS}}^{\sim} = -1 && (4.209)\\ \mathbf{I}_{\mathcal{DS}}^{-1} &= \mathbf{I}_{\mathcal{DS}}^{\sim} = -\mathbf{I}_{\mathcal{DS}}. && (4.210)\end{aligned}$$

According to the sign rule $(-1)^{r(10-1)}$ for the commutation of the inner product of two blades, the DCSA unit pseudoscalar $\mathbf{I}_{\mathcal{DS}}$ commutes with blades of even grade r, such as the DCSA 2-vector points, DCSA GIPNS 2,4,6,8-vector surfaces, and their dual DCSA GOPNS surfaces.

A DCSA GIPNS k-vector surface entity $\mathbf{X}$ is *dualized* into its *dual* DCSA GOPNS $(10-k)$-vector surface entity $\mathbf{X}^{*\mathcal{DS}}$ as

$$\mathbf{X}^{*\mathcal{DS}} = \mathbf{X}/\mathbf{I}_{\mathcal{DS}} = -\mathbf{X}\cdot\mathbf{I}_{\mathcal{DS}}. \tag{4.211}$$

A DCSA GOPNS k-vector surface entity $\mathbf{X}^{*\mathcal{DS}}$ is *undualized* into its *undual* DCSA GIPNS $(10-k)$-vector surface entity $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{X}^{*\mathcal{DS}}\mathbf{I}_{\mathcal{DS}} = \mathbf{X}^{*\mathcal{DS}}\cdot\mathbf{I}_{\mathcal{DS}}. \tag{4.212}$$

This definition of dual and undual preserves the sign on the entities, otherwise the dual applied twice changes signs.

It is understandable that many authors may call the GIPNS entities *dual* and the GOPNS entities *direct*, *standard*, or *undual*, but since in DCSA we cannot wedge DCSA points into all of the GOPNS entities, the GIPNS entities are considered the *undual* entities and the GOPNS entities are the *dual* entities. Most of the DCSA GOPNS entities can only be obtained by the dualization operation as duals.

In DCSTA, a DCSA (or DCSTA) GIPNS 2-vector *quadric* entity $\mathbf{X}$ (at zero velocity) and its DCSTA dual (§7.7.1), the DCSTA GOPNS $(12-2)$-vector quadric entity $\mathbf{X}^{*\mathcal{D}}$, are independent of time $w=ct$, but the DCSA dual, the DCSA GOPNS $(10-2)$-vector entity $\mathbf{X}^{*\mathcal{DS}}$, is at $w=ct=0$. A DCSA quadric $\mathbf{X}$ that has been boosted (§7.7.3) is a DCSTA quadric $\boldsymbol{X}$ that moves with the velocity of the boost and is length-contracted, consistent with special relativity length contraction $L=\sqrt{1-\beta^2}L_0$. The DCSA GIPNS 2-vector *cubic* and *quartic* entities are dependent on time w as DCSTA entities, but their DCSA duals are at time $w=0$.

4.5 DCSA computing using Gaalop

In [7], $\mathcal{G}_{8,2}$ DCGA computing using Gaalop is discussed, and listings of sample code are included. The computations are nearly the same for $\mathcal{G}_{2,8}$ DCSA, with just a few changes in signs.

4.6 Concluding remarks on DCSA

In conclusion, this section on $\mathcal{G}_{2,8}$ DCSA is directly based on [7] and there are actually very few differences between $\mathcal{G}_{2,8}$ DCSA and $\mathcal{G}_{8,2}$ DCGA other than sign changes. Therefore, this section has been quite redundant but is included for completeness. Charges of self-plagiarism shall be dismissed.

An important difference between DCSA and DCGA is apparent in vector reflections. In $\mathcal{G}_3$ *Algebra of Physical Space* (APS), the reflection of vector $\mathbf{v}$ in unit vector $\hat{\mathbf{u}}$ is

$$\hat{\mathbf{u}}(\mathbf{v}^{||\hat{\mathbf{u}}}+\mathbf{v}^{\perp\hat{\mathbf{u}}})\hat{\mathbf{u}} = \mathbf{v}^{||\hat{\mathbf{u}}}-\mathbf{v}^{\perp\hat{\mathbf{u}}}, \tag{4.213}$$

and in $\mathcal{G}_{0,3}$ *Space Algebra* (SA), the reflection is

$$\hat{\mathbf{u}}(\mathbf{v}^{||\hat{\mathbf{u}}}+\mathbf{v}^{\perp\hat{\mathbf{u}}})\hat{\mathbf{u}} = -\mathbf{v}^{||\hat{\mathbf{u}}}+\mathbf{v}^{\perp\hat{\mathbf{u}}}. \tag{4.214}$$

The APS reflection $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ reflects $\mathbf{v}$ in *line* $\hat{\mathbf{u}}$, but the SA reflection $\hat{\mathbf{u}}\mathbf{v}\hat{\mathbf{u}}$ reflects $\mathbf{v}$ in the *plane* $\hat{\mathbf{u}}$ that is through the origin and orthogonal to vector $\hat{\mathbf{u}}$.

In APS, a unit vector $\hat{\mathbf{u}}$, according to its action as a reflection operator, naturally represents a line through the origin in direction $\hat{\mathbf{u}}$. In CGA, $\hat{\mathbf{u}}$ is a CGA GIPNS plane entity $\mathbf{\Pi} = \hat{\mathbf{u}}$, or it can form a CGA GOPNS line entity $\mathbf{L}^* = \hat{\mathbf{u}} \wedge E = \hat{\mathbf{u}} \wedge \mathbf{e}_\infty \wedge \mathbf{e}_o$. Although $\hat{\mathbf{u}}$ is naturally an APS line entity according to its action as a reflection operator, $\hat{\mathbf{u}}$ is not a CGA line entity. By the reflection operation, there appears to be an incongruency between APS and CGA, where $\hat{\mathbf{u}}$ is a line entity in APS and a plane entity in CGA.

In SA, a unit vector $\hat{\mathbf{u}}$, according to its action as a reflection operator, naturally represents a plane through the origin perpendicular to $\hat{\mathbf{u}}$. In CSA, $\hat{\mathbf{u}}$ is a CSA GIPNS plane entity $\mathbf{\Pi} = \hat{\mathbf{u}}$. In SA and CSA, $\hat{\mathbf{u}}$ is a plane entity.

On the other hand, reflection could be defined as

$$\hat{\mathbf{u}}(\mathbf{v}^{||\hat{\mathbf{u}}} + \mathbf{v}^{\perp\hat{\mathbf{u}}})\hat{\mathbf{u}}^{-1} = \mathbf{v}^{||\hat{\mathbf{u}}} - \mathbf{v}^{\perp\hat{\mathbf{u}}}, \tag{4.215}$$

which is reflection in the line $\hat{\mathbf{u}}$ in APS and SA. However, using the inverse is often avoided in favor of using the reverse $\hat{\mathbf{u}}^{\sim} = \hat{\mathbf{u}}$, which makes a difference. More arguments could be made on specific formulas for reflections in planes or in lines, but these remarks conclude here.

5 Space-Time Algebra (STA)

Space-Time Algebra (STA) is introduced in the book *Space-Time Algebra* by DAVID HESTENES [16]. STA is also called DIRAC Algebra (DA). As explained in [16], the *space-time split* generates a PAULI Algebra (PA) on a unit bivector basis. DCSTA contains two STA $\mathcal{M}$ subalgebras, STA1 $\mathcal{M}^1$ and STA2 $\mathcal{M}^2$.

The $\mathcal{M}$ is for MINKOWSKI spacetime $(1,3)$ and is the subscript that denotes an element or operation in STA. The subscript $\mathcal{M}^1$ denotes an element or operation in STA1. The subscript $\mathcal{M}^2$ denotes an element or operation in STA2.

5.1 STA elements

5.1.1 Dirac gammas and Pauli sigmas in STA

The DIRAC gammas and PAULI sigmas can be defined in STA1 as

$$\boldsymbol{\gamma}_i = \begin{cases} \mathbf{e}_{i+1} & : i \in \{0,1,2,3\} \\ \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 & : i = 5 \end{cases} \tag{5.1}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_x = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 \tag{5.2}$$

$$\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_y = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 \tag{5.3}$$

$$\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_z = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0. \tag{5.4}$$

The STA elements can also be defined similarly in STA2. The DIRAC gammas and PAULI sigmas are represented as matrices in other literature, but they have multivector representations in STA. See reference [16] for more information about these representations.

The gammas are used to denote elements in STA $\mathcal{M}$, but it should be understood that all discussions of STA $\mathcal{M}$ apply similarly in STA1 $\mathcal{M}^1$ and STA2 $\mathcal{M}^2$ by changing subscripting and elements

$$\begin{array}{lllll} \text{STA} & \cong & \text{STA1} & \cong & \text{STA2} \\ \mathcal{M} & \cong & \mathcal{M}^1 & \cong & \mathcal{M}^2 \\ \boldsymbol{\gamma}_0 & \cong & \mathbf{e}_1 & \cong & \mathbf{e}_7 \\ \boldsymbol{\gamma}_1 & \cong & \mathbf{e}_2 & \cong & \mathbf{e}_8 \\ \boldsymbol{\gamma}_2 & \cong & \mathbf{e}_3 & \cong & \mathbf{e}_9 \\ \boldsymbol{\gamma}_3 & \cong & \mathbf{e}_4 & \cong & \mathbf{e}_{10}. \end{array} \tag{5.5}$$

5.1.2 STA unit pseudoscalar

The $\mathcal{G}_{1,3}$ STA 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 = \boldsymbol{\gamma}_5 \tag{5.6}$$

$$\mathbf{I}_{\mathcal{M}}^{\sim} = (-1)^{4(4-1)/2}\mathbf{I}_{\mathcal{M}} = \mathbf{I}_{\mathcal{M}} \tag{5.7}$$

$$\mathbf{I}_{\mathcal{M}}^2 = -1 \tag{5.8}$$

$$\mathbf{I}_{\mathcal{M}}^{-1} = -\mathbf{I}_{\mathcal{M}} = -\mathbf{I}_{\mathcal{M}}^{\sim}. \tag{5.9}$$

The $\mathcal{G}_{1,3}$ STA1 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}^1}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}^1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4. \tag{5.10}$$

The $\mathcal{G}_{1,3}$ STA2 4-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}^2}$ with signature $(+---)$ is

$$\mathbf{I}_{\mathcal{M}^2} = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}. \tag{5.11}$$

5.1.3 STA test vector

The symbolic STA *test vector* $\boldsymbol{t}_{\mathcal{M}}$ is defined on the basis of the DIRAC gammas [16] as

$$\boldsymbol{t} = \boldsymbol{t}_{\mathcal{M}} = w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 = ct\boldsymbol{\gamma}_0 + \mathbf{t}_{\mathcal{S}} = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{t}_{\mathcal{S}}. \tag{5.12}$$

The symbolic STA1 *test vector* $\boldsymbol{t}_{\mathcal{M}^1}$ is defined as

$$\boldsymbol{t}_{\mathcal{M}^1} = w\mathbf{e}_1 + x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_4 = ct\mathbf{e}_1 + \mathbf{t}_{\mathcal{S}^1} = \boldsymbol{o}_{\mathcal{M}^1}t + \mathbf{t}_{\mathcal{S}^1}. \tag{5.13}$$

The symbolic STA2 *test vector* $\boldsymbol{t}_{\mathcal{M}^2}$ is defined as

$$\boldsymbol{t}_{\mathcal{M}^2} = w\mathbf{e}_7 + x\mathbf{e}_8 + y\mathbf{e}_9 + z\mathbf{e}_{10} = ct\mathbf{e}_7 + \mathbf{t}_{\mathcal{S}^2} = \boldsymbol{o}_{\mathcal{M}^2}t + \mathbf{t}_{\mathcal{S}^2}. \tag{5.14}$$

The symbolic scalars w, x, y, and z are the conventional coordinates in spacetime. The *observer* with coordinate time t is identified with the worldline $\boldsymbol{o}_{\mathcal{M}}t = ct\boldsymbol{\gamma}_0$.

The symbolic test vector $\boldsymbol{t}_{\mathcal{M}}$ is useful in symbolic computations and will be embedded as the $\mathcal{G}_{2,4}$ CSTA *test point* $\boldsymbol{T}_{\mathcal{C}}$. $\mathcal{G}_{2,4}$ CSTA1 and CSTA2 test points $\boldsymbol{T}_{\mathcal{C}^1}$ and $\boldsymbol{T}_{\mathcal{C}^2}$, respectively, are multiplied to form the $\mathcal{G}_{4,8}$ DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{C}^1}\boldsymbol{T}_{\mathcal{C}^2} = \boldsymbol{T}_{\mathcal{C}^1} \wedge \boldsymbol{T}_{\mathcal{C}^2}$. The DCSTA point 2-vector value-*extraction elements* T_s are defined as inner product operators that extract values s from $\boldsymbol{T}_{\mathcal{D}}$ as $s = T_s \cdot \boldsymbol{T}_{\mathcal{D}}$. Linear combinations of the elements T_s form the 2-vector DCSTA entities for quadric surfaces and cyclides.

5.1.4 STA observer

An STA $\mathcal{M}$ *observer position* (or *worldline*) $\boldsymbol{o}_{\mathcal{M}}t$ at the observer's proper (coordinate) time t is

$$\boldsymbol{o}t = \boldsymbol{o}_{\mathcal{M}}t = ct\boldsymbol{\gamma}_0. \tag{5.15}$$

For all times t, $\boldsymbol{o}t$ symbolically represents the worldline of the observer that passes through the origin of spacetime. An observable worldline can be represented non-symbolically in $\mathcal{G}_{2,4}$ CSTA as a CSTA GIPNS 3-vector line entity $\boldsymbol{L}_{\mathcal{C}}$ (§6.4.11).

An STA1 $\mathcal{M}^1$ observer position $\boldsymbol{o}_{\mathcal{M}^1}t$ has the form

$$\boldsymbol{o}_{\mathcal{M}^1}t \;=\; ct\mathbf{e}_1. \tag{5.16}$$

An STA2 $\mathcal{M}^2$ observer position $\boldsymbol{o}_{\mathcal{M}^2}t$ has the form

$$\boldsymbol{o}_{\mathcal{M}^2}t \;=\; ct\mathbf{e}_7. \tag{5.17}$$

An STA *translated-observer position* $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$, translated by spatial position $\mathbf{p}_0$, has the form

$$\boldsymbol{o}^{\mathbf{p}_0}(t) = \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) \;=\; \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0 = ct\boldsymbol{\gamma}_0 + \mathbf{p}_0. \tag{5.18}$$

The observer worldline $\boldsymbol{o}_{\mathcal{M}}t$ of a translated-observer worldline $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ is

$$\boldsymbol{o}_{\mathcal{M}}t \;=\; \dot{\boldsymbol{o}}_{\mathcal{M}}^{\mathbf{p}_0}(t) = \frac{\mathrm{d}\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)}{\mathrm{d}t}t = ct\boldsymbol{\gamma}_0. \tag{5.19}$$

For a translated-observer position $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$, its spacetime velocity is $\boldsymbol{o}_{\mathcal{M}}$. The translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ represents the observer $\boldsymbol{o}_{\mathcal{M}}t$ at $\mathbf{p}_0$ and is the proper observer of any observable with initial position $\mathbf{p}_0$ at time $t=0$. Boosts, spacetime contractions (dilations), and other spacetime transformations can be translated relative to $\mathbf{p}_0$ (e.g., translated-boost, translated-dilator) when transforming an observable with initial position $\mathbf{p}_0$.

In special relativity, the observer $\boldsymbol{o}_{\mathcal{M}}t$ and an observable particle $\boldsymbol{p}_{\mathcal{M}}$ that is being observed must coincide at the observer's proper time $t=0$ (i.e., their worldlines should intersect in $\mathbf{p}_0$ at time $t=0$). If the particle and observer do not coincide at $t=0$, then the particle has an initial spatial position $\mathbf{p}_0$ at $t=0$ and

$$\boldsymbol{p}_{\mathcal{M}}(t) \;=\; \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t. \tag{5.20}$$

The translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0$ and particle then coincide at $t=0$, and $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ can be called the *proper observer* of $\boldsymbol{p}_{\mathcal{M}}(t)$. Hyperbolic rotations (boosts) that transform a particle must be relative to a proper observer that is coincident with the particle at $t=0$, which is the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$. The *translated-boost* (§6.6.9) relative to a translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ is similar to a rotation around a general line through a shifted or translated origin. A translated-boost is achieved as a translation of $\mathbf{p}_0$ to the origin (translaton by $-\mathbf{p}_0$), then a boost relative to $\boldsymbol{o}_{\mathcal{M}}t$, and then a translation back by $\mathbf{p}_0$. The translated-boost can be followed by a *translated-dilation* (§6.6.7) around $\mathbf{p}_0$ with dilation factor $1/\gamma$ for a *spacetime contraction* that exits the old observer frame of coordinate time t and enters the new observer frame of proper time τ ($\gamma = \mathrm{d}t/\mathrm{d}\tau$), where τ becomes the new coordinate time. The value of γ depends on the particular boosts and velocities involved, according to velocity additions or subtractions, where $\boldsymbol{o} = \gamma c\boldsymbol{\gamma}_0$ such that γ can always be extracted (after any number of boosts) from a spacetime velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ as $\boldsymbol{v}\cdot\boldsymbol{\gamma}_0/c = \gamma$. For one simple boost, γ is the Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$, but γ is different after successive boosts.

5.1.5 STA spatial velocity

An STA *spatial velocity* $\mathbf{v}_{\mathcal{S}}$ has the form

$$\mathbf{v} = \mathbf{v}_{\mathcal{S}} \;=\; v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3 = \beta c\hat{\mathbf{v}}. \tag{5.21}$$

An STA1 *spatial velocity* $\mathbf{v}_{\mathcal{S}^1}$ has the form

$$\mathbf{v}_{\mathcal{S}^1} \;=\; v_x\mathbf{e}_2 + v_y\mathbf{e}_3 + v_z\mathbf{e}_4. \tag{5.22}$$

An STA2 *spatial velocity* $\mathbf{v}_{\mathcal{S}^2}$ has the form

$$\mathbf{v}_{\mathcal{S}^2} = v_x\mathbf{e}_8 + v_y\mathbf{e}_9 + v_z\mathbf{e}_{10}. \tag{5.23}$$

STA spatial velocities are the same as SA spatial velocities. The v_x, v_y, and v_z are coordinate speeds in the conventional x, y, and z directions.

The non-negative *norm* of an SA spatial velocity $\mathbf{v}_{\mathcal{S}}$ is the speed

$$\|\mathbf{v}_{\mathcal{S}}\| = \sqrt{-\mathbf{v}_{\mathcal{S}}^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}. \tag{5.24}$$

In special relativity, speed cannot exceed light speed c,

$$0 \le \|\mathbf{v}_{\mathcal{S}}\| \le c. \tag{5.25}$$

The *unit* direction of an STA or SA *spatial velocity* $\mathbf{v}_{\mathcal{S}}$ is

$$\hat{\mathbf{v}}_{\mathcal{S}} = \frac{\mathbf{v}_{\mathcal{S}}}{\|\mathbf{v}_{\mathcal{S}}\|}. \tag{5.26}$$

5.1.6 STA spatial position

An STA *spatial position* $\mathbf{p}_{\mathcal{S}}$ has the form

$$\mathbf{p}(t) = \mathbf{p}_{\mathcal{S}}(t) = \mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t = \mathbf{p}_0 + \beta c\hat{\mathbf{v}}_{\mathcal{S}}t = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3. \tag{5.27}$$

An STA1 *spatial position* $\mathbf{p}_{\mathcal{S}^1}$ has the form

$$\mathbf{p}_{\mathcal{S}^1}(t) = \mathbf{p}_{0\mathcal{S}^1} + \mathbf{v}_{\mathcal{S}^1}t = \mathbf{p}_{0\mathcal{S}^1} + \beta c\hat{\mathbf{v}}_{\mathcal{S}^1}t = p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_3. \tag{5.28}$$

An STA2 *spatial position* $\mathbf{p}_{\mathcal{S}^2}$ has the form

$$\mathbf{p}_{\mathcal{S}^2}(t) = \mathbf{p}_{0\mathcal{S}^2} + \mathbf{v}_{\mathcal{S}^2}t = \mathbf{p}_{0\mathcal{S}^2} + \beta c\hat{\mathbf{v}}_{\mathcal{S}^2}t = p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. \tag{5.29}$$

In special relativity, the time t is called the *coordinate time* and is the *proper time* of the *observer* $\boldsymbol{o}_{\mathcal{M}}$. The spatial position $\mathbf{p}_{\mathcal{S}}$ is relative to the observer $\boldsymbol{o}_{\mathcal{M}}$ (§5.1.4) as the spacetime position (§5.1.8)

$$\begin{aligned} \boldsymbol{p}_{\mathcal{M}}(t) &= \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_{\mathcal{S}}(t) = \boldsymbol{o}_{\mathcal{M}}t + (\mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t) & (5.30)\\ &= (\boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0) + \mathbf{v}_{\mathcal{S}}t = \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) + \mathbf{v}_{\mathcal{S}}t & (5.31)\\ &= \mathbf{p}_0 + (\boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}})t = \mathbf{p}_0 + \boldsymbol{v}_{\mathcal{M}}t. & (5.32) \end{aligned}$$

The spacetime velocity is $\boldsymbol{v}_{\mathcal{M}}$ (§5.1.7). The spacetime position $\boldsymbol{p}_{\mathcal{M}}$ (§5.1.8) is relative to the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ (§5.1.4) such that $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_0$ at time $t = 0$.

5.1.7 STA spacetime velocity

An STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ has the form

$$\boldsymbol{v} = \boldsymbol{v}_{\mathcal{M}} = \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}} = c\boldsymbol{\gamma}_0 + \beta c\hat{\mathbf{v}}_{\mathcal{S}} = c\boldsymbol{\gamma}_0 + (v_x\boldsymbol{\gamma}_1 + v_y\boldsymbol{\gamma}_2 + v_z\boldsymbol{\gamma}_3). \tag{5.33}$$

An STA1 *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}^1}$ has the normalized form

$$\boldsymbol{v}_{\mathcal{M}^1} = \boldsymbol{o}_{\mathcal{M}^1} + \mathbf{v}_{\mathcal{S}^1} = c\mathbf{e}_1 + \beta c\hat{\mathbf{v}}_{\mathcal{S}^1} = c\mathbf{e}_1 + (v_x\mathbf{e}_2 + v_y\mathbf{e}_3 + v_z\mathbf{e}_4). \tag{5.34}$$

An STA2 *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}^2}$ has the normalized form

$$\boldsymbol{v}_{\mathcal{M}^2} = \boldsymbol{o}_{\mathcal{M}^2} + \mathbf{v}_{\mathcal{S}^2} = c\mathbf{e}_7 + \beta c\hat{\mathbf{v}}_{\mathcal{S}^2} = c\mathbf{e}_7 + (v_x\mathbf{e}_8 + v_y\mathbf{e}_9 + v_z\mathbf{e}_{10}). \tag{5.35}$$

In special relativity, a spacetime velocity $\boldsymbol{v}_{\mathcal{M}}$ is the sum of an observer spacetime velocity $\boldsymbol{o}_{\mathcal{M}}$ and a spatial velocity $\mathbf{v}_{\mathcal{S}}$ relative to the observer, where $0 \le \|\mathbf{v}_{\mathcal{S}}\| \le c$.

The *modulus* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$|\boldsymbol{v}_{\mathcal{M}}| = \sqrt{\boldsymbol{v}_{\mathcal{M}}^2} = \sqrt{c^2 + \mathbf{v}_{\mathcal{S}}^2} = \sqrt{c^2 - \|\mathbf{v}_{\mathcal{S}}\|^2}. \tag{5.36}$$

The square of a spacetime velocity $\boldsymbol{v}^2 = c^2 - \|\mathbf{v}\|^2$ may be positive, negative, or zero and represents a relative comparison of light speed to spatial speed. A spacetime velocity with positive signature $0 < \boldsymbol{v}^2$ is *timelike*, with negative signature $\boldsymbol{v}^2 < 0$ is *spacelike*, and with null signature $\boldsymbol{v}^2 = 0$ is *lightlike*.

The *conjugate* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\boldsymbol{v}_{\mathcal{M}}^{\dagger} = \boldsymbol{\gamma}_0 \boldsymbol{v}_{\mathcal{M}} \boldsymbol{\gamma}_0 = \boldsymbol{o}_{\mathcal{M}} - \mathbf{v}_{\mathcal{S}}. \tag{5.37}$$

The *norm* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\|\boldsymbol{v}_{\mathcal{M}}\| = \sqrt{\boldsymbol{v}_{\mathcal{M}} \cdot \boldsymbol{v}_{\mathcal{M}}^{\dagger}} = \sqrt{\boldsymbol{v}_{\mathcal{M}} \cdot (\boldsymbol{\gamma}_0 \boldsymbol{v}_{\mathcal{M}} \boldsymbol{\gamma}_0)} = \sqrt{c^2 - \mathbf{v}_{\mathcal{S}}^2} = \sqrt{c^2 + \|\mathbf{v}_{\mathcal{S}}\|^2}. \tag{5.38}$$

The *unit*, or *modulus-unit*, of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\hat{\boldsymbol{v}}_{\mathcal{M}} = \frac{\boldsymbol{v}_{\mathcal{M}}}{|\boldsymbol{v}_{\mathcal{M}}|} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{\boldsymbol{o}_{\mathcal{M}}^2 + \mathbf{v}_{\mathcal{S}}^2}} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{c^2 - v_x^2 - v_y^2 - v_z^2}}. \tag{5.39}$$

The *norm-unit* of an STA *spacetime velocity* $\boldsymbol{v}_{\mathcal{M}}$ is

$$\frac{\boldsymbol{v}_{\mathcal{M}}}{\|\boldsymbol{v}_{\mathcal{M}}\|} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{\boldsymbol{o}_{\mathcal{M}}^2 - \mathbf{v}_{\mathcal{S}}^2}} = \frac{\boldsymbol{v}_{\mathcal{M}}}{\sqrt{c^2 + v_x^2 + v_y^2 + v_z^2}}. \tag{5.40}$$

The overhat is on the modulus-unit of an STA spacetime vector $\boldsymbol{a}$ with $a_w \neq 0$ as $\hat{\boldsymbol{a}}$, but the overhat is on the norm-unit of an SA spatial vector $\mathbf{a}$ with $a_w = 0$ as $\hat{\mathbf{a}}$. In some contexts, it is explicitly noted when the overhat notation on spacetime vectors is taking the norm-unit.

There are two times associated with a spacetime velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$. The *coordinate time of* $\boldsymbol{v}$, denoted $t_{c\boldsymbol{v}}$,

$$t_{c\boldsymbol{v}} = t_{p\boldsymbol{o}} = t \tag{5.41}$$

is the proper time $t_{p\boldsymbol{o}}$ of the observer $\boldsymbol{o} = c\boldsymbol{\gamma}_0$ of $\boldsymbol{v}$. The time t is the conventional notation for coordinate time. The observable $\boldsymbol{v}$ has spacetime position $\boldsymbol{v}t$ (assuming $\mathbf{p}_0 = 0$, §5.1.8) in the frame of $\boldsymbol{o}$. The *proper time of* $\boldsymbol{v}$, denoted $t_{p\boldsymbol{v}}$,

$$t_{p\boldsymbol{v}} = t_{c\boldsymbol{o}_{\boldsymbol{v}}} = \tau \tag{5.42}$$

is the coordinate time $t_{c\boldsymbol{o}_{\boldsymbol{v}}}$ of the observer $\boldsymbol{o}_{\boldsymbol{v}} \cong \boldsymbol{o}$, which is $\boldsymbol{v}$ actively transformed relative to itself in its own rest frame as (see Fig. 5.1 in §5.2.3)

$$\boldsymbol{o}_{\boldsymbol{v}} = (\boldsymbol{v} \ominus \boldsymbol{v}) / \gamma_{\mathbf{v}}^{-1} \tag{5.43}$$

$$= (\gamma_{\mathbf{v}} \boldsymbol{v} \ominus \boldsymbol{v}). \tag{5.44}$$

The time τ is the conventional notation for proper time. The transformations of time by boosts can be cause for confusion. The notations $t_{c\boldsymbol{v}}$ and $t_{p\boldsymbol{v}}$ may help avoid some confusion. The (additive) "active" boosts $B_{\mathbf{v}}$ passively transform a *new* proper *time parameter* $t_{p\boldsymbol{v}} = \tau$ back into the old coordinate time $t_{p\boldsymbol{o}} = t$. The (subtractive or relative) "passive" boosts $B_{\mathbf{v}}^{\sim}$ passively transform the *same* old coordinate *time parameter* $t_{p\boldsymbol{o}} = t$ into a new proper time $t_{p\boldsymbol{v}} = \tau$. The "active" and "passive" boosts transform times in reverse of each other.

5.1.8 STA spacetime position

The STA *spacetime position* $\boldsymbol{p}_{\mathcal{M}}$ of SA spatial position $\mathbf{p}_{\mathcal{S}}$ relative to observer $\boldsymbol{o}_{\mathcal{M}}t$ is

$$\boldsymbol{p}(t) = \boldsymbol{p}_{\mathcal{M}}(t) = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_{\mathcal{S}}(t) = ct\boldsymbol{\gamma}_0 + (\mathbf{p}_0 + \mathbf{v}_{\mathcal{S}}t) \tag{5.45}$$

$$= (\boldsymbol{o}_{\mathcal{M}}t + \mathbf{p}_0) + \mathbf{v}_{\mathcal{S}}t = \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) + \mathbf{v}_{\mathcal{S}}t \tag{5.46}$$

$$= \mathbf{p}_0 + (\boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}})t = \mathbf{p}_0 + \boldsymbol{v}_{\mathcal{M}}t. \tag{5.47}$$

In special relativity, if the initial position $\mathbf{p}_0 \neq 0$, then the proper observer that coincides with $\boldsymbol{p}_{\mathcal{M}}$ at $t=0$ is the translated observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$. A spacetime boost of $\boldsymbol{p}_{\mathcal{M}}$ should be relative to the proper observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ by using a translated-boost around $\mathbf{p}_0$. CSTA has a versor for translated-boosts (§6.6.9).

The time t derivative of $\boldsymbol{p}_{\mathcal{M}}$ is

$$\dot{\boldsymbol{p}}_{\mathcal{M}} = \partial_t \boldsymbol{p}_{\mathcal{M}} = \frac{\partial \boldsymbol{p}_{\mathcal{M}}}{\partial t} = \boldsymbol{o}_{\mathcal{M}} + \mathbf{v}_{\mathcal{S}} = \boldsymbol{v}_{\mathcal{M}}. \tag{5.48}$$

The *modulus-unit*

$$\hat{\boldsymbol{p}}_{\mathcal{M}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{|\boldsymbol{p}_{\mathcal{M}}|} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{\boldsymbol{p}_{\mathcal{M}}^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{(ct)^2 - \|\mathbf{p}_{\mathcal{S}}\|^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{p_w^2 - p_x^2 - p_y^2 - p_z^2}} \tag{5.49}$$

and the *norm-unit*

$$\frac{\boldsymbol{p}_{\mathcal{M}}}{\|\boldsymbol{p}_{\mathcal{M}}\|} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{p}_{\mathcal{M}}^{\dagger}}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{(ct)^2 + \|\mathbf{p}_{\mathcal{S}}\|^2}} = \frac{\boldsymbol{p}_{\mathcal{M}}}{\sqrt{p_w^2 + p_x^2 + p_y^2 + p_z^2}} \tag{5.50}$$

of an STA spacetime position $\boldsymbol{p}_{\mathcal{M}}$ are similar to those of an STA spacetime velocity $\boldsymbol{v}_{\mathcal{M}}$.

The square of a spacetime position $\boldsymbol{p}_{\mathcal{M}}^2$ is the *spacetime interval* between the origin of spacetime and $\boldsymbol{p}_{\mathcal{M}}$. Likewise, $(\boldsymbol{p}_{\mathcal{M}_2} - \boldsymbol{p}_{\mathcal{M}_1})^2$ is the spacetime interval between $\boldsymbol{p}_{\mathcal{M}_1}$ and $p_{\mathcal{M}_2}$. The passive boost of a spacetime position $\boldsymbol{p}_{\mathcal{M}}$ to become relative to the frame of a new observer preserves the spacetime interval $\boldsymbol{p}_{\mathcal{M}}^2$.

5.2 STA operations

5.2.1 STA dualization

The STA *dual* $A_{\mathcal{M}}^{*\mathcal{M}}$ of an STA multivector $A_{\mathcal{M}}$ is

$$A_{\mathcal{M}}^* = A_{\mathcal{M}}^{*\mathcal{M}} = A_{\mathcal{M}} \mathbf{I}_{\mathcal{M}}^{-1} = -A_{\mathcal{M}} \mathbf{I}_{\mathcal{M}}. \tag{5.51}$$

The STA *undual* $A_{\mathcal{M}}$ of an STA multivector $A_{\mathcal{M}}^{*\mathcal{M}}$ is

$$A_{\mathcal{M}} = A_{\mathcal{M}}^* \mathbf{I}_{\mathcal{M}} = A_{\mathcal{M}} \mathbf{I}_{\mathcal{M}}^{-1} \mathbf{I}_{\mathcal{M}}. \tag{5.52}$$

The STA *unit pseudoscalar* $\mathbf{I}_{\mathcal{M}}$ (§5.1.2) is

$$\mathbf{I}_{\mathcal{M}} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \tag{5.53}$$

$$\mathbf{I}_{\mathcal{M}}^{-1} = -\mathbf{I}_{\mathcal{M}} = -\widetilde{\mathbf{I}}_{\mathcal{M}}. \tag{5.54}$$

5.2.2 STA rotor

The STA spatial rotation operator, or *rotor*, $R_{\mathcal{M}} = R_{\mathcal{S}}$ is the SA rotor $R_{\mathcal{S}}$ (§2.6).

The STA 2-versor spatial *rotor* $R_{\mathcal{S}}$ for rotation in SA space around the SA unit vector *axis* $\hat{\mathbf{x}}_{\mathcal{S}}$ by *angle* θ is

$$R_{\mathcal{S}} = e^{\frac{1}{2}\theta \hat{\mathbf{x}}_{\mathcal{S}}^{*\mathcal{S}}} = e^{\frac{1}{2}\theta \hat{\mathbf{x}}_{\mathcal{S}} \widetilde{\mathbf{I}}_{\mathcal{S}}} \tag{5.55}$$

$$= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right) \hat{\mathbf{x}}_{\mathcal{S}} \widetilde{\mathbf{I}}_{\mathcal{S}} \tag{5.56}$$

$$= \cos\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right) \hat{\mathbf{x}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}^{-1}. \tag{5.57}$$

The rotor operation

$$A'_{\mathcal{M}} = R_{\mathcal{S}} A_{\mathcal{M}} \widetilde{R}_{\mathcal{S}} \tag{5.58}$$

rotates any multivector $A_{\mathcal{M}}$ in STA as expected in the spatial SA components, but leaves the STA timelike components unchanged.

5.2.3 STA spacetime boost

5.2.3.1 Introduction

Although it would be sufficient to just define the *boost operator*

$$B_{\mathcal{M}}=B_{\mathbf{v}} \;=\; e^{\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0} \tag{5.59}$$

for a boost by spacetime velocity

$$\boldsymbol{v} \;=\; \boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta c\hat{\mathbf{v}} \tag{5.60}$$

with natural speed $\beta=\beta_{\mathbf{v}}$ and rapidity

$$\varphi=\varphi_{\mathbf{v}} \;=\; \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}(\|\mathbf{v}\|/c), \tag{5.61}$$

this section also attempts to discuss some of the basics of active and passive boosts, and velocity addition and subtraction.

5.2.3.2 The exponential function

The following functions and identities are frequently used to define versors.

$$\exp(A)=e^{A} \;=\; \sum_{n=0}^{\infty}\frac{A^n}{n!}=\cosh(A)+\sinh(A), \quad \text{where } A \text{ is any multivector [15]} \tag{5.62}$$

$$\cosh(A) \;=\; \sum_{n=0}^{\infty}\frac{A^{2n}}{(2n)!}=\frac{e^{A}+e^{-A}}{2} \tag{5.63}$$

$$\sinh(A) \;=\; \sum_{n=0}^{\infty}\frac{A^{2n+1}}{(2n+1)!}=\frac{e^{A}-e^{-A}}{2} \tag{5.64}$$

$$\tanh(A) \;=\; \frac{\sinh(A)}{\cosh(A)}=\frac{e^{A}-e^{-A}}{e^{A}+e^{-A}} \tag{5.65}$$

$$\cosh(iA) \;=\; \cos(A), \quad \text{where } i^2=-1 \text{ and } iA=Ai \tag{5.66}$$

$$\sinh(iA) \;=\; i\sin(A) \tag{5.67}$$

$$\cosh(jA) \;=\; \cosh(A), \quad \text{where } j^2=1 \text{ and } jA=Aj \tag{5.68}$$

$$\sinh(jA) \;=\; j\sinh(A) \tag{5.69}$$

$$\cosh(\varepsilon A) \;=\; 1, \quad \text{where } \varepsilon^2=0 \text{ and } \varepsilon A=A\varepsilon \tag{5.70}$$

$$\sinh(\varepsilon A) \;=\; \varepsilon A \tag{5.71}$$

$$\cosh\left(\frac{1}{2}\operatorname{atanh}(\beta_{\mathbf{v}})\right) \;=\; \frac{(1+\beta_{\mathbf{v}})+\sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.72}$$

$$\sinh\left(\frac{1}{2}\operatorname{atanh}(\beta_{\mathbf{v}})\right) \;=\; \frac{(1+\beta_{\mathbf{v}})-\sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.73}$$

$$\cosh(\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.74}$$

$$\sinh(\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{\beta_{\mathbf{v}}}{\sqrt{1-\beta_{\mathbf{v}}^2}}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.75}$$

$$\cosh(2\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{1+\beta_{\mathbf{v}}^2}{1-\beta_{\mathbf{v}}^2}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.76}$$

$$\sinh(2\operatorname{atanh}(\beta_{\mathbf{v}})) \;=\; \frac{2\beta_{\mathbf{v}}}{1-\beta_{\mathbf{v}}^2}, \quad \text{for } -1<\beta_{\mathbf{v}}<1 \tag{5.77}$$

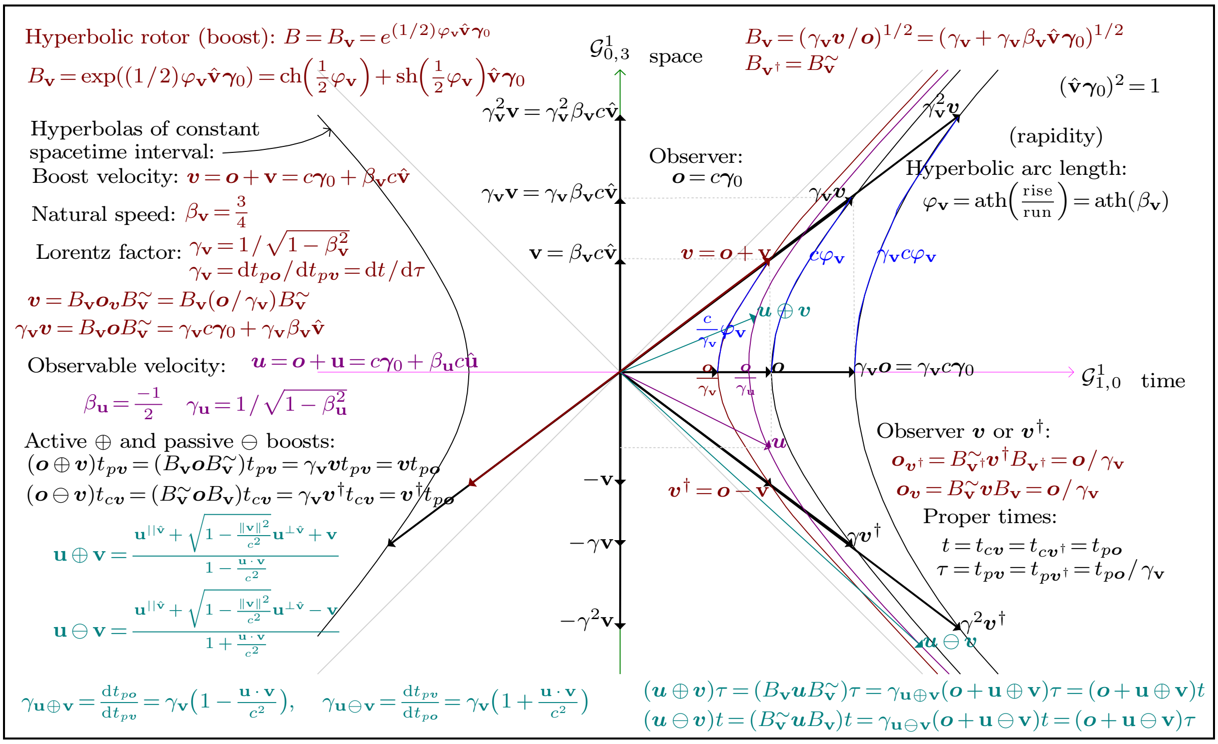

Figure 5.1. Spacetime diagram of observables $\boldsymbol{o}$, $\boldsymbol{v}$, and $\boldsymbol{u}$

Figure 5.1 shows the spacetime diagram of spacetime velocities for observer $\boldsymbol{o}$, boost observable $\boldsymbol{v}$, and observable $\boldsymbol{u}$. The time axis is horizontal and the space axis is vertical. The hyperbolic angle (rapidity) φ is positive anticlockwise. This orientation of the spacetime diagram of hyperbolic rotations by φ is analogous to circular rotations by an angle θ. By circular rotations, points are translated along a circle through a circular arc $r\theta$. By hyperbolic rotations, points are translated along a hyperbola through a hyperbolic arc $r\varphi$. For both circular and hyperbolic rotations, the radius r is an invariant distance (interval) from the origin to a point. A circular radius is a positive real scalar r, and a real hyperbolic radius (pseudoradius) is $r\in\left\{\frac{c}{\gamma},c,\gamma c\right\}$ for the hyperbolas of constant spacetime interval that are shown in the figure. The boost operator $B_{\mathbf{v}}$ rotates from $\boldsymbol{o}$ toward $\boldsymbol{v}$ by $\varphi_{\mathbf{v}}$, which corresponds to a change of speed by $\beta_{\mathbf{v}}c$ in the direction $\hat{\mathbf{v}}$. The speed $\beta_{\mathbf{v}}c$ of a worldline is its slope in the diagram, $\gamma_{\mathbf{v}}$ is the Lorentz (time dilation) factor, and c is the speed of light. In the rest frame of an observable, the observable is the observer having proper time τ, zero spatial speed $\beta c=0$, and a worldline $\boldsymbol{o}\tau=c\tau\gamma_0$. An observable worldline $\boldsymbol{v}t$ that is hyperbolically rotated into the time axis by a passive boost

$$\tilde{B_{\mathbf{v}}}(\boldsymbol{v}t)B_{\mathbf{v}} \quad = \quad \tilde{B_{\mathbf{v}}}\boldsymbol{v}B_{\mathbf{v}}t=\boldsymbol{o}t/\gamma_{\mathbf{v}}=\boldsymbol{o}\tau \tag{5.78}$$

gives the proper time τ rest frame worldline $\boldsymbol{o}\tau$ of the observable $\boldsymbol{v}$ relative to the observer $\boldsymbol{o}t/\gamma_{\mathbf{v}}$. A hyperbolic rotation (boost) of an observable velocity

$$\boldsymbol{u} \quad = \quad \boldsymbol{o}+\mathbf{u}=c\gamma_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}}, \qquad \text{for } 0\le\beta_{\mathbf{u}}<1, \tag{5.79}$$

preserves its spacetime interval

$$r_{\boldsymbol{u}} \quad = \quad |\boldsymbol{u}|=\sqrt{\boldsymbol{u}^2}=\sqrt{c^2-\beta_{\mathbf{u}}^2c^2}=c\sqrt{1-\beta_{\mathbf{u}}}=c/\gamma_{\mathbf{u}} \tag{5.80}$$

$$\quad = \quad |B_{\mathbf{v}}\boldsymbol{u}\tilde{B_{\mathbf{v}}}|=|\tilde{B_{\mathbf{v}}}\boldsymbol{u}B_{\mathbf{v}}|=|\boldsymbol{u}||B_{\mathbf{v}}\hat{\boldsymbol{u}}\tilde{B_{\mathbf{v}}}|=|\boldsymbol{u}||\tilde{B_{\mathbf{v}}}\hat{\boldsymbol{u}}B_{\mathbf{v}}|, \qquad \text{for } \boldsymbol{u}^2\neq0, \tag{5.81}$$

such that all boosted vectors remain on their hyperbola of constant spacetime interval. A null vector (at light speed) $\boldsymbol{u}^2=0$ is not on any hyperbola and cannot be boosted. A hyperbolic rotation by a negative hyperbolic angle is a passive transformation from current reference frame with time t into a new frame with time τ at the positive angle of rotation, representing a relativistic velocity subtraction $\mathbf{u}\ominus\mathbf{v}$. A hyperbolic rotation by a positive hyperbolic angle is an active transformation of a velocity vector into the boosted frame of the boost observable $\boldsymbol{v}$ with new coordinate time τ, representing a relativistic velocity addition $\mathbf{u}\oplus\mathbf{v}$. After a passive boost, the time is t, which passively transforms into a relative τ. After an active boost, the time is τ, which passively transforms into a relative t. The passive boost of a position $\boldsymbol{p}$ that has an initial position $\mathbf{p}_0$

$$\boldsymbol{p}(t) \;=\; \mathbf{p}_0+\dot{\boldsymbol{p}}t=\mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}})t \tag{5.82}$$

is valid, but it should be performed using a translated-boost (§6.6.9) as

$$\begin{aligned}\mathcal{C}^{-1}(B_{\mathbf{v}}^{\mathbf{p}_0\sim}\mathcal{C}(\boldsymbol{p})B_{\mathbf{v}}^{\mathbf{p}_0}) &= \mathbf{p}_0+B_{\mathbf{v}}^{\sim}\dot{\boldsymbol{p}}B_{\mathbf{v}}t & (5.83)\\ &= \mathbf{p}_0+B_{\mathbf{v}}^{\sim}(\boldsymbol{o}+\dot{\mathbf{p}})B_{\mathbf{v}}t & (5.84)\\ &= \mathbf{p}_0+\gamma_{\dot{\mathbf{p}}\ominus\mathbf{v}}(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v})t & (5.85)\\ &= \mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v})\tau, & (5.86)\end{aligned}$$

which preserves the initial position $\mathbf{p}_0$ at time $t=\tau=0$.

5.2.3.3 Derivation of boost operator

An "active" boost operation is a hyperbolic rotation operation in spacetime (Fig. 5.1) that passively turns (transforms) an observer spacetime velocity

$$\boldsymbol{o} \;=\; c\boldsymbol{\gamma}_0, \tag{5.87}$$

which has zero spatial velocity in its own frame with coordinate time t, into a relative spacetime velocity

$$\begin{aligned}\boldsymbol{o}' &= B_{\mathbf{v}}\boldsymbol{o}B_{\mathbf{v}}^{\sim}=B_{\mathbf{v}^\dagger}^{\sim}\boldsymbol{o}B_{\mathbf{v}^\dagger} & (5.88)\\ &= \boldsymbol{o}\oplus\boldsymbol{v}=\boldsymbol{o}\ominus\boldsymbol{v}^\dagger & (5.89)\\ &= \gamma_{\mathbf{v}}\boldsymbol{v}=\gamma_{\mathbf{v}}(\boldsymbol{o}+\mathbf{v}) & (5.90)\\ &= \gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}} & (5.91)\end{aligned}$$

that is relative to ($\ominus$) the new observer $\boldsymbol{v}^\dagger=\boldsymbol{o}-\mathbf{v}$ with proper time $\tau=t_{\boldsymbol{v}^\dagger}$, consistent with special relativity. The "active" boost operator is $B_{\mathbf{v}}$, which is a "passive" boost operator $B_{\mathbf{v}^\dagger}^{\sim}$. Boosts operate on spacetime velocities, not on positions. Following a velocity boost, *the new time parameter*, which is to be multiplied into a transformed velocity $\boldsymbol{o}'$ as a spacetime displacement $\boldsymbol{o}'\tau$, *is the proper time τ of $\boldsymbol{v}^\dagger$*, not the coordinate time t of $\boldsymbol{o}$. The time and spatial displacement of $\boldsymbol{o}'\tau$ is passively transformed into coordinate time $t=\gamma_{\mathbf{v}}\tau$ and spatial displacement $\mathbf{d}=\gamma_{\mathbf{v}}\tau\mathbf{v}=\mathbf{v}t$ as seen by the coordinate time t observer $\boldsymbol{o}$ and corresponds to (but is not) a time τ and displacement seen by $\boldsymbol{v}^\dagger$. That is,

$$\begin{aligned}\boldsymbol{o}'\tau &= (\gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}})\tau & (5.92)\\ &= ct\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}ct\hat{\mathbf{v}} & (5.93)\\ &= \boldsymbol{o}t+\mathbf{v}t & (5.94)\\ &= \boldsymbol{o}t+\mathbf{d}. & (5.95)\end{aligned}$$

A boost can be applied to any spacetime velocity, but its boosted speed can never exceed light speed c relative to any observer.

To derive the boost operator, we can start by defining the *ratio* of spacetime velocities of an observable (particle) $\boldsymbol{v}$ to its coordinate time t observer $\boldsymbol{o}$ as the *hyperbolic biradial* $\boldsymbol{v}/\boldsymbol{o} = \boldsymbol{v}\boldsymbol{o}^{-1}$ ("$\boldsymbol{v}$ by $\boldsymbol{o}$"). The term *biradial* was coined by HAMILTON in his original work on Quaternions [14].

The *hyperbolic biradial*

$$H = \boldsymbol{v}\boldsymbol{o}^{-1} = \frac{|\boldsymbol{v}|}{|\boldsymbol{o}|}\widehat{\boldsymbol{v}/\boldsymbol{o}} = \frac{\sqrt{c^2 + \mathbf{v}^2}}{c}\hat{H} \tag{5.96}$$

$$= \sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}\hat{H} = \sqrt{1 - \beta_{\mathbf{v}}^2}\hat{H} = \frac{1}{\gamma_{\mathbf{v}}}\hat{H} \tag{5.97}$$

is an operator that *turns* the spacetime velocity of the observer $\boldsymbol{o}$ into the spacetime velocity of the boost particle

$$\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} \tag{5.98}$$

as the one-sided versor operation

$$\boldsymbol{v} = H\boldsymbol{o}. \tag{5.99}$$

The *natural speed* $\beta_{\mathbf{v}}$ of the velocity $\mathbf{v}$ is

$$\beta_{\mathbf{v}} = \frac{\|\mathbf{v}\|}{c} = \frac{\sqrt{\mathbf{v} \cdot \mathbf{v}^{\dagger}}}{c} = \frac{\sqrt{-\mathbf{v}^2}}{c} = \frac{v}{c}. \tag{5.100}$$

The *Lorentz factor* (*spacetime dilation factor*) $\gamma_{\mathbf{v}}$ of the velocity $\mathbf{v}$ is

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \beta_{\mathbf{v}}^2}} = \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{5.101}$$

where $t = t_{c\boldsymbol{v}} = t_{p\boldsymbol{o}}$ is the coordinate time of $\boldsymbol{v}$ and $\tau = t_{p\boldsymbol{v}}$ is the proper time of $\boldsymbol{v}$.

The interval (pseudodistance) $c\tau$ is the hyperbolic arc length along the worldline of $\boldsymbol{v}$

$$c\mathrm{d}\tau = |\boldsymbol{v}|\mathrm{d}t = \sqrt{\boldsymbol{v}^2}\mathrm{d}t = \sqrt{c^2 - \beta_{\mathbf{v}}^2 c^2}\mathrm{d}t = \frac{c}{\gamma_{\mathbf{v}}}\mathrm{d}t \tag{5.102}$$

$$\tau = \int_0^t \frac{1}{\gamma_{\mathbf{v}}}\mathrm{d}t = \frac{t}{\gamma_{\mathbf{v}}}, \qquad \text{where } \tau = 0 \text{ when } t = 0. \tag{5.103}$$

Maximum $c\tau$ is when $\beta_{\mathbf{v}} = 0$, such that inertial observers experience maximum time.

The *length contraction*, to length L from an initial length L_0 in the direction of boost velocity $\mathbf{v}$, is given by

$$L = \frac{L_0}{\gamma_{\mathbf{v}}} = L_0\sqrt{1 - \beta_{\mathbf{v}}^2}. \tag{5.104}$$

The *dilation factor* of the velocity $\mathbf{v}$ is

$$d = \frac{1}{\gamma_{\mathbf{v}}} = \sqrt{1-\beta_{\mathbf{v}}^2}. \tag{5.105}$$

For a dilation factor d, the required natural speed is $\beta_{\mathbf{v}} = \sqrt{1-d^2}$. For $d \leq 1$, the dilation factor d can be called the *spacetime contraction factor*, which is the usual case. For $d > 1$, then $\beta_{\mathbf{v}}$ is an *imaginary* natural speed and it is possible to dilate lengths instead of contract lengths, but dilated lengths are only geometrical effects, not physics effects.

The *hyperbolic versor* $\hat{H}$ is the *unit hyperbolic biradial*

$$\begin{aligned}
\hat{H} &= \gamma_{\mathbf{v}} H = \gamma_{\mathbf{v}} \boldsymbol{v}\boldsymbol{o}^{-1} = \frac{\gamma_{\mathbf{v}}}{c}\boldsymbol{v}\boldsymbol{\gamma}_0 && (5.106)\\
&= \frac{\gamma_{\mathbf{v}}}{c}(\boldsymbol{v}\cdot\boldsymbol{\gamma}_0 + \boldsymbol{v}\wedge\boldsymbol{\gamma}_0) = \gamma_{\mathbf{v}} + \frac{\gamma_{\mathbf{v}}}{c}\mathbf{v}\boldsymbol{\gamma}_0 && (5.107)\\
&= \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\frac{\|\mathbf{v}\|}{c}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 = \gamma_{\mathbf{v}} + \gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 && (5.108)\\
&= \cosh(\varphi_{\mathbf{v}}) + \sinh(\varphi_{\mathbf{v}})\hat{\mathbf{v}}\boldsymbol{\gamma}_0 = \exp(\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0) = e^{\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}, && (5.109)
\end{aligned}$$

where

$$\begin{aligned}
j^2 = (\hat{\mathbf{v}}\boldsymbol{\gamma}_0)^2 = (\hat{\mathbf{v}}\wedge\boldsymbol{\gamma}_0)^2 &= 1 && (5.110)\\
\gamma_{\mathbf{v}} &= \cosh(\varphi_{\mathbf{v}}) && (5.111)\\
\gamma_{\mathbf{v}}\beta_{\mathbf{v}} &= \sinh(\varphi_{\mathbf{v}}) && (5.112)\\
\beta_{\mathbf{v}} &= \tanh(\varphi_{\mathbf{v}}) = \frac{\sinh(\varphi_{\mathbf{v}})}{\cosh(\varphi_{\mathbf{v}})} && (5.113)\\
\varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}). && (5.114)
\end{aligned}$$

Using half of the *hyperbolic angle (rapidity)* $\varphi_{\mathbf{v}}$, the *hyperbolic rotation operator (hyperbolic rotor or boost operator)* $B_{\mathbf{v}}$ is the *square root of the hyperbolic versor*

$$\begin{aligned}
B_{\mathbf{v}} = \hat{H}^{\frac{1}{2}} &= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right) = e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0} && (5.115)\\
&= \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right) + \sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}\boldsymbol{\gamma}_0 && (5.116)\\
&= \frac{(1+\beta_{\mathbf{v}}) + \sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}} + \frac{(1+\beta_{\mathbf{v}}) - \sqrt{1-\beta_{\mathbf{v}}^2}}{2\sqrt{1+\beta_{\mathbf{v}}}\sqrt[4]{1-\beta_{\mathbf{v}}^2}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0 \quad \text{for } -1 < \beta_{\mathbf{v}} < 1. && (5.117)
\end{aligned}$$

The hyperbolic rotation by a natural speed $|\beta_{\mathbf{v}}| = 1$, corresponding to rapidity $|\varphi_{\mathbf{v}}| = \infty$, is invalid since it represents reaching the light speed asymptote $\boldsymbol{l} = c\boldsymbol{\gamma}_0 + c\hat{\mathbf{v}}$ (a null vector), which can never be reached on the hyperbola of constant (invariant) spacetime interval.

While the *hyperbolic versor* $\hat{H}$ is a one-sided versor, the *hyperbolic rotor (boost)* $B_{\mathbf{v}}$ is a two-sided versor or "sandwiching" versor with its reverse $B_{\mathbf{v}}^{\sim} = B_{\mathbf{v}}^{-1}$, such that

$$\begin{aligned}
\boldsymbol{v} &= \hat{H}\boldsymbol{o}/\gamma_{\mathbf{v}} = B_{\mathbf{v}}^2\boldsymbol{o}/\gamma_{\mathbf{v}} && (5.118)\\
&= B_{\mathbf{v}}\boldsymbol{o}B_{\mathbf{v}}^{\sim}/\gamma_{\mathbf{v}}. && (5.119)
\end{aligned}$$

The one-sided versor operation is valid only for collinear (coplanar in spacetime) velocity boosts, while the two-sided versor operation is valid for general boosts of any velocity that need not be collinear with $\mathbf{v}$. This is similar to the difference in quaternion rotations between planar rotation using the one-sided versor operation $e^{\theta\hat{\mathbf{n}}}\mathbf{r}^{\perp\hat{\mathbf{n}}}$ and conical rotation using the two-sided versor "sandwich" operation $e^{\frac{1}{2}\theta\hat{\mathbf{n}}}\mathbf{r}e^{-\frac{1}{2}\theta\hat{\mathbf{n}}}=\mathbf{r}^{||\hat{\mathbf{n}}}+e^{\theta\hat{\mathbf{n}}}\mathbf{r}^{\perp\hat{\mathbf{n}}}$.

Although boosts are generally valid on spacetime velocities and not generally valid on spacetime positions since the time after a velocity boost is subject to interpretation, *it is valid* to passively boost a spacetime position of the form $\boldsymbol{u}t=(\boldsymbol{o}+\mathbf{u})t$, which is the product of the spacetime velocity $\boldsymbol{u}$ and its coordinate time $t=t_{p\boldsymbol{o}}=t_{c\boldsymbol{u}}$. After a passive boost, the time to be applied to the boosted velocity is still t, which can be correctly factored out of the passive boost operation as

$$B_{\mathbf{v}}^{\sim}(\boldsymbol{u}t)B_{\mathbf{v}} \;=\; B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t. \tag{5.120}$$

The passive boost is interpreted as transforming coordinate time t into the proper time $\tau=t_{p\boldsymbol{v}}$ of $\boldsymbol{v}$, and transforming distance into the distance relative to observable $\boldsymbol{v}$. Evaluating this passive position boost gives some useful results, as follows.

$$\begin{aligned}
B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t &= e^{-\frac{1}{2}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}(\boldsymbol{o}+\mathbf{u})e^{\frac{1}{2}\hat{\mathbf{v}}\boldsymbol{\gamma}_0}t=(\boldsymbol{u}\ominus\boldsymbol{v})t && (5.121)\\
&= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}(c\boldsymbol{\gamma}_0+\mathbf{u}^{||\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}})(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}t && (5.122)\\
&= (c\boldsymbol{\gamma}_0+\mathbf{u}^{||\hat{\mathbf{v}}})(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})t+\mathbf{u}^{\perp\hat{\mathbf{v}}}t && (5.123)\\
&= (c\boldsymbol{\gamma}_0+\mathbf{u}^{||\hat{\mathbf{v}}})(\gamma_{\mathbf{v}}+\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)t+\mathbf{u}^{\perp\hat{\mathbf{v}}}t && (5.124)\\
&= (\gamma_{\mathbf{v}}c\boldsymbol{\gamma}_0-\gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}+\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}+\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\boldsymbol{\gamma}_0)t+\mathbf{u}^{\perp\hat{\mathbf{v}}}t && (5.125)\\
&= \left(\left(\gamma_{\mathbf{v}}+\frac{1}{c}\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\right)c\boldsymbol{\gamma}_0+\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}}-\gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}\right)t && (5.126)\\
&= (\boldsymbol{o}'+\mathbf{u}')t=(\gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t. && (5.127)
\end{aligned}$$

The time transformation is

$$\begin{aligned}
\tau &= \gamma_{\mathbf{u}\ominus\mathbf{v}}t && (5.128)\\
&= \left(\gamma_{\mathbf{v}}+\frac{1}{c}\gamma_{\mathbf{v}}\beta_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}\hat{\mathbf{v}}\right)t && (5.129)\\
&= \gamma_{\mathbf{v}}\left(1+\frac{1}{c}\beta_{\mathbf{v}}(\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}^{-1}\hat{\mathbf{v}}\right)t && (5.130)\\
&= \gamma_{\mathbf{v}}\left(1+\frac{1}{c}\frac{\|\mathbf{v}\|}{c}\frac{(\mathbf{u}\cdot\mathbf{v})}{\|\mathbf{v}\|}\right)t && (5.131)\\
&= \gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)t. && (5.132)
\end{aligned}$$

The distance transformation is

$$\begin{aligned}
\mathbf{u}'t &= (\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t && (5.133)\\
&= (\mathbf{u}\ominus\mathbf{v})\tau && (5.134)\\
&= \frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}}-\gamma_{\mathbf{v}}\beta_{\mathbf{v}}c\hat{\mathbf{v}}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\tau && (5.135)\\
&= \frac{\gamma_{\mathbf{v}}\mathbf{u}^{||\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}}-\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}\ominus\mathbf{v}}}\tau && (5.136)\\
&= \frac{\mathbf{u}^{||\hat{\mathbf{v}}}+\sqrt{1-\beta_{\mathbf{v}}^2}\mathbf{u}^{\perp\hat{\mathbf{v}}}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau. && (5.137)
\end{aligned}$$

Summary of useful results:

$$\begin{aligned}
\boldsymbol{u}t &= (\boldsymbol{o}+\mathbf{u})t=(c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}})t & (5.138)\\
\boldsymbol{v}t &= (\boldsymbol{o}+\mathbf{v})t=(c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}})t & (5.139)\\
t &= t_{p\boldsymbol{o}}=t_{c\boldsymbol{v}}=t_{c\boldsymbol{u}}=t_{c\boldsymbol{u}\ominus\boldsymbol{v}}=t_{c\mathbf{u}\oplus\mathbf{v}} & (5.140)\\
\tau &= t_{p\boldsymbol{v}}=t_{c\boldsymbol{u}\oplus\boldsymbol{v}}=t_{c\mathbf{u}\ominus\mathbf{v}} & (5.141)\\
B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}t &= (\boldsymbol{u}\ominus\boldsymbol{v})t=(\gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v})t=(\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau & (5.142)\\
B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}\tau &= (\boldsymbol{u}\oplus\boldsymbol{v})\tau=(\gamma_{\mathbf{u}\oplus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}}\mathbf{u}\oplus\mathbf{v})\tau=(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})t & (5.143)\\
\gamma_{\mathbf{u}\ominus\mathbf{v}} &= \gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) & (5.144)\\
\gamma_{\mathbf{u}\oplus\mathbf{v}} &= \gamma_{\mathbf{v}}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) & (5.145)\\
\mathbf{u}\ominus\mathbf{v}\tau &= \frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\beta_{\mathbf{v}}^2}\mathbf{u}^{\perp\hat{\mathbf{v}}}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau=\frac{(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}+\sqrt{1-\beta_{\mathbf{v}}^2}(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\tau & (5.146)\\
\mathbf{u}\oplus\mathbf{v}t &= \frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\beta_{\mathbf{v}}^2}\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}t=\frac{(\mathbf{u}\cdot\mathbf{v})\mathbf{v}^{-1}+\sqrt{1-\beta_{\mathbf{v}}^2}(\mathbf{u}\wedge\mathbf{v})\mathbf{v}^{-1}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}t. & (5.147)
\end{aligned}$$

5.2.3.4 Active boost

The "active" boost $\boldsymbol{u}\oplus\boldsymbol{v}$ of spacetime velocity

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{o}+\mathbf{u}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}} & (5.148)\\
t_{c\boldsymbol{u}} &= t=t_{p\boldsymbol{o}}, & (5.149)
\end{aligned}$$

by spacetime velocity

$$\begin{aligned}
\boldsymbol{v} &= \boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}} & (5.150)\\
t_{c\boldsymbol{v}} &= t=t_{p\boldsymbol{o}} & (5.151)\\
t_{p\boldsymbol{v}} &= \tau, & (5.152)
\end{aligned}$$

is the "active" boost operation

$$\begin{aligned}
\boldsymbol{u}\oplus\boldsymbol{v} &= B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}=B_{\mathbf{v}^\dagger}^{\sim}\boldsymbol{u}B_{\mathbf{v}^\dagger} & (5.153)\\
&= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}=(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}} & (5.154)\\
&= \exp\left(\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)\boldsymbol{u}\exp\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}\right)=\exp\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}^\dagger\right)\boldsymbol{u}\exp\left(\frac{1}{2}\varphi\hat{\mathbf{v}}^\dagger\boldsymbol{\gamma}_0\right) & (5.155)\\
&= \boldsymbol{u}\oplus\boldsymbol{v} \quad = \quad \boldsymbol{u}\ominus\boldsymbol{v}^\dagger & (5.156)\\
&= \boldsymbol{u}\oplus(\boldsymbol{o}+\mathbf{v}) \quad = \quad \boldsymbol{u}\ominus(\boldsymbol{o}-\mathbf{v}) & (5.157)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}}\mathbf{u}\oplus\mathbf{v} \quad = \quad \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\mathbf{u}\ominus\mathbf{v}^\dagger & (5.158)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\oplus\mathbf{v}}\beta_{\mathbf{u}\oplus\mathbf{v}}c\widehat{\mathbf{u}\oplus\mathbf{v}} \quad = \quad \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}\beta_{\mathbf{u}\ominus\mathbf{v}^\dagger}c\widehat{\mathbf{u}\ominus\mathbf{v}^\dagger}. & (5.159)
\end{aligned}$$

The hyperbolic angle (rapidity) is

$$\begin{aligned}
\varphi=\varphi_{\mathbf{v}}=\varphi_{\mathbf{v}^\dagger} &= \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}\left(\frac{|\mathbf{v}|}{|\boldsymbol{o}|}\right)=\operatorname{atanh}\left(\frac{v}{c}\right)=\operatorname{atanh}\left(\frac{\text{rise}}{\text{run}}\right) & (5.160)\\
\varphi_{\mathbf{v}^\dagger}\mathbf{v}^\dagger &= -\varphi_{\mathbf{v}}\mathbf{v}. & (5.161)
\end{aligned}$$

The Lorentz factor for $\mathbf{v}$ is

$$\gamma_{\mathbf{v}}=\gamma_{\mathbf{v}^\dagger} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}=\frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. \qquad (5.162)$$

The spacetime dilation factor is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger} \;=\; \gamma_{\mathbf{v}}\Big(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\Big). \tag{5.163}$$

The spatial relativistic velocity addition is

$$\mathbf{u}\oplus\mathbf{v} = \mathbf{u}\ominus\mathbf{v}^\dagger \;=\; \frac{\mathbf{u}^{||\hat{\mathbf{v}}} + \sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}} + \mathbf{v}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. \tag{5.164}$$

The natural speed of the spatial velocity addition $\mathbf{u}\oplus\mathbf{v}$ is

$$\beta_{\mathbf{u}\oplus\mathbf{v}} = \beta_{\mathbf{u}\ominus\mathbf{v}^\dagger} \;=\; \frac{\|\mathbf{u}\oplus\mathbf{v}\|}{c} = \frac{\sqrt{(\mathbf{u}\oplus\mathbf{v})\cdot(\mathbf{u}\oplus\mathbf{v})^\dagger}}{c}. \tag{5.165}$$

For parallel velocities $\mathbf{u}||\mathbf{v}$, then

$$\beta_{\mathbf{u}\oplus\mathbf{v}} \;=\; \frac{\beta_{\mathbf{u}} + \beta_{\mathbf{v}}}{1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}}, \qquad \text{for } \mathbf{u}||\mathbf{v}; \mathbf{u} = \beta_{\mathbf{u}} c\hat{\mathbf{v}}. \tag{5.166}$$

For perpendicular velocities $\mathbf{u}\perp\mathbf{v}$, then

$$\beta_{\mathbf{u}\oplus\mathbf{v}} \;=\; \sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2} \qquad \text{for } \mathbf{u}\perp\mathbf{v}. \tag{5.167}$$

5.2.3.5 Time transformations for active boost

Although $\boldsymbol{u}\oplus\boldsymbol{v}$ and $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ are the same spacetime velocities $\boldsymbol{u}\oplus\boldsymbol{v} = \boldsymbol{u}\ominus\boldsymbol{v}^\dagger$, their associated time transformations are different. For $\boldsymbol{u}\oplus\boldsymbol{v}$, the boost operator is $(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}$ and $\boldsymbol{v}/\boldsymbol{o}$ transforms time t into τ internal to the boost operator, and then $\gamma_{\mathbf{v}}$ transforms time τ back into t external to the boost operator. Therefore, the time is τ for $\boldsymbol{u}\oplus\boldsymbol{v}$ and the displacement is

$$(\boldsymbol{u}\oplus\boldsymbol{v})\tau \;=\; (\boldsymbol{o} + \mathbf{u}\oplus\mathbf{v})\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau \tag{5.168}$$

$$\;=\; (\boldsymbol{o} + \mathbf{u}\oplus\mathbf{v})t \tag{5.169}$$

that passively transforms τ into t. For $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$, the boost operator is $(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}}$ and time transformations are reciprocal such that time τ transforms into t internal to the boost operator, and then time t transforms back into τ external to the boost operator. Therefore, the time is t for $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ and the displacement is

$$(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger)t \;=\; (\boldsymbol{o} + \mathbf{u}\ominus\mathbf{v}^\dagger)\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t \tag{5.170}$$

$$\;=\; (\boldsymbol{o} + \mathbf{u}\ominus\mathbf{v}^\dagger)\tau \tag{5.171}$$

that passively transforms t into τ.

An "active" boost $(\boldsymbol{u}\oplus\boldsymbol{v})\tau$ passively transforms time τ into $t = \gamma_{\mathbf{u}\oplus\mathbf{v}}\tau$, from the frame of $\boldsymbol{v}$ into the frame of $\boldsymbol{o}$. The boost $\boldsymbol{u}\oplus\boldsymbol{v}$ adds/moves $\boldsymbol{u}$ into the frame of $\boldsymbol{v}$ with time $\tau = t_{p\boldsymbol{v}}$ that passively transforms to coordinate time $t = t_{p\boldsymbol{o}}$ such that observer $\boldsymbol{o}$ sees a dilated time $t = \gamma_{\mathbf{u}\oplus\mathbf{v}}\tau$ and a velocity addition $\gamma_{\mathbf{u}\oplus\mathbf{v}}\tau\mathbf{u}\oplus\mathbf{v} = t\mathbf{u}\oplus\mathbf{v}$. The spacetime contraction $(\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}})t$ is an active velocity addition in the frame of $\boldsymbol{o}$.

A "passive" boost $(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger)t$ passively transforms time t into $\tau = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t$, from the frame of $\boldsymbol{o}$ into the frame of $\boldsymbol{v}^\dagger$. The boost $\boldsymbol{u}\ominus\boldsymbol{v}^\dagger$ subtracts/moves $\boldsymbol{u}$ from the frame of $\boldsymbol{v}^\dagger$ into the frame of $\boldsymbol{o}$ with time $t = t_{p\boldsymbol{o}}$ that passively transforms to time $\tau = t_{p\boldsymbol{v}^\dagger}$ such that observer $\boldsymbol{v}^\dagger$ sees a relative dilated time $\tau = \gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t$ and a relative velocity subtraction $\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger}t\mathbf{u}\ominus\mathbf{v}^\dagger = \tau\mathbf{u}\ominus\mathbf{v}^\dagger$. The spacetime contraction $(\boldsymbol{u}\ominus\boldsymbol{v}^\dagger/\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger})t$ is an active velocity subtraction in the frame of $\boldsymbol{o}$.

5.2.3.6 Passive boost

The "passive" boost is the reverse of the "active" boost. Therefore, the following is very similar to the "active" boost, but with everything going in reverse.

The "passive" boost $\boldsymbol{u}\ominus\boldsymbol{v}$ of spacetime velocity

$$\boldsymbol{u} = \boldsymbol{o}+\mathbf{u}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{u}}c\hat{\mathbf{u}} \tag{5.172}$$

$$t_{c\boldsymbol{u}} = t=t_{p\boldsymbol{o}}, \tag{5.173}$$

by spacetime velocity

$$\boldsymbol{v} = \boldsymbol{o}+\mathbf{v}=c\boldsymbol{\gamma}_0+\beta_{\mathbf{v}}c\hat{\mathbf{v}} \tag{5.174}$$

$$t_{c\boldsymbol{v}} = t=t_{p\boldsymbol{o}}, \tag{5.175}$$

is the "passive" boost operation

$$\begin{aligned}
\boldsymbol{u}\ominus\boldsymbol{v} &= B_{\mathbf{v}}^{\sim}\boldsymbol{u}B_{\mathbf{v}}=B_{\mathbf{v}^\dagger}\boldsymbol{u}B_{\mathbf{v}^\dagger}^{\sim} && (5.176)\\
&= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}=(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}}\boldsymbol{u}(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{-\frac{1}{2}} && (5.177)\\
&= \exp\!\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}\right)\boldsymbol{u}\exp\!\left(\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0\right)=\exp\!\left(\frac{1}{2}\varphi\hat{\mathbf{v}}^\dagger\boldsymbol{\gamma}_0\right)\boldsymbol{u}\exp\!\left(\frac{1}{2}\varphi\boldsymbol{\gamma}_0\hat{\mathbf{v}}^\dagger\right) && (5.178)\\
&= \boldsymbol{u}\ominus\boldsymbol{v} \quad = \quad \boldsymbol{u}\oplus\boldsymbol{v}^\dagger && (5.179)\\
&= \boldsymbol{u}\ominus(\boldsymbol{o}+\mathbf{v}) \quad = \quad \boldsymbol{u}\oplus(\boldsymbol{o}-\mathbf{v}) && (5.180)\\
&= \gamma_{\mathbf{u}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\ominus\mathbf{v}}\mathbf{u}\ominus\mathbf{v} \quad = \quad \gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\mathbf{u}\oplus\mathbf{v}^\dagger && (5.181)\\
&= \gamma_{\mathbf{u}\ominus\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\ominus\mathbf{v}}\beta_{\mathbf{u}\ominus\mathbf{v}}c\widehat{\mathbf{u}\ominus\mathbf{v}} \quad = \quad \gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\beta_{\mathbf{u}\oplus\mathbf{v}^\dagger}c\widehat{\mathbf{u}\oplus\mathbf{v}^\dagger}. && (5.182)
\end{aligned}$$

The hyperbolic angle (rapidity) is

$$\varphi=\varphi_{\mathbf{v}}=\varphi_{\mathbf{v}^\dagger} = \operatorname{atanh}(\beta_{\mathbf{v}})=\operatorname{atanh}\!\left(\frac{|\mathbf{v}|}{|\boldsymbol{o}|}\right)=\operatorname{atanh}\!\left(\frac{v}{c}\right)=\operatorname{atanh}\!\left(\frac{\text{rise}}{\text{run}}\right) \tag{5.183}$$

$$\varphi_{\mathbf{v}^\dagger}\mathbf{v}^\dagger = -\varphi_{\mathbf{v}}\mathbf{v}. \tag{5.184}$$

The Lorentz factor for $\mathbf{v}$ is

$$\gamma_{\mathbf{v}}=\gamma_{\mathbf{v}^\dagger} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}=\frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. \tag{5.185}$$

The spacetime dilation factor is

$$\gamma_{\mathbf{u}\oplus\mathbf{v}}=\gamma_{\mathbf{u}\ominus\mathbf{v}^\dagger} = \gamma_{\mathbf{v}}\!\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right). \tag{5.186}$$

The spatial relativistic velocity subtraction is

$$\mathbf{u}\ominus\mathbf{v}=\mathbf{u}\oplus\mathbf{v}^\dagger = \frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}}-\mathbf{v}}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. \tag{5.187}$$

The natural speed of the spatial velocity subtraction $\mathbf{u}\ominus\mathbf{v}$ is

$$\beta_{\mathbf{u}\ominus\mathbf{v}}=\beta_{\mathbf{u}\oplus\mathbf{v}^\dagger} = \frac{\|\mathbf{u}\ominus\mathbf{v}\|}{c}=\frac{\sqrt{(\mathbf{u}\ominus\mathbf{v})\cdot(\mathbf{u}\ominus\mathbf{v})^\dagger}}{c}. \tag{5.188}$$

For parallel velocities $\mathbf{u}||\mathbf{v}$, then

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \frac{\beta_{\mathbf{u}}-\beta_{\mathbf{v}}}{1-\beta_{\mathbf{u}}\beta_{\mathbf{v}}} \qquad \text{for } \mathbf{u}||\mathbf{v}; \mathbf{u}=\beta_{\mathbf{u}}c\hat{\mathbf{v}}. \tag{5.189}$$

For perpendicular velocities $\mathbf{u}\perp\mathbf{v}$, then

$$\beta_{\mathbf{u}\ominus\mathbf{v}} = \sqrt{(1-\beta_{\mathbf{v}}^2)\beta_{\mathbf{u}}^2+\beta_{\mathbf{v}}^2} \qquad \text{for } \mathbf{u}\perp\mathbf{v}. \tag{5.190}$$

5.2.3.7 Time transformations for passive boost

Although $\boldsymbol{u}\ominus\boldsymbol{v}$ and $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ are the same spacetime velocities $\boldsymbol{u}\ominus\boldsymbol{v}=\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$, their associated time transformations are different. For $\boldsymbol{u}\ominus\boldsymbol{v}$, the boost operator is $(\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{-\frac{1}{2}}$ and the reciprocal of $\boldsymbol{v}/\boldsymbol{o}$ transforms time τ into t internal to the boost operator, and then the reciprocal of $\gamma_{\mathbf{v}}$ transforms time t back into τ external to the boost operator. Therefore, the time is t for $\boldsymbol{u}\ominus\boldsymbol{v}$ and the displacement is

$$\begin{aligned}(\boldsymbol{u}\ominus\boldsymbol{v})t &= (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\gamma_{\mathbf{u}\ominus\mathbf{v}}t && (5.191)\\ &= (\boldsymbol{o}+\mathbf{u}\ominus\mathbf{v})\tau && (5.192)\end{aligned}$$

that passively transforms t into τ. For $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$, the boost operator is $(\gamma_{\mathbf{v}^\dagger}\boldsymbol{v}^\dagger/\boldsymbol{o})^{\frac{1}{2}}$ and time t transforms into τ internal to the boost operator, and then time τ transforms back into t external to the boost operator. Therefore, the time is τ for $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ and the displacement is

$$\begin{aligned}(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger)\tau &= (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}^\dagger)\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau && (5.193)\\ &= (\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v}^\dagger)t && (5.194)\end{aligned}$$

that passively transforms τ into t.

A "passive" boost $(\boldsymbol{u}\ominus\boldsymbol{v})t$ passively transforms time t into $\tau=\gamma_{\mathbf{u}\ominus\mathbf{v}}t$, from the frame of $\boldsymbol{o}$ into the frame of $\boldsymbol{v}$. The boost $\boldsymbol{u}\ominus\boldsymbol{v}$ subtracts/moves $\boldsymbol{u}$ from the frame of $\boldsymbol{v}$ into the frame of $\boldsymbol{o}$ with time $t=t_{p\boldsymbol{o}}$ that passively transforms to time $\tau=t_{p\boldsymbol{v}}$ such that observer $\boldsymbol{v}$ sees a relative dilated time $\tau=\gamma_{\mathbf{u}\ominus\mathbf{v}}t$ and a relative velocity subtraction $\gamma_{\mathbf{u}\ominus\mathbf{v}}t\mathbf{u}\ominus\mathbf{v}=\tau\mathbf{u}\ominus\mathbf{v}$. The spacetime contraction $(\boldsymbol{u}\ominus\boldsymbol{v}/\gamma_{\mathbf{u}\ominus\mathbf{v}})t$ is an active velocity subtraction in the frame of $\boldsymbol{o}$.

An "active" boost $(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger)\tau$ passively transforms time τ into $t=\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau$, from the frame of $\boldsymbol{v}^\dagger$ into the frame of $\boldsymbol{o}$. The boost $\boldsymbol{u}\oplus\boldsymbol{v}^\dagger$ adds/moves $\boldsymbol{u}$ into the frame of $\boldsymbol{v}^\dagger$ with time $\tau=t_{p\boldsymbol{v}^\dagger}$ that passively transforms to coordinate time $t=t_{p\boldsymbol{o}}$ such that observer $\boldsymbol{o}$ sees a dilated time $t=\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau$ and a velocity addition $\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger}\tau\mathbf{u}\oplus\mathbf{v}^\dagger=t\mathbf{u}\oplus\mathbf{v}^\dagger$. The spacetime contraction $(\boldsymbol{u}\oplus\boldsymbol{v}^\dagger/\gamma_{\mathbf{u}\oplus\mathbf{v}^\dagger})t$ is an active velocity addition in the frame of $\boldsymbol{o}$.

5.2.3.8 Generalization of spacetime contraction operation

A spacetime velocity $\boldsymbol{u}$ that has been boosted successively has the general form

$$\boldsymbol{u}' = \gamma c\boldsymbol{\gamma}_0+\gamma\beta c\hat{\mathbf{u}}'. \tag{5.195}$$

In general, the spacetime contraction is

$$\boldsymbol{u}'' = \boldsymbol{u}'/\gamma = c\frac{\boldsymbol{u}'}{\boldsymbol{u}'\cdot\boldsymbol{\gamma}_0}. \tag{5.196}$$

The spacetime velocity $\boldsymbol{u}''$ can be interpreted as some combination of active velocity additions and subtractions in the frame of observer $\boldsymbol{o}$. Or, $\boldsymbol{u}''$ can be interpreted as a spacetime velocity transformed into the frame of a new observer $\boldsymbol{v}\rightarrow\boldsymbol{o}_{\boldsymbol{v}}$ with time $\tau=t_{p\boldsymbol{v}}=t_{c\boldsymbol{u}''}$.

If $\boldsymbol{u}=\boldsymbol{o}+\mathbf{u}$ is in the contracted frame of observable $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ with proper time τ, where $\boldsymbol{u}$ sees $\boldsymbol{v}$ as its observer $\boldsymbol{o}$, but $\boldsymbol{v}$ sees its observer $\boldsymbol{o}$ as the coordinate time t observer, then $\boldsymbol{u}$ can be transformed into the contracted frame of the coordinate time t observer as the active velocity addition with spacetime contraction

$$\begin{aligned}
\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}} &= B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}/\gamma_{\mathbf{u}\oplus\mathbf{v}} &(5.197)\\
&= \left(\gamma_{\mathbf{u}\oplus\mathbf{v}}c\boldsymbol{\gamma}_0+\gamma_{\mathbf{u}\oplus\mathbf{v}}\beta_{\mathbf{u}\oplus\mathbf{v}}c\widehat{\mathbf{u}\oplus\mathbf{v}}\right)/\gamma_{\mathbf{u}\oplus\mathbf{v}} &(5.198)\\
&= \boldsymbol{o}+\mathbf{u}\oplus\mathbf{v} &(5.199)\\
&= \boldsymbol{o}+\frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}. &(5.200)
\end{aligned}$$

The transformation of this back into the contracted frame of $\boldsymbol{v}$ is

$$\begin{aligned}
B_{\mathbf{v}}^{\sim}(\boldsymbol{u}\oplus\boldsymbol{v}/\gamma_{\mathbf{u}\oplus\mathbf{v}})B_{\mathbf{v}}/\gamma &= B_{\mathbf{v}}^{\sim}(\boldsymbol{o}+\mathbf{u}\oplus\mathbf{v})B_{\mathbf{v}}/\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} &(5.201)\\
&= (\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}}\boldsymbol{o}+\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}}\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v})/\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} &(5.202)\\
&= \boldsymbol{o}+\mathbf{u}=\boldsymbol{u}. &(5.203)
\end{aligned}$$

where

$$\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v} = \mathbf{u} \tag{5.204}$$

but

$$\begin{aligned}
\gamma_{\mathbf{u}\oplus\mathbf{v}\ominus\mathbf{v}} &\neq \gamma_{\mathbf{u}} &(5.205)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}^{-1}. &(5.206)
\end{aligned}$$

The generalization, for transformation into the new contracted frame after passive boosts of $\boldsymbol{u}$, is to divide by the general spacetime contraction factor

$$\gamma = \frac{\boldsymbol{u}'\cdot\boldsymbol{\gamma}_0}{c} \tag{5.207}$$

as

$$\boldsymbol{u}'' = \boldsymbol{u}'/\gamma=c\frac{\boldsymbol{u}'}{\boldsymbol{u}'\cdot\boldsymbol{\gamma}_0}=\boldsymbol{o}+\mathbf{u}' \tag{5.208}$$

where $\boldsymbol{o}$ represents the new observer with new coordinate time $t_{p\boldsymbol{o}}=t_{c\boldsymbol{u}''}$.

5.2.3.9 Approximations

For boost speed $\|\mathbf{v}\|\ll c$ and initial speed $\|\mathbf{u}\|\ll c$, an active boost is approximately an addition of velocities

$$\mathbf{u}\oplus\mathbf{v} \approx \mathbf{u}+\mathbf{v}, \qquad \text{for } \|\mathbf{u}\|, \|\mathbf{v}\|\ll c. \tag{5.209}$$

and a passive boost is approximately the subtraction of velocities

$$\mathbf{u}\ominus\mathbf{v} \approx \mathbf{u}-\mathbf{v}, \qquad \text{for } \|\mathbf{u}\|, \|\mathbf{v}\|\ll c. \tag{5.210}$$

For a very small speed $(v=\beta_{\mathbf{v}}c)\ll c$, then $\beta_{\mathbf{v}}\ll 1$ and the rapidity $\varphi_{\mathbf{v}}$ is approximately equal to $\beta_{\mathbf{v}}$

$$\begin{aligned}
\varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}) &(5.211)\\
&\approx \beta_{\mathbf{v}}, \qquad \text{for } \beta_{\mathbf{v}}\ll 1. &(5.212)
\end{aligned}$$

For $\beta_{\mathbf{v}} \ll 1$, the proper velocity (celerity) $\varphi_{\mathbf{v}} c\hat{\mathbf{v}}$ is approximately equal to velocity

$$\varphi_{\mathbf{v}} c\hat{\mathbf{v}} \;\approx\; (\beta_{\mathbf{v}} c\hat{\mathbf{v}} = \mathbf{v}), \qquad \text{for } \beta_{\mathbf{v}} \ll 1. \tag{5.213}$$

For a small speed $v \ll c$, the Lorentz factor $\gamma_{\mathbf{v}} = 1/\sqrt{1 - v^2/c^2} \approx 1$.

Using the hyperbolic function composition identities, it can be shown that the boost operator that applies the boost $\beta_{\mathbf{v}}$ *twice* successively, and adds the rapidity $2\varphi_{\mathbf{v}}$, is

$$B_{\mathbf{v}} B_{\mathbf{v}} = B_{\beta_{\mathbf{v}}} B_{\beta_{\mathbf{v}}} \;=\; \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{1-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0 \tag{5.214}$$

and the boost operator that applies the boost $\frac{1}{2}\beta_{\mathbf{v}}$ *twice* successively is

$$B_{\frac{1}{2}\beta_{\mathbf{v}}} B_{\frac{1}{2}\beta_{\mathbf{v}}} \;=\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0. \tag{5.215}$$

The double boost operator $B_{\beta_{\mathbf{v}}} B_{\beta_{\mathbf{v}}}$ can be defined as successive reflections in two spacetime planes through the origin, where the first plane contains the observer $\boldsymbol{o}$ and the second plane contains the boost observable (particle) $\boldsymbol{v}$. The two planes bound the hyperbolic angle $\varphi_{\mathbf{v}}$ that turns from the first plane at $\beta_{\boldsymbol{o}} = 0$ into the second plane at $\beta_{\mathbf{v}}$, toward the direction in space of the boost velocity $\mathbf{v}$.

For very small $\beta_{\mathbf{v}} \ll c$, then $\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}}) \approx \beta_{\mathbf{v}}$ and then

$$B_{\mathbf{v}} \;=\; e^{\frac{1}{2}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = e^{\frac{1}{4}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} e^{\frac{1}{4}\varphi_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = B_{\frac{1}{2}\varphi_{\mathbf{v}}} B_{\frac{1}{2}\varphi_{\mathbf{v}}} \tag{5.216}$$

$$\approx\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0 = B_{\frac{1}{2}\beta_{\mathbf{v}}} B_{\frac{1}{2}\beta_{\mathbf{v}}} \tag{5.217}$$

$$\approx\; e^{\frac{1}{2}\beta_{\mathbf{v}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0} = \cosh\!\left(\frac{1}{2}\beta_{\mathbf{v}}\right) + \sinh\!\left(\frac{1}{2}\beta_{\mathbf{v}}\right) \hat{\mathbf{v}} \boldsymbol{\gamma}_0 \tag{5.218}$$

The good approximation of $B_{\mathbf{v}}$ for very small $\beta_{\mathbf{v}} \ll c$ is

$$B_{\mathbf{v}} \;\approx\; \frac{2}{\sqrt{4-\beta_{\mathbf{v}}^2}} + \frac{\beta_{\mathbf{v}}}{\sqrt{4-\beta_{\mathbf{v}}^2}} \hat{\mathbf{v}} \boldsymbol{\gamma}_0. \tag{5.219}$$

6 Conformal Space-Time Algebra (CSTA)

$\mathcal{G}_{2,4}$ Conformal Space-Time Algebra (CSTA) is introduced in [3] as the *spacetime conformal group*.

$\mathcal{G}_{2,4}$ CSTA is a straightforward extension and adaptation of the $\mathcal{G}_{4,1}$ Conformal Geometric Algebra (CGA). CGA is introduced by Hestenes, Li, and Rockwood in [25]. CGA is also discussed by Perwass in [20], and by Dorst, Fontijne, and Mann in [4].

$\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) $\mathcal{D}$ contains two copies of $\mathcal{G}_{2,4}$ CSTA $\mathcal{C}$, which are called CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$. Elements and operations in CSTA1 are subscripted with $\mathcal{C}^1$. Elements and operations in CSTA2 are subscripted with $\mathcal{C}^2$. Elements and operations in generic CSTA are subscripted with $\mathcal{C}$.

Most formulas are expressed in CSTA $\mathcal{C}$ and written explicitly in CSTA1 $\mathcal{C}^1$ and CSTA2 $\mathcal{C}^2$ only when helpful to see how the particular CSTA1 and CSTA2 elements are used in formulas. Most formulas in CSTA can be written in CSTA1 and CSTA2 by just changing subscripts. CSTA uses the origin $\mathbf{e}_{o\gamma}$ and infinity $\mathbf{e}_{\infty\gamma}$ points and the Dirac gammas $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, $\boldsymbol{\gamma}_3$ for the timelike $w = ct$ and spatial x, y, z axes, respectively. The generic CSTA point embedding is $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$.

6.1 CSTA unit pseudoscalar

The $\mathcal{G}_{2,4}$ CSTA 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}}$ with signature $(+---+-)$ is

$$\begin{aligned}
\mathbf{I}_{\mathcal{C}} &= \mathbf{I}_{\mathcal{M}}(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\mathbf{I}_{\mathcal{S}}(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}) = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3\mathbf{e}_+\mathbf{e}_- && (6.1)\\
\mathbf{I}_{\mathcal{C}}^{\sim} &= (-1)^{6(6-1)/2}\mathbf{I}_{\mathcal{C}} = -\mathbf{I}_{\mathcal{C}} && (6.2)\\
\mathbf{I}_{\mathcal{C}}^2 &= -1 && (6.3)\\
\mathbf{I}_{\mathcal{C}}^{-1} &= -\mathbf{I}_{\mathcal{C}} = \mathbf{I}_{\mathcal{C}}^{\sim}. && (6.4)
\end{aligned}$$

The $\mathcal{G}_{2,4}$ CSTA1 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}^1}$ with signature $(+---+-)$ is

$$\mathbf{I}_{\mathcal{C}^1} = \mathbf{I}_{\mathcal{M}^1}(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{I}_{\mathcal{S}^1}(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4(\mathbf{e}_{\infty1}\wedge\mathbf{e}_{o1}) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6. \quad (6.5)$$

The $\mathcal{G}_{2,4}$ CSTA2 6-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}^2}$ with signature $(+---+-)$ is

$$\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{\mathcal{M}^2}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{I}_{\mathcal{S}^2}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}(\mathbf{e}_{\infty2}\wedge\mathbf{e}_{o2}) = \mathbf{e}_7\mathbf{e}_8\mathbf{e}_9\mathbf{e}_{10}\mathbf{e}_{11}\mathbf{e}_{12}. \quad (6.6)$$

6.2 CSTA point

The CSTA null 1-vector *point* entity is very similar to the CGA null 1-vector *point* entity. The following subsections define the CSTA points at the origin and at infinity, and the CSTA point embedding.

6.2.1 Stereographic embedding and homogenization

The embedding of an $\mathcal{G}_{1,3}$ STA position *vector* $\boldsymbol{p}_{\mathcal{M}}$ into a $\mathcal{G}_{2,4}$ CSTA null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ is done in exactly the same way a $\mathcal{G}_3$ APS point $\mathbf{p}$ is embedded into a $\mathcal{G}_{4,1}$ CGA point $\mathbf{P}_{\mathcal{C}}$. There are many references that explain the stereographic embedding and homogenization, such as Perwass [20], Rosenhahn [22], and the paper on $\mathcal{G}_{8,2}$ DCGA [7].

6.2.2 CSTA point at the origin

The CSTA null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o\gamma} = \frac{1}{2}(-\mathbf{e}_+ + \mathbf{e}_-) \quad (6.7)$$

where $\mathbf{e}_+$ is the stereographic unit and $\mathbf{e}_-$ is the homogeneous unit, and

$$\mathbf{e}_+ = \begin{cases} \mathbf{e}_5 & : \text{ in CSTA1} \\ \mathbf{e}_{11} & : \text{ in CSTA2} \end{cases} \quad (6.8)$$

$$\mathbf{e}_- = \begin{cases} \mathbf{e}_6 & : \text{ in CSTA1} \\ \mathbf{e}_{12} & : \text{ in CSTA2.} \end{cases} \quad (6.9)$$

The CSTA1 null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o1} = \frac{1}{2}(-\mathbf{e}_5 + \mathbf{e}_6). \quad (6.10)$$

The CSTA2 null 1-vector *point at the origin* is defined as

$$\mathbf{e}_{o2} = \frac{1}{2}(-\mathbf{e}_{11} + \mathbf{e}_{12}). \quad (6.11)$$

The CSTA null 1-vector *point at the origin* $\mathbf{e}_{o\gamma}$ represents either $\mathbf{e}_{o1}$ or $\mathbf{e}_{o2}$.

6.2.3 CSTA point at infinity

The CSTA null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty\gamma} = \mathbf{e}_+ + \mathbf{e}_-. \quad (6.12)$$

The CSTA1 null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty 1} = \mathbf{e}_5 + \mathbf{e}_6. \tag{6.13}$$

The CSTA2 null 1-vector *point at infinity* is defined as

$$\mathbf{e}_{\infty 2} = \mathbf{e}_{11} + \mathbf{e}_{12}. \tag{6.14}$$

The CSTA null 1-vector *point at infinity* $\mathbf{e}_{\infty\gamma}$ represents either $\mathbf{e}_{\infty 1}$ or $\mathbf{e}_{\infty 2}$.

6.2.4 CSTA point embedding

The generic CSTA null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ entity is the embedding of an STA *position* $\boldsymbol{p}_{\mathcal{M}}$ as

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \boldsymbol{p}_{\mathcal{M}} + \frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2 \mathbf{e}_{\infty\gamma} + \mathbf{e}_{o\gamma}. \tag{6.15}$$

The CSTA1 null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}^1}$ entity is the embedding of an STA1 *position* $\boldsymbol{p}_{\mathcal{M}^1}$ as

$$\boldsymbol{P}_{\mathcal{C}^1} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1}) = \boldsymbol{p}_{\mathcal{M}^1} + \frac{1}{2}\boldsymbol{p}_{\mathcal{M}^1}^2 \mathbf{e}_{\infty 1} + \mathbf{e}_{o1}. \tag{6.16}$$

The CSTA2 null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}^2}$ entity is the embedding of an STA2 *position* $\boldsymbol{p}_{\mathcal{M}^2}$ as

$$\boldsymbol{P}_{\mathcal{C}^2} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}) = \boldsymbol{p}_{\mathcal{M}^2} + \frac{1}{2}\boldsymbol{p}_{\mathcal{M}^2}^2 \mathbf{e}_{\infty 2} + \mathbf{e}_{o2}. \tag{6.17}$$

The embedding function $\mathcal{C}$ is implemented as a piecewise embedding function that embeds an STA, STA1, or STA2 vector into the corresponding CSTA, CSTA1, or CSTA2 point. The generic CSTA embedding will used to avoid duplication in generic discussions that can apply just as well in either CSTA1 or CSTA2 by only changing the subscripts accordingly.

The CSTA point $\boldsymbol{P}_{\mathcal{C}}$ is similar to a CGA point $\mathbf{P}_{\mathcal{C}}$ as in [7] *when* $\boldsymbol{P}_{\mathcal{C}}$ is the embedding of a spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ and we hold $w = ct = 0$, which gives the $\mathcal{G}_{1,4}$ CSA null 1-vector point $\mathbf{P}_{\mathcal{CS}}$.

As a GOPNS entity, a CSTA point $\boldsymbol{P}_{\mathcal{C}}$ simply represents the *point* (§6.5.2), as expected.

As a GIPNS entity, a finite CSTA point $\boldsymbol{P}_{\mathcal{C}}$, excluding $\mathbf{e}_{\infty\gamma}$, actually represents a *hypercone* (§6.4.2) in spacetime of the form

$$(w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0 \tag{6.18}$$

where

$$\boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 = p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}. \tag{6.19}$$

In general, a *hyper*surface in an n-D space has an $(n-1)$-D surface. A cone or other surface in 3D space has a 2D surface, but a hypercone or other hypersurface in 4D spacetime has a 3D surface. A hypersurface is treated and conceptualized in most respects the same as a 2D surface, but it embeds extended dimensions and its mathematical forms contain an additional term per extended dimension.

The hypercone is a result of the MINKOWSKI spacetime metric $(1, 3)$, which can be seen in the hypercone equation. For comparison to $\mathcal{G}_{4,1}$ CGA, a CGA point embeds a 3D Euclidean vector with metric $(3, 0)$ and represents an implicit surface equation of a sphere with zero radius

$$(x - p_x)^2 + (y - p_y)^2 + (z - p_z)^2 = 0. \tag{6.20}$$

In 3D spacetime with only two spatial dimensions by holding $z - p_z = 0$, the hypercone reduces to the circular cone

$$(x - p_x)^2 + (y - p_y)^2 - (w - p_w)^2 \;=\; 0 \tag{6.21}$$

which is an expanding circle in the xy-plane as the time-like coordinate $w = ct$ increases past p_w. The hypercone is an expanding sphere in space that is expanding with time t in radius

$$r \;=\; w - p_w = ct - p_w \tag{6.22}$$

at the speed of light c. The point begins expanding after time $t = p_w / c$ and is contracting before that time.

A CSTA point, as an expanding sphere, represents a *light-cone* in spacetime that is centered at the vertex point $\boldsymbol{p}_{\mathcal{M}}$. In spacetime, the light-cone is a spherical hypercone, which is a cone with a 3D hypersurface. A surface is usually 2D, but a hypersurface is imagined as a surface while it is actually a higher-dimensional space. The light-cone is often depicted as a *cone* in a 3D spacetime of two spatial dimensions and a time-like dimension, wherein the cone is a *circular wave front* of light that expands in space as time t increases. The expanding radius $r = ct - p_w$ of the wave front is centered at a point light source $\boldsymbol{p}_{\mathcal{M}}$. A CSTA point represents a *spherical wave front* of light in space, or light-cone in spacetime, centered at a point light source $\boldsymbol{p}_{\mathcal{M}}$ that flashes at time $t = p_w / c$.

6.2.5 CSTA point normalization

A homogeneous CSTA point embedding with scalar weight s is

$$s\boldsymbol{P}_{\mathcal{C}} = s\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) \;=\; s\boldsymbol{p}_{\mathcal{M}} + s\frac{1}{2}\boldsymbol{p}_{\mathcal{M}}^2\mathbf{e}_{\infty\gamma} + s\mathbf{e}_{o\gamma}. \tag{6.23}$$

A *normalized* point is scaled to weight $s = 1$.

The *normalization* of a weighted CSTA point $s\boldsymbol{P}_{\mathcal{C}}$ is

$$\boldsymbol{P}_{\mathcal{C}} \;=\; \frac{(s\boldsymbol{P}_{\mathcal{C}})}{-(s\boldsymbol{P}_{\mathcal{C}}) \cdot \mathbf{e}_{\infty\gamma}} = \frac{s\boldsymbol{P}_{\mathcal{C}}}{s}. \tag{6.24}$$

Many formulas require points and other entities to be unit weight. The normalization of an entity can be particular to the type of the entity.

A normalized point can be denoted

$$\hat{\boldsymbol{P}}_{\mathcal{C}} \;=\; \frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}} \cdot \mathbf{e}_{\infty\gamma}}. \tag{6.25}$$

6.2.6 CSTA point projection (inverse embedding)

The projection of CSTA point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ to STA position $\boldsymbol{p}_{\mathcal{M}}$ is

$$\boldsymbol{p}_{\mathcal{M}} = \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{C}}) \;=\; \left(\frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}} \cdot \mathbf{e}_{\infty\gamma}} \cdot \mathbf{I}_{\mathcal{M}}\right)\mathbf{I}_{\mathcal{M}}^{-1} \tag{6.26}$$

$$\;=\; (\hat{\boldsymbol{P}}_{\mathcal{C}} \cdot \mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1}. \tag{6.27}$$

6.2.7 CSTA test point

The symbolic CSTA *test point* $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ is the embedding of the symbolic STA *test vector*

$$\boldsymbol{t}_{\mathcal{M}} \;=\; w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 = ct\boldsymbol{\gamma}_0 + \mathbf{t}_{\mathcal{S}} = \boldsymbol{o}_{\mathcal{M}}t + \mathbf{t}_{\mathcal{S}}. \tag{6.28}$$

The symbolic CSTA1 *test point* $\boldsymbol{T}_{\mathcal{C}^1}=\mathcal{C}(\boldsymbol{t}_{\mathcal{M}^1})$ is the embedding of the symbolic STA1 *test vector*

$$\boldsymbol{t}_{\mathcal{M}^1} = w\mathbf{e}_1+x\mathbf{e}_2+y\mathbf{e}_3+z\mathbf{e}_4=ct\mathbf{e}_1+\mathbf{t}_{\mathcal{S}^1}=\boldsymbol{o}_{\mathcal{M}^1}t+\mathbf{t}_{\mathcal{S}^1}. \tag{6.29}$$

The symbolic CSTA2 *test point* $\boldsymbol{T}_{\mathcal{C}^2}=\mathcal{C}(\boldsymbol{t}_{\mathcal{M}^2})$ is the embedding of the symbolic STA2 *test vector*

$$\boldsymbol{t}_{\mathcal{M}^2} = w\mathbf{e}_7+x\mathbf{e}_8+y\mathbf{e}_9+z\mathbf{e}_{10}=ct\mathbf{e}_7+\mathbf{t}_{\mathcal{S}^2}=\boldsymbol{o}_{\mathcal{M}^2}t+\mathbf{t}_{\mathcal{S}^2}. \tag{6.30}$$

The symbolic scalars w, x, y, and z are the conventional coordinates in spacetime. The time-like coordinate $w = ct$ represents the distance traveled by light in time t. The *observer*, as defined for special relativity, is identified as the symbolic time-like velocity $\boldsymbol{o}_{\mathcal{M}}$.

CSTA1 and CSTA2 test points $\boldsymbol{T}_{\mathcal{C}^1}$ and $\boldsymbol{T}_{\mathcal{C}^2}$, respectively, are wedged to form the $\mathcal{G}_{4,8}$ DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{C}^1}\wedge\boldsymbol{T}_{\mathcal{C}^2}$. The DCSTA point value-*extraction elements* T_s are defined as elements that extract values from the DCSTA test point $\boldsymbol{T}_{\mathcal{D}}$ as $s=\boldsymbol{T}_{\mathcal{D}}\cdot T_s$.

6.3 CSTA point value-extraction elements

The CSTA 1-vector point value-*extraction elements* C_s extract the value s from a test point $\boldsymbol{T}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ as $s=\boldsymbol{T}_{\mathcal{C}}\cdot C_s$. The CSTA value-extraction elements are

$$C_1 = -\mathbf{e}_{\infty\gamma} \tag{6.31}$$
$$C_w = \boldsymbol{\gamma}_0 \tag{6.32}$$
$$C_t = \frac{1}{c}C_w \tag{6.33}$$
$$C_x = -\boldsymbol{\gamma}_1 \tag{6.34}$$
$$C_y = -\boldsymbol{\gamma}_2 \tag{6.35}$$
$$C_z = -\boldsymbol{\gamma}_3 \tag{6.36}$$
$$C_{\boldsymbol{t}^2} = -2\mathbf{e}_{o\gamma}. \tag{6.37}$$

These elements are straightforward to verify. When $w=ct$, the extraction C_t gives t. The extraction

$$\boldsymbol{T}_{\mathcal{C}}\cdot C_{\boldsymbol{t}^2} = \boldsymbol{t}_{\mathcal{M}}^2=|\boldsymbol{t}_{\mathcal{M}}|^2=w^2-r^2=(ct)^2-x^2-y^2-z^2 \tag{6.38}$$

is the squared modulus, or interval from the origin, of the STA test vector $\boldsymbol{t}_{\mathcal{M}}$.

The CSTA geometric inner product null space (GIPNS) 1-vector surface entities can be defined in terms of these extraction elements by writing their implicit surface functions. Two of these entities are the CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ and the CSTA GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{C}}$. A hyperhyperboloid can degenerate into a hypercone, which is a CSTA GIPNS null 1-vector point entity $\boldsymbol{P}_{\mathcal{C}}$. The CSTA GIPNS 1-vector entities $\boldsymbol{\Sigma}_{\mathcal{C}}$ and $\boldsymbol{E}_{\mathcal{C}}$ are similar in form to the CGA sphere $\mathbf{S}$ and plane $\boldsymbol{\Pi}$. The other CSTA GIPNS entities are of grades 2 to 5 and are formed as intersections (wedges) of hyperpseudospheres and hyperplanes or by specific formulas.

6.4 CSTA GIPNS entities

The $\mathcal{G}_{2,4}$ CSTA GIPNS entities are similar to $\mathcal{G}_{4,1}$ CGA GIPNS entities, but with some changes to account for the anti-Euclidean signature of $\mathcal{G}_{0,3}$ SA and the pseudo-Euclidean spacetime signature of $\mathcal{G}_{1,3}$ STA in a 4-D spacetime. The CSTA GIPNS entities of forms similar to CGA GIPNS entities are representing hypersurfaces in 4-D spacetime.

6.4.1 Geometric inner product null space (GIPNS)

Geometric inner product null space (GIPNS) entities are introduced by Perwass in [20], and are reviewed by this author in [7] and [9].

6.4.2 CSTA GIPNS 1-vector hypercone

The implicit quadric surface equation for a circular *hypercone* is

$$(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0. \tag{6.39}$$

The CSTA GIPNS null 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ is the point embedding

$$\boldsymbol{K}_{\mathcal{C}} \;=\; \boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) \tag{6.40}$$

with center vertex point $\boldsymbol{p}_{\mathcal{M}}$. The hypercone is a sphere in space that expands from a point at $\boldsymbol{p}_{\mathcal{M}}$ with squared radius

$$r^2 \;=\; (w-p_w)^2=(ct-p_w)^2. \tag{6.41}$$

6.4.3 CSTA GIPNS 1-vector hyperplane

A hyperplane is a linear subspace of dimension $(n-1)$ in a space of dimension n. In 4D spacetime, a hyperplane is a 3D subspace. The hyperplane space can be a Minkowski spacetime $(1,2)$ or an anti-Euclidean space $(0,3)$.

An implicit surface equation for a hyperplane in spacetime through the origin can be written

$$\begin{aligned} \boldsymbol{t}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}} \;&=\; && (6.42)\\ n_w w - n_x x - n_y y - n_z z \;&=\; 0. && (6.43) \end{aligned}$$

The STA vector

$$\boldsymbol{n}_{\mathcal{M}} \;=\; n_w\boldsymbol{\gamma}_0+n_x\boldsymbol{\gamma}_1+n_y\boldsymbol{\gamma}_2+n_z\boldsymbol{\gamma}_3 \tag{6.44}$$

is the *normal vector* to the hyperplane. Only the direction of $\boldsymbol{n}_{\mathcal{M}}$ is significant, and its magnitude can be arbitrary. However, as a normalization of the scale, normal vectors with unit magnitudes (norms) $\boldsymbol{n}/\sqrt{\boldsymbol{n}\cdot\boldsymbol{n}^\dagger}$ are sometimes required. The STA test vector $\boldsymbol{t}_{\mathcal{M}}$ is

$$\boldsymbol{t}_{\mathcal{M}} \;=\; w\boldsymbol{\gamma}_0+x\boldsymbol{\gamma}_1+y\boldsymbol{\gamma}_2+z\boldsymbol{\gamma}_3. \tag{6.45}$$

The equation holds good for any point $\boldsymbol{t}_{\mathcal{M}}$ on the hyperplane through the origin orthogonal to $\boldsymbol{n}_{\mathcal{M}}$. Using the CSTA point value-extraction elements (§6.3), the hyperplane implicit surface function can be written as the CSTA GIPNS entity

$$\begin{aligned} n_w w - n_x x - n_y y - n_z z \;&\rightarrow && (6.46)\\ n_w C_w - n_x C_x - n_y C_y - n_z C_z \;&= && (6.47)\\ n_w\boldsymbol{\gamma}_0+n_x\boldsymbol{\gamma}_1+n_y\boldsymbol{\gamma}_2+n_z\boldsymbol{\gamma}_3 \;&= && (6.48)\\ \boldsymbol{n}_{\mathcal{M}} \;&. && (6.49) \end{aligned}$$

The CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ through the origin with normal vector $\boldsymbol{n}_{\mathcal{M}}$ is defined as

$$\boldsymbol{E}_{\mathcal{C}} \;=\; \boldsymbol{n}_{\mathcal{M}}. \tag{6.50}$$

The hyperplane through the origin $\boldsymbol{n}_{\mathcal{M}}$ can be translated from the origin to a point $\boldsymbol{d}_{\mathcal{M}}$ using the translator (§6.6.4) operation

$$\begin{aligned} T_{\mathcal{C}}\boldsymbol{n}_{\mathcal{M}}\widetilde{T_{\mathcal{C}}} &= & (6.51)\\ \left(1-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}\right)\boldsymbol{n}_{\mathcal{M}}\left(1-\frac{1}{2}\mathbf{e}_{\infty\gamma}\boldsymbol{d}_{\mathcal{M}}\right) &= & (6.52)\\ \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{d}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma} &\quad . & (6.53)\end{aligned}$$

The CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ with normal vector $\boldsymbol{n}_{\mathcal{M}}$ is defined as

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma} & (6.54)\\ &\simeq \boldsymbol{E}^{*}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} & (6.55)\end{aligned}$$

and is equal to the CSTA undual of the dual CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}^{*}_{\mathcal{C}}$ (§6.5.13) up to a homogeneous scalar factor. The normal vector $\boldsymbol{n}_{\mathcal{M}}$ can have any magnitude, and $\boldsymbol{p}_{\mathcal{M}}$ can be any point on the hyperplane. The hyperplane $\boldsymbol{E}_{\mathcal{C}}$ has a form that is similar to a $\mathcal{G}_{4,1}$ CGA plane $\mathbf{\Pi}$, and when we hold $w=ct=0$, then the form gives the $\mathcal{G}_{1,4}$ CSA GIPNS 1-vector plane $\mathbf{\Pi}_{\mathcal{CS}}=\mathbf{n}_{\mathcal{S}}+(\mathbf{p}_{\mathcal{S}}\cdot\mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty\gamma}$.

If $\boldsymbol{n}_{\mathcal{M}}$ is normal (perpendicular) to the hyperplane and also a point on the hyperplane, then the hyperplane can be defined as

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= \boldsymbol{n}_{\mathcal{M}}+\boldsymbol{n}^{2}_{\mathcal{M}}\mathbf{e}_{\infty\gamma} & (6.56)\\ &= \boldsymbol{n}_{\mathcal{M}}+d^{2}\mathbf{e}_{\infty\gamma} & (6.57)\end{aligned}$$

where $d=\sqrt{\boldsymbol{n}^{2}_{\mathcal{M}}}$ is the hyperbolic distance (modulus) of $\boldsymbol{n}_{\mathcal{M}}$ from the origin. The modulus d may be a real or imaginary number, but the *spacetime interval* d^2 of $\boldsymbol{n}_{\mathcal{M}}$ from the origin is a real scalar. For $\boldsymbol{n}_{\mathcal{M}}$ both normal to and on the hyperplane, the squared modulus $d^2=\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}}$ from the origin, as well as d itself, is constant for all points $\boldsymbol{p}_{\mathcal{M}}$ on the hyperplane. By using the squared modulus d^2, it is possible to avoid imaginary numbers. The actual magnitude of $\boldsymbol{n}_{\mathcal{M}}$ does not affect the representation of the hyperplane surface since the hyperplane entity is a homogeneous entity that may be arbitrarily scaled by any non-zero scalar without affecting the surface that is represented. The scaling of entities affects metrical distance calculations between entities, and the formulas for distances between entities must include methods for normalizing the scale of entities.

The hyperplane $\boldsymbol{E}_{\mathcal{C}}$ is the set of STA points $\boldsymbol{t}_{\mathcal{M}}$

$$\mathbb{NI}_{G}(\boldsymbol{E}_{\mathcal{C}}) = \{\ \boldsymbol{t}_{\mathcal{M}}\ :\ \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})\cdot\boldsymbol{E}_{\mathcal{C}}=0\ \} \tag{6.58}$$

of the geometric inner product null space of $\boldsymbol{E}_{\mathcal{C}}$, denoted $\mathbb{NI}_{G}(\boldsymbol{E}_{\mathcal{C}})$ [20]. A similar set holds for all other GIPNS entities.

If $\boldsymbol{n}_{\mathcal{M}}$ is a null vector, then the hyperplane $\boldsymbol{E}_{\mathcal{C}}$ degenerates into the representation of a CSTA GIPNS null 1-vector *light-line (null line)* entity $\mathbb{L}_{\mathcal{C}}$ (§6.4.4) parallel to $\boldsymbol{n}_{\mathcal{M}}$ and through the point $\boldsymbol{p}_{\mathcal{M}}$. The null line entity $\mathbb{L}_{\mathcal{C}}$ includes the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

The intersections of two, three, four, or five hyperplanes $\boldsymbol{E}_{\mathcal{C}_i}$ can define entities for planes, lines, flat points, and the point at infinity, respectively as follows:

Two hyperplanes intersect as a CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ (§6.4.10)

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2} &= & (6.59)\\ (\boldsymbol{n}_1+d_1^2\mathbf{e}_{\infty\gamma})\wedge(\boldsymbol{n}_2+d_2^2\mathbf{e}_{\infty\gamma}) &= & (6.60)\\ \boldsymbol{n}_1\wedge\boldsymbol{n}_2+(d_2^2\boldsymbol{n}_1-d_1^2\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma} &= & (6.61)\\ \boldsymbol{D}^{*\mathcal{M}}-(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} &= \mathbf{\Pi}_{\mathcal{C}} & (6.62)\end{aligned}$$

where $\boldsymbol{D}^{*\mathcal{M}} = \boldsymbol{n}_1 \wedge \boldsymbol{n}_2$ is the STA dual of the plane $\boldsymbol{\Pi}_\mathcal{C}$ direction bivector $\boldsymbol{D}$.

Three hyperplanes intersect as a CSTA GIPNS 3-vector *line* $\boldsymbol{L}_\mathcal{C}$ (§6.4.11)

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \wedge \boldsymbol{E}_{\mathcal{C}_3} &= &(6.63)\\
(\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 + (d_2^2\boldsymbol{n}_1 - d_1^2\boldsymbol{n}_2)\mathbf{e}_{\infty\gamma}) \wedge (\boldsymbol{n}_3 + d_3^2\mathbf{e}_{\infty\gamma}) &= &(6.64)\\
\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 + (d_2^2\boldsymbol{n}_1 - d_1^2\boldsymbol{n}_2) \wedge \mathbf{e}_{\infty\gamma} \wedge \boldsymbol{n}_3 + d_3^2\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \mathbf{e}_{\infty\gamma} &= &(6.65)\\
\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 + (d_1^2\boldsymbol{n}_2 \wedge \boldsymbol{n}_3 - d_2^2\boldsymbol{n}_1 \wedge \boldsymbol{n}_3 + d_3^2\boldsymbol{n}_1 \wedge \boldsymbol{n}_2)\mathbf{e}_{\infty\gamma} &= &(6.66)\\
\boldsymbol{d}^{*\mathcal{M}} + (\boldsymbol{p}_\mathcal{M} \cdot \boldsymbol{d}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} &= \boldsymbol{L}_\mathcal{C} &(6.67)
\end{aligned}$$

where $\boldsymbol{d}^{*\mathcal{M}} = \boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3$ is the STA dual of the line $\boldsymbol{L}_\mathcal{C}$ direction vector $\boldsymbol{d}$.

Four hyperplanes intersect as a CST GIPNS 4-vector *flat point* (§6.4.16)

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \wedge \boldsymbol{E}_{\mathcal{C}_3} \wedge \boldsymbol{E}_{\mathcal{C}_4} &= &(6.68)\\
(\boldsymbol{n}_1 \wedge \boldsymbol{n}_2 \wedge \boldsymbol{n}_3 + (d_1^2\boldsymbol{n}_2 \wedge \boldsymbol{n}_3 - d_2^2\boldsymbol{n}_1 \wedge \boldsymbol{n}_3 + d_3^2\boldsymbol{n}_1 \wedge \boldsymbol{n}_2)\mathbf{e}_{\infty\gamma}) \wedge (\boldsymbol{n}_4 + d_4^2\mathbf{e}_{\infty\gamma}) &= &(6.69)\\
\alpha\mathbf{I}_\mathcal{M} - \alpha(\boldsymbol{p}_\mathcal{M} \cdot \mathbf{I}_\mathcal{M})\mathbf{e}_{\infty\gamma} &\simeq &(6.70)\\
\mathbf{I}_\mathcal{M} + \boldsymbol{p}_\mathcal{M}^*\mathbf{e}_{\infty\gamma} &= \mathbb{P}_\mathcal{C}. &(6.71)
\end{aligned}$$

The CSTA dual of the flat point is

$$\begin{aligned}
(\mathbf{I}_\mathcal{M} + \boldsymbol{p}_\mathcal{M}^*\mathbf{e}_{\infty\gamma})\mathbf{I}_\mathcal{C}^{-1} &= &(6.72)\\
\mathbf{e}_{\infty\gamma} \wedge \boldsymbol{P}_\mathcal{C} &= \mathbb{P}_\mathcal{C}^*. &(6.73)
\end{aligned}$$

Five hyperplanes intersect as the CSTA GIPNS 5-vector *point* at infinity $\mathbf{e}_{\infty\gamma}^\star$ (§6.4.17)

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \wedge \boldsymbol{E}_{\mathcal{C}_3} \wedge \boldsymbol{E}_{\mathcal{C}_4} \wedge \boldsymbol{E}_{\mathcal{C}_5} &= &(6.74)\\
(\mathbf{I}_\mathcal{M} + \boldsymbol{p}_\mathcal{M}^*\mathbf{e}_{\infty\gamma}) \wedge (\boldsymbol{n}_5 + d_5^2\mathbf{e}_{\infty\gamma}) &= &(6.75)\\
\boldsymbol{p}_\mathcal{M}^* \wedge \mathbf{e}_{\infty\gamma} \wedge \boldsymbol{n}_5 + d_5^2\mathbf{I}_\mathcal{M} \wedge \mathbf{e}_{\infty\gamma} &\simeq &(6.76)\\
\mathbf{I}_\mathcal{M} \wedge \mathbf{e}_{\infty\gamma} &= &(6.77)\\
\mathbf{I}_\mathcal{M} \wedge \mathbf{e}_+ + \mathbf{I}_\mathcal{M} \wedge \mathbf{e}_- &= &(6.78)\\
-\mathbf{I}_\mathcal{C} \cdot \mathbf{e}_- - \mathbf{I}_\mathcal{C} \cdot \mathbf{e}_+ &= &(6.79)\\
-\mathbf{I}_\mathcal{C} \cdot \mathbf{e}_{\infty\gamma} &= &(6.80)\\
-(-1)^{1(6-1)}\mathbf{e}_{\infty\gamma} \cdot \mathbf{I}_\mathcal{C} &= &(6.81)\\
\mathbf{e}_{\infty\gamma}\mathbf{I}_\mathcal{C} &= \mathbf{e}_{\infty\gamma}^\star. &(6.82)
\end{aligned}$$

6.4.4 CSTA GIPNS null 1-vector light-line (null line)

As a CSTA GIPNS entity, a null vector $\boldsymbol{n}_\mathcal{M} = \alpha(\boldsymbol{\gamma}_0 + \hat{\mathbf{n}})$ represents the line through the origin in the direction of $\boldsymbol{n}_\mathcal{M}$. This is a degenerate case of the hyperplane $\boldsymbol{E}_\mathcal{C}$ (§6.4.3) with null vector $\boldsymbol{n}_\mathcal{M}$ as the hyperplane normal vector.

The CSTA GIPNS null 1-vector *light-line (null line)* $\mathbb{L}_\mathcal{C}$ through the origin in the direction of the null vector $\boldsymbol{n}_\mathcal{M}$ is defined as

$$\mathbb{L}_\mathcal{C} = \boldsymbol{n}_\mathcal{M}. \qquad (6.83)$$

The null line $\mathbb{L}_\mathcal{C} = \boldsymbol{n}_\mathcal{M}$ can be translated in spacetime from the origin to an arbitrary point $\boldsymbol{p}_\mathcal{M}$ using a translation operation (§6.6.4). The CSTA GIPNS null 1-vector *null line* $\mathbb{L}_\mathcal{C}$ through the point $\boldsymbol{p}_\mathcal{M}$ in the direction of null vector $\boldsymbol{n}_\mathcal{M}$ is defined, via translation, as

$$\mathbb{L}_\mathcal{C} = \boldsymbol{n}_\mathcal{M} + (\boldsymbol{p}_\mathcal{M} \cdot \boldsymbol{n}_\mathcal{M})\mathbf{e}_{\infty\gamma}. \qquad (6.84)$$

The null line $\mathbb{L}_{\mathcal{C}}$ includes the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

6.4.5 CSTA GIPNS 1-vector hyperhyperboloid of one sheet

The implicit quadric surface equation for a circular *hyperhyperboloid of one sheet* is

$$r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_z)^2 \;=\; 0 \tag{6.85}$$

where r_0 is the initial radius of the expanding sphere in space with time-varying radius

$$r \;=\; \sqrt{r_0^2+(w-p_w)^2}=\sqrt{r_0^2+(ct-p_w)^2} \tag{6.86}$$

and center position

$$\boldsymbol{p}_{\mathcal{M}} \;=\; p_w\boldsymbol{\gamma}_0+p_x\boldsymbol{\gamma}_1+p_y\boldsymbol{\gamma}_2+p_z\boldsymbol{\gamma}_3=p_w\boldsymbol{\gamma}_0+\mathbf{p}_{\mathcal{S}} \tag{6.87}$$

in 4-D spacetime.

When $w-p_w=0$, the surface is a sphere with radius r_0. The circular hyperhyperboloid of one sheet can also be called a ***hyperpseudosphere***. Like a sphere, a hyperpseudosphere does not include the point at infinity.

In 3-D spacetime with only two spatial dimensions by holding $(z-p_z)=0$, the circular hyperhyperboloid of one sheet reduces to the circular *hyperboloid of one sheet*

$$\frac{(x-p_x)^2}{r_0^2}+\frac{(y-p_y)^2}{r_0^2}-\frac{(w-p_w)^2}{r_0^2} \;=\; 1. \tag{6.88}$$

When $(w-p_w)=0$, the hyperboloid of one sheet is a circle in the xy-plane with initial radius r_0 at initial time $t=p_w/c$, or $w=p_w$. The circle radius $r=\sqrt{r_0^2+(w-p_w)^2}$ is expanding after time $t=p_w/c$ and is contracting before that time. The radius is expanding with time t at the rate

$$\dot{r}=\frac{\partial r}{\partial t} \;=\; \frac{1}{2}r^{-1}2(ct-p_w)c=\frac{ct-p_w}{r}c=\frac{ct-p_w}{\sqrt{(ct-p_w)^2+r_0^2}}c. \tag{6.89}$$

The initial rate at time $t=p_w/c$ is $\dot{r}(p_w/c)=0$ and increases to $\dot{r}(\infty)=c$ as $t\to\infty$. In natural units, $c=1$ and the hyperhyperboloid of one sheet is asymptotically the hypercone of a spherically expanding point $\boldsymbol{P}_{\mathcal{C}}$. The acceleration of the radius r is

$$\ddot{r}=\frac{\partial\dot{r}}{\partial t} \;=\; \partial_t\frac{ct-p_w}{r}c=\frac{cr-\dot{r}(ct-p_w)}{r^2}c=\frac{c^2-\dot{r}^2}{r}. \tag{6.90}$$

The initial acceleration at $t=p_w/c$ is $\ddot{r}(p_w/c)=c^2/r_0$ and decreases to $\ddot{r}(\infty)=0$ as $t\to\infty$. In natural units, $c=1$ and $\ddot{r}$ is a measure of circular, or spherical, curvature at time t.

Using the CSTA point value-extraction elements (§6.3) the hyperhyperboloid of one sheet implicit surface function entity can be written

$$r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_x)^2 \;\to \tag{6.91}$$
$$(r_0^2+\boldsymbol{p}_{\mathcal{M}}^2)C_1+C_{\boldsymbol{t}^2}-2p_wC_w+2p_xC_x+2p_yC_y+2p_zC_z \;=\; \tag{6.92}$$
$$-(r_0^2+\boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma}-2\mathbf{e}_{o\gamma}-2p_w\boldsymbol{\gamma}_0-2p_x\boldsymbol{\gamma}_1-2p_y\boldsymbol{\gamma}_2-2p_z\boldsymbol{\gamma}_3 \;=\; \tag{6.93}$$
$$-2\boldsymbol{p}_{\mathcal{M}}-(r_0^2+\boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma}-2\mathbf{e}_{o\gamma} \quad . \tag{6.94}$$

Normalizing $\mathbf{e}_{o\gamma}$ by scaling $-1/2$ gives

$$\boldsymbol{p}_{\mathcal{M}}+\frac{1}{2}(r_0^2+\boldsymbol{p}_{\mathcal{M}}^2)\mathbf{e}_{\infty\gamma}+\mathbf{e}_{o\gamma} \;=\; \tag{6.95}$$
$$\boldsymbol{P}_{\mathcal{C}}+\frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma} \quad . \tag{6.96}$$

The CSTA GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\mathbf{\Sigma}_{\mathcal{C}}$ in spacetime with initial radius r_0 centered at CSTA point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is defined as

$$\mathbf{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma} \tag{6.97}$$

$$\simeq \mathbf{\Sigma}_{\mathcal{C}}^* \mathbf{I}_{\mathcal{C}} \tag{6.98}$$

and equals the CSTA undual of the dual CSTA GOPNS 5-vector *hyperpseudosphere* $\mathbf{\Sigma}_{\mathcal{C}}^*$ up to a homogeneous scalar factor.

The CSTA hyperpseudosphere $\mathbf{\Sigma}_{\mathcal{C}}$ is similar to a CGA sphere $\mathbf{S}$ discussed in [7] *when* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$ is the embedding of a spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ with $w = ct = 0$, which gives the $\mathcal{G}_{1,4}$ CSA GIPNS 1-vector sphere $\mathbf{S}_{\mathcal{CS}} = \mathbf{P}_{\mathcal{CS}} + \frac{1}{2} r^2 \mathbf{e}_{\infty\gamma}$. When $r_0 = 0$, $\mathbf{\Sigma}_{\mathcal{C}} = \boldsymbol{K}_{\mathcal{C}}$ is a *hypercone*, which is the CSTA point embedding $\boldsymbol{P}_{\mathcal{C}} = \boldsymbol{K}_{\mathcal{C}}$ as a GIPNS entity.

Two hyperpseudospheres can intersect in a growing spatial circle, which is a CSTA GIPNS 2-vector spacetime hyperboloid or pseudosphere $\boldsymbol{S}_{\mathcal{C}}$. Three hyperpseudospheres can intersect in a growing spatial point pair, which is a CSTA GIPNS 3-vector spacetime hyperbola or pseudocircle $\boldsymbol{C}_{\mathcal{C}}$. Four hyperpseudospheres can intersect in a CSTA GIPNS 4-vector spacetime point pair $\mathbb{P}_{\mathcal{C}}$.

6.4.6 CSTA GIPNS 1-vector hyperhyperboloid of two sheets

The implicit quadric surface equation for a circular *hyperhyperboloid of two sheets* is

$$-r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0. \tag{6.99}$$

The CSTA GIPNS 1-vector *hyperhyperboloid of two sheets (imaginary hyperpseudosphere)* is

$$\mathbf{\Xi}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} - \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma}. \tag{6.100}$$

The imaginary radius is $\sqrt{-1} r_0$.

The intersection of $\mathbf{\Xi}_{\mathcal{C}}$ and hyperplane $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0 + p_w \mathbf{e}_{\infty\gamma}$ holds $w = p_w$ and produces an imaginary sphere. The intersection of $\mathbf{\Xi}_{\mathcal{C}}$ and hyperplane $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_3 - p_z \mathbf{e}_{\infty\gamma}$ holds $z = p_z$ and produces a hyperboloid of two sheets in wxy-spacetime opening up and down the w-axis. The intersection of $\mathbf{\Xi}_{\mathcal{C}}$ and spacetime plane $\mathbf{\Pi}_{\mathcal{C}}$ produces a hyperbola in the spacetime plane that opens up and down the time axis.

6.4.7 CSTA GIPNS 2-vector spatial sphere

The implicit quadric surface equation for a *hyperpseudosphere* is

$$r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0. \tag{6.101}$$

If we set the time coordinate w, then we get the implicit surface equation for a *sphere* in xyz-space

$$(x - p_x)^2 + (y - p_y)^2 + (z - p_z)^2 - r^2 = 0 \tag{6.102}$$

with radius

$$r = \sqrt{r_0^2 + (w - p_w)^2}. \tag{6.103}$$

To set w, we can intersect a hyperpseudosphere

$$\mathbf{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2} r_0^2 \mathbf{e}_{\infty\gamma} \tag{6.104}$$

with radius r_0 centered at

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}) \tag{6.105}$$

with the hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0 + w\mathbf{e}_{\infty\gamma} \tag{6.106}$$

of xyz-space at w. The sphere of radius r_0 centered at $\mathbf{p}_{\mathcal{S}}$ is at the time $w = p_w$.

The CSTA GIPNS 2-vector *sphere* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with radius r_0 is the intersection

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} & (6.107)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma}\right) \wedge (\boldsymbol{\gamma}_0 + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{\gamma}_0)\mathbf{e}_{\infty\gamma}) & (6.108)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma}\right) \wedge (\boldsymbol{\gamma}_0 + p_w\mathbf{e}_{\infty\gamma}) & (6.109)\\ &\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} & (6.110) \end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *sphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

6.4.8 CSTA GIPNS 2-vector spacetime hyperboloid of one sheet

The implicit quadric surface equation for a *hyperpseudosphere* is

$$r_0^2 + (w - p_w)^2 - (x - p_x)^2 - (y - p_y)^2 - (z - p_z)^2 = 0. \tag{6.111}$$

If we set one spatial coordinate, for instance z, then we get the implicit surface equation for a circular *hyperboloid of one sheet* in wxy-spacetime

$$\begin{aligned} (x - p_x)^2 + (y - p_y)^2 - (w - p_w)^2 - (r_0^2 - (z - p_z)^2) &= & (6.112)\\ \frac{(x - p_x)^2}{r^2} + \frac{(y - p_y)^2}{r^2} - \frac{(w - p_w)^2}{r^2} - 1 &= 0 & (6.113) \end{aligned}$$

with central radius

$$r = \sqrt{r_0^2 - (z - p_z)^2} \tag{6.114}$$

and circular conic section radius at w, z

$$r_c = \sqrt{(w - p_w)^2 + (r_0^2 - (z - p_z)^2)} = \sqrt{(w - p_w)^2 + r^2}. \tag{6.115}$$

The spacetime hyperboloid of one sheet is also called a *pseudosphere*. Like a sphere, a pseudosphere does not include the point at infinity. The pseudosphere in wxy-spacetime is a *circle* in xy-space that changes in radius with w, z. To set z, we can intersect a hyperpseudosphere

$$\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma} \tag{6.116}$$

with radius r_0 centered at

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}}) \tag{6.117}$$

with the hyperplane

$$\begin{aligned} \boldsymbol{E}_{\mathcal{C}} &= z\boldsymbol{\gamma}_3 - z^2\mathbf{e}_{\infty\gamma} & (6.118)\\ &\simeq \boldsymbol{\gamma}_3 - z\mathbf{e}_{\infty\gamma} & (6.119) \end{aligned}$$

of wxy-space at $z\boldsymbol{\gamma}_3$. To have pseudosphere central radius r_0, set $z = p_z$. The xy-plane circle of radius r_0 centered at $\mathbf{p}_{\mathcal{S}}$ is at the time $w = p_w$. More generally, the hyperplane can be through point $\boldsymbol{p}_{\mathcal{M}}$ with spatial normal vector $\mathbf{n}_{\mathcal{S}}$ as

$$\boldsymbol{E}_{\mathcal{C}} = \mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}. \tag{6.120}$$

The CSTA GIPNS 2-vector *spacetime hyperboloid of one sheet (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with central radius r_0 in the spatial plane orthogonal to normal vector $\mathbf{n}_{\mathcal{S}}$ is the intersection

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} && (6.121)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty}\right) \wedge (\mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}) && (6.122)\\ &\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (6.123) \end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *pseudosphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

The CSTA GIPNS null 2-vector *spacetime cone (null cone)* is the pseudosphere $\boldsymbol{S}_{\mathcal{C}}$ with central radius $r = 0$.

The spacetime hyperboloid of one sheet is always in a 3D spacetime and is not a purely spatial surface. The spatial part is circles in planes parallel to the $\mathbf{n}_{\mathcal{S}}^{*}$-plane centered at $\mathbf{p}_{\mathcal{S}}$ with radius $r_c = \sqrt{(w - p_w)^2 + r_0^2}$. The CSTA GIPNS 3-vector *circle* entity $\boldsymbol{C}_{\mathcal{C}}$ with radius r_0 is obtained by another intersection with the hyperplane at $p_w\boldsymbol{\gamma}_0$.

6.4.9 CSTA GIPNS 2-vector spacetime hyperboloid of two sheets

The CSTA GIPNS 2-vector *spacetime hyperboloid of two sheets (imaginary pseudosphere)* $\boldsymbol{S}_{\mathcal{C}}$ centered at $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with central radius r_0 in the spatial plane orthogonal to normal vector $\mathbf{n}_{\mathcal{S}}$ is the intersection

$$\begin{aligned} \boldsymbol{S}_{\mathcal{C}} &= \boldsymbol{\Xi}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} && (6.124)\\ &= \left(\boldsymbol{P}_{\mathcal{C}} - \frac{1}{2}r_0^2\mathbf{e}_{\infty}\right) \wedge (\mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty}) && (6.125)\\ &\simeq \boldsymbol{S}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} && (6.126) \end{aligned}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *imaginary pseudosphere* $\boldsymbol{S}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor. The two sheets open up and down the w-axis and have circular sections in the $\mathbf{n}_{\mathcal{S}}^{*}$-plane.

6.4.10 CSTA GIPNS 2-vector plane

A plane in spacetime can be defined by two orthogonal unit-norm direction vectors

$$\begin{aligned} \boldsymbol{d}_{\mathcal{M}_1} &= \frac{\boldsymbol{d}_{\mathcal{M}_1}}{\|\boldsymbol{d}_{\mathcal{M}_1}\|} = d_{w_1}\boldsymbol{\gamma}_0 + d_{x_1}\boldsymbol{\gamma}_1 + d_{y_1}\boldsymbol{\gamma}_2 + d_{z_1}\boldsymbol{\gamma}_3 && (6.127)\\ \boldsymbol{d}_{\mathcal{M}_2} &= \frac{\boldsymbol{d}_{\mathcal{M}_2}}{\|\boldsymbol{d}_{\mathcal{M}_2}\|} = d_{w_2}\boldsymbol{\gamma}_0 + d_{x_2}\boldsymbol{\gamma}_1 + d_{y_2}\boldsymbol{\gamma}_2 + d_{z_2}\boldsymbol{\gamma}_3 && (6.128)\\ \boldsymbol{d}_{\mathcal{M}_1} \cdot \boldsymbol{d}_{\mathcal{M}_2}^{\dagger} &= 0 && (6.129) \end{aligned}$$

and a point

$$\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3 \tag{6.130}$$

on the plane. The direction of the plane is represented by the normalized unit bivector

$$\boldsymbol{D} = \boldsymbol{d}_{\mathcal{M}_1} \wedge \boldsymbol{d}_{\mathcal{M}_2} \tag{6.131}$$

$$= \frac{\boldsymbol{D}}{\sqrt{\boldsymbol{D} \cdot \boldsymbol{D}^{\dagger}}} = \frac{\boldsymbol{D}}{\sqrt{\boldsymbol{D} \cdot (\boldsymbol{\gamma}_0 \boldsymbol{D}^{\sim} \boldsymbol{\gamma}_0)}}. \tag{6.132}$$

The notation $\mathbf{A}_{\mathcal{M}}^{\dagger} = \boldsymbol{\gamma}_0 \widetilde{\mathbf{A}_{\mathcal{M}}} \boldsymbol{\gamma}_0$ is the anti-Euclidean space conjugation, or SA space conjugation, which is necessary for the case where $\boldsymbol{D}$ is a null bivector. For blade $\mathbf{A}_{\mathcal{M}}$ in spacetime, the *conjugate* [20] has the property

$$\mathbf{A}_{\mathcal{M}} \cdot \mathbf{A}_{\mathcal{M}}^{\dagger} = \mathbf{A}_{\mathcal{M}} \cdot \left(\boldsymbol{\gamma}_0 \widetilde{\mathbf{A}_{\mathcal{M}}} \boldsymbol{\gamma}_0 \right) = \|\mathbf{A}_{\mathcal{M}}\|^2. \tag{6.133}$$

Any test point

$$\boldsymbol{t}_{\mathcal{M}} = w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 \tag{6.134}$$

on the plane must satisfy the plane equation

$$(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}) \wedge \boldsymbol{D} = 0 \tag{6.135}$$

which can also be written in the dual form

$$(\boldsymbol{t}_{\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}}) \cdot \boldsymbol{D}^{*\mathcal{M}} = \boldsymbol{t}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}} - \boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}} = 0. \tag{6.136}$$

The dual form plane equation is vector-valued and the components represent a system of implicit surface equations for an intersection of hyperplanes that gives the plane.

The CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ in the planar direction of the unit bivector $\boldsymbol{D}$ in spacetime can be defined as

$$\mathbf{\Pi}_{\mathcal{C}} = \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} \tag{6.137}$$

$$\simeq \mathbf{\Pi}_{\mathcal{C}}^{*} \mathbf{I}_{\mathcal{C}} \tag{6.138}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 4-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}^{*}$ up to a homogeneous scalar factor.

The CSTA translation operation on any CSTA entity can be defined as its successive reflections in two parallel CSTA planes. The CSTA 2-versor *translator* (translation operator) $T_{\mathcal{C}}$ can be defined by two parallel planes $\mathbf{\Pi}_{\mathcal{C}_1}$ and $\mathbf{\Pi}_{\mathcal{C}_2}$ that are separated by a spacetime displacement vector $\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}$ from $\mathbf{\Pi}_{\mathcal{C}_1}$ to $\mathbf{\Pi}_{\mathcal{C}_2}$ as

$$T_{\mathcal{C}} = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1}. \tag{6.139}$$

The translator versor operation on a CSTA point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$, for example, is

$$\boldsymbol{P}_{\mathcal{C}}' = T_{\mathcal{C}} \boldsymbol{P}_{\mathcal{C}} \widetilde{T_{\mathcal{C}}} = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1} \boldsymbol{P}_{\mathcal{C}} \widetilde{\mathbf{\Pi}_{\mathcal{C}_1}} \widetilde{\mathbf{\Pi}_{\mathcal{C}_2}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}} + \boldsymbol{d}_{\mathcal{M}}). \tag{6.140}$$

The successive reflections in two parallel planes translates by *twice* the spacetime displacement between the parallel planes.

The *rotor* (spatial rotation operator) $R_{\mathcal{S}}$ for a rotation by *twice* the angle between two non-parallel spatial planes $\mathbf{\Pi}_{\mathcal{C}_1}$ and $\mathbf{\Pi}_{\mathcal{C}_2}$ can be defined as

$$R_{\mathcal{S}} = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1}. \tag{6.141}$$

The spatial rotation operator $R_{\mathcal{S}}$ is equivalent to the SA rotor and is the same spatial rotor that is used in STA and CSTA. The spatial rotor $R_{\mathcal{S}}$ can spatially rotate any multivector by *versor outermorphism* [20] that rotates all vectors within outer products.

The double *boost* operator $B_{\mathbf{v}}B_{\mathbf{v}}$, where $B_{\mathbf{v}} = (\gamma_{\mathbf{v}}\boldsymbol{v}\,/\,\boldsymbol{o})^{\frac{1}{2}}$ and $\boldsymbol{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}}c\hat{\mathbf{v}}$, that adds the double rapidity $2\varphi_{\mathbf{v}} = 2\mathrm{atanh}(\beta_{\mathbf{v}})$ in the direction of $\boldsymbol{v}$ can be defined as the successive reflections in two non-parallel spacetime planes. The first plane $\mathbf{\Pi}_{\mathcal{C}_1}$ should represent the observer as a plane through the origin and observer $\boldsymbol{o}$ that spans the time axis and another spatial axis perpendicular to $\mathbf{v}$. The second plane $\mathbf{\Pi}_{\mathcal{C}_2}$ should represent the boost velocity $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ by passing through the origin and $\boldsymbol{v}$ and spanning the same direction perpendicular to $\mathbf{v}$ as the first plane. The planes should be unit scale by using unit bivectors to define the plane directions. The two planes contain the hyperbolic angle $\varphi_{\mathbf{v}}$ that turns positive from $\mathbf{\Pi}_{\mathcal{C}_1}$ toward $\mathbf{\Pi}_{\mathcal{C}_2}$ into the direction of $\mathbf{v}$. The double boost $B_{\mathbf{v}}B_{\mathbf{v}}$ of a spacetime velocity $\boldsymbol{u}$ is obtained by the successive reflections

$$\boldsymbol{u}' = \mathbf{\Pi}_{\mathcal{C}_2}\mathbf{\Pi}_{\mathcal{C}_1}\boldsymbol{u}\mathbf{\Pi}_{\mathcal{C}_1}^{\sim}\mathbf{\Pi}_{\mathcal{C}_2}^{\sim} = B_{\mathbf{v}}B_{\mathbf{v}}\boldsymbol{u}B_{\mathbf{v}}^{\sim}B_{\mathbf{v}}^{\sim}. \tag{6.142}$$

6.4.11 CSTA GIPNS 3-vector line

6.4.11.1 Implicit surface equation of line

An implicit equation for a line in spacetime through two points can be written as

$$(\boldsymbol{t} - \boldsymbol{p}_1)\cdot(\boldsymbol{p}_2 - \boldsymbol{p}_1)^{*\mathcal{M}} = 0 \tag{6.143}$$

where $\boldsymbol{t}$ is the CSTA test point. The equation holds good for any $\boldsymbol{t}$ on the line of the two points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. The unit norm direction $\boldsymbol{d}$ of the line can be written as

$$\boldsymbol{d} = \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\sqrt{(\boldsymbol{p}_2 - \boldsymbol{p}_1)\cdot(\boldsymbol{p}_2 - \boldsymbol{p}_1)^{\dagger}}} \tag{6.144}$$

$$= \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\sqrt{(\boldsymbol{p}_2 - \boldsymbol{p}_1)\cdot(\boldsymbol{\gamma}_0(\boldsymbol{p}_2 - \boldsymbol{p}_1)\boldsymbol{\gamma}_0)}} \tag{6.145}$$

$$= \frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{\|\boldsymbol{p}_2 - \boldsymbol{p}_1\|}. \tag{6.146}$$

The unit norm trivector dual to the line direction is

$$\boldsymbol{d}^{*\mathcal{M}} = \boldsymbol{d}\mathbf{I}_{\mathcal{M}}^{-1}. \tag{6.147}$$

The implicit equation can be rewritten as

$$(\boldsymbol{t} - \boldsymbol{p})\cdot\boldsymbol{d}^{*\mathcal{M}} = \tag{6.148}$$

$$\boldsymbol{t}\cdot\boldsymbol{d}^{*\mathcal{M}} - \boldsymbol{p}\cdot\boldsymbol{d}^{*\mathcal{M}} = 0 \tag{6.149}$$

where $\boldsymbol{p}$ is any point on the line.

6.4.11.2 Definition of line entity

The CSTA 3-vector *line* entity $\boldsymbol{L}_{\mathcal{C}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ in the direction of unit norm vector $\boldsymbol{d}_{\mathcal{M}}$ can be defined as

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{d}^{*\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{d}^{*\mathcal{M}})\wedge\mathbf{e}_{\infty\gamma} \tag{6.150}$$

$$\simeq \boldsymbol{L}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}} \tag{6.151}$$

which is equal to the CSTA undual of the dual CSTA GOPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ (§6.5.6) up to a homogeneous scalar factor. If the line direction $\boldsymbol{d}$ is a null vector, then the line entity $\boldsymbol{L}_{\mathcal{C}}$ is a null 3-vector representing a null line (light-line), otherwise it is a non-null 3-vector representing a timelike or spacelike line. The point at infinity $\mathbf{e}_{\infty\gamma}$ is on all 3-vector lines $\boldsymbol{L}_{\mathcal{C}}$.

6.4.11.3 Observable representation

The observable worldline

$$\boldsymbol{p}_{\mathcal{M}}^{\mathbf{p}_0}(t) = \mathbf{p}_0 + \boldsymbol{v}t \tag{6.152}$$
$$= \mathbf{p}_0 + \boldsymbol{o}t + \mathbf{v}t \tag{6.153}$$
$$= \boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t) + \mathbf{v}t, \tag{6.154}$$

which intersects the worldline of the translated observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ at

$$\mathbf{p}_0 = o_x\boldsymbol{\gamma}_1 + o_y\boldsymbol{\gamma}_2 + o_z\boldsymbol{\gamma}_3 \tag{6.155}$$

when the coordinate time is $t = t_{\boldsymbol{po}} = t_{\boldsymbol{cv}} = 0$, can be represented as the CSTA GIPNS 3-vector *line* $\boldsymbol{L_p}$ that is constructed as

$$\boldsymbol{L_p} = \boldsymbol{v}^{*\mathcal{M}} + (\mathbf{p}_0 \cdot \boldsymbol{v}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma}. \tag{6.156}$$

The observable worldline $\boldsymbol{L_p}$ can be operated on by all of the CSTA versors. For example, the translated-boost $B_{\mathcal{C}\mathbf{u}}^{\mathbf{p}_0}$ (§6.6.9) of $\boldsymbol{L_p}$ can boost $\boldsymbol{L_p}$ into the frame of $\boldsymbol{u} = \dot{\boldsymbol{o}}_{\mathcal{M}}^{\mathbf{p}_0}(t) + \mathbf{u}$ with proper time $\tau = t_{\boldsymbol{pu}} = t_{\boldsymbol{cv}\oplus\boldsymbol{u}}$ relative to the translated-observer $\boldsymbol{o}_{\mathcal{M}}^{\mathbf{p}_0}(t)$ as a relativistic velocity addition $\boldsymbol{v} \oplus \boldsymbol{u}$ while the initial position remains $\mathbf{p}_0$. Other, even more complicated, spacetime transformations can be achieved by compositions of the CSTA versors applied to an observable line $\boldsymbol{L_p}$.

The position *point* $\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t)$ of the observable $\boldsymbol{L_p}$ at time t is represented by the CSTA GIPNS 4-vector *flat point* (§6.4.16)

$$\mathbb{P}_{\mathcal{C}} = \boldsymbol{L_p} \wedge \boldsymbol{E}_{\mathcal{C}} \tag{6.157}$$
$$\simeq (\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t) \wedge \mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} \tag{6.158}$$

with time hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0 + t\mathbf{e}_{\infty}. \tag{6.159}$$

The *flat point projection* (Eq. 6.239) of the CSTA GOPNS 2-vector *flat point* $\mathbb{P}_{\mathcal{C}}^* \simeq \mathbb{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}$ (§6.5.5) can *project* the CSTA position *point* $\boldsymbol{P}_{\mathcal{C}}^{\mathbf{p}_0}(t)$ as the STA position *vector* $\boldsymbol{p}_{\mathcal{M}}^{\mathbf{p}_0}(t)$.

The rapidity of $\boldsymbol{L_p}$ is given by

$$\varphi = \operatorname{acosh}\left(\frac{\boldsymbol{L_p} \cdot (\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3)}{\sqrt{\boldsymbol{L}_{\boldsymbol{p}}^2}}\right) \tag{6.160}$$

and the natural speed β of $\boldsymbol{L_p}$ is

$$\beta = \tanh(\varphi). \tag{6.161}$$

The formula for rapidity φ is similar to the standard formula for the angle θ between two Euclidean plane vectors $\mathbf{v}$ and $\mathbf{u}$, $\theta = \operatorname{acos}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)$, except that this plane is a Minkowski spacetime plane with hyperbolic angle φ.

The spacetime velocity $\boldsymbol{v}$ of $\boldsymbol{L_p}$ is the line direction

$$\boldsymbol{v} = \boldsymbol{d} = ((\boldsymbol{L_p} \cdot \mathbf{I}_{\mathcal{M}})\mathbf{I}_{\mathcal{M}}^{-1})\mathbf{I}_{\mathcal{M}} = \boldsymbol{L_p} \cdot \mathbf{I}_{\mathcal{M}}. \tag{6.162}$$

If the observable $\boldsymbol{L_p}$ is boosted by translated-boost operations, then $\boldsymbol{v}$ is boosted according to all the same results and interpretations as $\boldsymbol{v}$ boosted by STA boost operations centered on the observer $\boldsymbol{o}$.

One advantage of using the CSTA worldline representation $\boldsymbol{L_p}$ is the ability to easily incorporate initial positions $\mathbf{p}_0$ and use translated spacetime operations with the CSTA translator (§6.6.4). Other advantages may include the ability to compute various intersections of a worldline with other CSTA spacetime entities.

6.4.12 CSTA GIPNS 3-vector spatial circle

The CSTA GIPNS 1-vector *hyperpseudosphere* with radius r_0

$$\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0^2\mathbf{e}_{\infty\gamma} \tag{6.163}$$

centered at

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3) \tag{6.164}$$

can be intersected with two CSTA GIPNS 1-vector *hyperplanes*

$$\begin{aligned}\boldsymbol{E}_{\mathcal{C}_1} &= \mathbf{n}_{\mathcal{S}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{n}_{\mathcal{S}})\mathbf{e}_{\infty\gamma} &(6.165)\\ \boldsymbol{E}_{\mathcal{C}_2} &= \boldsymbol{\gamma}_0 + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{\gamma}_0)\mathbf{e}_{\infty\gamma} &(6.166)\end{aligned}$$

to obtain a circle with radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}}$ in the spatial plane through $\boldsymbol{p}_{\mathcal{M}}$ with direction bivector $\mathbf{N}_{\mathcal{S}} = \mathbf{n}_{\mathcal{S}}^* = -\mathbf{n}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}$.

The CSTA GIPNS 3-vector *circle* entity $\boldsymbol{C}_{\mathcal{C}}$ centered at $\boldsymbol{p}_{\mathcal{M}}$ with radius r_0 at time p_w in the plane of bivector $\mathbf{N}_{\mathcal{S}} = \mathbf{n}_{\mathcal{S}}^* = -\mathbf{n}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{-1}$ dual to normal vector $\mathbf{n}_{\mathcal{S}}$ can be formed as

$$\begin{aligned}\boldsymbol{C}_{\mathcal{C}} &= \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} &(6.167)\\ &= \boldsymbol{S}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_2} &(6.168)\end{aligned}$$

Without setting the time $w = p_w$ by intersecting $\boldsymbol{E}_{\mathcal{C}_2}$, the circle changes radius with time as the CSTA GIPNS 2-vector *hyperboloid (pseudosphere)* $\boldsymbol{S}_{\mathcal{C}} = \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1}$ (§6.4.8).

The CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ can also be represented as the intersection of the CSTA GIPNS 1-vector *hyperpseudosphere* $\boldsymbol{\Sigma}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ as

$$\boldsymbol{C}_{\mathcal{C}} = \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} \tag{6.169}$$

where the hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ is the same as above and sets the center $\boldsymbol{p}_{\mathcal{M}}$ and radius r_0, and the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ with spatial direction bivector $\mathbf{N}_{\mathcal{S}}$ through point $\boldsymbol{p}_{\mathcal{M}}$ is

$$\begin{aligned}\boldsymbol{\Pi}_{\mathcal{C}} &= \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} &(6.170)\\ &= \mathbf{N}_{\mathcal{S}}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \mathbf{N}_{\mathcal{S}}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma} &(6.171)\\ &= \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2}. &(6.172)\end{aligned}$$

The CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ is equal to the CSTA undual of the dual CSTA GOPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}^*$

$$\boldsymbol{C}_{\mathcal{C}} \simeq \boldsymbol{C}_{\mathcal{C}}^*\mathbf{I}_{\mathcal{C}} \tag{6.173}$$

up to a homogeneous scalar factor.

A CSTA GIPNS 3-vector *circle* $\boldsymbol{C}_{\mathcal{C}}$ can also be formed as the intersection of a CSTA GIPNS 1-vector *imaginary hyperpseudosphere* $\boldsymbol{\Xi}_{\mathcal{C}}$ or CSTA GIPNS 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *spatial plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ as

$$\begin{aligned}\boldsymbol{C}_{\mathcal{C}} &= \boldsymbol{\Xi}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} &(6.174)\\ &= \boldsymbol{K}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}}. &(6.175)\end{aligned}$$

6.4.13 CSTA GIPNS 3-vector spacetime hyperbola (pseudocircle)

The *circle* $\boldsymbol{C}_{\mathcal{C}}$ with radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}$ in spatial plane $\mathbf{N}_{\mathcal{S}} = \mathbf{n}_{\mathcal{S}}^*$ is formed by intersecting the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ of $\mathbf{N}_{\mathcal{S}}$ through $\boldsymbol{p}_{\mathcal{M}}$ with the hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ of radius r_0 at $\boldsymbol{p}_{\mathcal{M}}$. Similarly, the *pseudocircle* $\boldsymbol{C}_{\mathcal{C}}$ with central radius r_0 centered at $\boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + \mathbf{p}_{\mathcal{S}}$ in MINKOWSKI spacetime plane $\boldsymbol{D}_{\mathcal{M}} = \boldsymbol{\gamma}_0\mathbf{d}_{\mathcal{S}}$ is formed by intersecting the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ of $\boldsymbol{D}_{\mathcal{M}}$ through $\boldsymbol{p}_{\mathcal{M}}$ with the hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ of radius r_0 at $\boldsymbol{p}_{\mathcal{M}}$. The hyperbola opens up and down the spatial vector axis $\mathbf{d}_{\mathcal{S}}$ for a hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$, and it opens up and down the time axis $\boldsymbol{\gamma}_0$ for an imaginary hyperpseudosphere $\Xi_{\mathcal{C}}$.

The CSTA GIPNS 3-vector *spacetime hyperbola (pseudocircle)* $\boldsymbol{C}_{\mathcal{C}}$ can be defined as

$$\boldsymbol{C}_{\mathcal{C}} = \boldsymbol{\Sigma}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} \tag{6.176}$$

where the hyperpseudosphere $\boldsymbol{\Sigma}_{\mathcal{C}}$ sets the central position $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ and initial radius r_0 as

$$\boldsymbol{\Sigma}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} + \frac{1}{2}r_0\mathbf{e}_{\infty\gamma} \tag{6.177}$$

and the plane $\boldsymbol{\Pi}_{\mathcal{C}}$ sets the MINKOWSKI spacetime plane $\boldsymbol{D} = \boldsymbol{\gamma}_0\mathbf{d}_{\mathcal{S}}$ of spatial unit direction vector $\mathbf{d}_{\mathcal{S}}$ and time direction $\boldsymbol{\gamma}_0$ as

$$\boldsymbol{\Pi}_{\mathcal{C}} = \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}}) \wedge \mathbf{e}_{\infty\gamma}. \tag{6.178}$$

The hyperbola can be visualized as a point pair on the spatial line $\mathbf{d}_{\mathcal{S}}$, centered on $\mathbf{p}_{\mathcal{S}}$, and separated by an initial distance $2r = 2r_0$ at time $w = p_w$. As time w changes away from the initial time p_w, the radius r increases to $r = \sqrt{r_0^2 + (w - p_w)^2}$. The CSTA GIPNS 4-vector spacetime point pair can be obtained as

$$\mathbf{2}_{\mathcal{C}} = \boldsymbol{S}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}} \tag{6.179}$$

where $\boldsymbol{E}_{\mathcal{C}}$ is the xyz-space hyperplane

$$\boldsymbol{E}_{\mathcal{C}} = \gamma_0 + w\mathbf{e}_{\infty\gamma} \tag{6.180}$$

at the time w for the point pair with radius r around $\mathbf{p}_{\mathcal{S}}$ on the line direction $\mathbf{d}_{\mathcal{S}}$. The hyperplane sets the time w component of the points in the spacetime point pair. The points appear to move apart spatially with time away from p_w.

A CSTA GIPNS 3-vector *spacetime hyperbola (pseudocircle)* $\boldsymbol{C}_{\mathcal{C}}$ can also be formed as the intersection of a CSTA GIPNS 1-vector *imaginary hyperpseudosphere* $\Xi_{\mathcal{C}}$ or CSTA GIPNS 1-vector *hypercone* $\boldsymbol{K}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *spacetime plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ as

$$\begin{aligned}\boldsymbol{C}_{\mathcal{C}} &= \Xi_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} &(6.181)\\ &= \boldsymbol{K}_{\mathcal{C}} \wedge \boldsymbol{\Pi}_{\mathcal{C}} &(6.182)\end{aligned}$$

which open up and down the time w-axis.

6.4.14 CSTA GIPNS 4-vector point pair

The CSTA GIPNS 4-vector *point pair* $\mathbf{2}_{\mathcal{C}}$ is

$$\begin{aligned}\mathbf{2}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{C}_1} \cdot \boldsymbol{P}_{\mathcal{C}_2}^* &(6.183)\\ &= -\boldsymbol{P}_{\mathcal{C}_1} \cdot (\boldsymbol{P}_{\mathcal{C}_2}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{C}_1} \cdot (\boldsymbol{P}_{\mathcal{C}_2}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2})\mathbf{I}_{\mathcal{C}} &(6.184)\\ &= \mathbf{2}_{\mathcal{C}}^*\mathbf{I}_{\mathcal{C}} &(6.185)\end{aligned}$$

which is exactly the CSTA undual of the dual CSTA GOPNS 2-vector *point pair* $\mathbf{2}^*_{\mathcal{C}}$.

If the two points are *relatively lightlike*, then the point pair is actually the CSTA GIPNS null 4-vector *light-line (null line)* $\mathcal{L}_{\mathcal{C}}$ that is exactly undual to the dual CSTA GOPNS null 2-vector *light-line (null line)* $\mathcal{L}^*_{\mathcal{C}}$ (§6.5.4). The point pair $\mathbf{2}_{\mathcal{C}}$ of two not relatively lightlike points is *non-null*.

If one of the two points is $\mathbf{e}_{\infty\gamma}$, then the point pair is actually the CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ that is exactly undual to the dual CSTA GOPNS 2-vector *flat point* $\mathbb{P}^*_{\mathcal{C}}$ (§6.5.5). A flat point is non-null.

6.4.15 CSTA GIPNS null 4-vector light-line (null line)

The CSTA GIPNS null 4-vector *null line (light-line)* $\mathcal{L}_{\mathcal{C}}$ is exactly the undual $\mathcal{L}_{\mathcal{C}} = \mathcal{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}$ of the dual CSTA GOPNS null 2-vector *null line (light-line)* $\mathcal{L}^*_{\mathcal{C}}$ (§6.5.4).

The CSTA GIPNS null 4-vector *light-line (null line)* $\mathcal{L}_{\mathcal{C}}$ is

$$\begin{aligned}
\mathcal{L}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{L}_1}\cdot\boldsymbol{P}^*_{\mathcal{L}_2} && (6.186)\\
&= -\boldsymbol{P}_{\mathcal{L}_1}\cdot(\boldsymbol{P}_{\mathcal{L}_2}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{L}_1}\cdot(\boldsymbol{P}_{\mathcal{L}_2}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2})\mathbf{I}_{\mathcal{C}} && (6.187)\\
&= \mathcal{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} && (6.188)
\end{aligned}$$

where $\boldsymbol{P}_{\mathcal{L}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i}) = \mathcal{C}(p_{w_i}\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}_i})$ denotes points that are *relatively lightlike* in spacetime positions. The two relatively lightlike points $\boldsymbol{P}_{\mathcal{L}_1}$ and $\boldsymbol{P}_{\mathcal{L}_2}$ are on a light-line in spacetime having equal changes in time components $|p_{w_1} - p_{w_2}|$ to space components $\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\|$,

$$\begin{aligned}
\frac{|p_{w_1} - p_{w_2}|}{\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\|} &= 1 && (6.189)\\
\partial_t\|\mathbf{p}_{\mathcal{S}_1} - \mathbf{p}_{\mathcal{S}_2}\| = \partial_t|p_{w_1} - p_{w_2}| &= c. && (6.190)
\end{aligned}$$

Light speed c is required to travel between the two points, or any two points on a light-line, in spacetime. The vector $\boldsymbol{p}_{\mathcal{M}_1} - \boldsymbol{p}_{\mathcal{M}_2}$ is a null vector in spacetime, and any two points in spacetime with a null difference vector are relatively lightlike.

A null 4-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ can be converted into a non-null 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ as

$$\boldsymbol{L}_{\mathcal{C}} = (\mathcal{L}^*_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} = \boldsymbol{L}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}. \qquad (6.191)$$

6.4.16 CSTA GIPNS 4-vector flat point

The CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ is

$$\begin{aligned}
\mathbb{P}_{\mathcal{C}} &= -\boldsymbol{P}_{\mathcal{C}}\cdot\mathbf{e}^*_{\infty\gamma} && (6.192)\\
&= -\boldsymbol{P}_{\mathcal{C}}\cdot(\mathbf{e}_{\infty\gamma}\mathbf{I}_{\mathcal{C}}^{-1}) = \boldsymbol{P}_{\mathcal{C}}\cdot(\mathbf{e}_{\infty\gamma}\mathbf{I}_{\mathcal{C}}) = (\boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma})\mathbf{I}_{\mathcal{C}} && (6.193)\\
&= \mathbb{P}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} && (6.194)
\end{aligned}$$

which is exactly the undual of the dual CSTA GOPNS 2-vector *flat point* $\mathbb{P}^*_{\mathcal{C}}$.

A flat spatial point $\mathbb{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})\wedge\mathbf{e}_{\infty\gamma}$ at $w = 0$ can be represented as the intersection of a CSTA GIPNS 2-vector plane $\mathbf{\Pi}_{\mathcal{C}}$ and CSTA GIPNS 3-vector line $\boldsymbol{L}_{\mathcal{C}}$ that are in the common xyz-space hyperplane at any times in spacetime as

$$\begin{aligned}
\mathbb{P}_{\mathcal{C}} &= \boldsymbol{\gamma}_0\wedge(\boldsymbol{\gamma}_0\cdot\mathbf{\Pi}_{\mathcal{C}})\wedge(\boldsymbol{\gamma}_0\cdot\boldsymbol{L}_{\mathcal{C}}) && (6.195)\\
&= \mathbb{P}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} && (6.196)
\end{aligned}$$

where the common hyperplane of xyz-space $\boldsymbol{E}_{\mathcal{C}} = \boldsymbol{\gamma}_0$ is contracted out of the plane and line before they are intersected, and then $\boldsymbol{\gamma}_0$ is intersected back into the result. The time components of the plane and line do not affect the result, which is spatial intersection at $w=0$. The flat spatial point $\mathbb{P}_{\mathcal{C}}$ can exactly match the undual $\mathbb{P}^*_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}$. The flat spatial point represents the point $\boldsymbol{P}_{\mathcal{C}}$ of intersection on the plane where the line passes through, and it also represents $\mathbf{e}_\infty$ where the plane and line also intersect. A flat spacetime point as intersections may also be possible but is not considered here.

6.4.17 CSTA GIPNS 5-vector point

The CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is the

- CSTA *GIPNS* null 1-vector *hypercone* $\boldsymbol{P}_{\mathcal{C}}$ centered at $\boldsymbol{p}_{\mathcal{M}}$
- CSTA *GOPNS* null 1-vector *point* $\boldsymbol{P}_{\mathcal{C}}$ representing the point $\boldsymbol{p}_{\mathcal{M}}$.

Therefore, the CSTA GIPNS null 5-vector *point* $\boldsymbol{P}^{\star}_{\mathcal{C}}$ is the undual

$$\boldsymbol{P}^{\star}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}} \tag{6.197}$$

which also introduces a notation for the undual operation. The undual notation has been omitted on other undual entities. For the 5-vector point, the undual notation avoids a notational conflict since $\boldsymbol{P}_{\mathcal{C}}$ is the dual, not $\boldsymbol{P}^*_{\mathcal{C}}$.

For STA vectors, the undual is

$$\boldsymbol{\gamma}^{\star}_0 = \boldsymbol{\gamma}_0\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \tag{6.198}$$
$$\boldsymbol{\gamma}^{\star}_1 = \boldsymbol{\gamma}_1\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 \tag{6.199}$$
$$\boldsymbol{\gamma}^{\star}_2 = \boldsymbol{\gamma}_2\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_3\boldsymbol{\gamma}_1 \tag{6.200}$$
$$\boldsymbol{\gamma}^{\star}_3 = \boldsymbol{\gamma}_3\mathbf{I}_{\mathcal{M}} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 \tag{6.201}$$

which is consistent, in this case, with *Hodge dual* that is denoted $\star A$ in other literature.

The CSTA GIPNS null 5-vector *point* $\boldsymbol{P}^{\star}_{\mathcal{C}}$ can also be represented as the intersection of a hypercone $\boldsymbol{P}_{\mathcal{C}}$ with the four hyperplanes $\boldsymbol{E}_{\mathcal{C}_i}$ through the hypercone vertex $\boldsymbol{p}_{\mathcal{M}}$ as

$$\boldsymbol{P}^{\star}_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} \wedge \boldsymbol{E}_{\mathcal{C}_1} \wedge \boldsymbol{E}_{\mathcal{C}_2} \wedge \boldsymbol{E}_{\mathcal{C}_3} \wedge \boldsymbol{E}_{\mathcal{C}_4} \tag{6.202}$$

where

$$\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}) = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3) \tag{6.203}$$
$$\boldsymbol{E}_{\mathcal{C}_1} = \boldsymbol{\gamma}_0 + p_w\mathbf{e}_{\infty\gamma} \tag{6.204}$$
$$\boldsymbol{E}_{\mathcal{C}_2} = -\boldsymbol{\gamma}_1 + p_x\mathbf{e}_{\infty\gamma} \tag{6.205}$$
$$\boldsymbol{E}_{\mathcal{C}_3} = -\boldsymbol{\gamma}_2 + p_y\mathbf{e}_{\infty\gamma} \tag{6.206}$$
$$\boldsymbol{E}_{\mathcal{C}_4} = -\boldsymbol{\gamma}_3 + p_z\mathbf{e}_{\infty\gamma}. \tag{6.207}$$

Each hyperplane fixes one coordinate to hold a value.

6.5 CSTA GOPNS entities

In $\mathcal{G}_{2,4}$ CSTA, *five or less points* can be wedged into CSTA GOPNS entities, allowing a greater variety of entities than in $\mathcal{G}_{4,1}$ CGA, which uses four or less points. The CSTA 6-vector unit pseudoscalar and the CGA 5-vector unit pseudoscalar are dualization operators that can also be interpreted as GOPNS entities that represent the whole 4-D spacetime in CSTA or the whole 3-D space CGA.

The familiar flat and round GOPNS entities of $\mathcal{G}_{4,1}$ CGA have a similar representation in $\mathcal{G}_{2,4}$ CSTA as the wedge four or less points that are the embeddings of spatial points $\mathbf{P}_{\mathcal{C}} = \mathcal{C}(\mathbf{p}_{\mathcal{S}})$. These flat and round CGA-like entities are at $w = ct = 0$ in spacetime unless translated to time $w \neq 0$, and can be constructed as the intersections of CSTA GIPNS flat and hyperbolic entities with the CSTA GIPNS hyperplane $\boldsymbol{\gamma}_0 + w\mathbf{e}_\infty$ for time w. These CGA-like entities are the entities of the $\mathcal{G}_{1,4}$ Conformal Space Algebra (CSA) that is without time components. $\mathcal{G}_{1,4}$ CSA is similar to $\mathcal{G}_{4,1}$ CGA, except that there are differences in the signs of some similar CGA expressions. For example, the distance between two $\mathcal{G}_{1,4}$ CSA points is $d = \sqrt{2\mathbf{P}_{\mathcal{CS}_1} \cdot \mathbf{P}_{\mathcal{CS}_2}}$, while in $\mathcal{G}_{4,1}$ CGA it is $d = \sqrt{-2\mathbf{P}_{\mathcal{C}_1} \cdot \mathbf{P}_{\mathcal{C}_2}}$. As a subalgebra of CSTA, the CSA versors are the CSTA versors for spatial operations, which excludes the spacetime boost versors. The dilator operation, or successive inversions in two concentric spheres for a dilation by factor r_2^2 / r_1^2, can *isotropically dilate* CSA entities in space *and time*. The translator operation, or successive reflections in parallel spacetime planes, can translate CSA entities in space and time. The rotor operation, or successive reflections in non-parallel spatial planes, can rotate CSA entities in space, leaving the time unaffected.

The GOPNS entities are called *dual* to the *undual* GIPNS entities, but this naming is quite often reversed in other literature. This naming is chosen to be consistent with DCSTA entities, where the DCSTA GIPNS entities are unduals and the DCSTA GOPNS entities are duals.

6.5.1 Geometric outer product null space (GOPNS)

Geometric outer product null space (GOPNS) entities are introduced by Perwass in [20], and are reviewed by this author in [7] and [9].

The $\mathcal{G}_{2,4}$ CSTA unit pseudoscalar $\mathbf{I}_{\mathcal{C}}$ is grade 6, and it can be interpreted to be a GOPNS 6-vector entity that represents the entire 4D spacetime. Otherwise, the CSTA GOPNS surface entities are formed as the wedge of *five or less* CSTA GOPNS null 1-vector points $\bigwedge \boldsymbol{P}_{\mathcal{C}_i}$ (§6.5.2) on the surface that span the surface. In $\mathcal{G}_{4,1}$ CGA, the CGA GOPNS entities are formed as the wedge of four or less points. Compared to CGA, CSTA has a larger set of GOPNS entities.

The subset of $\mathcal{G}_{2,4}$ CSTA GOPNS entities that are similar to $\mathcal{G}_{4,1}$ CGA GOPNS entities are the $\mathcal{G}_{1,4}$ CSA GOPNS entities, which are defined as the wedge of four or less CSTA spatial points $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$, or $\mathcal{G}_{1,4}$ CSA null 1-vector points $\mathbf{P}_{\mathcal{CS}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$, that are on the surface and that also *span the surface* of the entity. The CSA GOPNS entities are constructed as wedges of spatial points by the same forms as in $\mathcal{G}_{4,1}$ CGA. The spatial CSA entities are located at time $w = ct = 0$ in the CSTA spacetime, but can be translated (§6.6.4) to exist at any time $w = ct = p_w$. The $\mathcal{G}_{1,4}$ CSA entities and $\mathcal{G}_{4,1}$ CGA entities represent the same surfaces, but there are some sign changes. For example, the distance d between two spatial points $\mathbf{P}_{\mathcal{CS}_1}$ and $\mathbf{P}_{\mathcal{CS}_2}$ is now

$$d = \sqrt{2\mathbf{P}_{\mathcal{CS}_1} \cdot \mathbf{P}_{\mathcal{CS}_2}} \tag{6.208}$$

while in ordinary $\mathcal{G}_{4,1}$ CGA, $d = \sqrt{-2\mathbf{P}_{\mathcal{C}_1} \cdot \mathbf{P}_{\mathcal{C}_2}}$.

6.5.2 CSTA GOPNS 1-vector point

As a GOPNS entity, the CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represents the *point* of the embedded STA position $\boldsymbol{p}_{\mathcal{M}}$. The GOPNS test

$$\boldsymbol{T}_{\mathcal{C}} \wedge \boldsymbol{P}_{\mathcal{C}} = 0 \tag{6.209}$$

holds good *if and only if* (iff)

$$\boldsymbol{T}_{\mathcal{C}} \equiv \boldsymbol{P}_{\mathcal{C}}. \tag{6.210}$$

As a GIPNS entity, a point $\boldsymbol{P}_{\mathcal{C}}$ represents a null *hypercone* (§6.4.2) in spacetime. The GIPNS test

$$\boldsymbol{T}_{\mathcal{C}} \cdot \boldsymbol{P}_{\mathcal{C}} = 0 \tag{6.211}$$

holds good for any point $\boldsymbol{T}_{\mathcal{C}}$ on the hypercone with vertex $\boldsymbol{P}_{\mathcal{C}}$. A point $\boldsymbol{T}_{\mathcal{C}}$ on the hypercone is a point that is located at a *lightlike* (null vector) displacement from the vertex $\boldsymbol{P}_{\mathcal{C}}$. The hypercone is a sphere in space, centered at $\boldsymbol{P}_{\mathcal{C}}$, with time-varying radius $r = w - p_w = ct - p_w$.

A CSTA null 1-vector *point embedding* $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ represents

$$\boldsymbol{T}_{\mathcal{C}} \circ \boldsymbol{P}_{\mathcal{C}} = \begin{cases} \text{null } \textit{hypercone} \text{ centered at vertex } \boldsymbol{p}_{\mathcal{M}} & : \circ \text{ is } \cdot \\ \text{null } \textit{point} \text{ at } \boldsymbol{p}_{\mathcal{M}}. & : \circ \text{ is } \wedge \end{cases} \tag{6.212}$$

6.5.3 CSTA GOPNS 2-vector point pair

The CSTA GOPNS 2-vector *point pair* $\mathbf{2}^*_{\mathcal{C}}$ is the wedge of two finite CSTA points that are *not relatively lightlike* (i.e., $(\boldsymbol{p}_{\mathcal{M}_2} - \boldsymbol{p}_{\mathcal{M}_1})^2 \neq 0$)

$$\mathbf{2}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \tag{6.213}$$
$$= \mathbf{2}_{\mathcal{C}} \mathbf{I}^{-1}_{\mathcal{C}} \tag{6.214}$$

and is the CSTA dual of the CSTA GIPNS 4-vector *point pair* $\mathbf{2}_{\mathcal{C}}$. Two points are relatively lightlike if they are separated by a null vector displacement. The GOPNS test

$$\boldsymbol{T}_{\mathcal{C}} \cdot \mathbf{2}^*_{\mathcal{C}} = 0 \tag{6.215}$$

holds good for the point pair $\mathbf{2}^*_{\mathcal{C}}$ of two finite points that are not relatively lightlike *if and only if* (iff)

$$\boldsymbol{T}_{\mathcal{C}} \in \{\boldsymbol{P}_{\mathcal{C}_1}, \boldsymbol{P}_{\mathcal{C}_2}\}. \tag{6.216}$$

A valid *point pair* $\mathbf{2}^*_{\mathcal{C}}$ represents the two distinct points as a single entity.

The *point pair decomposition* [4]

$$\hat{\boldsymbol{P}}_{\mathcal{C}_\pm} = \frac{\mathbf{2}^*_{\mathcal{C}} \mp \sqrt{(\mathbf{2}^*_{\mathcal{C}})^2}}{-\mathbf{e}_{\infty\gamma} \cdot \mathbf{2}^*_{\mathcal{C}}} = (\mathbf{2}^*_{\mathcal{C}} \mp \sqrt{\mathbf{2}^*_{\mathcal{C}} \cdot \mathbf{2}^*_{\mathcal{C}}})(-\mathbf{e}_{\infty\gamma} \cdot \mathbf{2}^*_{\mathcal{C}})^{-1} \tag{6.217}$$

gives the two normalized (unit scale) finite points $\hat{\boldsymbol{P}}_{\mathcal{C}_+}$ and $\hat{\boldsymbol{P}}_{\mathcal{C}_-}$ of the point pair $\mathbf{2}^*_{\mathcal{C}}$.

A *light-line (null line)* $\mathcal{L}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{L}_1} \wedge \boldsymbol{P}_{\mathcal{L}_2}$ (§6.5.4) is the wedge of two *relatively lightlike points* $\boldsymbol{P}_{\mathcal{L}_i}$ and represents the line of the two points, excluding $\mathbf{e}_{\infty\gamma}$. A *flat point* $\mathbb{P}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}} \wedge \mathbf{e}_{\infty\gamma}$ (§6.5.5) is the wedge of one finite point $\boldsymbol{P}_{\mathcal{C}}$ and the point at infinity $\mathbf{e}_{\infty\gamma}$.

6.5.4 CSTA GOPNS 2-vector light-line (null line)

An STA null *lightlike position* vector $\boldsymbol{l}_{\mathcal{M}}$ relative to the origin has the form

$$\boldsymbol{l}_{\mathcal{M}} = ct(\boldsymbol{\gamma}_0 + \hat{\mathbf{n}}_{\mathcal{S}}) = w(\boldsymbol{\gamma}_0 + \hat{\mathbf{n}}_{\mathcal{S}}) = ct\boldsymbol{n}_{\mathcal{M}}. \tag{6.218}$$

It can be verified that a vector of the form of $\boldsymbol{n}_{\mathcal{M}}=\boldsymbol{\gamma}_0+\hat{\mathbf{n}}_{\mathcal{S}}$ is a null vector $\boldsymbol{n}_{\mathcal{M}}^2=0$, where $\hat{\mathbf{n}}_{\mathcal{S}}$ is any spatial unit direction vector. A lightlike position relative to an STA position vector $\boldsymbol{p}_{\mathcal{M}}$ is

$$\boldsymbol{p}_{\mathcal{L}} \;=\; \boldsymbol{p}_{\mathcal{M}}+\boldsymbol{l}_{\mathcal{M}}. \tag{6.219}$$

Let any three collinear positions $\boldsymbol{p}_{\mathcal{L}_i}$ and their CSTA point embeddings $\boldsymbol{P}_{\mathcal{L}_i}$ be

$$\boldsymbol{P}_{\mathcal{L}_1}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_1}) \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_1\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_1\boldsymbol{n}_{\mathcal{M}}) \tag{6.220}$$
$$\boldsymbol{P}_{\mathcal{L}_2}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_2}) \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_2\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_2\boldsymbol{n}_{\mathcal{M}}) \tag{6.221}$$
$$\boldsymbol{P}_{\mathcal{L}_3}=\mathcal{C}(\boldsymbol{p}_{\mathcal{L}_3}) \;=\; \mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+w_3\boldsymbol{n}_{\mathcal{M}})=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}}+ct_3\boldsymbol{n}_{\mathcal{M}}). \tag{6.222}$$

These three points, called *relatively lightlike* points, are along a light-line in the null direction $\boldsymbol{n}_{\mathcal{M}}$ on a light-cone with vertex $\boldsymbol{p}_{\mathcal{M}}$. Two points, $\boldsymbol{p}_{\mathcal{L}_1}$ and $\boldsymbol{p}_{\mathcal{L}_2}$, are relatively lightlike if their difference vector $\boldsymbol{l}_{\mathcal{M}}=c(t_2-t_1)\boldsymbol{n}_{\mathcal{M}}=\boldsymbol{p}_{\mathcal{L}_2}-\boldsymbol{p}_{\mathcal{L}_1}$ is a null vector $\boldsymbol{n}_{\mathcal{M}}^2=0$. It can be verified that for any three collinear relatively lightlike points

$$\boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2}\wedge\boldsymbol{P}_{\mathcal{L}_3} \;=\; 0. \tag{6.223}$$

Therefore, the *light-line* $\mathcal{L}_{\mathcal{C}}$ in the direction of $\boldsymbol{n}_{\mathcal{M}}$ through the point $\boldsymbol{p}_{\mathcal{M}}$ is characterized by the wedge of any two points on the light-line. The point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on a light-line that is represented like a point pair.

The CSTA GOPNS null 2-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ is the wedge of any two *relatively lightlike* points on the light-line

$$\mathcal{L}_{\mathcal{C}}^* \;=\; \boldsymbol{P}_{\mathcal{L}_1}\wedge\boldsymbol{P}_{\mathcal{L}_2} \tag{6.224}$$
$$\simeq\; \mathcal{L}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \tag{6.225}$$

and is the CSTA dual of the CSTA null 4-vector *light-line* $\mathcal{L}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The light-line $\mathcal{L}_{\mathcal{C}}^*$ does *not* include the point at infinity $\mathbf{e}_{\infty\gamma}$. A light-line exists only in spacetime. In general, the two points of $\mathcal{L}_{\mathcal{C}}^*$ are along a light-line, which is a line through spacetime with slope $m=\pm1$ of time to space distance on a light-cone. A light-line is also called a *null line* since

$$(\mathcal{L}_{\mathcal{C}}^*)^2 \;=\; 0. \tag{6.226}$$

For any two *coplanar light-lines* $\mathcal{L}_{\mathcal{C}_1}^*$ and $\mathcal{L}_{\mathcal{C}_2}^*$, the lines share a light-cone vertex $\boldsymbol{p}_{\mathcal{M}}$ and

$$\mathcal{L}_{\mathcal{C}_1}^*\wedge\mathcal{L}_{\mathcal{C}_2}^* \;=\; 0 \tag{6.227}$$

which is a result that holds in general for all coplanar lines.

A *light-line* $\mathcal{L}_{\mathcal{C}}^*$ is a special type of line that requires only two points to define the line. A light-line is also called a *lightlike line*. Other lines in spacetime are *timelike* or *spacelike* lines and require the wedge of three collinear points to define them as GOPNS entities.

The CSTA GOPNS 3-vector line $\boldsymbol{L}_{\mathcal{C}}^*$ (§6.5.6) always includes $\mathbf{e}_{\infty\gamma}$ on the line. A light-line $\mathcal{L}_{\mathcal{C}}^*$ can be extended to include $\mathbf{e}_{\infty\gamma}$ as a null 3-vector line $\boldsymbol{L}_{\mathcal{C}}^*$. A *lightlike line* $\mathcal{L}_{\mathcal{C}}^*$ is converted into a CSTA GOPNS null 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}^*$ as

$$\boldsymbol{L}_{\mathcal{C}}^* \;=\; \mathcal{L}_{\mathcal{C}}^*\wedge\mathbf{e}_{\infty\gamma}. \tag{6.228}$$

See also, the CSTA GIPNS null 1-vector *light-line (null line)* (§6.4.4), which also includes $\mathbf{e}_{\infty\gamma}$ on the null line.

The two points of a lightlike line $\mathcal{L}_{\mathcal{C}}^*$ cannot be decomposed from the line entity.

6.5.5 CSTA GOPNS 2-vector flat point

A CSTA *flat point* $\mathbb{P}_{\mathcal{C}}^{*}$ is the wedge of a *finite* CSTA point $\boldsymbol{P}_{\mathcal{C}}$ and the CSTA *point at infinity* $\mathbf{e}_{\infty\gamma}$

$$\mathbb{P}_{\mathcal{C}}^{*} = \boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})\wedge\mathbf{e}_{\infty\gamma} \quad (6.229)$$

$$\simeq \mathbb{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \quad (6.230)$$

and equals the CSTA dual of the CSTA 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ up to a homogeneous scalar factor.

As introduced in [4] in the context of $\mathcal{G}_{4,1}$ CGA, a flat point is the intersection point of a plane and line in space. However, a plane and line both also include the point at infinity. Therefore, a flat point represents the *two points* where a line and plane intersect in space. In $\mathcal{G}_{2,4}$ CSTA, a line and plane are in spacetime and may intersect at a spacetime flat point. The CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ is the intersection of two hyperplanes

$$\boldsymbol{\Pi}_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_1}\wedge\boldsymbol{E}_{\mathcal{C}_2} \quad (6.231)$$

and the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ is the intersection of three hyperplanes

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_3}\wedge\boldsymbol{E}_{\mathcal{C}_4}\wedge\boldsymbol{E}_{\mathcal{C}_5}. \quad (6.232)$$

In CGA, the intersection of a line and plane is simply $\mathbf{L}\wedge\boldsymbol{\Pi}$, but this form cannot work as simply in CSTA. There can be *zero, one, or two* hyperplanes that are the same in the line and plane. If *zero* are the same, then $\boldsymbol{\Pi}_{\mathcal{C}}\wedge\boldsymbol{L}_{\mathcal{C}}\neq 0$ and the intersection is $\boldsymbol{\Pi}_{\mathcal{C}}\wedge\boldsymbol{L}_{\mathcal{C}}\simeq\mathbf{e}_{\infty\gamma}^{\star}$ (§6.4.17). If *two* are the same, then $\boldsymbol{L}_{\mathcal{C}}=\boldsymbol{\Pi}_{\mathcal{C}}\wedge\boldsymbol{E}_{\mathcal{C}}$ and the intersection is $\boldsymbol{L}_{\mathcal{C}}$. If *one* hyperplane is the same, then the intersection is a finite spacetime point $\boldsymbol{P}_{\mathcal{C}}$ and $\mathbf{e}_{\infty\gamma}$, which are represented as a CSTA GOPNS 2-vector *flat point* $\mathbb{P}_{\mathcal{C}}^{*}=\boldsymbol{P}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}$. In all three cases, a line and plane intersect at $\mathbf{e}_{\infty\gamma}$.

Assume, for now, that there is only *one* common hyperplane

$$\boldsymbol{E}_{\mathcal{C}}=\boldsymbol{E}_{\mathcal{C}_1}=\boldsymbol{E}_{\mathcal{C}_3} = \boldsymbol{n}_{\mathcal{M}}+(\boldsymbol{p}_{\mathcal{M}}\cdot\boldsymbol{n}_{\mathcal{M}})\mathbf{e}_{\infty\gamma}. \quad (6.233)$$

We expect to obtain a CSTA GIPNS 4-vector *flat point* $\mathbb{P}_{\mathcal{C}}$ as the intersection of the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ and CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$. If we contract $\boldsymbol{E}_{\mathcal{C}}$ into the line or the plane and then wedge them, then we get the 4-vector flat point. The pseudoscalar of the spacetime projections of $\boldsymbol{\Pi}_{\mathcal{C}}$ and $\boldsymbol{L}_{\mathcal{C}}$ is $\mathbf{I}_{\mathcal{M}}$, which can be used to project their directional blades. The conjugate normal vector $\boldsymbol{n}_{\mathcal{M}}^{\dagger}$ of the common hyperplane $\boldsymbol{E}_{\mathcal{C}}$ is given by the spacetime *meet* product $\vee_{\mathcal{M}}$ of the plane and line as

$$\boldsymbol{n}_{\mathcal{M}}^{\dagger} = (\boldsymbol{\Pi}_{\mathcal{C}}\vee_{\mathcal{M}}\boldsymbol{L}_{\mathcal{C}})^{\dagger} \quad (6.234)$$

$$= \boldsymbol{\gamma}_0(((\boldsymbol{\Pi}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{M}}^{-1})\wedge(\boldsymbol{L}_{\mathcal{C}}\cdot\mathbf{I}_{\mathcal{M}}^{-1}))\cdot\mathbf{I}_{\mathcal{M}})\boldsymbol{\gamma}_0. \quad (6.235)$$

The conjugate normal vector $\boldsymbol{n}_{\mathcal{M}}^{\dagger}$ can be used to contract $\boldsymbol{E}_{\mathcal{C}}$ in either the plane or line, which then allows intersections of the plane and line to be formed as the two flat points

$$\mathbb{P}_{\mathcal{C}_1} = (\boldsymbol{n}_{\mathcal{M}}^{\dagger}\cdot\boldsymbol{\Pi}_{\mathcal{C}})\wedge\boldsymbol{L}_{\mathcal{C}} \quad (6.236)$$

$$\mathbb{P}_{\mathcal{C}_2} = \boldsymbol{\Pi}_{\mathcal{C}}\wedge(\boldsymbol{n}_{\mathcal{M}}^{\dagger}\cdot\boldsymbol{L}_{\mathcal{C}}). \quad (6.237)$$

For *one* common hyperplane $\boldsymbol{E}_{\mathcal{C}}$, as assumed, then $\mathbb{P}_{\mathcal{C}_1}=\pm\mathbb{P}_{\mathcal{C}_2}$. Now, if *two* hyperplanes are common in the plane and line, then the spacetime meet produces zero and the flat points are zero. These results allow the following definition for the intersection $\boldsymbol{\Pi}_{\mathcal{C}}\cap\boldsymbol{L}_{\mathcal{C}}$ of a line and plane.

The CSTA GIPNS intersection $\mathbf{\Pi}_{\mathcal{C}} \cap \boldsymbol{L}_{\mathcal{C}}$ of a CSTA GIPNS 2-vector *plane* $\mathbf{\Pi}_{\mathcal{C}}$ and CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ can be defined as

$$\mathbf{\Pi}_{\mathcal{C}} \cap \boldsymbol{L}_{\mathcal{C}} = \begin{cases} \mathbf{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} \simeq \mathbf{e}_{\infty\gamma}^{\star} & : \ \mathbf{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} \neq 0 \\ \mathbb{P}_{\mathcal{C}_1} = \pm \mathbb{P}_{\mathcal{C}_2} & : \ \mathbf{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} = 0, \mathbf{\Pi}_{\mathcal{C}} \vee_{\mathcal{M}} \boldsymbol{L}_{\mathcal{C}} \neq 0 \\ \boldsymbol{L}_{\mathcal{C}} & : \ \mathbf{\Pi}_{\mathcal{C}} \wedge \boldsymbol{L}_{\mathcal{C}} = 0, \mathbf{\Pi}_{\mathcal{C}} \vee_{\mathcal{M}} \boldsymbol{L}_{\mathcal{C}} = 0. \end{cases} \tag{6.238}$$

The intersection is valid for any null or non-null 3-vector line $\boldsymbol{L}_{\mathcal{C}}$ and any spacetime plane $\mathbf{\Pi}_{\mathcal{C}}$.

The point $\boldsymbol{P}_{\mathcal{C}}$ of a flat point $\mathbb{P}_{\mathcal{C}}^{*} = \mathbb{P}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1}$ is projected [4] as

$$\boldsymbol{p}_{\mathcal{M}} = \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{C}}) = \frac{(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot (\mathbf{e}_{o\gamma} \wedge \mathbb{P}_{\mathcal{C}}^{*})}{-(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot \mathbb{P}_{\mathcal{C}}^{*}} = \frac{-\mathbb{P}_{\mathcal{C}}^{*}}{(\mathbf{e}_{o\gamma} \wedge \mathbf{e}_{\infty\gamma}) \cdot \mathbb{P}_{\mathcal{C}}^{*}} \cdot \mathbf{e}_{o\gamma} - \mathbf{e}_{o\gamma}. \tag{6.239}$$

6.5.6 CSTA GOPNS 3-vector line

The CSTA GOPNS line $\boldsymbol{L}_{\mathcal{C}}^{*}$ is similar to the CGA GOPNS line $\mathbf{L}^{*}$ discussed in [7]. In general, any line in spacetime can be represented as the wedge of three well-chosen points on the line. A CSTA GOPNS *2-vector lightlike line* $\mathcal{L}_{\mathcal{C}}^{*}$ (§6.5.4) is represented by the wedge of just two points but it does not include the point at infinity $\mathbf{e}_{\infty\gamma}$ on the line.

A CSTA GOPNS null 3-vector **lightlike** *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ is the wedge of *any two relatively lightlike points* $\boldsymbol{P}_{\mathcal{L}_i}$ on the line and the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$

$$\boldsymbol{L}_{\mathcal{C}}^{*} = \boldsymbol{P}_{\mathcal{L}_1} \wedge \boldsymbol{P}_{\mathcal{L}_2} \wedge \mathbf{e}_{\infty\gamma} = \mathcal{L}_{\mathcal{C}}^{*} \wedge \mathbf{e}_{\infty\gamma}. \tag{6.240}$$

A CSTA GOPNS non-null 3-vector **timelike or spacelike** *line* $\boldsymbol{L}_{\mathcal{C}}^{*}$ can be the wedge of *any two points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the line and the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$ *or* the wedge of *any three collinear points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the line

$$\boldsymbol{L}_{\mathcal{C}}^{*} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \mathbf{e}_{\infty\gamma} \tag{6.241}$$
$$\simeq \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \tag{6.242}$$
$$\simeq \boldsymbol{L}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{6.243}$$

and is equal to the CSTA dual of the CSTA GIPNS 3-vector *line* $\boldsymbol{L}_{\mathcal{C}}$ (§6.4.11) up to a homogeneous scalar factor.

The 3-vector line entity can be used to represent an observable (§6.4.11).

6.5.7 CSTA GOPNS 3-vector spatial circle

The system of implicit surface equations for a *spatial circle* with radius r_0 centered at (p_x, p_y, p_z) in the xy-plane at $z = p_z$ is

$$(x - p_x)^2 + (y - p_y)^2 - r_0^2 = 0 \tag{6.244}$$
$$z - p_z = 0 \tag{6.245}$$
$$w - p_w = 0. \tag{6.246}$$

The CSTA spatial circle entity $\boldsymbol{C}_{\mathcal{C}}$ represents a system of implicit surface equations of this form for the intersection of a circular cylinder and plane. The center position of the circle

$$\boldsymbol{p}_{\mathcal{M}} = p_w\gamma_0 + p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \tag{6.247}$$

includes a time component $p_w\gamma_0$ that indicates *when* the circle exists.

The CSTA GOPNS spatial circle $\boldsymbol{C}_{\mathcal{C}}^{*}$ is similar to the CGA GOPNS circle $\mathbf{C}^{*}$ discussed in [7] and is the wedge of *any three points* on the circle in space at the same time. Three points are always *coplanar cocircular* points. Three *collinear* points are on a circle of infinite radius, which is a line.

The CSTA GOPNS 3-vector *spatial circle* $\boldsymbol{C}_{\mathcal{C}}^{*}$ is the wedge of *any three* CSTA *points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(p_w\boldsymbol{\gamma}_0 + \mathbf{p}_{\mathcal{S}_i})$ *at the same time* p_w on the circle

$$\boldsymbol{C}_{\mathcal{C}}^{*} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \tag{6.248}$$

$$\simeq \boldsymbol{C}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{6.249}$$

and is the CSTA dual of the CSTA GIPNS 3-vector *spatial circle* $\boldsymbol{C}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The wedge of three points that are not all at the same time may produce a *spacetime hyperbola* (§6.5.8). The circle is produced for three points at the same time.

6.5.8 CSTA GOPNS 3-vector spacetime hyperbola (pseudocircle)

The system of implicit surface equations for a *spacetime circular hyperbola* in the xw-plane centered at (p_w, p_x, p_y, p_z), with central radius r, opening up and down the w-axis is

$$(x - p_x)^2 + r^2 - (w - p_w)^2 = 0 \tag{6.250}$$

$$y - p_y = 0 \tag{6.251}$$

$$z - p_z = 0. \tag{6.252}$$

The spacetime circular hyperbola can also be called a *pseudocircle*. The CSTA pseudocircle entity represents a system of implicit surface equations of this form. This hyperbola is not general, but circular. To get the expected shape, the points have to be chosen carefully. At $x = p_x$, $w = p_w \pm r$. At $w = p_w + \sqrt{2}r$, $x = p_x \pm r$. The axes may be transposed. Spatial rotations, spacetime translations, and spacetime isotropic dilations permit the pseudocircle to be in any MINKOWSKI space-time plane, at any spacetime center point, and with any central radius. The hyperbola is generally a conic section of a related circular hyperboloid (§6.5.10) cut through a spacetime plane and has lightlike asymptotes. By cutting the related hyperboloid in different spacetime planes, it is possible to get hyperbolas that open up and down the time or space axis. The hyperboloids are w-axis (time-axis) aligned with circles in the xy-planes.

The CSTA GOPNS 3-vector *spacetime circular hyperbola* $\boldsymbol{C}_{\mathcal{C}}^{*}$ is the wedge of *any three non-collinear* CSTA *spacetime points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ on the spacetime circular hyperbola

$$\boldsymbol{C}_{\mathcal{C}}^{*} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \tag{6.253}$$

$$\simeq \boldsymbol{C}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{6.254}$$

and is the CSTA dual of the CSTA GIPNS 3-vector *spacetime hyperbola* $\boldsymbol{C}_{\mathcal{C}}$ up to a homogeneous scalar factor. Similar to a circle, the point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on the pseudocircle.

The spacetime hyperbola $\boldsymbol{C}_{\mathcal{C}}^{*}$ becomes a *light-line pair* of a light-cone when the three non-collinear points are *relatively lightlike* points $\boldsymbol{P}_{\mathcal{L}_i}$ (§6.5.4). The points are relatively lightlike when any two points are relatively lightlike, forming one of the light-lines. The perpendicular line through the third point is the other light-line. The light-cone vertex is the point of intersection of the two light-lines, which could be one of the points.

In general, the wedge of three non-collinear CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ produces a *spatial circle* (§6.5.7) at $w = ct = 0$.

6.5.9 CSTA GOPNS 4-vector spatial sphere

The CSTA GOPNS spatial sphere $\boldsymbol{S}_{\mathcal{C}}^{*}$ is similar to the CGA GOPNS sphere $\mathbf{S}^{*}$ discussed in [7].

The CSTA GOPNS 4-vector *spatial sphere* $\boldsymbol{S}^*_{\mathcal{C}}$ is the wedge of four CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ on the sphere surface that span the sphere

$$\boldsymbol{S}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \wedge \boldsymbol{P}_{\mathcal{C}_4} \tag{6.255}$$

$$\simeq \boldsymbol{S}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{6.256}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *spatial sphere* $\boldsymbol{S}_{\mathcal{C}}$ up to a homogeneous scalar factor. To span the sphere, the points cannot be all coplanar. The spatial sphere $\boldsymbol{S}^*_{\mathcal{C}}$ holds $w = ct = 0$ and is a sphere in space that exists at time $t = 0$, but it can be translated to any time $w = p_w$ (or to any spacetime position) using the CSTA translator (§6.6.4).

6.5.10 CSTA GOPNS 4-vector spacetime hyperboloid (pseudosphere)

The implicit quadric surface equation of a *spacetime circular hyperboloid* of one sheet with circular sections in the xy-plane and central radius r is

$$(w - p_w)^2 + r^2 - (x - p_x)^2 - (y - p_y) = 0. \tag{6.257}$$

The spacetime circular hyperboloid can also be called a *pseudosphere*. Spatial rotations, spacetime translations, and spacetime isotropic dilations permit the pseudosphere to be in any spatial plane, at any spacetime center point, and with any central radius.

The CSTA GOPNS 4-vector *spacetime circular hyperboloid of one sheet (pseudosphere)* $\boldsymbol{S}^*_{\mathcal{C}}$ is the wedge of four CSTA *spacetime points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ on the surface that span the surface

$$\boldsymbol{S}^*_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}_1} \wedge \boldsymbol{P}_{\mathcal{C}_2} \wedge \boldsymbol{P}_{\mathcal{C}_3} \wedge \boldsymbol{P}_{\mathcal{C}_4} \tag{6.258}$$

$$\simeq \mathcal{L}^*_{\mathcal{C}_1} \wedge \mathcal{L}^*_{\mathcal{C}_2} \tag{6.259}$$

$$\simeq \boldsymbol{S}_{\mathcal{C}} \mathbf{I}_{\mathcal{C}}^{-1} \tag{6.260}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *spacetime circular hyperboloid of one sheet* $\boldsymbol{S}_{\mathcal{C}}$ up to a homogeneous scalar factor. Similar to a sphere, the point at infinity $\mathbf{e}_{\infty\gamma}$ is *not* a point on the pseudosphere. Two *non-coplanar* light-lines $\mathcal{L}^*_{\mathcal{C}_1}$ and $\mathcal{L}^*_{\mathcal{C}_2}$ can span a pseudosphere as asymptotes that are tangent to the hyperboloid.

The pseudosphere $\boldsymbol{S}^*_{\mathcal{C}}$ becomes a *light-cone*, also called a null cone, when the four points are *relatively lightlike* points $\boldsymbol{P}_{\mathcal{L}_i}$. The points are relatively lightlike when any three points are relatively lightlike to the fourth point, which is the vertex center point of the light-cone. The wedge of the light-cone vertex and another point is a light-line $\mathcal{L}^*_{\mathcal{C}}$, and the light-cone is spanned by three light-lines sharing the vertex.

In general, the wedge of four non-coplanar CSTA *spatial points* $\boldsymbol{P}_{\mathcal{C}_i} = \mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ produces a *spatial sphere* that holds $w = ct = 0$.

It is also possible to produce the CSTA GOPNS 4-vector *spacetime hyperboloid of two sheets (imaginary pseudosphere)* as the wedge of four well-chosen points that span the surface.

6.5.11 CSTA GOPNS 4-vector plane

The CSTA GOPNS plane $\mathbf{\Pi}^*_{\mathcal{C}}$ is similar to the CGA GOPNS plane $\mathbf{\Pi}^*$ discussed in [7]. In CGA, a plane $\mathbf{\Pi}^*$ is the wedge of *any four coplanar non-collinear non-cocircular points* on the plane. The four well-chosen points that define a plane in CGA are nearly the same for $\mathbf{\Pi}^*_{\mathcal{C}}$, but light-lines $\mathcal{L}^*_{\mathcal{C}}$ (§6.5.4) introduce an additional constraint on the choice of the four coplanar points in spacetime.

Three non-collinear finite points $\boldsymbol{P}_{\mathcal{C}_i}$ are co(pseudo)circular and define a finite (pseudo)circle $\boldsymbol{C}^*_{\mathcal{C}}$. The fourth point can be the point at infinity $\mathbf{e}_{\infty\gamma}$ or some other coplanar non-co(pseudo)circular finite point $\boldsymbol{P}_{\mathcal{C}_4}$.

Three collinear, not relatively lightlike, finite points $\boldsymbol{P}_{\mathcal{C}_i}$ define a line $\boldsymbol{L}^*_{\mathcal{C}}$. The fourth point *cannot* be the point at infinity $\mathbf{e}_{\infty\gamma}$ since it is collinear. The fourth point can be some other non-collinear finite point $\boldsymbol{P}_{\mathcal{C}_4}$.

Two relatively lightlike finite points $\boldsymbol{P}_{\mathcal{L}_i}$ define a light-line $\mathcal{L}^*_{\mathcal{C}}$. The other two points can be $\mathbf{e}_{\infty\gamma}$ and a coplanar non-collinear finite point $\boldsymbol{P}_{\mathcal{C}_4}$. The other two points can also be not relatively lightlike finite points $\boldsymbol{P}_{\mathcal{C}_3}$ and $\boldsymbol{P}_{\mathcal{C}_4}$ that are coplanar non-collinear points to $\mathcal{L}^*_{\mathcal{C}}$.

The CSTA GOPNS 4-vector *plane* $\boldsymbol{\Pi}^*_{\mathcal{C}}$ is the wedge of *four well-chosen points* $\boldsymbol{P}_{\mathcal{C}_i}$ on the plane in space or spacetime

$$\begin{aligned}
\boldsymbol{\Pi}^*_{\mathcal{C}} &= \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4}=\boldsymbol{C}_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (6.261)\\
&\simeq \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\mathbf{e}_{\infty\gamma}=\boldsymbol{C}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma} && (6.262)\\
&\simeq \boldsymbol{L}^*_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (6.263)\\
&\simeq \mathcal{L}^*_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (6.264)\\
&\simeq \mathcal{L}^*_{\mathcal{C}}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4} && (6.265)\\
&\simeq \boldsymbol{\Pi}_{\mathcal{C}}\mathbf{I}^{-1}_{\mathcal{C}} && (6.266)
\end{aligned}$$

and is the CSTA dual of the CSTA GIPNS 2-vector *plane* $\boldsymbol{\Pi}_{\mathcal{C}}$ (§6.4.10) up to a homogeneous scalar factor. The four points must be well-chosen as explained above.

The entity $\boldsymbol{\Pi}^*_{\mathcal{C}}$ is a *plane in space* that holds $w=ct=0$ when its points $\boldsymbol{P}_{\mathcal{C}_i}=\mathcal{C}(\mathbf{p}_{\mathcal{S}_i})$ are the embeddings of spatial points $\mathbf{p}_{\mathcal{S}_i}$ in 3D SA space. In the general case of points $\boldsymbol{P}_{\mathcal{C}_i}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ in spacetime, the entity $\boldsymbol{\Pi}^*_{\mathcal{C}}$ is a *plane in spacetime*. The plane entity is generally valid in both space and spacetime.

As explained in §6.4.10, the rotor $R_{\mathcal{C}}$, translator $T_{\mathcal{C}}$, and boost $B_{\mathcal{C}}$ can be defined as reflections in planes. Reflections in either the GIPNS plane $\boldsymbol{\Pi}_{\mathcal{C}}$ or GOPNS plane $\boldsymbol{\Pi}^*_{\mathcal{C}}$ are both valid on all entities. The dilator $D_{\mathcal{C}}$ (§6.6.6) can be defined as inversions in hyperpseudospheres (§6.4.5).

6.5.12 CSTA GOPNS 5-vector hyperhyperboloid

The implicit quadric surface equation for a circular *hyperhyperboloid of one sheet (hyperpseudosphere)* is

$$r_0^2+(w-p_w)^2-(x-p_x)^2-(y-p_y)^2-(z-p_x)^2 = 0 \qquad (6.267)$$

where r_0 is the initial radius of the expanding sphere in space with time-varying radius

$$r = \sqrt{r_0^2+(w-p_w)^2}=\sqrt{r_0^2+(ct-p_w)^2} \qquad (6.268)$$

and center position

$$\boldsymbol{p}_{\mathcal{M}} = p_w\boldsymbol{\gamma}_0+p_x\boldsymbol{\gamma}_1+p_y\boldsymbol{\gamma}_2+p_z\boldsymbol{\gamma}_3 \qquad (6.269)$$

in spacetime.

The hyperhyperboloid can be spanned by five surface points that do not form entities for any (pseudo)sphere, plane, line, or (pseudo)circle. Planes and lines are avoided by excluding the point at infinity. Spheres and circles are avoided by using only one or two points in any circle on the surface. The choice of points is otherwise arbitrary. For example, using an arbitrary scalar $l\neq 0$, three values of time

$$w \in \{p_w+l, p_w+2l, p_w-3l\} \qquad (6.270)$$

and corresponding values of radius

$$r \in \left\{\sqrt{r_0^2+l^2}, \sqrt{r_0^2+4l^2}, \sqrt{r_0^2+9l^2}\right\} \tag{6.271}$$

can be chosen. Then, use at most two surface points per value of w. The hyperhyperboloid, a sphere that expands with time, has the five surface points that span the surface

$$\boldsymbol{P}_{\mathcal{C}_1} = \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}}+l\boldsymbol{\gamma}_0+\sqrt{r_0^2+l^2}\boldsymbol{\gamma}_1\Big) \tag{6.272}$$

$$\boldsymbol{P}_{\mathcal{C}_2} = \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}}+2l\boldsymbol{\gamma}_0-\sqrt{r_0^2+4l^2}\boldsymbol{\gamma}_2\Big) \tag{6.273}$$

$$\boldsymbol{P}_{\mathcal{C}_3} = \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}}+2l\boldsymbol{\gamma}_0-\sqrt{r_0^2+4l^2}\boldsymbol{\gamma}_3\Big) \tag{6.274}$$

$$\boldsymbol{P}_{\mathcal{C}_4} = \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}}-3l\boldsymbol{\gamma}_0+\sqrt{r_0^2+9l^2}\boldsymbol{\gamma}_1\Big) \tag{6.275}$$

$$\boldsymbol{P}_{\mathcal{C}_5} = \mathcal{C}\Big(\boldsymbol{p}_{\mathcal{M}}-3l\boldsymbol{\gamma}_0+\sqrt{r_0^2+9l^2}\boldsymbol{\gamma}_2\Big). \tag{6.276}$$

These points are just an example of five well-chosen points on the surface that span the surface, and other points could be chosen.

The CSTA GOPNS 5-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{C}}^*$ is the wedge of five CSTA points $\boldsymbol{P}_{\mathcal{C}_i}$ on the surface that span the surface

$$\boldsymbol{\Sigma}_{\mathcal{C}}^* = \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4}\wedge\boldsymbol{P}_{\mathcal{C}_5} \tag{6.277}$$

$$\simeq \boldsymbol{\Sigma}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \tag{6.278}$$

and is the CSTA dual of the CSTA GIPNS 1-vector *hyperhyperboloid* $\boldsymbol{\Sigma}_{\mathcal{C}}$ up to a homogeneous scalar factor.

The hyperhyperboloid with $r_0=0$ degenerates into the CSTA GOPNS null 5-vector *hypercone* $\boldsymbol{P}_{\mathcal{C}}^*=\boldsymbol{\Sigma}_{\mathcal{C}}^*(\boldsymbol{p}_{\mathcal{M}}, r_0=0)$. The CSTA GIPNS null 1-vector hypercone $\boldsymbol{P}_{\mathcal{C}}=\boldsymbol{K}_{\mathcal{C}}$ at $\boldsymbol{p}_{\mathcal{M}}$ is the *point embedding* $\boldsymbol{P}_{\mathcal{C}}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$. The undual $\boldsymbol{P}_{\mathcal{C}}^{\star}=\boldsymbol{P}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}=\boldsymbol{P}_{\mathcal{C}}^{*}\mathbf{I}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}=-\boldsymbol{P}_{\mathcal{C}}^{*}$ is the CSTA GIPNS null 5-vector *point* $\boldsymbol{P}_{\mathcal{C}}^{\star}$.

It is also possible to produce the CSTA GOPNS 5-vector *hyperhyperboloid of two sheets (imaginary hyperpseudosphere)* $\boldsymbol{\Xi}_{\mathcal{C}}^*$ as the wedge of five well-chosen points that span the surface.

6.5.13 CSTA GOPNS 5-vector hyperplane

A hyperplane is a subspace of dimension $(n-1)$ in a space of dimension n. In 4D spacetime, a hyperplane is a 3D subspace at a fixed coordinate along a fourth perpendicular axis. Intersecting with a hyperplane serves to set or fix one coordinate. The signature of the hyperplane space can be $(2,1)$ or $(3,0)$.

The wedge of three non-collinear CSTA points $\boldsymbol{P}_{\mathcal{C}_i}=\mathcal{C}(\boldsymbol{p}_{\mathcal{M}_i})$ spans a spatial circle or spacetime pseudocircle $\boldsymbol{C}_{\mathcal{C}}^*$. Adding the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$, then the four CSTA points span a spatial plane or spacetime plane $\boldsymbol{\Pi}_{\mathcal{C}}^*$. Adding a fifth CSTA point that is non-coplanar to the other four points, then the five points span a 3D space. The wedge of five well-chosen CSTA points is the CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}^*$ that represents a 3D subspace of the 4D spacetime.

The CSTA GOPNS 5-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}^*$ is the wedge the CSTA point at infinity $\mathbf{e}_{\infty\gamma}$ and four CSTA points $\boldsymbol{P}_{\mathcal{C}_i}$ on the surface that span the hyperplane

$$\boldsymbol{E}_{\mathcal{C}}^* = \boldsymbol{P}_{\mathcal{C}_1}\wedge\boldsymbol{P}_{\mathcal{C}_2}\wedge\boldsymbol{P}_{\mathcal{C}_3}\wedge\boldsymbol{P}_{\mathcal{C}_4}\wedge\mathbf{e}_{\infty\gamma} \tag{6.279}$$

$$= \boldsymbol{E}_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} \tag{6.280}$$

and is the CSTA dual of the CSTA GIPNS 1-vector *hyperplane* $\boldsymbol{E}_{\mathcal{C}}$ (§6.4.3) up to a homogeneous scalar factor.

6.6 CSTA operations

6.6.1 CSTA dualization

The CSTA *dual* $A_{\mathcal{C}}^{*\mathcal{C}}$ of a CSTA multivector $A_{\mathcal{C}}$ is

$$A_{\mathcal{C}}^{*\mathcal{C}} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{\sim}. \tag{6.281}$$

The CSTA *undual* $A_{\mathcal{C}}$ of a CSTA multivector $A_{\mathcal{C}}^{*\mathcal{C}}$ is

$$A_{\mathcal{C}} = A_{\mathcal{C}}^{*\mathcal{C}}\mathbf{I}_{\mathcal{C}} = A_{\mathcal{C}}\mathbf{I}_{\mathcal{C}}^{-1}\mathbf{I}_{\mathcal{C}}. \tag{6.282}$$

The CSTA *unit pseudoscalar* $\mathbf{I}_{\mathcal{C}}$ (§6.1) is

$$\mathbf{I}_{\mathcal{C}} = \gamma_0\gamma_1\gamma_2\gamma_3\mathbf{e}_+\mathbf{e}_- \tag{6.283}$$

$$\mathbf{I}_{\mathcal{C}}^{-1} = -\mathbf{I}_{\mathcal{C}} = \mathbf{I}_{\mathcal{C}}^{\sim}. \tag{6.284}$$

6.6.2 CSTA spatial projection

The $\mathcal{G}_{1,4}$ CSA1 *spatial projection* $A_{\mathcal{CS}^1}$ of a $\mathcal{G}_{2,4}$ CSTA1 multivector $A_{\mathcal{C}^1}$ is

$$A_{\mathcal{CS}^1} = (A_{\mathcal{C}^1}\cdot\mathbf{I}_{\mathcal{CS}^1})\mathbf{I}_{\mathcal{CS}^1}^{-1} \tag{6.285}$$

where the $\mathcal{G}_{1,4}$ Conformal Space Algebra 1 (CSA1) *unit pseudoscalar* $\mathbf{I}_{\mathcal{CS}^1}$ is

$$\mathbf{I}_{\mathcal{CS}^1} = \mathbf{e}_1\cdot\mathbf{I}_{\mathcal{C}^1} = \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6. \tag{6.286}$$

The $\mathcal{G}_{1,4}$ CSA2 *spatial projection* $A_{\mathcal{CS}^2}$ of a $\mathcal{G}_{2,4}$ CSTA2 multivector $A_{\mathcal{C}^2}$ is

$$A_{\mathcal{CS}^2} = (A_{\mathcal{C}^2}\cdot\mathbf{I}_{\mathcal{CS}^2})\mathbf{I}_{\mathcal{CS}^2}^{-1} \tag{6.287}$$

where the $\mathcal{G}_{1,4}$ Conformal Space Algebra 2 (CSA2) *unit pseudoscalar* $\mathbf{I}_{\mathcal{CS}^2}$ is

$$\mathbf{I}_{\mathcal{CS}^2} = \mathbf{e}_7\cdot\mathbf{I}_{\mathcal{C}^2} = \mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12}. \tag{6.288}$$

The spatial projections $A_{\mathcal{CS}}$ drop the time components of $A_{\mathcal{C}}$, and may be useful for extracting geometrical results in space.

6.6.3 CSTA spatial rotor

The spatial rotor R is the same in $\mathcal{G}_{0,3}$ SA $\mathcal{S}$, $\mathcal{G}_{1,3}$ STA $\mathcal{M}$, $\mathcal{G}_{1,4}$ CSA $\mathcal{CS}$, and $\mathcal{G}_{2,4}$ CSTA $\mathcal{C}$.

The CSTA 2-versor *spatial rotor* $R_{\mathcal{C}}$ is equal to the SA *rotor* $R_{\mathcal{S}}$

$$R = R_{\mathcal{S}} = R_{\mathcal{M}} = R_{\mathcal{CS}} = R_{\mathcal{C}} = e^{\frac{1}{2}\theta\hat{\mathbf{n}}_{\mathcal{S}}^{*\mathcal{S}}} \tag{6.289}$$

$$= \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{n}}_{\mathcal{S}}\mathbf{I}_{\mathcal{S}}^{\sim} \tag{6.290}$$

where the SA unit vector $\hat{\mathbf{n}}_{\mathcal{S}}$ is the axis of rotation, and θ is the angle of rotation around the axis by the right-hand rule on a system of right-handed axes. The *rotor operation* $R_{\mathcal{C}}A_{\mathcal{C}}R_{\mathcal{C}}^{\sim}$ on any CSTA entity $A_{\mathcal{C}}$ spatially rotates the entity in the usual way in space, leaving any time component unchanged.

The CSTA *rotor operation* that spatially rotates any CSTA entity $A_{\mathcal{C}}$ by angle θ around SA axis $\hat{\mathbf{n}}_{\mathcal{S}}$ is defined as

$$A'_{\mathcal{C}} = R_{\mathcal{C}} A_{\mathcal{C}} \widetilde{R_{\mathcal{C}}}. \tag{6.291}$$

The rotor R can be defined as

$$R = \mathbf{\Pi}_{\mathcal{C}_2} \mathbf{\Pi}_{\mathcal{C}_1}, \tag{6.292}$$

which is the successive reflections in two non-parallel *spatial* CSTA GIPNS planes (§6.4.10) that intersect in the rotation axis $\hat{\mathbf{n}}_{\mathcal{S}}$ through the origin. The angle of rotation θ is *twice* the angle subtended by the two planes. More generally, the two *spatial* planes can intersect in an arbitrary *spatial* CSA GIPNS 2-vector line

$$\begin{aligned} \mathbf{L}_{\mathcal{CS}} &= -\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}} && (6.293)\\ &= (-\boldsymbol{\gamma}_0 \cdot \mathbf{\Pi}_{\mathcal{C}_2}) \wedge (-\boldsymbol{\gamma}_0 \cdot \mathbf{\Pi}_{\mathcal{C}_1}) && (6.294)\\ &= \mathbf{\Pi}_{\mathcal{CS}_2} \wedge \mathbf{\Pi}_{\mathcal{CS}_1} && (6.295)\end{aligned}$$

as the rotation axis (§6.6.5).

As a 2-versor, the rotor R can also be defined as

$$R = \boldsymbol{E}_{\mathcal{C}_2} \boldsymbol{E}_{\mathcal{C}_1}, \tag{6.296}$$

which is the successive reflections in two non-parallel CSTA GIPNS 1-vector hyperplanes (§6.4.3). The rotation is by *twice* the angle subtended by the two *spatial* hyperplane normal vectors, from $\mathbf{n}_{\mathcal{S}_1}$ toward $\mathbf{n}_{\mathcal{S}_2}$. The right-handed rotation axis is the SA undual $\mathbf{n} = -(\mathbf{n}_{\mathcal{S}_2} \wedge \mathbf{n}_{\mathcal{S}_1})\mathbf{I}_{\mathcal{S}}$. If the two hyperplanes are both centered on $\mathbf{p}_{\mathcal{S}}$, then the rotation axis is the line through $\mathbf{p}_{\mathcal{S}}$ in the direction of $\mathbf{n}$.

6.6.4 CSTA translator

The CSTA 2-versor *translator* $T_{\mathcal{C}}$, adapted from the CGA translator, is defined as

$$\begin{aligned} T_{\mathcal{C}} &= e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}} \mathbf{e}_{\infty\gamma}} && (6.297)\\ &= 1 - \frac{1}{2}\boldsymbol{d}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma}. && (6.298)\end{aligned}$$

The translation vector $\boldsymbol{d}_{\mathcal{M}}$ is an STA *spacetime displacement* vector. Translations through space and of time are possible.

The CSTA *translator operation* that translates CSTA entity $A_{\mathcal{C}}$ in spacetime by STA spacetime displacement $\boldsymbol{d}_{\mathcal{M}}$ is the two-sided versor "sandwich" operation

$$A'_{\mathcal{C}} = T_{\mathcal{C}} A_{\mathcal{C}} \widetilde{T_{\mathcal{C}}}. \tag{6.299}$$

The translator $T_{\mathcal{C}}$ can be defined as

$$T_{\mathcal{C}} = \boldsymbol{E}_{\mathcal{C}_2} \boldsymbol{E}_{\mathcal{C}_1}, \tag{6.300}$$

which is the successive reflections in two parallel CSTA GIPNS 1-vector *hyperplanes* (§6.4.3). The translation is by *twice* the spacetime displacement between the two hyperplanes. As a proof, assume that the translation direction $\boldsymbol{n}$ is unit modulus

$$\begin{aligned} \boldsymbol{n} = \hat{\boldsymbol{n}} &= \frac{\boldsymbol{d}_{\mathcal{M}}}{\sqrt{\boldsymbol{d}_{\mathcal{M}}^2}} = \frac{\boldsymbol{d}_{\mathcal{M}}}{|\boldsymbol{d}_{\mathcal{M}}|}, \qquad \text{for } \boldsymbol{d}_{\mathcal{M}}^2 \neq 0, && (6.301)\\ \boldsymbol{n}^2 &= 1, && (6.302)\\ \boldsymbol{n}^{-1} &= \boldsymbol{n}, && (6.303)\end{aligned}$$

and that $\boldsymbol{p}_1$ is a point on hyperplane $\boldsymbol{E}_{\mathcal{C}_1}$, and $\boldsymbol{p}_2$ is a point on $\boldsymbol{E}_{\mathcal{C}_2}$ (n.b., $|\boldsymbol{d}_{\mathcal{M}}| \in \mathbb{C}$). Then,

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{C}_2}\boldsymbol{E}_{\mathcal{C}_1} &= (\boldsymbol{n} + (\boldsymbol{p}_2 \cdot \boldsymbol{n})\mathbf{e}_{\infty\gamma})(\boldsymbol{n} + (\boldsymbol{p}_1 \cdot \boldsymbol{n})\mathbf{e}_{\infty\gamma}) && (6.304)\\
&= \boldsymbol{n}^2 + (\boldsymbol{p}_1 \cdot \boldsymbol{n})\boldsymbol{n}\mathbf{e}_{\infty\gamma} - (\boldsymbol{p}_2 \cdot \boldsymbol{n})\boldsymbol{n}\mathbf{e}_{\infty\gamma} && (6.305)\\
&= \boldsymbol{n}^2 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{n})\boldsymbol{n}\mathbf{e}_{\infty\gamma} && (6.306)\\
&= \boldsymbol{n}^{-2}(\boldsymbol{n}^2 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{n})\boldsymbol{n}\mathbf{e}_{\infty\gamma}) && (6.307)\\
&= 1 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{n})\boldsymbol{n}^{-1}\mathbf{e}_{\infty\gamma} && (6.308)\\
&= 1 - (\mathcal{P}_{\boldsymbol{n}}(\boldsymbol{p}_2) - \mathcal{P}_{\boldsymbol{n}}(\boldsymbol{p}_1))\mathbf{e}_{\infty\gamma} && (6.309)\\
&= 1 - \boldsymbol{d}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma}. && (6.310)
\end{aligned}$$

The translator (RHS Eq. 6.310) is also valid for a null translation vector $\boldsymbol{d}_{\mathcal{M}}^2 = 0$, even though the computation of its unit direction $\boldsymbol{n} = \hat{\boldsymbol{n}}$ is invalid. The translator does not depend on $\boldsymbol{n}$ *when* $\boldsymbol{d}_{\mathcal{M}}$ *is a given*, rather than computed by the projections $\mathcal{P}_{\boldsymbol{n}}$. Equation 6.306 degenerates to 0 when $(\boldsymbol{n} = \boldsymbol{d}_{\mathcal{M}})^2 = 0$, and the theory of reflections in parallel hyperplanes fails for translation in null spaces. The translator extends the theory of reflections in parallel hyperplanes for translation in non-null and null spaces.

Similar to reflections in two hyperplanes, the translator $T_{\mathcal{C}}$ can also be defined as

$$T_{\mathcal{C}} = \boldsymbol{\Pi}_{\mathcal{C}_2}\boldsymbol{\Pi}_{\mathcal{C}_1}, \qquad (6.311)$$

which is the successive reflections in two *parallel* CSTA GIPNS 2-vector *planes* (§6.4.10) that are separated by $\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}$, half the spacetime displacement $\boldsymbol{d}_{\mathcal{M}}$ of the translator. The translation is by *twice* the spacetime displacement between the planes. The orientation of the displacement $\boldsymbol{d}_{\mathcal{M}}$ is from $\boldsymbol{\Pi}_{\mathcal{C}_1}$ toward $\boldsymbol{\Pi}_{\mathcal{C}_2}$. As a proof, consider two planes having the same unit directional 2-blade

$$\boldsymbol{D} = \hat{\boldsymbol{D}} = \frac{\boldsymbol{D}}{\sqrt{\boldsymbol{D}\boldsymbol{D}}} = \frac{\boldsymbol{D}}{|\boldsymbol{D}|}, \qquad \text{for } \boldsymbol{D}^2 \neq 0, \qquad (6.312)$$

for parallel planes passing through two different points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ (n.b., $|\boldsymbol{D}| \in \mathbb{C}$). Then,

$$\begin{aligned}
\boldsymbol{\Pi}_{\mathcal{C}_2}\boldsymbol{\Pi}_{\mathcal{C}_1} &= (\boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_2 \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma})(\boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_1 \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma}) && (6.313)\\
&= (\boldsymbol{D}^{*\mathcal{M}})^2 - \boldsymbol{D}^{*\mathcal{M}}(\boldsymbol{p}_1 \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} - (\boldsymbol{p}_2 \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma}\boldsymbol{D}^{*\mathcal{M}} && (6.314)\\
&= (\boldsymbol{D}^{*\mathcal{M}})^2 + (\boldsymbol{p}_1 \cdot \boldsymbol{D}^{*\mathcal{M}})\boldsymbol{D}^{*\mathcal{M}}\mathbf{e}_{\infty\gamma} - (\boldsymbol{p}_2 \cdot \boldsymbol{D}^{*\mathcal{M}})\boldsymbol{D}^{*\mathcal{M}}\mathbf{e}_{\infty\gamma} && (6.315)\\
&= (\boldsymbol{D}^{*\mathcal{M}})^2 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{D}^{*\mathcal{M}})\boldsymbol{D}^{*\mathcal{M}}\mathbf{e}_{\infty\gamma} && (6.316)\\
&= (\boldsymbol{D}^{*\mathcal{M}})^{-2}((\boldsymbol{D}^{*\mathcal{M}})^2 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{D}^{*\mathcal{M}})\boldsymbol{D}^{*\mathcal{M}}\mathbf{e}_{\infty\gamma}) && (6.317)\\
&= 1 - ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \cdot \boldsymbol{D}^{*\mathcal{M}})(\boldsymbol{D}^{*\mathcal{M}})^{-1}\mathbf{e}_{\infty\gamma} && (6.318)\\
&= 1 - ((\mathcal{P}_{\boldsymbol{D}^{*\mathcal{M}}}(\boldsymbol{p}_2) - \mathcal{P}_{\boldsymbol{D}^{*\mathcal{M}}}(\boldsymbol{p}_1))\mathbf{e}_{\infty\gamma} && (6.319)\\
&= 1 - \boldsymbol{d}_{\mathcal{M}} \wedge \mathbf{e}_{\infty\gamma}. && (6.320)
\end{aligned}$$

This result is the translator, which is valid for any *given* spacetime displacement $\boldsymbol{d}_{\mathcal{M}}$, but the theory of reflections in planes fails when $\boldsymbol{D}^2 = 0$ for translation in a null space. The translator extends the reflections theory for translations in non-null and null spaces.

Since the scale of hyperplane $\boldsymbol{E}_{\mathcal{C}}$ and plane $\boldsymbol{\Pi}_{\mathcal{C}}$ entities does not affect the surface representation, it is not necessary to use unit directions $\boldsymbol{n}$ and $\boldsymbol{D}$. Therefore, when these directions are not null directions, the reflections in hyperplanes and planes are valid, and complex (imaginary) numbers $\mathbb{C}$ can be avoided. As a normalization, it is useful to use unit norm directions.

6.6.5 CSTA translated-rotor

The CSTA *translated-rotor (spatial line rotor)* $R^{\mathbf{p}}_{\mathcal{C}} = L_{\mathcal{C}}$ for rotation by angle $\theta = 2\|\mathbf{d}\|$ around spatial line $\mathbf{L}_{\mathcal{CS}} = -\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}}$ through point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}} = \mathbf{p}$ with unit direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}} = \hat{\mathbf{d}}$ is defined as

$$\begin{aligned}
R^{\mathbf{p}}_{\mathbf{d}} = L_{\mathcal{C}} &= T_{\mathbf{p}} R_{\mathbf{d}} T^{\sim}_{\mathbf{p}} && (6.321)\\
&= e^{-\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}} e^{\mathbf{d}^{*\mathcal{S}}} e^{\frac{1}{2}\mathbf{p}\mathbf{e}_{\infty\gamma}} && (6.322)\\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) T_{\mathbf{p}} \hat{\mathbf{d}}^{*\mathcal{S}} T^{\sim}_{\mathbf{p}} && (6.323)\\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\left(\hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}} - \left(\mathbf{p}_{\mathcal{S}} \cdot \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}}\right) \wedge \mathbf{e}_{\infty\gamma}\right) && (6.324)\\
&= e^{\frac{1}{2}\theta \mathbf{L}_{\mathcal{CS}}} && (6.325)\\
&= e^{-\frac{1}{2}\theta \boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}}} && (6.326)\\
&= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)(-\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}}). && (6.327)
\end{aligned}$$

The $\mathcal{G}_{1,4}$ CSA GIPNS 2-vector *spatial line* $\mathbf{L}_{\mathcal{CS}}$ has the form of the $\mathcal{G}_{4,1}$ CGA line $\mathbf{L}$ [7]. The $\mathcal{G}_{2,4}$ CSTA GIPNS 3-vector *spacetime line* (§6.4.11)

$$\boldsymbol{L}_{\mathcal{C}} = \boldsymbol{d}^{*\mathcal{M}} + (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{d}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} \qquad (6.328)$$

should be a purely spatial line, with a spatial *unit vector* direction $\boldsymbol{d} = \boldsymbol{d}_{\mathcal{M}} = \hat{\mathbf{d}}_{\mathcal{S}}$ through a spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$. The direction of rotation follows the right-hand rule, which is anticlockwise θ radians around the spatial direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$.

To understand how the spatial line rotor $L_{\mathcal{C}}$ is derived, consider the following. The STA dual of the unit spatial direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ of the line $\boldsymbol{L}_{\mathcal{C}}$ is

$$\begin{aligned}
\boldsymbol{d}^{*\mathcal{M}} &= && (6.329)\\
\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}^{-1}_{\mathcal{M}} &= && (6.330)\\
-\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{M}} &= && (6.331)\\
\boldsymbol{\gamma}_0 \hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}} &\ . && (6.332)
\end{aligned}$$

Therefore, the line $\boldsymbol{L}_{\mathcal{C}}$ in spatial direction $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ through spatial point $\boldsymbol{p}_{\mathcal{M}} = \mathbf{p}_{\mathcal{S}}$ is

$$\begin{aligned}
\boldsymbol{L}_{\mathcal{C}} &= \boldsymbol{\gamma}_0 \hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}} + \left(\mathbf{p}_{\mathcal{S}} \cdot \left(\boldsymbol{\gamma}_0 \wedge \left(\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}\right)\right)\right) \wedge \mathbf{e}_{\infty\gamma} && (6.333)\\
&= \boldsymbol{\gamma}_0 \wedge \left(\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}\right) - \boldsymbol{\gamma}_0 \wedge \left(\mathbf{p}_{\mathcal{S}} \cdot \left(\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}\right)\right) \wedge \mathbf{e}_{\infty\gamma}. && (6.334)
\end{aligned}$$

The spatial line rotor $L_{\mathcal{C}}$ uses the generator

$$\begin{aligned}
-\boldsymbol{\gamma}_0 \cdot \boldsymbol{L}_{\mathcal{C}} &= && (6.335)\\
-\left(\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}} - \left(\mathbf{p}_{\mathcal{S}} \cdot \left(\hat{\mathbf{d}}_{\mathcal{S}} \mathbf{I}_{\mathcal{S}}\right)\right) \wedge \mathbf{e}_{\infty\gamma}\right) &= && (6.336)\\
\hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}} - \left(\mathbf{p}_{\mathcal{S}} \cdot \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}}\right) \wedge \mathbf{e}_{\infty\gamma} &= \mathbf{L}_{\mathcal{CS}} = \mathbf{L} && (6.337)
\end{aligned}$$

which has the same form as a spatial $\mathcal{G}_{4,1}$ CGA GIPNS 2-vector line $\mathbf{L}$, but is a line entity in the similar $\mathcal{G}_{1,4}$ Conformal Space Algebra (CSA) $\mathcal{CS}$. If the line $\mathbf{L}$ were at (or through) the origin, then it would be $\mathbf{L} = \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}}$, which should be a *unit bivector* as a rotor generator. Therefore, $\boldsymbol{d} = \hat{\mathbf{d}}_{\mathcal{S}}$ should be a unit SA direction. But, the line $\mathbf{L}$ is translated from the origin to point $\mathbf{p}_{\mathcal{S}}$ as

$$\mathbf{L} = T_{\mathbf{p}} \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}} T^{\sim}_{\mathbf{p}} = \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}} - \left(\mathbf{p}_{\mathcal{S}} \cdot \hat{\mathbf{d}}^{*\mathcal{S}}_{\mathcal{S}}\right) \wedge \mathbf{e}_{\infty\gamma} \qquad (6.338)$$

by CSTA translator $T_{\mathbf{p}}$ (§6.6.4) for translation by $\mathbf{p}_{\mathcal{S}}$.

The SA dualization (§2.2) of an SA unit direction vector $\hat{\mathbf{d}}_{\mathcal{S}}$ is defined to create a rotor unit bivector generator $\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}$ (§2.6) that is isomorphic to a quaternion versor without a reversal in orientation or sign and obeying the usual right-hand rule for rotation orientation around an axis $\hat{\mathbf{d}}_{\mathcal{S}}$ through the origin.

Now, consider a rotor R and translators T and $T^{\sim}$ by $\mathbf{p}_{\mathcal{S}}$ and $-\mathbf{p}_{\mathcal{S}}$, respectively,

$$R = e^{\frac{1}{2}\theta\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}} \tag{6.339}$$
$$T = e^{-\frac{1}{2}\mathbf{p}_{\mathcal{S}}\wedge\mathbf{e}_{\infty\gamma}} \tag{6.340}$$
$$T^{\sim} = e^{\frac{1}{2}\mathbf{p}_{\mathcal{S}}\wedge\mathbf{e}_{\infty\gamma}} \tag{6.341}$$

and their composition that translates $\mathbf{p}_{\mathcal{S}}$ to the origin, then rotates around the line of the unit vector $\hat{\mathbf{d}}_{\mathcal{S}}$ through the origin, and then translates the origin back to $\mathbf{p}_{\mathcal{S}}$, which is applied to an entity $\mathbf{E}$ as

$$TRT^{\sim}\mathbf{E}TR^{\sim}T^{\sim}\ . \tag{6.342}$$

The versor $TRT^{\sim}$ is a spatially *translated rotor*

$$TRT^{\sim} = \tag{6.343}$$
$$T\left(\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}\right)T^{\sim} = \tag{6.344}$$
$$\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)T\hat{\mathbf{d}}_{\mathcal{S}}^{*\mathcal{S}}T^{\sim} = \tag{6.345}$$
$$\cos\left(\frac{1}{2}\theta\right)+\sin\left(\frac{1}{2}\theta\right)\mathbf{L} = \tag{6.346}$$
$$e^{\frac{1}{2}\theta\mathbf{L}} = \tag{6.347}$$
$$e^{\frac{1}{2}\theta(-\boldsymbol{\gamma}_0\cdot\boldsymbol{L}_{\mathcal{C}})} = L_{\mathcal{C}}. \tag{6.348}$$

This composition, the translated rotor $L_{\mathcal{C}}$, is a versor for rotation around the spatial line $\boldsymbol{L}_{\mathcal{C}}$ by the angle θ. The spatial line $\boldsymbol{L}_{\mathcal{C}}$ should be unit scale, with the spatial line direction given by a unit vector $\boldsymbol{d}=\hat{\mathbf{d}}_{\mathcal{S}}$ that passes through the spatial point $\boldsymbol{p}_{\mathcal{M}}=\mathbf{p}_{\mathcal{S}}$.

All of the CSTA GIPNS 1-vector entities can be transformed by the CSTA versors that are defined in this section, including $L_{\mathcal{C}}$ in this subsection. All of the CSTA GIPNS k-vector entities can be constructed as the wedge, or intersection, of five or less CSTA GIPNS 1-vector entities. By versor outermorphism, all of the CSTA GIPNS k-vector entities can be correctly transformed by the CSTA versors. By the CSTA dualization transformation of the CSTA GIPNS entities into CSTA GOPNS entities, all of the CSTA GOPNS k-vector entities can also be correctly transformed by the CSTA versors.

6.6.6 CSTA isotropic dilator

The CSTA 2-versor *isotropic dilator* $D_{\mathcal{C}}$, adapted from the CGA dilator, is defined as

$$D_{\mathcal{C}} = \frac{1}{2}(1+d)+\frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}. \tag{6.349}$$

The scalar d is the *dilation factor*. The CSTA isotropic dilator $D_{\mathcal{C}}$ is a *spacetime dilator*, which includes the dilation of the time and space components of an entity by the factor d.

The CSTA *isotropic dilation operation* that isotropically dilates CSTA entity $A_{\mathcal{C}}$ by factor d in spacetime is the two-sided versor "sandwich" operation

$$A'_{\mathcal{C}} = D_{\mathcal{C}}A_{\mathcal{C}}\widetilde{D_{\mathcal{C}}}. \tag{6.350}$$

It can be verified algebraically that the dilator $D_{\mathcal{C}}$ correctly dilates by factor d any CSTA GIPNS 1-vector entity. By versor outermorphism, $D_{\mathcal{C}}$ also correctly dilates any CSTA GIPNS k-vector entity, which can always be formed as the wedge of k CSTA GIPNS 1-vector entities. By CSTA dualization of GIPNS entities to GOPNS entities, all CSTA GOPNS entities are also dilated correctly by the dilator.

The dilator $D_{\mathcal{C}}$ can be derived from successive inversions in two CSTA GIPNS 1-vector *hyperpseudospheres* $\mathbf{\Sigma}_{\mathcal{C}_1}$ and $\mathbf{\Sigma}_{\mathcal{C}_2}$ (§6.4.5) centered on the origin $\mathbf{e}_{o\gamma}$ with radius $r_1 = 1$ and $r_2 = \sqrt{d}$, respectively, as

$$D_{\mathcal{C}} \;=\; -\mathbf{\Sigma}_{\mathcal{C}_2}\mathbf{\Sigma}_{\mathcal{C}_1} \simeq \mathbf{\Sigma}_{\mathcal{C}_2}\mathbf{\Sigma}_{\mathcal{C}_1}. \tag{6.351}$$

The minus sign can be dropped since it cancels in the versor operation. $D_{\mathcal{C}}$ dilates relative to (around) the origin $\mathbf{e}_{o\gamma}$, but it can be translated by $\boldsymbol{p}_{\mathcal{M}}$ using a translator $T_{\mathcal{C}}$ to make the CSTA *translated-dilator* $D_{\mathcal{C}}^{\boldsymbol{p}}$ (§6.6.7) around point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$.

6.6.7 CSTA translated-dilator

The CSTA 2-versor *translated-dilator* $D_{\mathcal{C}}^{\boldsymbol{p}}$ that dilates by factor d around $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ is

$$\begin{aligned}
D_{\mathcal{C}}^{\boldsymbol{p}} &= T_{\mathcal{C}} D_{\mathcal{C}} T_{\mathcal{C}}^{\sim} && (6.352)\\
&= T_{\mathcal{C}}\left(\frac{1}{2}(1+d) + \frac{1}{2}(1-d)\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma}\right)T_{\mathcal{C}}^{\sim} && (6.353)\\
&= \frac{1}{2}(1+d) + \frac{1}{2}(1-d)T_{\mathcal{C}}(\mathbf{e}_{\infty\gamma}\wedge\mathbf{e}_{o\gamma})T_{\mathcal{C}}^{\sim} && (6.354)\\
&= \frac{1}{2}(1+d) + \frac{1}{2}(1-d)\big(\mathbf{e}_{\infty\gamma}\wedge\hat{\boldsymbol{P}}_{\mathcal{C}}\big). && (6.355)
\end{aligned}$$

The flat point (§6.5.5) in reverse orientation

$$\hat{\mathbb{P}}_{\mathcal{C}}^{\sim} \;=\; \mathbf{e}_{\infty\gamma}\wedge\hat{\boldsymbol{P}}_{\mathcal{C}} \tag{6.356}$$

should be unit scale

$$\begin{aligned}
\big(\hat{\mathbb{P}}_{\mathcal{C}}^{\sim}\big)^2 &= \hat{\mathbb{P}}_{\mathcal{C}}\hat{\mathbb{P}}_{\mathcal{C}} = 1 && (6.357)\\
\hat{\boldsymbol{P}}_{\mathcal{C}} &= \frac{\boldsymbol{P}_{\mathcal{C}}}{-\boldsymbol{P}_{\mathcal{C}}\cdot\mathbf{e}_{\infty\gamma}}. && (6.358)
\end{aligned}$$

The orientation of $\hat{\mathbb{P}}_{\mathcal{C}}^{\sim}$ is important since its reverse makes reverse operations. For $d > 0$, the translated dilator can also be formulated as

$$\begin{aligned}
D_{\mathcal{C}}^{\boldsymbol{p}} &= e^{\operatorname{atanh}\left(\frac{1-d}{1+d}\right)\mathbf{e}_{\infty\gamma}\wedge\hat{\boldsymbol{P}}_{\mathcal{C}}} && (6.359)\\
&= e^{-\frac{1}{2}\ln(d)\mathbf{e}_{\infty\gamma}\wedge\hat{\boldsymbol{P}}_{\mathcal{C}}}. && (6.360)
\end{aligned}$$

Using the unit scale flat point in standard orientation $\hat{\mathbb{P}}_{\mathcal{C}} = \hat{\boldsymbol{P}}_{\mathcal{C}}\wedge\mathbf{e}_{\infty\gamma}$ (per §6.5.5), the translated dilator can be written as

$$\begin{aligned}
D_{\mathcal{C}}^{\boldsymbol{p}} &= e^{\frac{1}{2}\ln(d)\hat{\mathbb{P}}_{\mathcal{C}}} && (6.361)\\
&= \cosh\left(\frac{1}{2}\ln(d)\right) + \sinh\left(\frac{1}{2}\ln(d)\right)\hat{\mathbb{P}}_{\mathcal{C}}, \quad \text{for } d > 0. && (6.362)
\end{aligned}$$

The translated-dilator $D_{\mathcal{C}}^{\boldsymbol{p}}$ can be derived from successive inversions in two CSTA GIPNS 1-vector *hyperpseudospheres* $\mathbf{\Sigma}_{\mathcal{C}_1}$ and $\mathbf{\Sigma}_{\mathcal{C}_2}$ (§6.4.5) centered on $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p}_{\mathcal{M}})$ with radius $r_1 = 1$ and $r_2 = \sqrt{d}$, respectively, as

$$D_{\mathcal{C}}^{\boldsymbol{p}} \;=\; -\mathbf{\Sigma}_{\mathcal{C}_2}\mathbf{\Sigma}_{\mathcal{C}_1} \simeq \mathbf{\Sigma}_{\mathcal{C}_2}\mathbf{\Sigma}_{\mathcal{C}_1} \tag{6.363}$$

The minus sign can be dropped since it cancels in a versor operation. Any CSTA entity $\mathbf{E}$ with center position $\boldsymbol{p}_{\mathcal{M}}$ can be dilated in situ as

$$\mathbf{E}' \;=\; D_{\mathcal{C}}^{\boldsymbol{p}} \mathbf{E} D_{\mathcal{C}}^{\boldsymbol{p}\sim}. \tag{6.364}$$

The identity

$$\operatorname{atanh}\left(\frac{1-d}{1+d}\right) \;=\; -\frac{1}{2}\ln(d), \tag{6.365}$$

which can also be written as

$$\ln(d) \;=\; 2\operatorname{atanh}\left(\frac{d-1}{d+1}\right), \tag{6.366}$$

may not be familiar, but can be derived or verified as follows.

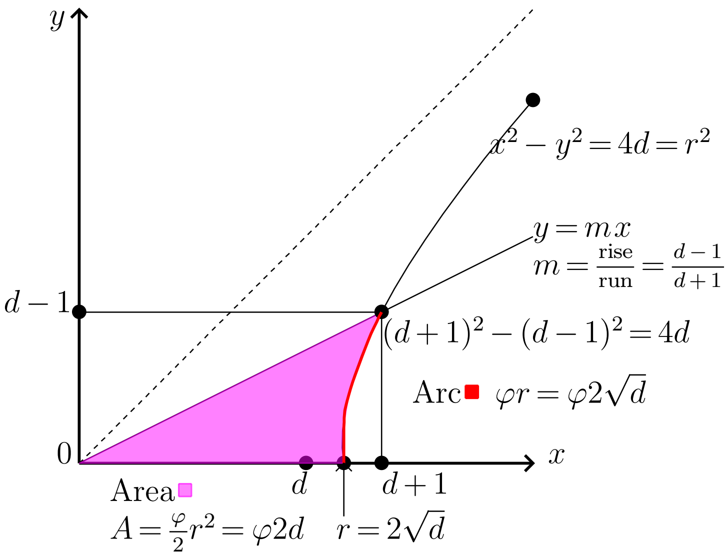

Figure 6.1. Area $A = \varphi r^2/2 = \varphi 2d$ of hyperbolic angle φ

Referring to Figure 6.1, and noting the analogy between the trigonometry of circles and the trigonometry of hyperbolas (spacetime pseudocircles), then

$$2\operatorname{atanh}\left(\frac{d-1}{d+1}\right) \;=\; \tag{6.367}$$

$$2\operatorname{atanh}\left(\frac{r\sinh(\varphi)}{r\cosh(\varphi)}\right) = 2\varphi = 2\frac{1}{2d}A \;=\; \tag{6.368}$$

$$\frac{1}{d}\int_0^{d-1}\left(\sqrt{4d+y^2} - \frac{d+1}{d-1}y\right)\mathrm{d}y \;=\; \tag{6.369}$$

$$\frac{1}{d}\left(\left(\frac{y}{2}\sqrt{4d+y^2} + \frac{4d}{2}\ln\left(y+\sqrt{4d+y^2}\right)\right)^{[26]} - \left(\frac{d+1}{d-1}\right)\frac{y^2}{2}\right)_0^{d-1} \;=\; \tag{6.370}$$

$$\frac{1}{d}\left(\frac{4d}{2}\ln(d-1+d+1)\right) - \frac{1}{d}\left(\frac{4d}{2}\ln\left(2\sqrt{d}\right)\right) \;=\; \tag{6.371}$$

$$2\ln(2d) - 2\ln\left(2\sqrt{d}\right) = 2\ln\left(\frac{2d}{2\sqrt{d}}\right) = 2\ln\left(\sqrt{d}\right) \;=\; \ln(d) \tag{6.372}$$

6.6.8 CSTA spacetime boost

CSTA inherits the STA boost operator $B_{\mathcal{M}}$ (§5.2.3) as the CSTA boost operator $B_{\mathcal{C}}=B_{\mathcal{M}}$. The boost $B_{\mathcal{C}}=B_{\mathbf{v}}$ by a spacetime velocity $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ (with proper time $\tau=t_{\boldsymbol{pv}}$) can be applied to any CSTA spacetime surface entity. An STA spacetime surface point $\boldsymbol{p}$ (with coordinate time $t=t_{\boldsymbol{cu}}=t_{\boldsymbol{po}}$) of a CSTA spacetime entity $\boldsymbol{E}$ represents an observable spacetime position of the form

$$\begin{aligned}\boldsymbol{p}(t) &= \mathbf{p}_0+\dot{\boldsymbol{p}}t &(6.373)\\ &= \mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}})t. &(6.374)\end{aligned}$$

The boost of the entity $B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}$ is congruent to the set of all boosted surface points $B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim}$. For GIPNS entity $\boldsymbol{E}$, the set of spacetime surface points is

$$\mathbb{NI}_G(\boldsymbol{E}) = \{\ \boldsymbol{p}\ :\ \mathcal{C}(\boldsymbol{p})\cdot\boldsymbol{E}=0\ \} \qquad (6.375)$$

and the set of the boosted entity is

$$\mathbb{NI}_G(B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}) = \{\ B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim}\ :\ \mathcal{C}(B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim})\cdot(B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim})=0\ \}. \qquad (6.376)$$

A point $\boldsymbol{p}$ is boosted as

$$\begin{aligned}B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim} &= B_{\mathbf{v}}(\mathbf{p}_0+\dot{\boldsymbol{p}}t)B_{\mathbf{v}}^{\sim}=\boldsymbol{p}\oplus\boldsymbol{v} &(6.377)\\ &= B_{\mathbf{v}^\dagger}^{\sim}(\mathbf{p}_0+\boldsymbol{o}t+\dot{\mathbf{p}}t)B_{\mathbf{v}^\dagger}=\boldsymbol{p}\ominus\boldsymbol{v}^\dagger &(6.378)\\ &= B_{\mathbf{v}^\dagger}^{\sim}(-\boldsymbol{o}+\boldsymbol{o}+\mathbf{p}_0)B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\dot{\mathbf{p}})B_{\mathbf{v}^\dagger}t &(6.379)\\ &= -B_{\mathbf{v}^\dagger}^{\sim}\boldsymbol{o}B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\mathbf{p}_0)B_{\mathbf{v}^\dagger}+B_{\mathbf{v}^\dagger}^{\sim}(\boldsymbol{o}+\dot{\mathbf{p}})B_{\mathbf{v}^\dagger}t &(6.380)\\ &= -\gamma_{\mathbf{v}}(\boldsymbol{o}+\mathbf{v})+\gamma_{\mathbf{p}_0\ominus\mathbf{v}^\dagger}(\boldsymbol{o}+\mathbf{p}_0\ominus\mathbf{v}^\dagger)+\gamma_{\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger}(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger)t &(6.381)\\ &= \mathbf{p}_0\ominus\boldsymbol{v}^\dagger+(\dot{\boldsymbol{p}}\ominus\boldsymbol{v}^\dagger)t &(6.382)\\ &= \mathbf{p}_0\ominus\boldsymbol{v}^\dagger+(\boldsymbol{o}+\dot{\mathbf{p}}\ominus\mathbf{v}^\dagger)\tau &(6.383)\\ &= \mathbf{p}_0\oplus\boldsymbol{v}+(\boldsymbol{o}+\dot{\mathbf{p}}\oplus\mathbf{v})t. &(6.384)\end{aligned}$$

This boost can be interpreted at least two ways: (Eq. 6.383) as $\boldsymbol{p}$ relative to $\boldsymbol{v}^\dagger=\boldsymbol{o}-\mathbf{v}$ and expressed in the frame of $\boldsymbol{v}^\dagger$ as a passive change of frame, or (Eq. 6.384) as $\boldsymbol{p}$ actively boosted up into the frame of $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ but passively expressed in the frame of $\boldsymbol{o}$ as an active relativistic velocity addition. The boost of a spacetime surface entity $B_{\mathbf{v}}\boldsymbol{E}B_{\mathbf{v}}^{\sim}$, representing the set of all boosted spacetime surface points $B_{\mathbf{v}}\boldsymbol{p}B_{\mathbf{v}}^{\sim}$, has a similar interpretation: that the entity is either (Eq. 6.383) relative to $\boldsymbol{v}^\dagger$ in its frame as a passive frame change, or (Eq. 6.384) boosted up into the frame of $\boldsymbol{v}$ but expressed (viewed) in the frame of $\boldsymbol{o}$ as an active relativistic velocity addition.

The CSTA *boost operator* $B_{\mathcal{C}}$ for a natural speed $\beta_{\mathbf{v}}$ in the SA *unit* direction $\hat{\mathbf{v}}_{\mathcal{S}}$ is defined as

$$\begin{aligned}B_{\mathcal{C}}=B_{\mathcal{M}}=B_{\mathbf{v}} &= (\gamma_{\mathbf{v}}\boldsymbol{v}/\boldsymbol{o})^{\frac{1}{2}}=e^{\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}}\boldsymbol{\gamma}_0} &(6.385)\\ &= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}}\boldsymbol{\gamma}_0\right) &(6.386)\\ &= \cosh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)+\sinh\left(\frac{1}{2}\varphi_{\mathbf{v}}\right)\hat{\mathbf{v}}_{\mathcal{S}}\wedge\boldsymbol{\gamma}_0. &(6.387)\end{aligned}$$

For more information, see the STA boost operator (§5.2.3).

6.6.9 CSTA translated-boost

The CSTA 2-versor *translated-boost* $B_{\mathcal{C}}^{\boldsymbol{d}}$ by $\varphi = \operatorname{atanh}(\beta)$ in direction $\hat{\mathbf{v}}$ centered at $\boldsymbol{d}_{\mathcal{M}}$ is defined as

$$\begin{aligned}
B_{\mathcal{C}}^{\boldsymbol{d}} = T_{\mathcal{C}} B_{\mathcal{C}} \widetilde{T_{\mathcal{C}}} &= e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} e^{\frac{1}{2}\varphi\hat{\mathbf{v}}\boldsymbol{\gamma}_0} e^{\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} && (6.388)\\
&= \cosh\left(\frac{1}{2}\varphi\right) + \sin\left(\frac{1}{2}\varphi\right) e^{-\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} \hat{\mathbf{v}}\boldsymbol{\gamma}_0 e^{\frac{1}{2}\boldsymbol{d}_{\mathcal{M}}\mathbf{e}_{\infty\gamma}} && (6.389)\\
&= \cosh\left(\frac{1}{2}\varphi\right) + \sin\left(\frac{1}{2}\varphi\right)(\hat{\mathbf{v}}\boldsymbol{\gamma}_0 - (\boldsymbol{d}_{\mathcal{M}} \cdot (\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma}) && (6.390)\\
&= e^{\frac{1}{2}\varphi(\hat{\mathbf{v}}\boldsymbol{\gamma}_0 - (\boldsymbol{d}_{\mathcal{M}} \cdot (\hat{\mathbf{v}}\boldsymbol{\gamma}_0))\mathbf{e}_{\infty\gamma})} && (6.391)\\
&= e^{\frac{1}{2}\varphi\boldsymbol{\Pi}_{\mathcal{C}}} && (6.392)
\end{aligned}$$

where the plane (§6.4.10)

$$\boldsymbol{\Pi}_{\mathcal{C}} = \boldsymbol{D}^{*\mathcal{M}} - (\boldsymbol{p}_{\mathcal{M}} \cdot \boldsymbol{D}^{*\mathcal{M}})\mathbf{e}_{\infty\gamma} \quad (6.393)$$

has *unit bivector* direction $\boldsymbol{D} = \hat{\mathbf{v}}\boldsymbol{\gamma}_0\mathbf{I}_{\mathcal{M}}$ through point $\boldsymbol{p}_{\mathcal{M}} = \boldsymbol{d}_{\mathcal{M}}$.

6.6.10 CSTA differential operators

Some of the CSTA point *value-extraction elements* C_s (§6.3) have inverses. These inverses allow the following CSTA 2-vector *differential elements* to be defined as

$$\begin{aligned}
D_w^{\mathcal{C}} &= C_1 C_w^{-1} = \boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (6.394)\\
D_t^{\mathcal{C}} &= C_1 C_t^{-1} = c\boldsymbol{\gamma}_0 \wedge \mathbf{e}_{\infty\gamma} && (6.395)\\
D_x^{\mathcal{C}} &= C_1 C_x^{-1} = \boldsymbol{\gamma}_1 \wedge \mathbf{e}_{\infty\gamma} && (6.396)\\
D_y^{\mathcal{C}} &= C_1 C_y^{-1} = \boldsymbol{\gamma}_2 \wedge \mathbf{e}_{\infty\gamma} && (6.397)\\
D_z^{\mathcal{C}} &= C_1 C_z^{-1} = \boldsymbol{\gamma}_3 \wedge \mathbf{e}_{\infty\gamma}. && (6.398)
\end{aligned}$$

The CSTA differential elements are *free vectors* [4], which are translation-invariant and represent directions without location.

Using the commutator product $\times$, the CSTA *differential operators* are defined as

$$\begin{aligned}
\partial_w^{\mathcal{C}} &= \frac{\partial}{\partial w} = D_w^{\mathcal{C}} \times && (6.399)\\
\partial_t^{\mathcal{C}} &= \frac{\partial}{\partial t} = D_t^{\mathcal{C}} \times && (6.400)\\
\partial_x^{\mathcal{C}} &= \frac{\partial}{\partial x} = D_x^{\mathcal{C}} \times && (6.401)\\
\partial_y^{\mathcal{C}} &= \frac{\partial}{\partial y} = D_y^{\mathcal{C}} \times && (6.402)\\
\partial_z^{\mathcal{C}} &= \frac{\partial}{\partial z} = D_z^{\mathcal{C}} \times . && (6.403)
\end{aligned}$$

The differential elements and operators in CSTA1 and CSTA2 can be denoted $D_s^{\mathcal{C}_1}$ and $\partial_s^{\mathcal{C}_1}$, and $D_s^{\mathcal{C}_2}$ and $\partial_s^{\mathcal{C}_2}$, respectively. The CSTA differential operators can be used for entity analysis. A different, but similar, set of differential elements and operators are defined in DCSTA (§7.8.1).

A CSTA directional derivative operator $D_{\boldsymbol{n}}^{\mathcal{C}} \times$, in the direction of a vector

$$\boldsymbol{n}_{\mathcal{M}} = n_w\boldsymbol{\gamma}_0 + n_x\boldsymbol{\gamma}_1 + n_y\boldsymbol{\gamma}_2 + n_z\boldsymbol{\gamma}_3, \quad (6.404)$$

can be formed as the linear combination

$$D_{\boldsymbol{n}}^{\mathcal{C}} \times = (n_w D_w^{\mathcal{C}} + n_x D_x^{\mathcal{C}} + n_y D_y^{\mathcal{C}} + n_z D_z^{\mathcal{C}}) \times . \quad (6.405)$$

The simplest example of using a CSTA differential operator $D_{\boldsymbol{n}}^{\mathcal{C}}\times$ is to take the derivative of any CSTA GIPNS 1-vector entity $\boldsymbol{\Sigma}_{\mathcal{C}}$ in the $\boldsymbol{n}$-direction as

$$\partial_{\boldsymbol{n}}^{\mathcal{C}}\boldsymbol{\Sigma}_{\mathcal{C}} = \frac{\partial\boldsymbol{\Sigma}_{\mathcal{C}}}{\partial\boldsymbol{n}} = D_{\boldsymbol{n}}^{\mathcal{C}}\times\boldsymbol{\Sigma}_{\mathcal{C}}. \tag{6.406}$$

7 Double Conformal Space-Time Algebra (DCSTA)

The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) is a straightforward extension of the $\mathcal{G}_{2,8}$ Double Conformal Space Algebra (DCSA), which is similar to the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) that is introduced in the original research monograph [7] and in the published short paper [8], and discussed further in the papers [5] and [6]. There are only some differences in signs between $\mathcal{G}_{2,8}$ DCSA and $\mathcal{G}_{8,2}$ DCGA, such that all the results of DCGA transfer to DCSA with only some sign changes.

The key idea of DCSTA is that any CSTA1 entity or versor $A_{\mathcal{C}^1}$ and its double $A_{\mathcal{C}^2}$ in CSTA2 can be multiplied to form the corresponding DCSTA entity or versor $A_{\mathcal{D}}$, where

$$A_{\mathcal{D}} = A_{\mathcal{C}^1}A_{\mathcal{C}^2} = A_{\mathcal{C}^1}\wedge A_{\mathcal{C}^2}. \tag{7.1}$$

According to the *outermorphism* property for transformation operators [20], or versors that operate as the two-sided versor "sandwich" operation, any doubled versor $V_{\mathcal{D}}$, which can be for rotation $R_{\mathcal{D}}$ (§7.7.6), translation $T_{\mathcal{D}}$ (§7.7.7), dilation $D_{\mathcal{D}}$ (§7.7.8), boost $B_{\mathcal{D}}$ (§7.7.3), or any of their compositions, operates on a doubled entity $E_{\mathcal{D}}$ as

$$\begin{aligned}
E'_{\mathcal{D}} &= V_{\mathcal{D}}E_{\mathcal{D}}\widetilde{V_{\mathcal{D}}} && (7.2)\\
&= V_{\mathcal{C}^1}V_{\mathcal{C}^2}(E_{\mathcal{C}^1}\wedge E_{\mathcal{C}^2})\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}} && (7.3)\\
&= (V_{\mathcal{C}^1}V_{\mathcal{C}^2}E_{\mathcal{C}^1}\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}})\wedge(V_{\mathcal{C}^1}V_{\mathcal{C}^2}E_{\mathcal{C}^2}\widetilde{V_{\mathcal{C}^2}}\widetilde{V_{\mathcal{C}^1}}) && (7.4)\\
&= (V_{\mathcal{C}^1}E_{\mathcal{C}^1}\widetilde{V_{\mathcal{C}^1}})\wedge(V_{\mathcal{C}^2}E_{\mathcal{C}^2}\widetilde{V_{\mathcal{C}^2}}) && (7.5)\\
&= E'_{\mathcal{C}^1}\wedge E'_{\mathcal{C}^2} = E'_{\mathcal{C}^1}E'_{\mathcal{C}^2}. && (7.6)
\end{aligned}$$

Therefore, the CSTA1 entity $E_{\mathcal{C}^1}$ is correctly transformed by the CSTA1 versor $V_{\mathcal{C}^1}$, and similarly for the CSTA2 entity $E_{\mathcal{C}^2}$. The product of the two correctly transformed CSTA entities is the correctly transformed DCSTA entity. For example, the DCSTA point entity $\boldsymbol{P}_{\mathcal{D}}$ (§7.2) is correctly transformed by all of the DCSTA versors. The DCSTA point value-extraction elements T_s (§7.2.3) extract correctly transformed values from a point $\boldsymbol{P}_{\mathcal{D}}$, leading to the ability to form entities in the general form of Darboux cyclides that can be correctly transformed by all of the DCSTA versors.

As a subalgebra of DCSTA, all the results of DCSA (or DCGA) are available in DCSTA. DCSTA extends DCSA with the pseudospatial time axis $(w = ct)\boldsymbol{\gamma}_0$, a variety of spacetime entities, and the spacetime boost (hyperbolic rotor) operator. In DCSTA, the DCSA GIPNS 2-vector quadric surfaces are surfaces in spacetime at zero velocity that can be boosted into any velocity. The boosted quadric surfaces display spacetime contraction effects.

DCSTA includes many operations on quadric surface entities, including

- Rotation in space
- Translation in spacetime
- Isotropic dilation in spacetime
- Anisotropic dilation (directed length dilation) in space

- Spacetime active boosts of velocity with length contraction effect
- Spacetime passive boosts relative to a new observer frame
- Intersection with standard entities that are doubled CSTA entities.

The general DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§7.5) has the general form of a Darboux cyclide in spacetime, which has degenerate forms that include Dupin cyclides, horned Dupin cyclides, parabolic cyclides, and the quadric surfaces. In DCSTA, the Darboux cyclide surface entities can be formed, similar to in DCSA or DCGA, as linear combinations of the spatial DCSTA point value-extraction elements T_s (§7.2.3) that represent spatial cyclide surfaces in the 3D $\mathcal{G}_{0,3}$ SA space (at zero velocity). Darboux cyclide entities can also be formed from spacetime value-extraction elements to represent spacetime cyclides in a 3D $\mathcal{G}_{1,2}$ STA spacetime, where the spacetime cyclides are called pseudocyclides, pseudoquadrics, etc.

The DCSTA quadric surfaces support anisotropic length contraction and dilation (§7.7.9) since these forms can be written in terms of the DCSTA value-extraction elements. On the other hand, the higher-degree surfaces, which include cubic parabolic cyclides and quartic Darboux and Dupin cyclides, do not support anisotropic length contraction and dilation forms. Any DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ represents an implicit surface function in spacetime $F(w,x,y,z)$ and supports function differentiation $\partial_{\boldsymbol{n}}F$ (or $2F\partial_{\boldsymbol{n}}F$ for doubled entities $E_{\mathcal{D}}\widehat{=}F^2$) using the differential operations (§7.8) for

- Differentiation with respect to $w=ct$, t, x, y, or z
- Directional derivative with respect to a unit-norm direction $\boldsymbol{n}$ in spacetime.

The DCSTA forms of conic sections can also support the operations for

- Orthographic projection
- Perspective projection

as discussed in the paper [5].

7.1 DCSTA unit pseudoscalar

The DCSTA 12-vector *unit pseudoscalar* $\mathbf{I}_{\mathcal{D}}$ with signature $(+---+-+---+-)$ is

$$\begin{aligned} \mathbf{I}_{\mathcal{D}} &= \mathbf{I}_{\mathcal{C}^1}\wedge\mathbf{I}_{\mathcal{C}^2}=\bigwedge_{i=1}^{12}\mathbf{e}_i && (7.7)\\ &= \mathbf{I}_{\mathcal{D}}^{\sim}=\mathbf{I}_{\mathcal{D}}^{-1} && (7.8)\\ \mathbf{I}_{\mathcal{D}}^2 &= 1. && (7.9)\end{aligned}$$

7.2 DCSTA point

7.2.1 DCSTA point embedding

The DCSTA null 2-vector *point embedding* $\boldsymbol{P}_{\mathcal{D}}=\mathcal{D}(\boldsymbol{p})$ of STA position vector

$$\begin{aligned} \boldsymbol{p}=\boldsymbol{p}_{\mathcal{M}} &= p_w\boldsymbol{\gamma}_0+p_x\boldsymbol{\gamma}_1+p_y\boldsymbol{\gamma}_2+p_z\boldsymbol{\gamma}_3 && (7.10)\\ &= p_w\boldsymbol{\gamma}_0+\mathbf{p}_{\mathcal{S}} && (7.11)\\ &= \mathbf{p}_0+\dot{\boldsymbol{p}}t && (7.12)\\ &= \mathbf{p}_0+(\boldsymbol{o}+\dot{\mathbf{p}})t && (7.13)\end{aligned}$$

is defined as the doubling of the CSTA point $\boldsymbol{P}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{p})$ as

$$\begin{aligned} \boldsymbol{P}_{\mathcal{D}} &= \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^1}) \wedge \mathcal{C}(\boldsymbol{p}_{\mathcal{M}^2}) & (7.14)\\ &= \boldsymbol{P}_{\mathcal{C}^1} \wedge \boldsymbol{P}_{\mathcal{C}^2} & (7.15)\\ &= \mathcal{D}(\boldsymbol{p}) & (7.16) \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{p}_{\mathcal{M}^1} &= p_w \mathbf{e}_1 + p_x \mathbf{e}_2 + p_y \mathbf{e}_3 + p_z \mathbf{e}_4 = p_w \mathbf{e}_1 + \mathbf{p}_{\mathcal{S}^1} & (7.17)\\ \boldsymbol{p}_{\mathcal{M}^2} &= p_w \mathbf{e}_7 + p_x \mathbf{e}_8 + p_y \mathbf{e}_9 + p_z \mathbf{e}_{10} = p_w \mathbf{e}_7 + \mathbf{p}_{\mathcal{S}^2}. & (7.18) \end{aligned}$$

In general, the doubled CSTA entities, as DCSTA entities, represent the same entities as defined in CSTA (§6).

7.2.1.1 DCSTA origin point

The DCSTA null 2-vector *point at the origin* is

$$\mathbf{e}_o = \mathbf{e}_{o1} \wedge \mathbf{e}_{o2}. \quad (7.19)$$

7.2.1.2 DCSTA infinity point

The DCSTA null 2-vector *point at infinity* is

$$\mathbf{e}_\infty = \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2}. \quad (7.20)$$

7.2.1.3 DCSTA point is DCSTA GIPNS hypercone

As a GIPNS entity, the DCSTA *point embedding* $\boldsymbol{P}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{p})$ represents a spacetime hypercone (lightcone) centered on vertex $\boldsymbol{p}$ and is the DCSTA GIPNS null 2-vector standard *hypercone* $\boldsymbol{P}_{\mathcal{D}} = \boldsymbol{K}_{\mathcal{D}}$.

7.2.1.4 DCSTA point is DCSTA GOPNS point

As a GOPNS entity, the DCSTA *point embedding* $\boldsymbol{P}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{p})$ represents the point $\boldsymbol{p}$ and is the DCSTA GOPNS null 2-vector standard *point* $\boldsymbol{P}_{\mathcal{D}}$.

A *point embedding* $\boldsymbol{P}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{p})$ will often be called a *point*, but it should be understood that it is a GOPNS point and not a GIPNS point. For the purpose of *testing* any GIPNS or GOPNS surface entity for a surface point $\boldsymbol{p}$, the point embedding $\boldsymbol{P}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{p})$ is the *test point entity*. Point $\boldsymbol{p}$ is on the surface of GIPNS entity $\boldsymbol{E}$ iff $\mathcal{D}(\boldsymbol{p}) \cdot \boldsymbol{E} = 0$. Point $\boldsymbol{p}$ is on the surface of GOPNS entity $\boldsymbol{E}^*$ iff $\mathcal{D}(\boldsymbol{p}) \wedge \boldsymbol{E}^* = 0$.

7.2.2 DCSTA point projection (inverse embedding)

The projection of DCSTA point $\boldsymbol{P}_{\mathcal{D}}$ back to STA1 vector $\boldsymbol{p}_{\mathcal{M}^1}$ is

$$\begin{aligned} \boldsymbol{p}_{\mathcal{M}^1} &= \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 2}) & (7.21)\\ &= \left(\frac{\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 2}}{-(\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 2}) \cdot \mathbf{e}_{\infty 1}} \cdot \mathbf{I}_{\mathcal{M}^1} \right) \cdot \mathbf{I}_{\mathcal{M}^1}^{-1}. & (7.22) \end{aligned}$$

The projection of DCSTA point $\boldsymbol{P}_{\mathcal{D}}$ back to STA2 vector $\boldsymbol{p}_{\mathcal{M}^2}$ is

$$\begin{aligned} \boldsymbol{p}_{\mathcal{M}^2} &= \mathcal{C}^{-1}(\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 1}) & (7.23)\\ &= \left(\frac{\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 1}}{-(\boldsymbol{P}_{\mathcal{D}} \cdot \mathbf{e}_{\infty 1}) \cdot \mathbf{e}_{\infty 2}} \cdot \mathbf{I}_{\mathcal{M}^2} \right) \cdot \mathbf{I}_{\mathcal{M}^2}^{-1}. & (7.24) \end{aligned}$$

7.2.3 DCSTA point value-extraction elements

The DCSTA *test point* $\boldsymbol{T}_{\mathcal{D}} = \mathcal{D}(\boldsymbol{t})$ is the point embedding of the STA test vector

$$\begin{aligned} \boldsymbol{t} &= w\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3 & (7.25)\\ &= ct\boldsymbol{\gamma}_0 + x\boldsymbol{\gamma}_1 + y\boldsymbol{\gamma}_2 + z\boldsymbol{\gamma}_3. & (7.26) \end{aligned}$$

The DCSTA 2-vector *extraction elements* T_s are defined as

$$
\begin{aligned}
T_w &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1} \wedge \mathbf{e}_7) && (7.27)\\
T_t &= \frac{1}{c} T_w && (7.28)\\
T_x &= \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{\infty 1}) && (7.29)\\
T_y &= \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{\infty 1}) && (7.30)\\
T_z &= \frac{1}{2}(\mathbf{e}_{\infty 2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{\infty 1}) && (7.31)
\end{aligned}
$$

$$
\begin{aligned}
T_{w^2} &= \mathbf{e}_7 \wedge \mathbf{e}_1 && (7.32)\\
T_{t^2} &= \frac{1}{c^2} T_{w^2} && (7.33)\\
T_{x^2} &= \mathbf{e}_8 \wedge \mathbf{e}_2 && (7.34)\\
T_{y^2} &= \mathbf{e}_9 \wedge \mathbf{e}_3 && (7.35)\\
T_{z^2} &= \mathbf{e}_{10} \wedge \mathbf{e}_4 && (7.36)
\end{aligned}
$$

$$
\begin{aligned}
T_{wx} &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_8 + \mathbf{e}_2 \wedge \mathbf{e}_7) && (7.37)\\
T_{wy} &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_9 + \mathbf{e}_3 \wedge \mathbf{e}_7) && (7.38)\\
T_{wz} &= \frac{1}{2}(\mathbf{e}_1 \wedge \mathbf{e}_{10} + \mathbf{e}_4 \wedge \mathbf{e}_7) && (7.39)\\
T_{tx} &= \frac{1}{c} T_{wx} && (7.40)\\
T_{ty} &= \frac{1}{c} T_{wy} && (7.41)\\
T_{tz} &= \frac{1}{c} T_{wz} && (7.42)\\
T_{xy} &= \frac{1}{2}(\mathbf{e}_9 \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_3) && (7.43)\\
T_{yz} &= \frac{1}{2}(\mathbf{e}_{10} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_4) && (7.44)\\
T_{zx} &= \frac{1}{2}(\mathbf{e}_8 \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_2) && (7.45)
\end{aligned}
$$

$$
\begin{aligned}
T_{w\boldsymbol{t}^2} &= \mathbf{e}_1 \wedge \mathbf{e}_{o2} + \mathbf{e}_{o1} \wedge \mathbf{e}_7 && (7.46)\\
T_{t\boldsymbol{t}^2} &= \frac{1}{c} T_{w\boldsymbol{t}^2} && (7.47)\\
T_{x\boldsymbol{t}^2} &= \mathbf{e}_{o2} \wedge \mathbf{e}_2 + \mathbf{e}_8 \wedge \mathbf{e}_{o1} && (7.48)\\
T_{y\boldsymbol{t}^2} &= \mathbf{e}_{o2} \wedge \mathbf{e}_3 + \mathbf{e}_9 \wedge \mathbf{e}_{o1} && (7.49)\\
T_{z\boldsymbol{t}^2} &= \mathbf{e}_{o2} \wedge \mathbf{e}_4 + \mathbf{e}_{10} \wedge \mathbf{e}_{o1} && (7.50)
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= -\mathbf{e}_\infty && (7.51)\\
T_{\boldsymbol{t}^2} &= \mathbf{e}_{o2} \wedge \mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{o1} && (7.52)\\
T_{\boldsymbol{t}^4} &= -4\mathbf{e}_o. && (7.53)
\end{aligned}
$$

The value s is extracted from a point $\boldsymbol{T}_{\mathcal{D}}$ as

$$
s = T_s \cdot \boldsymbol{T}_{\mathcal{D}} = \boldsymbol{T}_{\mathcal{D}} \cdot T_s. \quad (7.54)
$$

The DCSTA 2-vector extraction elements T_s are *inner product extraction operators*.

The DCSTA 2-vector extraction elements T_s are used to define the DCSTA GIPNS 2-vector entities that are similar to those that can be defined in DCSA or DCGA. The general DCSTA GIPNS 2-vector spacetime surface entity $\mathbf{\Omega}$ is a linear combination of the DCSTA extraction elements T_s. For example, an ellipsoid entity $\mathbf{E}$ can be defined as

$$\mathbf{E} = T_{x^2}/a^2 + T_{y^2}/b^2 + T_{z^2}/c^2 - T_1. \tag{7.55}$$

In general, any DCSTA GIPNS 2-vector *spatial quadric* entity $\mathbf{Q}$, formed similarly to the ellipsoid entity $\mathbf{E}$ as a linear combination of spatial extraction elements T_s, is independent of time $w = ct$ and exists for all time at its current spatial position, which can be translated using the DCSTA translator (§7.7.7). A spatial quadric surface entity $\mathbf{Q}$ is "pseudocylindrical" with the pseudospatial time w-axis as a type of hypercylinder entity. The interpretation in spacetime is that the spatial quadric $\mathbf{Q}$ is at zero velocity $\beta = 0$. The DCSTA 4-versor *boost* operator $B_{\mathbf{v}}$ (§7.7.3) can actively boost any spatial quadric $\mathbf{Q}$ into any velocity $\mathbf{v} = \beta_{\mathbf{v}} c\hat{\mathbf{v}}$ (into the rest frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$) as a *spacetime quadric* $\boldsymbol{Q} = B_{\mathbf{v}}\mathbf{Q}\tilde{B_{\mathbf{v}}}$. Only a DCSTA GIPNS 2-vector spatial quadric surface entity $\mathbf{Q}$ can be formed independently of time $w = ct$ as a linear combination of spatial extraction elements T_s. The DCSTA GIPNS 2-vector cubic (parabolic cyclide) and quartic (Darboux and Dupin cyclide) entities use the extraction elements T_s that include $\boldsymbol{t}$ as its square $\boldsymbol{t}^2$ or square square $\boldsymbol{t}^4$ and are dependent on time $w = ct$, with the interpretation that the spatial cubic and quartic entities exist at time $w = ct = 0$ or are translated to exist at some time $w = p_w = ct_w$.

7.2.4 DCSTA point value-extraction pseudo-inverse elements

The pseudo-inverse of A is denoted A^+ and has the relation

$$A \cdot A^+ = 1. \tag{7.56}$$

If A^{-1} exists, it may be equal to A^+. The inverse or pseudo-inverse of an extraction element T_s can be useful for formulating certain other elements and operators, such as *pseudo-integral operators* (§7.9). The pseudo-inverses of some of the extraction elements are

$$T_{w^2}^{-1} = T_{w^2}^{+} = -T_{w^2} \tag{7.57}$$
$$T_{t^2}^{-1} = T_{t^2}^{+} = -c^2 T_{w^2} \tag{7.58}$$
$$T_{x^2}^{-1} = T_{x^2}^{+} = -T_{x^2} \tag{7.59}$$
$$T_{y^2}^{-1} = T_{y^2}^{+} = -T_{y^2} \tag{7.60}$$
$$T_{z^2}^{-1} = T_{z^2}^{+} = -T_{z^2} \tag{7.61}$$

$$T_w^+ = T_{w\boldsymbol{t}^2} \tag{7.62}$$
$$T_t^+ = c^2 T_{t\boldsymbol{t}^2} \tag{7.63}$$
$$T_x^+ = -T_{x\boldsymbol{t}^2} \tag{7.64}$$
$$T_y^+ = -T_{y\boldsymbol{t}^2} \tag{7.65}$$
$$T_z^+ = -T_{z\boldsymbol{t}^2} \tag{7.66}$$

$$T_{wx}^+ = 2T_{wx} \tag{7.67}$$
$$T_{wy}^+ = 2T_{wy} \tag{7.68}$$
$$T_{wz}^+ = 2T_{wz} \tag{7.69}$$
$$T_{tx}^+ = 2c^2 T_{tx} \tag{7.70}$$
$$T_{ty}^+ = 2c^2 T_{ty} \tag{7.71}$$
$$T_{tz}^+ = 2c^2 T_{tz} \tag{7.72}$$

$$T^+_{xy} = -2T_{xy} \quad (7.73)$$
$$T^+_{yz} = -2T_{yz} \quad (7.74)$$
$$T^+_{zx} = -2T_{zx} \quad (7.75)$$
$$T^+_1 = -\frac{1}{4}T_{\boldsymbol{t}^4} \quad (7.76)$$
$$T^+_{\boldsymbol{t}^2} = -\frac{1}{2}T_{\boldsymbol{t}^2} \quad (7.77)$$
$$T^+_{\boldsymbol{t}^4} = -\frac{1}{4}T_1. \quad (7.78)$$

7.3 DCSTA GIPNS standard entities

The DCSTA GIPNS *standard* surface entities are the doubling of the CSTA GIPNS entities. The wedge of corresponding CSTA1 and CSTA2 GIPNS entities, $\boldsymbol{X}_{\mathcal{C}^1}$ and $\boldsymbol{X}_{\mathcal{C}^2}$, forms the DCSTA GIPNS standard entity $\boldsymbol{X}_{\mathcal{D}} = \boldsymbol{X}_{\mathcal{C}^1} \wedge \boldsymbol{X}_{\mathcal{C}^2}$ representing the same surface. The following subsections provide some explicit examples of the doubling.

The DCSTA GIPNS standard entities have special properties and can act as operators for reflections and intersections. All DCSTA entities can be reflected in the standard entities. The reflection in a standard sphere is called inversion in a sphere. All DCSTA entities can be intersected with standard entities. A DCSTA GIPNS *intersection* entity is a wedge of GIPNS entities, similar to a DCGA GIPNS intersection entity and with similar limitations on what combinations of entities can be wedged to form a valid intersection entity. The basic examples of intersection entities are the DCSTA GIPNS 6-vector *standard line* $\boldsymbol{L}_{\mathcal{D}} = \boldsymbol{\Pi}_{\mathcal{D}} \wedge \boldsymbol{E}_{\mathcal{D}}$ (§7.3.7) and DCSTA GIPNS 6-vector *standard (pseudo)circle* $\boldsymbol{C}_{\mathcal{D}} = \boldsymbol{\Pi}_{\mathcal{D}} \wedge \boldsymbol{\Sigma}_{\mathcal{D}}$ (§7.3.8).

Any DCSTA k-vector standard entity, as a doubling of a CSTA entity, is a k-blade since it can be factored into the outer product of k vectors. The DCSTA non-standard entities (§7.5) are generally not blades.

7.3.1 DCSTA GIPNS null 2-vector hypercone

The DCSTA GIPNS null 2-vector *standard hypercone* $\boldsymbol{K}_{\mathcal{D}}$ is defined as

$$\boldsymbol{K}_{\mathcal{D}} = \boldsymbol{K}_{\mathcal{C}^1} \wedge \boldsymbol{K}_{\mathcal{C}^2} = \boldsymbol{P}_{\mathcal{C}^1} \wedge \boldsymbol{P}_{\mathcal{C}^2} = \boldsymbol{P}_{\mathcal{D}} \quad (7.79)$$

which is the wedge of the same *point embedding (hypercone)* (§6.2) in CSTA1 and CSTA2.

7.3.2 DCSTA GIPNS 2-vector standard hyperplane

The DCSTA GIPNS 2-vector *standard hyperplane* $\boldsymbol{E}_{\mathcal{D}}$ is defined as

$$\boldsymbol{E}_{\mathcal{D}} = \boldsymbol{E}_{\mathcal{C}^1} \wedge \boldsymbol{E}_{\mathcal{C}^2} \quad (7.80)$$

which is the wedge of the same *hyperplane* (§6.4.3) embedded in CSTA1 and CSTA2.

7.3.3 DCSTA GIPNS 2-vector standard hyperhyperboloid of one sheet

The DCSTA GIPNS 2-vector *standard hyperhyperboloid of one sheet (hyperpseudosphere)* $\boldsymbol{\Sigma}_{\mathcal{D}}$ is defined as

$$\boldsymbol{\Sigma}_{\mathcal{D}} = \boldsymbol{\Sigma}_{\mathcal{C}^1} \wedge \boldsymbol{\Sigma}_{\mathcal{C}^2} \quad (7.81)$$

which is the wedge of the same *hyperpseudosphere* (§6.4.5) embedded in CSTA1 and CSTA2.

7.3.4 DCSTA GIPNS 2-vector standard hyperhyperboloid of two sheets

The DCSTA GIPNS 2-vector *standard hyperhyperboloid of two sheets (imaginary hyper-pseudosphere)* $\boldsymbol{\Xi}_{\mathcal{D}}$ is defined as

$$\boldsymbol{\Xi}_{\mathcal{D}} \;=\; \boldsymbol{\Xi}_{\mathcal{C}^1} \wedge \boldsymbol{\Xi}_{\mathcal{C}^2} \tag{7.82}$$

which is the wedge of the same *imaginary hyperpseudosphere* (§6.4.6) embedded in CSTA1 and CSTA2.

7.3.5 DCSTA GIPNS 4-vector standard sphere or pseudosphere

The DCSTA GIPNS 4-vector *standard sphere or pseudosphere* $\boldsymbol{S}_{\mathcal{D}}$ is defined as

$$\boldsymbol{S}_{\mathcal{D}} \;=\; \boldsymbol{S}_{\mathcal{C}^1} \wedge \boldsymbol{S}_{\mathcal{C}^2} \tag{7.83}$$

which is the wedge of the same *sphere* (§6.4.7) or *pseudosphere* (§6.4.8) embedded in CSTA1 and CSTA2.

7.3.6 DCSTA GIPNS 4-vector standard plane

The DCSTA GIPNS 4-vector *standard plane* $\boldsymbol{\Pi}_{\mathcal{D}}$ is defined as

$$\boldsymbol{\Pi}_{\mathcal{D}} \;=\; \boldsymbol{\Pi}_{\mathcal{C}^1} \wedge \boldsymbol{\Pi}_{\mathcal{C}^2} \tag{7.84}$$

which is the wedge of the same *plane* (§6.4.10) embedded in CSTA1 and CSTA2.

7.3.7 DCSTA GIPNS 6-vector standard line

The DCSTA GIPNS 6-vector *standard line* $\boldsymbol{L}_{\mathcal{D}}$ is defined as

$$\boldsymbol{L}_{\mathcal{D}} \;=\; \boldsymbol{L}_{\mathcal{C}^1} \wedge \boldsymbol{L}_{\mathcal{C}^2} \tag{7.85}$$

which is the wedge of the same *line* (§6.4.11) embedded in CSTA1 and CSTA2.

7.3.8 DCSTA GIPNS 6-vector standard circle or pseudocircle

The DCSTA GIPNS 6-vector *standard circle or pseudocircle* $\boldsymbol{C}_{\mathcal{D}}$ is defined as

$$\boldsymbol{C}_{\mathcal{D}} \;=\; \boldsymbol{C}_{\mathcal{C}^1} \wedge \boldsymbol{C}_{\mathcal{C}^2} \tag{7.86}$$

which is the wedge of the same *circle* (§6.4.12) or *pseudocircle* (§6.4.13) embedded in CSTA1 and CSTA2.

7.3.9 DCSTA GIPNS 8-vector standard point pair

The DCSTA GIPNS 8-vector *standard point pair* $\mathbf{2}_{\mathcal{D}}$ is defined as

$$\mathbf{2}_{\mathcal{D}} \;=\; \mathbf{2}_{\mathcal{C}^1} \wedge \mathbf{2}_{\mathcal{C}^2} \tag{7.87}$$

which is the wedge of the same *point pair* (§6.4.14) embedded in CSTA1 and CSTA2.

7.3.10 DCSTA GIPNS null 10-vector standard point

The DCSTA GIPNS null 10-vector *standard point* $\boldsymbol{P}^{\star}_{\mathcal{D}}$ is defined as

$$\boldsymbol{P}^{\star}_{\mathcal{D}} \;=\; \boldsymbol{P}^{\star}_{\mathcal{C}^1} \wedge \boldsymbol{P}^{\star}_{\mathcal{C}^2} \tag{7.88}$$

which is the wedge of the same *GIPNS point* (§6.4.17) embedded in CSTA1 and CSTA2.

7.4 DCSTA GOPNS standard entities

The DCSTA GOPNS $(12-k)$-blade *standard entity* $\boldsymbol{X}^{*\mathcal{D}}$ is the DCSTA dual (§7.7.1) of the DCSTA GIPNS k-blade *standard entity* $\boldsymbol{X}$ (§7.3),

$$\boldsymbol{X}^{*\mathcal{D}} = \boldsymbol{X}\mathbf{I}_{\mathcal{D}}, \tag{7.89}$$

which represents the same surface.

Any DCSTA GOPNS $(12-k)$-blade *standard entity* $\boldsymbol{X}^{*\mathcal{D}}$ can also be formed as the wedge of $(12-k)/2$ DCSTA GOPNS null 2-vector *points* (§7.2) by the same formulas as in CSTA (§6.5),

$$\boldsymbol{X}^{*\mathcal{D}} = \bigwedge \boldsymbol{P}_{\mathcal{D}_i}, \qquad \text{for } 1 \leq i \leq 6. \tag{7.90}$$

Only the DCSTA GOPNS *standard entities* can be formed as the wedge of DCSTA points. The DCSTA GOPNS 10-vector *non-standard entities* (§7.5) are generally not blades and cannot be formed as the wedge of DCSTA points.

7.5 DCSTA GIPNS 2-vector non-standard surface entities

The DCSTA GIPNS 2-vector *non-standard surface entities* $\boldsymbol{\Omega}$ are defined as linear combinations

$$\boldsymbol{\Omega} = \sum \alpha_i T_i \tag{7.91}$$

of the DCSTA 2-vector *value-extraction elements* T_s (§7.2.3).

In a straightforward way, an entity $\boldsymbol{\Omega}$, which has the general form of a spacetime Darboux (pseudo)cyclide, is formulated in terms of the value-extraction elements T_s to represent an implicit surface function

$$F(w,x,y,z) = \boldsymbol{T}_{\mathcal{D}} \cdot \boldsymbol{\Omega}. \tag{7.92}$$

A *pseudocyclide* or *pseudoquadric* has the form of a spatial cyclide or quadric, but one of the spatial axes is replaced by the pseudospatial time w-axis.

In terms of the value-extraction elements T_s, these entities $\boldsymbol{\Omega}$ are defined exactly as they are in the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) that is introduced in [7]. The reader should refer to [7] for additional details that are not repeated in this paper.

Any DCSTA GIPNS 2-vector *non-standard surface entity* $\boldsymbol{\Omega}$, or its dual GOPNS 10-vector entity

$$\boldsymbol{\Omega}^{*\mathcal{D}} = \boldsymbol{\Omega}\mathbf{I}_{\mathcal{D}}, \tag{7.93}$$

can be translated in spacetime using the DCSTA *translator* $T_{\mathcal{D}}$ (§7.7.7), spatially rotated in space using the DCSTA spatial *rotor* $R_{\mathcal{D}}$ (§7.7.6), and isotropically dilated in spacetime using the DCSTA *isotropic dilator* $D_{\mathcal{D}}$ (§7.7.8).

Any DCSTA GIPNS 2-vector, or dual GOPNS 10-vector, non-standard *quadric* surface entity $\boldsymbol{Q}$ can also be boosted in spacetime using the DCSTA spacetime *boost* operator (§7.7.3) with the interpretation that the *quadric* surface relativistically gains the spacetime velocity $\mathbf{v}$ of the active boost $B_{\mathbf{v}}$. A *spatial quadric surface* (at zero velocity) $\mathbf{Q} = \mathbf{Q}_{\mathcal{DS}}$ at position $\mathbf{p}$ can also be *anisotropically dilated* (§7.7.9) in position $\mathbf{p}$ by a factor d in a specific direction $\hat{\mathbf{v}}$ in space using the DCSTA *translated-boost* $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ (§7.7.4) with an imaginary natural boost speed $\beta_{\mathbf{v}}$ that is followed by a DCSTA *spatial projection* $\mathbf{Q}' = (\boldsymbol{Q} \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}$ (§7.7.2). The spatial projection discards all time components to recover a spatial quadric surface $\mathbf{Q}'$ that is again at zero velocity.

7.6 DCSTA conic section entities

It should be straightforward to adapt the DCGA conic sections into DCSTA and its spatial subalgebra DCSA. The reader is referred to the paper [5] for details on conic sections in DCGA and possible applications that include orthographic and perspective projection of conic sections.

7.7 DCSTA operations

7.7.1 DCSTA dualization

The dual DCSTA GOPNS $(12-k)$-vector surface entity $\mathbf{Q}^{*\mathcal{D}}$ of any DCSTA GIPNS k-vector surface entity $\mathbf{Q}$ is obtained by the DCSTA dualization as

$$\mathbf{Q}^{*\mathcal{D}} = \mathbf{Q}\mathbf{I}_{\mathcal{D}} = \mathbf{Q}\cdot\mathbf{I}_{\mathcal{D}}. \tag{7.94}$$

The undual operation is

$$\mathbf{Q} = \mathbf{Q}^{*\mathcal{D}}\cdot\mathbf{I}_{\mathcal{D}}. \tag{7.95}$$

The dual and undual operations are the repeated application of the same dualization operation. Therefore, DCSTA dualization is an involution.

The DCSTA GIPNS k-vector surface entity $\mathbf{Q}$ and its dual DCSTA GOPNS $(12-k)$-vector surface entity $\mathbf{Q}^{*\mathcal{D}}$ represent the same surface.

7.7.2 DCSTA spatial projection

The DCSTA *spatial projection* of a DCSTA entity $\mathbf{\Omega}_{\mathcal{D}}$ is defined as

$$\mathbf{\Omega}_{\mathcal{DS}} = (\mathbf{\Omega}_{\mathcal{D}}\cdot\mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, \tag{7.96}$$

which is the projection into the $\mathcal{G}_{2,8}$ DCSA subalgebra, where

$$\begin{aligned}\mathbf{I}_{\mathcal{DS}} &= (\mathbf{e}_1\wedge\mathbf{e}_7)\cdot\mathbf{I}_{\mathcal{D}} = \mathbf{I}_{\mathcal{S}^1}\mathbf{e}_5\mathbf{e}_6\mathbf{I}_{\mathcal{S}^2}\mathbf{e}_{11}\mathbf{e}_{12} && (7.97)\\ &= -\widetilde{\mathbf{I}}_{\mathcal{DS}} = -\mathbf{I}_{\mathcal{DS}}^{-1} && (7.98)\end{aligned}$$

is the DCSA unit pseudoscalar. The projection produces the $\mathcal{G}_{2,8}$ DCSA entity $\mathbf{\Omega}_{\mathcal{DS}}$ representing the $\mathcal{G}_{4,8}$ DCSTA entity $\mathbf{\Omega}_{\mathcal{D}}$ at time $w=ct=0$.

The DCSA null 2-vector *point* is defined as

$$\begin{aligned}\mathbf{P}_{\mathcal{DS}} &= \mathcal{DS}(\mathbf{p}_{\mathcal{S}}) && (7.99)\\ &= \mathcal{CS}^1(\mathbf{p}_{\mathcal{S}^1})\wedge\mathcal{CS}^2(\mathbf{p}_{\mathcal{S}^2}) = \mathcal{CS}^1(\mathbf{p}_{\mathcal{S}^1})\mathcal{CS}^2(\mathbf{p}_{\mathcal{S}^2}) && (7.100)\\ &= \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1})\wedge\mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) = \mathcal{C}^1(\mathbf{p}_{\mathcal{S}^1})\mathcal{C}^2(\mathbf{p}_{\mathcal{S}^2}) && (7.101)\\ &= \mathcal{D}(\mathbf{p}_{\mathcal{S}}) && (7.102)\\ &= \left(\mathbf{p}_{\mathcal{S}^1} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^1}^2\mathbf{e}_{\infty 1} + \mathbf{e}_{o1}\right)\wedge\left(\mathbf{p}_{\mathcal{S}^2} + \frac{1}{2}\mathbf{p}_{\mathcal{S}^2}^2\mathbf{e}_{\infty 2} + \mathbf{e}_{o2}\right), && (7.103)\end{aligned}$$

which is just the doubled embedding of a $\mathcal{G}_{0,3}$ SA spatial point of the form

$$\mathbf{p}_{\mathcal{S}} = p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3 \tag{7.104}$$

without a time $w = p_w$ component.

The DCSA GIPNS entity $\mathbf{\Omega}_{\mathcal{DS}}$, or dual GOPNS entity $\mathbf{\Omega}_{\mathcal{DS}}^{*\mathcal{DS}}$, that is produced by the DCSTA spatial projection should be tested against only spatial DCSA points $\mathbf{P}_{\mathcal{DS}} = \mathcal{D}(\mathbf{p}_{\mathcal{S}})$ that represent spatial positions at time $w=ct=0$.

The DCSTA *spatial projection* is specifically defined for the spatial projection of any DCSTA GIPNS 2-vector *quadric surface entity* $\boldsymbol{Q}$ (§7.5) as

$$\mathbf{Q} = \mathbf{Q}_{\mathcal{DS}} = (\boldsymbol{Q} \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, \tag{7.105}$$

which is a DCSA quadric surface entity that represents the DCSTA quadric $\boldsymbol{Q}$ at time $w = ct = 0$. The DCSTA anisotropic (directed or non-uniform) dilation operation (§7.7.9) is defined for any DCSA quadric entity $\mathbf{Q}$ formed as a linear combination of quadric extraction elements T_s (§7.2.3) without a time component. The anisotropic dilation is implemented as a boost by a natural speed $\beta = \sqrt{1-d^2}$ that may be imaginary for a dilation factor $d > 1$. Directed dilation using an imaginary β results in imaginary time components that are only artifacts of the directed dilation operation and should usually be discarded. The DCSTA spatial projection of a DCSTA quadric $\boldsymbol{Q}$, which may be a spatial DCSA quadric $\mathbf{Q}$ that has been boosted for directed dilation, discards any time components and produces the quadric at time $w = ct = 0$ as a DCSA quadric $\mathbf{Q}$. The DCSA quadric $\mathbf{Q}$ is again a DCSTA quadric $\boldsymbol{Q}$ at zero velocity $\beta = 0$.

7.7.3 DCSTA spacetime boost

The DCSTA 4-versor *boost* operator is defined as

$$B_{\mathcal{D}} = B_{\mathcal{C}^1} \wedge B_{\mathcal{C}^2}, \tag{7.106}$$

which is the doubling of the CSTA 2-versor *boost* $B_{\mathcal{C}}$ (§6.6.8) in CSTA1 and CSTA2.

For a DCSTA 4-versor boost, the notation $B_{\mathbf{v}}$ is defined as

$$B_{\mathbf{v}} = B_{\mathcal{D}\mathbf{v}} = B_{\mathcal{C}^1\mathbf{v}} \wedge B_{\mathcal{C}^2\mathbf{v}} \tag{7.107}$$

$$= \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}^1}\mathbf{e}_1\right) \wedge \exp\left(\frac{1}{2}\varphi_{\mathbf{v}}\hat{\mathbf{v}}_{\mathcal{S}^2}\mathbf{e}_7\right), \tag{7.108}$$

which is an active boost into the frame of the observable spacetime velocity

$$\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} \tag{7.109}$$

$$= c\gamma_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}}. \tag{7.110}$$

The observable worldline $\boldsymbol{v}t$, with initial position $\mathbf{p}_0 = 0$ (the spacetime origin) at time $t = 0$, is also being called the *observable* $\boldsymbol{v}$ or the *frame* $\boldsymbol{v}$. The frame of $\boldsymbol{v}$ is carried at the position $\boldsymbol{v}t$ of the observable.

All DCSTA entities can be boosted. A boosted surface entity represents the set of all boosted surface points. For a full discussion of the boost operator, see the CSTA boost $B_{\mathcal{C}}$ (§6.6.8).

A DCSTA GIPNS 2-vector *quadric* surface entity $\boldsymbol{Q}$ (§7.5) is *boosted* as

$$\boldsymbol{Q}' = B_{\mathcal{D}}\boldsymbol{Q}B_{\mathcal{D}}^{\sim}. \tag{7.111}$$

The quadric surface $\boldsymbol{Q}$ should have initial position $\mathbf{p}_0 = 0$ at time $w = ct = 0$. If $\mathbf{p}_0 \neq 0$ at time $w = ct = 0$, then the quadric $\boldsymbol{Q}$ with position $\mathbf{p} = \mathbf{p}_0$ can be boosted using the DCSTA 4-versor *translated-boost* operator (§7.7.4)

$$B_{\mathcal{D}}^{\mathbf{p}_0} = B_{\mathcal{C}^1}^{\mathbf{p}_0} B_{\mathcal{C}^2}^{\mathbf{p}_0} \tag{7.112}$$

that is centered on the initial position $\mathbf{p}_0$. The translated-boost of $\boldsymbol{Q}$ is

$$\boldsymbol{Q}' = B_{\mathcal{D}}^{\mathbf{p}_0}\boldsymbol{Q}B_{\mathcal{D}}^{\mathbf{p}_0\sim}, \tag{7.113}$$

where the position of $\boldsymbol{Q}'$ is preserved as $\mathbf{p}=\mathbf{p}_0$ at time $w=ct=0$.

Since entities are homogeneous and any general spacetime dilation factor γ of the boosts can always be divided out (as a normalization of the entity) without affecting the surface representation, the boost of a quadric surface has the interpretation that the quadric surface undergoes a relativistic velocity addition for an active boost and a relativistic velocity subtraction for a passive boost. A boosted quadric displays the spacetime (length) contraction effect of the boost, which is possible since general quadrics can be represented by the DCSTA extraction elements T_s (§7.2.3). For active boosts $B_{\mathbf{v}}$, the length contraction of displacements is as seen by the observable $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ in its contracted spacetime frame. For passive boosts $B_{\mathbf{v}}^{\sim} = B_{-\mathbf{v}}$, the length contraction of displacements is as seen by the observable $\boldsymbol{v}^{\dagger} = \boldsymbol{o} - \mathbf{v}$ in its contracted spacetime frame, as if the boost is the active boost $B_{-\mathbf{v}}$. The boost is viewed in the active orientation, and the quadric is boosted up into the frame of the active boost observable $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ for $B_{\mathbf{v}}$, or the frame of the active boost observable $\boldsymbol{v}^{\dagger} = \boldsymbol{o} - \mathbf{v}$ for $B_{\mathbf{v}}^{\sim} = B_{\mathbf{v}^{\dagger}} = B_{-\mathbf{v}}$. In either case, the length contraction factor is the same, $1/\gamma_{\mathbf{v}} = 1/\gamma_{-\mathbf{v}} = \sqrt{1-\beta_{\mathbf{v}}^2}$, and the quadric surface relativistically gains the velocity $\mathbf{v}$ or $-\mathbf{v}$ of the frame it is boosted into.

For example, consider a spatial quadric $\mathbf{Q} = \mathbf{Q}_{\mathcal{DS}}$ that is initially at zero velocity $\beta_0 = 0$ and at position $\mathbf{p}_0 = 0$ (the origin) in the rest frame of the conventional coordinate time t observable $\boldsymbol{o}$. It is then boosted as

$$\boldsymbol{Q} \;=\; B_{\mathbf{v}} \mathbf{Q} B_{\mathbf{v}}^{\sim} = \mathbf{Q} \oplus \boldsymbol{v}. \tag{7.114}$$

$\boldsymbol{Q}$ is $\mathbf{Q}$ boosted or moved *into the rest frame* of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$. Local to the frame of $\boldsymbol{v}$, $\boldsymbol{Q}$ is still at the origin and still at rest as it was in the frame of $\boldsymbol{o}$. $\boldsymbol{Q}$ is carried along in the frame of $\boldsymbol{v}$ at the frame velocity $\mathbf{v} = \beta_{\mathbf{v}} c \hat{\mathbf{v}}$ and with position displacement $\mathbf{v}t$ in the frame of coordinate time t observable $\boldsymbol{o}$. Since the entity $\boldsymbol{Q}$ is homogeneous, it displays as the contracted quadric, by contraction factor $\sqrt{1-\beta_{\mathbf{v}}^2}$ in direction $\hat{\mathbf{v}}$, as seen by the observable $\boldsymbol{v}$ in its own contracted frame, but $\boldsymbol{Q}$ moves with coordinate time t in the frame of observable $\boldsymbol{o}$ at the velocity $\mathbf{v}$ as seen by $\boldsymbol{o}$ while $\boldsymbol{Q}$ is carried at rest and at the origin *in the frame of* $\boldsymbol{v}$ that moves at velocity $\mathbf{v}$.

The boosted quadric surface $\boldsymbol{Q}$ can be symbolically evaluated [24] for its implicit surface function F as

$$\mathcal{D}(\boldsymbol{t}_{\mathcal{M}}) \cdot \boldsymbol{Q} \;=\; F(w, x, y, z), \tag{7.115}$$

which can be graphed in xyz-space at any selected time $w = ct$. The graph of $\boldsymbol{Q}$ should show it to be centered in space at a position consistent with the elapsed coordinate time t in the frame of the observable $\boldsymbol{o}$, and the shape of the quadric surface should show a length contraction effect consistent with the speed of the boost that has been applied.

7.7.4 DCSTA translated-boost

The DCSTA 4-versor *translated-boost* operator $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ is the doubling of the CSTA 2-versor translated-boost $B_{\mathcal{C}\mathbf{v}}^{\mathbf{p}}$ (§6.6.9) and defined as

$$B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}} \;=\; B_{\mathcal{C}^1\mathbf{v}}^{\mathbf{p}} \wedge B_{\mathcal{C}^2\mathbf{v}}^{\mathbf{p}} = B_{\mathcal{C}^1\mathbf{v}}^{\mathbf{p}} B_{\mathcal{C}^2\mathbf{v}}^{\mathbf{p}} \tag{7.116}$$

$$\;=\; \exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}} \mathbf{\Pi}_{\mathcal{C}^1}\right) \wedge \exp\!\left(\frac{1}{2}\varphi_{\mathbf{v}} \mathbf{\Pi}_{\mathcal{C}^2}\right), \tag{7.117}$$

where

$$\mathbf{\Pi}_{\mathcal{C}^1} \;=\; \boldsymbol{D}_{\mathcal{C}^1}^{*\mathcal{M}^1} - (\mathbf{p}_{0\mathcal{S}^1} \cdot \boldsymbol{D}_{\mathcal{C}^1}^{*\mathcal{M}^1}) \mathbf{e}_{\infty 1} \tag{7.118}$$

$$\mathbf{\Pi}_{\mathcal{C}^2} \;=\; \boldsymbol{D}_{\mathcal{C}^2}^{*\mathcal{M}^2} - (\mathbf{p}_{0\mathcal{S}^2} \cdot \boldsymbol{D}_{\mathcal{C}^2}^{*\mathcal{M}^2}) \mathbf{e}_{\infty 2} \tag{7.119}$$

and

$$\begin{aligned}
\boldsymbol{D}_{\mathcal{C}^1} &= \hat{\mathbf{v}}_{\mathcal{S}^1}\mathbf{e}_1\mathbf{I}_{\mathcal{M}^1} && (7.120)\\
\boldsymbol{D}_{\mathcal{C}^2} &= \hat{\mathbf{v}}_{\mathcal{S}^2}\mathbf{e}_7\mathbf{I}_{\mathcal{M}^2} && (7.121)\\
\mathbf{p}_{\mathcal{S}^1} &= p_x\mathbf{e}_2 + p_y\mathbf{e}_3 + p_z\mathbf{e}_4 && (7.122)\\
\mathbf{p}_{\mathcal{S}^2} &= p_x\mathbf{e}_8 + p_y\mathbf{e}_9 + p_z\mathbf{e}_{10}. && (7.123)
\end{aligned}$$

The translated-boost is centered on the spatial point

$$\mathbf{p} = \mathbf{p}_0 = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3 \quad (7.124)$$

and actively boosts into the rest frame of the translated observable with worldline

$$\begin{aligned}
\boldsymbol{v}^{\mathbf{p}_0}(t) &= \mathbf{p}_0 + (\boldsymbol{o} + \mathbf{v})t && (7.125)\\
&= \mathbf{p}_0 + (c\gamma_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}})t. && (7.126)
\end{aligned}$$

If $\mathbf{p}_0 = 0$, then $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}_0} = B_{\mathcal{D}\mathbf{v}}$ (§7.7.3).

For example, the point $\mathbf{p}_0 = \mathbf{p}$ can be the center *position* $\mathbf{p}$ of a spatial quadric surface $\mathbf{Q}$, or it can be the *initial point* $\mathbf{p}_0$ at time $w = ct = 0$ of a DCSTA GIPNS 6-vector standard *line* $\boldsymbol{L}_{\mathcal{D}}$ (§7.3.7) that represents an observable worldline.

The translated-boost operator $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ is used in the definition of the DCSTA *anisotropic dilation operation* (§7.7.9), which is valid for the directed scaling of DCSTA quadric surface entities.

7.7.5 DCSTA spacetime reframe (reverse boost)

The DCSTA reframe operation is the reversed application of the active boost operation (§7.7.3) and is often interpreted as a passive transformation relative to a new frame of reference. For a full discussion of the boost operator, see the CSTA boost (§6.6.8).

A DCSTA GIPNS 2-vector quadric surface entity $\boldsymbol{Q}$ (§7.5) is reframed (*passively transformed*) into (*relative to*) the frame of observable $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ as

$$\begin{aligned}
\boldsymbol{Q}' &= B_{\mathbf{v}}^{\sim}\boldsymbol{Q}B_{\mathbf{v}} = \boldsymbol{Q} \ominus \boldsymbol{v} && (7.127)\\
&= B_{-\mathbf{v}}\boldsymbol{Q}B_{-\mathbf{v}}^{\sim} = B_{\mathbf{v}^\dagger}\boldsymbol{Q}B_{\mathbf{v}^\dagger}^{\sim} = \boldsymbol{Q} \oplus \boldsymbol{v}^\dagger. && (7.128)
\end{aligned}$$

One way to interpret the reframe is that $\boldsymbol{Q}$ is initially (being carried) in the frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$, where it sees $\boldsymbol{v}$ as its conventional coordinate time $\tau = t_{p\boldsymbol{v}}$ observer $\boldsymbol{o}_{\boldsymbol{v}}$, and it is actively boosted down or moved into the frame of the actual coordinate time $t = t_{p\boldsymbol{o}}$ observer $\boldsymbol{o}$ as $\boldsymbol{Q}'$. From the perspective of observer $\boldsymbol{o}$, $\boldsymbol{Q}'$ relativistically loses the velocity $\mathbf{v}$ as a velocity subtraction.

Another way to interpret the reframe is that $\boldsymbol{Q}$ is initially in the frame of $\boldsymbol{o}$, and it is actively boosted up into the frame of $\boldsymbol{v}^\dagger = \boldsymbol{o} - \mathbf{v}$. The frame of $\boldsymbol{v} = \boldsymbol{o} + \mathbf{v}$ and the frame of $\boldsymbol{v}^\dagger$ are conjugate frames that carry boosted entities in opposite directions with the same proper time $\tau = t_{p\boldsymbol{v}} = t_{p\boldsymbol{v}^\dagger}$ relative to the coordinate time $t = t_{p\boldsymbol{o}}$ of observer $\boldsymbol{o}$. In many respects, conjugate frames are the same frame, and they also see the same length contraction factor $\sqrt{1 - \beta_{\mathbf{v}}^2}$. Entities that are actively boosted or moved into the frame of $\boldsymbol{v}^\dagger$ are how entities carried in the frame of $\boldsymbol{o}$ appear to $\boldsymbol{v}$ when they are passively transformed (not actively boosted or moved) into the frame of $\boldsymbol{v}$ by a relative or passive boost.

If an entity $\boldsymbol{Q}$ has an initial position $\mathbf{p}_0 \neq 0$ at time $w = ct = 0$, then a passive translated-boost $B_{\mathbf{v}}^{\mathbf{p}_0\sim}$ (§7.7.4) should be used.

7.7.6 DCSTA rotor

The DCSTA 4-versor *rotor* $R_{\mathcal{D}}$ is defined as

$$R_{\mathcal{D}} = R_{\mathcal{C}^1} \wedge R_{\mathcal{C}^2} \tag{7.129}$$

which is the wedge of the same *rotor* (§6.6.3) in CSTA1 and CSTA2.

Any DCSTA entity $\mathbf{Q}$ is rotated by the rotor operation

$$\mathbf{Q}' = R_{\mathcal{D}}\mathbf{Q}\widetilde{R_{\mathcal{D}}}. \tag{7.130}$$

The CSTA 2-versor *line rotor (translated rotor)* $R^{\mathbf{p}}_{\mathcal{C}} = L_{\mathcal{C}}$ (§6.6.5) can also be doubled into the DCSTA 4-versor *translated rotor* $R^{\mathbf{p}}_{\mathcal{D}} = L_{\mathcal{D}} = L_{\mathcal{C}^1} \wedge L_{\mathcal{C}^2}$.

7.7.7 DCSTA translator

The DCSTA 4-versor *translator* $T_{\mathcal{D}}$ is defined as

$$T_{\mathcal{D}} = T_{\mathcal{C}^1} \wedge T_{\mathcal{C}^2} \tag{7.131}$$

which is the wedge of the same *translator* (§6.6.4) in CSTA1 and CSTA2.

A DCSTA entity $\mathbf{Q}$ is translated by the translator operation

$$\mathbf{Q}' = T_{\mathcal{D}}\mathbf{Q}\widetilde{T_{\mathcal{D}}}. \tag{7.132}$$

7.7.8 DCSTA isotropic dilator

The DCSTA 4-versor *isotropic dilator* $D_{\mathcal{D}}$ is defined as

$$D_{\mathcal{D}} = D_{\mathcal{C}^1} \wedge D_{\mathcal{C}^2} \tag{7.133}$$

which is the wedge of the same *isotropic dilator* (§6.6.6) in CSTA1 and CSTA2.

A DCSTA entity $\mathbf{Q}$ is isotropically dilated by the dilator operation

$$\mathbf{Q}' = D_{\mathcal{D}}\mathbf{Q}\widetilde{D_{\mathcal{D}}}. \tag{7.134}$$

The DCSTA 4-versor *translated dilator* $D^{\boldsymbol{p}}_{\mathcal{D}}$ that dilates relative to the center point $\boldsymbol{p}_{\mathcal{M}}$ is the doubling of the CSTA 2-versor *translated dilator* $D^{\boldsymbol{p}}_{\mathcal{C}}$ (§6.6.7) as

$$D^{\boldsymbol{p}}_{\mathcal{D}} = D^{\boldsymbol{p}}_{\mathcal{C}^1} \wedge D^{\boldsymbol{p}}_{\mathcal{C}^2}. \tag{7.135}$$

7.7.9 DCSTA anisotropic dilator

7.7.9.1 Introduction

A *spatial* DCSTA GIPNS 2-vector *quadric surface* entity $\mathbf{Q}$ (§7.5) at center *position*

$$\mathbf{p} = \mathbf{p}_0 = \mathbf{p}_{\mathcal{S}} = p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3, \tag{7.136}$$

formed similar to a DCGA GIPNS 2-vector quadric as a linear combination of *quadratic* DCSTA point value-*extraction elements* T_s (§7.2.3) without time components, can be anisotropically dilated (for non-uniform, directed scaling) in situ at $\mathbf{p} = \mathbf{p}_0$, by dilation factor d in a unit direction $\hat{\mathbf{v}} = \hat{\mathbf{v}}_{\mathcal{S}}$, as a *translated-boost* $B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}}$ (§7.7.4) that is followed by a DCSTA *spatial projection* using $\mathbf{I}_{\mathcal{DS}}$ (§7.7.2). The natural speed $\beta_{\mathbf{v}}$ of the translated-boost, for the dilation factor d, is

$$\beta_{\mathbf{v}} = \sqrt{1 - d^2}, \tag{7.137}$$

which is an imaginary number for $d>1$. The rapidity of the translated-boost is

$$\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}}) = \operatorname{atanh}\left(\sqrt{1-d^2}\right), \tag{7.138}$$

which can also be imaginary when $\beta_{\mathbf{v}}$ is imaginary. The translated-boost spacetime velocity is

$$\boldsymbol{v} = \boldsymbol{o} + \mathbf{v} = c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}}, \tag{7.139}$$

which can be an imaginary spacetime velocity when $\beta_{\mathbf{v}}$ is imaginary. A spatial quadric $\mathbf{Q}$ (at zero velocity) is boosted by the translated-boost $B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}}$ to be in the moving observable frame as $\boldsymbol{Q}$ and is carried along the observable worldline at velocity $\mathbf{v}$ with position

$$\begin{aligned} \boldsymbol{v}^{\mathbf{p}_0}(t) &= \mathbf{p}_0 + (\boldsymbol{o} + \mathbf{v})t & (7.140)\\ &= \mathbf{p}_0 + (c\boldsymbol{\gamma} + \beta_{\mathbf{v}} c\hat{\mathbf{v}})t. & (7.141) \end{aligned}$$

The observed dilation of quadric surface $\mathbf{Q}$ in the moving frame as $\boldsymbol{Q}$ is by factor

$$d = \sqrt{1-\beta_{\mathbf{v}}^2} \tag{7.142}$$

in the direction $\hat{\mathbf{v}}$, where $d>1$ for imaginary $\beta_{\mathbf{v}}$. The length dilation in the direction $\hat{\mathbf{v}}$ of the boost follows from the standard formula for special relativity length contraction

$$L = \frac{L_0}{\gamma_{\mathbf{v}}} = L_0\sqrt{1-\beta_{\mathbf{v}}^2} = L_0 d. \tag{7.143}$$

The moving, and anisotropically dilated, quadric $\boldsymbol{Q}$ can be evaluated at time $w = ct = 0$ to observe the anisotropic dilation at its position $\mathbf{p}$. By projecting $\boldsymbol{Q}$ into the spatial $\mathcal{G}_{2,8}$ DCSA subalgebra, an anisotropically dilated spatial quadric $\mathbf{Q}'$ at zero velocity is recovered.

7.7.9.2 Definition

The DCSTA *anisotropic dilator operation*, on a spatial DCSTA GIPNS 2-vector *quadric surface* $\mathbf{Q}$ (§7.5) with position $\mathbf{p}$, for dilation factor d in the direction $\hat{\mathbf{v}}$, is defined as

$$\begin{aligned} \mathbf{Q}' = \mathbf{Q}'_{\mathcal{DS}} &= ((B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}} \mathbf{Q} B_{\mathcal{D}\mathbf{v}}^{\mathbf{p}\sim}) \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1} & (7.144)\\ &= (\boldsymbol{Q} \cdot \mathbf{I}_{\mathcal{DS}})\mathbf{I}_{\mathcal{DS}}^{-1}, & (7.145) \end{aligned}$$

where the observable of the translated-boost (§7.7.4) is

$$\begin{aligned} \boldsymbol{v}^{\mathbf{p}}(t) &= \mathbf{p} + (\boldsymbol{o} + \mathbf{v})t & (7.146)\\ &= \mathbf{p} + (c\boldsymbol{\gamma}_0 + \beta_{\mathbf{v}} c\hat{\mathbf{v}})t & (7.147) \end{aligned}$$

with

$$\begin{aligned} \beta_{\mathbf{v}} &= \sqrt{1-d^2} & (7.148)\\ \varphi_{\mathbf{v}} &= \operatorname{atanh}(\beta_{\mathbf{v}}) & (7.149)\\ \mathbf{p} &= p_x\boldsymbol{\gamma}_1 + p_y\boldsymbol{\gamma}_2 + p_z\boldsymbol{\gamma}_3. & (7.150) \end{aligned}$$

7.7.9.3 Discussion

The dilated entity $\mathbf{Q}'_{\mathcal{DS}}$ can be symbolically evaluated as an implicit surface function

$$F(x,y,z) \;=\; \boldsymbol{T}_{\mathcal{D}} \cdot \mathbf{Q}_{\mathcal{DS}} \tag{7.151}$$

that graphs at position $\mathbf{p}$ as anisotropically dilated in the unit direction $\hat{\mathbf{v}}_{\mathcal{S}}$ by factor d.

The spatial quadric $\mathbf{Q}'$ has zero velocity and exists at all times at the same position $\mathbf{p}$. If $\mathbf{Q}'$ should exist only at a specific time $w=p_w$, then this can be represented as an intersection of $\mathbf{Q}'$ with a DCSTA 2-vector standard *hyperplane* $\boldsymbol{E}_{\mathcal{D}}$ (§7.3.2),

$$\begin{aligned}
\boldsymbol{E}_{\mathcal{D}} &= \boldsymbol{E}_{\mathcal{C}^1} \wedge \boldsymbol{E}_{\mathcal{C}^2} && (7.152)\\
\boldsymbol{E}_{\mathcal{C}^1} &= p_w\mathbf{e}_1 + p_w^2\mathbf{e}_{\infty 1} && (7.153)\\
\boldsymbol{E}_{\mathcal{C}^2} &= p_w\mathbf{e}_7 + p_w^2\mathbf{e}_{\infty 2}, && (7.154)
\end{aligned}$$

that fixes the time $w=ct=p_w$. The intersection is the GIPNS 4-vector intersection entity

$$\mathbf{Q}'^{\,w=p_w}_{\langle 4\rangle} \;=\; \mathbf{Q}' \wedge \boldsymbol{E}_{\mathcal{D}}. \tag{7.155}$$

The entity $\mathbf{Q}'^{\,w=p_w}_{\langle 4\rangle}$ can be translated in spacetime using the DCSTA translator (§7.7.7).

The entity $\mathbf{Q}'$ can subsequently be boosted from zero velocity into a real velocity $\mathbf{v}=\mathbf{v}_{\mathcal{S}}=\beta_{\mathbf{v}}c\hat{\mathbf{v}}_{\mathcal{S}}$ with natural speed $0\le\beta_{\mathbf{v}}<1$ by the action of an active DCSTA translated-boost operation (§7.7.4)

$$\mathbf{Q}'_{\mathbf{v}} \;=\; B^{\mathbf{p}}_{\mathcal{D}\mathbf{v}}\mathbf{Q}'B^{\mathbf{p}\sim}_{\mathcal{D}\mathbf{v}}. \tag{7.156}$$

The boosted quadric entity $\mathbf{Q}'_{\mathbf{v}}$ exists at all times t, but its spacetime position

$$\begin{aligned}
\boldsymbol{p}^{\mathrm{P}}(t) &= \mathbf{p} + \dot{\boldsymbol{p}}t && (7.157)\\
&= \mathbf{p} + (\boldsymbol{o} + \dot{\mathbf{p}})t && (7.158)\\
&= \mathbf{p} + (\boldsymbol{o} + \mathbf{v})t && (7.159)
\end{aligned}$$

has velocity $\mathbf{v}$ relative to the observer $\boldsymbol{o}$, and its geometric surface shape undergoes a contraction by factor $d=\sqrt{1-\beta_{\mathbf{v}}^2}$ along the direction of its velocity $\mathbf{v}$. The contraction is consistent with special relativity.

7.8 DCSTA differential calculus

The DCSTA differential calculus is a straightforward extension of the DCGA differential calculus that is introduced in the paper [6].

7.8.1 DCSTA differential elements

Some of the DCSTA point *value-extraction elements* T_s (§7.2.3) have inverses. These inverses allow the following DCSTA 2-vector *differential elements* to be defined as

$$\begin{aligned}
D_w &= 2T_wT_{w^2}^{-1} && (7.160)\\
D_t &= 2T_tT_{t^2}^{-1} && (7.161)\\
D_x &= 2T_xT_{x^2}^{-1} && (7.162)\\
D_y &= 2T_yT_{y^2}^{-1} && (7.163)\\
D_z &= 2T_zT_{z^2}^{-1}. && (7.164)
\end{aligned}$$

7.8.2 DCSTA antisymmetric differential operators

The DCSTA antisymmetric *differential operators* are defined as

$$\partial_w = \frac{\partial}{\partial w} \;=\; D_w \times \tag{7.165}$$

$$\partial_t = \frac{\partial}{\partial t} \;=\; D_t \times \tag{7.166}$$

$$\partial_x = \frac{\partial}{\partial x} \;=\; D_x \times \tag{7.167}$$

$$\partial_y = \frac{\partial}{\partial y} \;=\; D_y \times \tag{7.168}$$

$$\partial_z = \frac{\partial}{\partial z} \;=\; D_z \times \tag{7.169}$$

where the symbol $\times$ is the antisymmetric *commutator product*. For any multivectors A and B, the commutator product is

$$A \times B \;=\; \frac{1}{2}(AB - BA) = -B \times A. \tag{7.170}$$

Any DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§7.5), defined in terms of the extraction elements T_s (§7.2.3), can be differentiated as

$$\partial_{\boldsymbol{n}}\mathbf{\Omega} = \frac{\partial \mathbf{\Omega}}{\partial \boldsymbol{n}} \;=\; D_{\boldsymbol{n}} \times \mathbf{\Omega}, \tag{7.171}$$

where $D_{\boldsymbol{n}}$ is one of the differential elements or is a linear combination of differential elements (§7.8.1).

Higher-order mixed partial derivatives can also be computed as successive differential operations. For example,

$$\frac{d^2\mathbf{\Omega}}{\partial x \partial y} \;=\; D_x \times (D_y \times \mathbf{\Omega}) = D_y \times (D_x \times \mathbf{\Omega}). \tag{7.172}$$

As required of partial differential operators, the sequence in which the derivatives are computed does not affect the result.

7.8.3 DCSTA directional derivative operator

The DCSTA $\boldsymbol{n}$-directional derivative operator is defined as

$$\partial_{\boldsymbol{n}} = \frac{\partial}{\partial \boldsymbol{n}} \;=\; (n_w D_w + n_x D_x + n_y D_y + n_z D_z) \times \tag{7.173}$$

where $\boldsymbol{n}$ is a unit norm spacetime direction

$$\boldsymbol{n} \;=\; \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{n}}{\sqrt{\boldsymbol{n} \cdot \boldsymbol{n}^{\dagger}}} = n_w \gamma_0 + n_x \gamma_1 + n_y \gamma_2 + n_z \gamma_3. \tag{7.174}$$

7.8.4 DCSTA time derivative operator

The DCSTA time t derivative operator is

$$\partial_t = \frac{\partial}{\partial t} \;=\; D_t \times . \tag{7.175}$$

The time t derivative of any DCSTA GIPNS 2-vector *spacetime entity* $\mathbf{\Omega}$ (§7.5) is

$$\dot{\mathbf{\Omega}} = \partial_t \mathbf{\Omega} = \frac{\partial \mathbf{\Omega}}{\partial t} \;=\; D_t \times \mathbf{\Omega}. \tag{7.176}$$

The DCSTA 2-vector *spacetime entity* $\mathbf{\Omega}$ (§7.5) is the most general DCSTA GIPNS 2-vector *non-standard surface entity* that is formed as a linear combination of the DCSTA 2-vector *extraction elements* T_s (§7.2.3).

7.9 DCSTA pseudo-integral calculus

In the paper [6], the DCGA pseudo-integral calculus is introduced. A straightforward adaptation and extension into DCSTA is possible by using the DCSTA *extraction pseudo-inverse elements* T_s^+ (§7.2.4).

8 DCSTA computing with SymPy

DCSTA computing with *SymPy* (http://sympy.org) [24] is possible by using the *Geometric Algebra Module for Sympy* ($\mathcal{GA}$lgebra) by Alan Bromborsky (https://github.com/brombo/galgebra) [2]. This section provides sample code listings and example computations in DCSTA using $\mathcal{GA}$lgebra. The *Anaconda* and *SciPy* python distributions both include SymPy and the *Mayavi* [21] data visualization package. The current version of the $\mathcal{GA}$lgebra module for SymPy can be downloaded and installed from **GitHub**. The *Jupyter Notebook* web application (http://jupyter.org) is recommended for running the sample code and example computations.

8.1 Sample code

The sample code that is listed in the following subsections can be inserted into cells of a Jupyter notebook file and executed in the order listed. The sample code initializes the $\mathcal{GA}$lgebra modules and defines functions and symbols for DCSTA computing. The example computations use the functions and symbols that are defined in the sample code. The sample code is provided as is for experimental testing and educational purposes only!

8.1.1 Imports

Import the SymPy and $\mathcal{GA}$lgebra modules:

```
from sympy import *
from sympy.printing import *
from galgebra.ga import *
from galgebra.mv import *
from galgebra.lt import *
from galgebra.metric import *
from galgebra.printer import *
init_printing()
```

8.1.2 Basis vectors

$\mathcal{G}_{4,8}$ DCSTA requires twelve unit vectors (§1), which can be setup as follows:

```
(e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11,e12) = MV.setup(
    'e*1|2|3|4|5|6|7|8|9|10|11|12',
    metric=[1,-1,-1,-1,1,-1, 1,-1,-1,-1,1,-1]
)
```

8.1.3 Points at the origin and at infinity

The CSTA1, CSTA2 (§6.2), and DCSTA (§7.2) points at the origin and at infinity are defined as follows:

```
(eo1,ei1,eo2,ei2,eo,ei) = symbols(
    'e_o1 e_i1 e_o2 e_i2 e_o e_i'
)
# CSTA1 points
eo1 = Pow(2,-1)*(-e5+e6)
ei1 = (e5+e6)
# CSTA2 points
eo2 = Pow(2,-1)*(-e11+e12)
ei2 = (e11+e12)
# DCSTA points
eo = eo1^eo2
ei = ei1^ei2
```

8.1.4 Unit pseudoscalars

The unit pseudoscalars in $\mathcal{G}_{0,3}$ SA1 (§2.1), $\mathcal{G}_{1,3}$ STA1 (§5.1.2), $\mathcal{G}_{2,4}$ CSTA1 (§6.1), $\mathcal{G}_{0,3}$ SA2, $\mathcal{G}_{1,3}$ STA2, $\mathcal{G}_{2,4}$ CSTA2, and $\mathcal{G}_{4,8}$ DCSTA (§7.1), respectively, are defined as follows:

```
(I31,I41,I61,I32,I42,I62,ID IDS) = symbols(
    'I_31 I_41 I_61 I_32 I_42 I_62 I_D I_DS'
)
# SA1 unit pseudoscalar
I31 = e2^e3^e4
# STA1 unit pseudoscalar
I41 = e1^I31
# CSTA1 unit pseudoscalar
I61 = I41^ei1^eo1
# SA2 unit pseudoscalar
I32 = e8^e9^e10
# STA2 unit pseudoscalar
I42 = e7^I32
# CSTA2 unit pseudoscalar
I62 = I42^ei2^eo2
# DCSTA unit pseudoscalar
ID = I61^I62
# G2,8 DCSA (spatial) unit pseudoscalar
IDS = (e1^e7)|ID
```

The last value, `IDS`, is the $\mathcal{G}_{2,8}$ DCSA unit pseudoscalar for an algebra that is very similar to $\mathcal{G}_{8,2}$ DCGA. The `IDS` unit pseudoscalar is used to project entities (§7.7.2) into a purely spatial algebra that drops the two time dimensions $\mathbf{e}_1$ and $\mathbf{e}_7$. When these time dimensions are dropped, or rejected, by a projection of an entity onto `IDS`, then the entity is effectively located at $w = ct = 0$ in spacetime as a spatial DCSA entity that should be tested against spatial DCSA points. The projection onto `IDS` is useful after a directed scaling (anisotropic dilation) of a quadric surface (§7.7.9).

8.1.5 Point embeddings

CSTA1 point embedding (§6.2):

```
def EV1(v):
    # Embed STA1 vector v as CSTA1 point.
    v1 = v
    return ( v1 + Pow(2,-1)*(v1*v1)*ei1 + eo1 )
```

CSTA2 point embedding (§6.2):

```
def EV2(v):
    # Embed STA1 vector v as CSTA2 point.
    # STA1 vector v is converted to an STA2 vector v2.
    v2 = (v|e1)*e7 - ( (v|e2)*e8 + (v|e3)*e9 + (v|e4)*e10 )
    return ( v2 + Pow(2,-1)*(v2*v2)*ei2 + eo2 )
```

DCSTA point embedding (§7.2):

```
def EV(v):
    # Embed STA1 vector v as DCSTA point.
    return ( EV1(v)^EV2(v) )
```

8.1.6 Point projections

CSTA1 point projection (§6.2.6) to an STA1 vector:

```
def PV1(V1):
    # Project CSTA1 point to STA1 vector.
    # 1) Normalize point.
    # 2) Use multivector projection to project vector part.
    return Pow(scalar(-V1|ei1),-1)*(V1|I41)*I41.inv()
```

CSTA2 point projection (§6.2.6) to an STA1 vector:

```
def PV2(V2):
    # Project CSTA2 point to STA1 vector.
    # 1) Normalize point.
    # 2) Use multivector projection to project vector part.
    # 3) Convert into main STA1 space.
    v2 = Pow(scalar(-V2|ei2),-1)*(V2|I42)*I42.inv()
    return ( (v2|e7)*e1 + (-v2|e8)*e2 + (-v2|e9)*e3 + (-v2|e10)*e4 )
```

DCSTA point projection (§7.2.2) to an STA1 vector:

```
def PV(V):
    # Project DCSTA point V to an STA1 vector.
    # 1) Contract DCSTA point into CSTA1 point using ei2.
    # 2) Project CSTA1 point V1 to an STA1 vector.
    V1 = V|ei2
    return PV1(V1)
```

8.1.7 Symbolic vectors and points

Symbols for coordinates, parameters, and vectors:

```
w,x,y,z,c,t,g,b = symbols('w x y z c t g b')
pw,px,py,pz = symbols('p_w p_x p_y p_z')
rx,ry,rz = symbols('r_x r_y r_z')
nw,nx,ny,nz = symbols('n_w n_x n_y n_z')
vx,vy,vz = symbols('v_x v_y v_z')
v,v1,v2,V,V1,V2 = symbols('v v1 v2 V V1 V2')
```

The pw,px,py,pz are used as symbolic position coordinates for the center position of surface entities. The rx,ry,rz are used as symbolic radii parameters of implicit quadric surface functions. The nw,nx,ny,nz may be used as symbolic coordinates of a normalized unit vector **n**. The vx,vy,vz may be used to hold the velocity components of a velocity vector **v**. The symbol c is used as the symbolic speed of light, and symbol t is time.

Symbolic values, vectors, and points:

```
w = c*t
v = w*e1 + x*e2 + y*e3 + z*e4
v1 = v
v2 = w*e7 + x*e8 + y*e9 + z*e10
V1 = EV1(v)
V2 = EV2(v)
V = EV(v)
```

The embedding of the symbolic STA1 and STA2 vectors v1 and v2 are symbolic CSTA1 and CSTA2 points V1 and V2, respectively. The DCSTA embedding of a symbolic STA1 vector v is the symbolic DCSTA point V. In symbolic calculations, these symbolic point embeddings V1, V2, and V are useful to check results.

8.1.8 CSTA extraction elements

A CSTA point *value-extraction element* C_s (§6.3) extracts the value s from an embedded CSTA point $\boldsymbol{T}_{\mathcal{C}} = \mathcal{C}(\boldsymbol{t}_{\mathcal{M}})$ as

$$s = \boldsymbol{T}_{\mathcal{C}} \cdot C_s. \tag{8.1}$$

```
C11,C1t,C1w,C1x,C1y,C1z,C1t2 = symbols(
    'C1_1 C1_t C1_w C1_x C1_y C1_z C1_t2'
)
C21,C2t,C2w,C2x,C2y,C2z,C2t2 = symbols(
    'C2_1 C2_t C2_w C2_x C2_y C2_z C2_t2'
)
# CSTA1 (C1) point value-extraction elements
C11 = -ei1
C1w = e1
C1t = Pow(c,-1)*C1w
C1x = -e2
C1y = -e3
C1z = -e4
C1t2 = -2*eo1
# CSTA2 (C2) point value-extraction elements
C21 = -ei2
C2w = e7
C2t = Pow(c,-1)*C2w
C2x = -e8
C2y = -e9
C2z = -e10
C2t2 = -2*eo2
```

For example, in CSTA1, the element symbol `C1t2` extracts $\boldsymbol{t}^2_{\mathcal{M}}$ from an embedded point $\boldsymbol{T}_{\mathcal{C}}$.

8.1.9 CSTA differential elements

Using the commutator product $\times$, a CSTA differential element $D^{\mathcal{C}}_{\boldsymbol{n}}$ (§6.6.10) can take a partial derivative of an implicit surface function that is represented by a CSTA entity $\mathbf{E}$ as

$$\partial^{\mathcal{C}}_{\boldsymbol{n}}\mathbf{E} \;=\; \frac{\partial \mathbf{E}}{\partial \boldsymbol{n}} = D^{\mathcal{C}}_{\boldsymbol{n}} \times \mathbf{E}. \tag{8.2}$$

The simplest example is when $\mathbf{E}$ is any CSTA GIPNS 1-vector entity. The CSTA differential element $D^{\mathcal{C}}_{\boldsymbol{n}}$, for differentiating in the direction $\boldsymbol{n}$, can be a linear combination of the CSTA differential elements.

```
C1Dw,C1Dt,C1Dx,C1Dy,C1Dz,C2Dw,C2Dt,C2Dx,C2Dy,C2Dz = symbols(
     'C1Dw C1Dt C1Dx C1Dy C1Dz C2Dw C2Dt C2Dx C2Dy C2Dz'
)
# CSTA1 (C1) differential elements
C1Dw = C11*C1w.inv()
C1Dt = C11*C1t.inv()
C1Dx = C11*C1x.inv()
C1Dy = C11*C1y.inv()
C1Dz = C11*C1z.inv()
# CSTA2 (C2) differential elements
C2Dw = C21*C2w.inv()
C2Dt = C21*C2t.inv()
C2Dx = C21*C2x.inv()
C2Dy = C21*C2y.inv()
C2Dz = C21*C2z.inv()
```

8.1.10 DCSTA extraction elements

The DCSTA point value-extraction elements T_s (§7.2.3) are used to extract the value s from a DCSTA point $\boldsymbol{T}_{\mathcal{D}}$ as

$$s \;=\; \boldsymbol{T}_{\mathcal{D}} \cdot T_s. \tag{8.3}$$

The extraction elements are defined in code as:

```
(
    Tw,Tx,Ty,Tz,
    Tww,Txx,Tyy,Tzz,
    Txy,Tyz,Tzx,Twx,Twy,Twz,
    Twt2,Txt2,Tyt2,Tzt2,
    Tt,Ttt,Ttx,Tty,Ttz,Ttt2,
    T1,Tt2,Tt4
) = symbols(
        'Tw Tx Ty Tz '
        'Tww Txx Tyy Tzz '
        'Txy Tyz Tzx Twx Twy Twz '
        'Twt2 Txt2 Tyt2 Tzt2 '
        'Tt Ttt Ttx Tty Ttz Ttt2 '
        'T1 Tt2 Tt4'
)
# Coordinates; linear extractions
Tw = Pow(2,-1)*((e1^ei2)+(ei1^e7))
Tt = Pow(c,-1)*Tw
Tx = -Pow(2,-1)*((e2^ei2)+(ei1^e8))
Ty = -Pow(2,-1)*((e3^ei2)+(ei1^e9))
Tz = -Pow(2,-1)*((e4^ei2)+(ei1^e10))
# Squares; quadratic extractons
Tww = e7^e1
Ttt = Pow(c,-2)*Tww
Txx = e8^e2
Tyy = e9^e3
Tzz = e10^e4
# Cross terms; quadratic extractions
Twx = Pow(2,-1)*((e1^e8)+(e2^e7))
Twy = Pow(2,-1)*((e1^e9)+(e3^e7))
Twz = Pow(2,-1)*((e1^e10)+(e4^e7))
Ttx = Pow(c,-1)*Twx
Tty = Pow(c,-1)*Twy
Ttz = Pow(c,-1)*Twz
Txy = Pow(2,-1)*((e8^e3)+(e9^e2))
Tyz = Pow(2,-1)*((e10^e3)+(e9^e4))
Tzx = Pow(2,-1)*((e10^e2)+(e8^e4))
# Coordinates * squared test vector; cubic extractions
Twt2 = (e1^eo2)+(eo1^e7)
Ttt2 = Pow(c,-1)*Twt2
Txt2 = (eo2^e2)+(e8^eo1)
Tyt2 = (eo2^e3)+(e9^eo1)
Tzt2 = (eo2^e4)+(e10^eo1)
# Unit scalar extraction
T1 = -ei
# Squared test vector; quadratic extraction
Tt2 = (eo2^ei1)+(ei2^eo1)
# Squared squared test vector; quartic extraction
Tt4 = -4*eo
```

8.1.11 DCSTA extraction pseudo-inverse elements

A DCSTA extraction pseudo-inverse element (§7.2.4) has the property

$$T_s \cdot T_s^+ \;=\; 1. \tag{8.4}$$

The extraction pseudo-inverse elements are defined in code as:

```
(
    iTww,iTxx,iTyy,iTzz,iTtt,
    iTw,iTx,iTy,iTz,iTt,
    iTxy,iTyz,iTzx,iTwx,iTwy,iTwz,iTtx,iTty,iTtz,
    iT1,iTt2,iTt4 ) = symbols(
    'i_Tww i_Txx i_Tyy i_Tzz i_Ttt '
    'i_Tw i_Tx i_Ty i_Tz iTt '
    'i_Txy i_Tyz i_Tzx i_Twx i_Twy i_Twz i_Ttx i_Tty i_Ttz '
    'i_T1 i_Tt2 i_Tt4'
)
iTww = -Tww
iTxx = -Txx
iTyy = -Tyy
iTzz = -Tzz
iTtt = -Pow(c,2)*Tww
iTw = Twt2
iTx = -Txt2
iTy = -Tyt2
iTz = -Tzt2
iTt = Pow(c,2)*Ttt2
iTxy = -2*Txy
iTyz = -2*Tyz
iTzx = -2*Tzx
iTwx = 2*Twx
iTwy = 2*Twy
iTwz = 2*Twz
iTtx = 2*Pow(c,2)*Ttx
iTty = 2*Pow(c,2)*Tty
iTtz = 2*Pow(c,2)*Ttz
iT1 = -Pow(4,-1)*Tt4
iTt2 = -Pow(2,-1)*Tt2
iTt4 = -Pow(4,-1)*T1
```

8.1.12 DCSTA differential elements

The DCSTA differential (§7.8.1) and pseudo-integral elements (§7.9) are:

```
(
    Dw,Dx,Dy,Dz,Dt,
    Iw,Ix,Iy,Iz,It
) = symbols(
    'D_w D_x D_y D_z D_t '
    'I_w I_x I_y I_z I_t'
)
# Differential elements
Dw = 2*Tw*Tww.inv()
Dx = 2*Tx*Txx.inv()
Dy = 2*Ty*Tyy.inv()
Dz = 2*Tz*Tzz.inv()
Dt = 2*Tt*Ttt.inv()
# Pseudo-integral elements
Iw = Pow(2,-1)*Tww*iTw
Ix = Pow(2,-1)*Txx*iTx
Iy = Pow(2,-1)*Tyy*iTy
Iz = Pow(2,-1)*Tzz*iTz
It = Pow(2,-1)*Ttt*iTt
```

In recent versions of the $\mathcal{GA}$lgebra module [2], the commutator product $A \times B$ is coded as `(A>>B)`, and the anti-commutator product $A \bar{\times} B$ is coded as `(A<<B)`. The parentheses are required to ensure that the precedence rules for Python operators do not interfere. For example, the derivative of a DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§7.5) with respect to t is written $\partial_t\mathbf{\Omega} = \dot{\mathbf{\Omega}} = D_t \times \mathbf{\Omega}$, and if $\mathbf{\Omega}$ is assigned to variable `E`, then the derivative is coded as `(Dt>>E)` and evaluated symbolically as `(V|(Dt>>E))`. The operation `(A|B)` is the inner product.

8.1.13 DCSTA directional derivative operator

The DCSTA $\boldsymbol{n}$-directional derivative operator (§7.8.3) is defined in code as:

```
def Dn(w,x,y,z):
    n = sqrt(w**2 + x**2 + y**2 + z**2)
    return Pow(n,-1)*(w*Dw + x*Dx + y*Dy + z*Dz)
```

Only the direction of the spacetime vector

$$\boldsymbol{n} = w\mathbf{e}_1 + x\mathbf{e}_2 + y\mathbf{e}_3 + z\mathbf{e}_3 \tag{8.5}$$

is significant. The $\boldsymbol{n}$-directional derivative uses the norm-unit of $\boldsymbol{n}$, which is

$$\frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{n}}{\sqrt{n_w^2 + n_x^2 + n_y^2 + n_z^2}}. \tag{8.6}$$

The directional derivative of a DCSTA GIPNS 2-vector surface entity `E` (§7.5) is coded as `(Dn(w,x,y,z)>>E)`.

8.1.14 DCSTA pseudo-integral operator

The DCSTA $\boldsymbol{n}$-directional pseudo-integral operator (§7.9) is defined in code as:

```
def In(w,x,y,z):
    n = sqrt(w**2 + x**2 + y**2 + z**2)
    return Pow(n,-1)*(w*Iw + x*Ix + y*Iy + z*Iz)
```

The directional pseudo-integral of a DCSTA GIPNS 2-vector surface entity `E` (§7.5) is coded as `(In(w,x,y,z)>>E)`.

8.1.15 DCSTA GIPNS 2-vector surface entities

The following subsections define, in code, many of the same surface entities that are discussed in the paper on $\mathcal{G}_{8,2}$ DCGA [7]. The most general DCSTA GIPNS 2-vector surface entity $\mathbf{\Omega}$ (§7.5) is the linear combination of the value-extraction elements T_s (§8.1.10). The value-extraction elements can form a general DCSTA GIPNS 2-vector *quadric surface* entity $\mathbf{Q}$ that supports anisotropic dilations (§7.7.9). The value-extraction elements T_s (§7.2.3) can form particular cubic surfaces known as parabolic cyclides and particular quartic surfaces known as Dupin and Darboux cyclides that do not support anisotropic dilations. All of the DCSTA GIPNS 2-vector surfaces $\mathbf{\Omega}$ can be boosted (§7.7.3) into a velocity in spacetime, but only the quadric surface entities can correctly display length contraction or dilation effects.

8.1.16 DCSTA GIPNS 2-vector toroid

The DCSTA GIPNS 2-vector *toroid* is coded as:

```
def GIPNS_Toroid(R,r):
    # Torus centered at the origin circling the z-axis.
    # R is the major radius
    # r is the minor radius
    # R=0 degenerates into exactly -4*Sphere(0,r)
    # R=r=0 degenerates into exactly -4*eo
    # r=0 degenerates into non-standard circle radius R
    # Note, -Tt2 since signatures are negative
    return (
        Tt4 +
        -Tt2*2*(R**2 - r**2) +
        T1*(R**2 - r**2)**2 +
        (Txx + Tyy)*(-4)*R**2
    )
```

The `Toroid` is evaluated at $w = ct = 0$ to obtain the same torus as in $\mathcal{G}_{8,2}$ DCGA:

```
EV(c*0*e1+x*e2+y*e3+z*e4)|GIPNS_Toroid(R,r)
```

At other times $t \neq 0$, the minor radius r of the toroid grows with time t, which could be researched further. For instance, the toroid with major radius R and minor radius $r = 0$, which is a circle of radius R, will grow (at the natural speed of light $c = 1$) into the toroid with major radius R and minor radius $r = R$ at time $w = ct = R$, which is a horn torus. In spacetime, the toroid entity is a toroidal wavefront starting from a circle, and if also

$R=0$ then it is exactly a spherical wavefront or lightcone as the GIPNS hypercone $-4\mathbf{e}_o$ (GOPNS origin point).

8.1.17 DCSTA GIPNS 2-vector Dupin cyclide

The DCSTA GIPNS 2-vector *Dupin cyclide* $\mathbf{\Phi}$ is coded as:

```
def GIPNS_DupinCyclide(R,r1,r2):
    # DupinCyclide generalizes the torus.
    # Types of cyclide:
    #  Ring cyclide when (r1+r2)<2R
    #  Spindle cyclide when (r1+r2)>2R
    # Types of torus:
    #  Horn torus when (r1=r2)=R
    #  Ring torus when (r1=r2)<R
    #  Spindle torus when (r1=r2)>R
    #
    # R is major radius in the xy-plane.
    # r1 and r2 are minor radii.
    # r1 is the radius of sphere centered at x=+R.
    # r2 is the radius of sphere centered at x=-R.
    # When r1=r2, we get exactly a Toroid(R,r=r1=r2).
    # When r1+r2=2R, we get the union of two spheres
    # that touch in a tangent point, exactly.
    #
    # Note: -Tt2 since signatures are negative.
    a = R
    u = (r1+r2)*Pow(2,-1)
    c = (r1-r2)*Pow(2,-1)
    b = sqrt(a**2-c**2)
    return (
        Tt4 +
        -2*(b**2-u**2)*Tt2 +
        (b**2-u**2)**2*T1 +
        -4*(a**2*Txx - 2*a*c*u*Tx + c**2*u**2*T1) +
        -4*b**2*Tyy
    )
```

The `DupinCyclide` is evaluated at $t=0$ to obtain the same Dupin cyclide as in $\mathcal{G}_{8,2}$ DCGA:

```
EV(c*0*e1+x*e2+y*e3+z*e4)|DupinCyclide(R,r1,r2)
```

Similar to the `Toroid`, the minor radii of the `DupinCyclide` grow with time in spacetime.

8.1.18 DCSTA GIPNS 2-vector horned Dupin cyclide

The DCSTA GIPNS 2-vector *horned Dupin cyclide* $\mathbf{\Gamma}$ is coded as:

```
def GIPNS_HornedDupinCyclide(R,r1,r2):
    # Compared to DupinCyclide, just exchange values of
    # u and c to get horned Dupin cyclide.
    # For r1=r2: symmetrical, with horn points on y-axis.
    # For (r1+r2)<2R: horned ring cyclide.
    # For (r1+r2)>2R: horned spindle cyclide.
    # For (r1+r2)=2R: union of two spheres exactly.
    a = R
    u = (r1+r2)*Pow(2,-1)
    c = (r1-r2)*Pow(2,-1)
    b = sqrt(a**2-u**2)
    return (
        Tt4 +
        -2*(b**2-c**2)*Tt2 +
        (b**2-c**2)**2*T1 +
        -4*(a**2*Txx - 2*a*c*u*Tx + c**2*u**2*T1) +
        -4*b**2*Tyy
    )
```

Similar to the `Toroid`, the minor radii of the `HornedDupinCyclide` grow with time in spacetime. When $r_1 = r_2$ and some time $w = ct = p_w$ is chosen, it is possible to produce an ovoidal ring cyclide called a *Blum cyclide* of the form

$$A(x^2+y^2+z^2)^2+Fx^2+Gy^2+Hz^2+O \;=\; 0$$

with scalar coefficients A, F, G, H, O. When $r_1 \neq r_2$, an asymmetrical form of the ovoidal ring cyclide can be produced. As time w increases from $w=0$ to $w=p_w$, the radius at the horn points increases to p_w when $c=1$. The rate of radius increase at the horn points is at light speed c.

8.1.19 DCSTA GIPNS 2-vector ellipsoid

The DCSTA GIPNS 2-vector *ellipsoid* is coded as:

```
def GIPNS_Ellipsoid(px,py,pz,rx,ry,rz):
    # Axis-aligned ellipsoid.
    return (
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -Tx*2*px*Pow(rx**2,-1) +
        -Ty*2*py*Pow(ry**2,-1) +
        -Tz*2*pz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

An ellipsoid, or any other *quadric surface* entity (§7.5), that has an initial position (p_x, p_y, p_z) can be boosted using a translated-boost (§7.7.4). After the boost operation(s) on a quadric surface entity, the boosted entity can be evaluated at any time t, where the entity has a moving position and displays a length contraction effect.

8.1.20 DCSTA GIPNS 2-vector elliptic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic cylinder.
    return (
        T1*(py**2*Pow(ry**2,-1)+pz**2*Pow(rz**2,-1)-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

8.1.21 DCSTA GIPNS 2-vector elliptic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic cylinder.
    return (
        T1*(px**2*Pow(rx**2,-1)+pz**2*Pow(rz**2,-1)-1) +
        Txx*Pow(rx**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

8.1.22 DCSTA GIPNS 2-vector elliptic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic cylinder* is coded as:

```
def GIPNS_ECylinderZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic cylinder.
    return (
        T1*(px**2*Pow(rx**2,-1)+py**2*Pow(ry**2,-1)-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1)
    )
```

8.1.23 DCSTA GIPNS 2-vector elliptic cone, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic cone.
    return (
        -T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

8.1.24 DCSTA GIPNS 2-vector elliptic cone, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic cone.
    return (
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1)
    )
```

8.1.25 DCSTA GIPNS 2-vector elliptic cone, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic cone* is coded as:

```
def GIPNS_ConeZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic cone.
    return (
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1)
    )
```

8.1.26 DCSTA GIPNS 2-vector elliptic paraboloid, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidX(px,py,pz,rx,ry,rz):
    # x-axis aligned elliptic paraboloid.
    return (
        -2*pz*Tz*Pow(rz**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -Tx*Pow(rx,-1) +
        Tzz*Pow(rz**2,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*px*Pow(rx,-1)
    )
```

8.1.27 DCSTA GIPNS 2-vector elliptic paraboloid, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidY(px,py,pz,rx,ry,rz):
    # y-axis aligned elliptic paraboloid.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*pz*Tz*Pow(rz**2,-1) +
        -Ty*Pow(ry,-1) +
        Txx*Pow(rx**2,-1) +
        Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        T1*py*Pow(ry,-1)
    )
```

8.1.28 DCSTA GIPNS 2-vector elliptic paraboloid, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned elliptic paraboloid* is coded as:

```
def GIPNS_ParaboloidZ(px,py,pz,rx,ry,rz):
    # z-axis aligned elliptic paraboloid.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

8.1.29 DCSTA GIPNS 2-vector hyperbolic paraboloid

The DCSTA GIPNS 2-vector *z-axis aligned hyperbolic paraboloid* is coded as:

```
def GIPNS_HParaboloidZ(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperbolic paraboloid.
    # A saddle-like shape
    #  that "saddles" x-axis
    #  and "straddles" y-axis
    #  with "up" direction as z-axis.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

8.1.30 DCSTA GIPNS 2-vector hyperboloid of one sheet

The DCSTA GIPNS 2-vector *hyperboloid of one sheet* is coded as:

```
def GIPNS_Hyperboloid1(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperboloid of one sheet.
    # An hourglass-like shape that
    #  is elliptic in the xy-plane
    #  and hyperbolic in xz and yz planes.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

8.1.31 DCSTA GIPNS 2-vector hyperboloid of two sheets

The DCSTA GIPNS 2-vector *hyperboloid of two sheets* is coded as:

```
def GIPNS_Hyperboloid2(px,py,pz,rx,ry,rz):
    # z-axis aligned hyperboloid of two sheets.
    # A shape like two dishes that
    #  are elliptic in the xy-plane
    #  and hyperbolic in xz and yz planes.
    return (
        Tx*2*px*Pow(rx**2,-1) +
        Ty*2*py*Pow(ry**2,-1) +
        -Tz*2*pz*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        Tzz*Pow(rz**2,-1) +
        -T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

8.1.32 DCSTA GIPNS 2-vector parabolic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderX(px,py,pz,rx,ry,rz):
    # Cylinder along x-axis with
    # constant parabola cross-section in yz-plane.
    return (
        -2*py*Ty*Pow(ry**2,-1) +
        -Tz*Pow(rz,-1) +
        Tyy*Pow(ry**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

8.1.33 DCSTA GIPNS 2-vector parabolic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderY(px,py,pz,rx,ry,rz):
    # Cylinder along y-axis with
    # constant parabola cross-section in xz-plane.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -Tz*Pow(rz,-1) +
        Txx*Pow(rx**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*pz*Pow(rz,-1)
    )
```

8.1.34 DCSTA GIPNS 2-vector parabolic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned parabolic cylinder* is coded as:

```
def GIPNS_PCylinderZ(px,py,pz,rx,ry,rz):
    # Cylinder along z-axis with
    # constant parabola cross-section in xy-plane.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        -Ty*Pow(ry,-1) +
        Txx*Pow(rx**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        T1*py*Pow(ry,-1)
    )
```

8.1.35 DCSTA GIPNS 2-vector hyperbolic cylinder, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderX(px,py,pz,rx,ry,rz):
    # Cylinder along x-axis with
    # constant hyperbola cross-section in yz-plane
    # opening up and down the y-axis.
    return (
        -Ty*2*py*Pow(ry**2,-1) +
         Tz*2*pz*Pow(rz**2,-1) +
         Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
         T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1) +
        -T1
    )
```

8.1.36 DCSTA GIPNS 2-vector hyperbolic cylinder, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderY(px,py,pz,rx,ry,rz):
    # Cylinder along y-axis with
    # constant hyperbola cross-section in xz-plane
    # opening up and down the z-axis.
    return (
        -Tz*2*pz*Pow(rz**2,-1) +
         Tx*2*px*Pow(rx**2,-1) +
         Tzz*Pow(rz**2,-1) +
        -Txx*Pow(rx**2,-1) +
         T1*pz**2*Pow(rz**2,-1) +
        -T1*px**2*Pow(rx**2,-1) +
        -T1
    )
```

8.1.37 DCSTA GIPNS 2-vector hyperbolic cylinder, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned hyperbolic cylinder* is coded as:

```
def GIPNS_HCylinderZ(px,py,pz,rx,ry,rz):
    # Cylinder along z-axis with
    # constant hyperbola cross-section in xy-plane
    # opening up and down the x-axis.
    return (
        -Tx*2*px*Pow(rx**2,-1) +
         Ty*2*py*Pow(ry**2,-1) +
         Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
         T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1) +
        -T1
    )
```

8.1.38 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to x-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp x$-axis is coded as:

```
def GIPNS_PPlanesPairX(px1,px2):
    # Parallel planes pair, x=px1 and x=px2.
    return ( Txx - (px1+px2)*Tx + px1*px2*T1 )
```

8.1.39 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to y-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp y$-axis is coded as:

```
def GIPNS_PPlanesPairY(py1,py2):
    # Parallel planes pair, y=py1 and y=py2.
    return ( Tyy - (py1+py2)*Ty + py1*py2*T1 )
```

8.1.40 DCSTA GIPNS 2-vector parallel planes pair, perpendicular to z-axis

The DCSTA GIPNS 2-vector *parallel planes pair* $\perp z$-axis is coded as:

```
def GIPNS_PPlanesPairZ(pz1,pz2):
    # Parallel planes pair, z=pz1 and z=pz2.
    return ( Tzz - (pz1+pz2)*Tz + pz1*pz2*T1 )
```

8.1.41 DCSTA GIPNS 2-vector non-parallel planes pair, x-axis aligned

The DCSTA GIPNS 2-vector *x-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairX(py,pz,ry,rz):
    # The non-parallel planes pair aligned with x-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the yz-plane. The lines
    # intersect at (py,pz), and the slopes of the two
    # lines are +rz/ry and -rz/ry.
    return (
        -2*py*Ty*Pow(ry**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Tyy*Pow(ry**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*py**2*Pow(ry**2,-1) +
        -T1*pz**2*Pow(rz**2,-1)
    )
```

8.1.42 DCSTA GIPNS 2-vector non-parallel planes pair, y-axis aligned

The DCSTA GIPNS 2-vector *y-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairY(px,pz,rx,rz):
    # The non-parallel planes pair aligned with y-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the xz-plane. The lines
    # intersect at (px,pz), and the slopes of the two
    # lines are +rz/rx and -rz/rx.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*pz*Tz*Pow(rz**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tzz*Pow(rz**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*pz**2*Pow(rz**2,-1)
    )
```

8.1.43 DCSTA GIPNS 2-vector non-parallel planes pair, z-axis aligned

The DCSTA GIPNS 2-vector *z-axis aligned non-parallel planes pair* is coded as:

```
def GIPNS_XPlanesPairZ(px,py,rx,ry):
    # The non-parallel planes pair aligned with z-axis
    # is a type of cylinder with constant cross section
    # that is a pair of lines in the xy-plane. The lines
    # intersect at (px,py), and the slopes of the two
    # lines are +ry/rx and -ry/rx.
    return (
        -2*px*Tx*Pow(rx**2,-1) +
        2*py*Ty*Pow(ry**2,-1) +
        Txx*Pow(rx**2,-1) +
        -Tyy*Pow(ry**2,-1) +
        T1*px**2*Pow(rx**2,-1) +
        -T1*py**2*Pow(ry**2,-1)
    )
```

8.1.44 CSTA1 GIPNS 1-vector hyperplane

The CSTA1 GIPNS 1-vector *hyperplane* (§6.4.3) is coded as:

```
def GIPNS_HPlane1(p,n):
    # CSTA1 1-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    return ( n + (p|n)*ei1 )
```

8.1.45 CSTA2 GIPNS 1-vector hyperplane

The CSTA2 GIPNS 1-vector *hyperplane* (§6.4.3) is coded as:

```
def GIPNS_HPlane2(p,n):
    # CSTA2 1-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    p2 = (p1|e1)*e7 + (-p1|e2)*e8 + (-p1|e3)*e9 + (-p1|e4)*e10
    n2 = (n1|e1)*e7 + (-n1|e2)*e8 + (-n1|e3)*e9 + (-n1|e4)*e10
    return ( n2 + (p2|n2)*ei2 )
```

8.1.46 DCSTA GIPNS 2-vector hyperplane

The DCSTA GIPNS 2-vector *hyperplane* (§7.3.2) is coded as:

```
def GIPNS_HPlane(p,n):
    # DCSTA 2-vector hyperplane
    # p is any point on the hyperplane
    # n is the normal vector
    # the magnitude of n is not significant
    return ( GIPNS_HPlane1(p,n)^GIPNS_HPlane2(p,n) )
```

8.1.47 CSTA1 GIPNS 1-vector hyperhyperboloid of one sheet

The CSTA1 GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§6.4.5) is coded as:

```
def GIPNS_HPSphere1(p,r):
    # CSTA1 1-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( EV1(p) + Pow(2,-1)*r**2*ei1 )
```

8.1.48 CSTA2 GIPNS 1-vector hyperhyperboloid of one sheet

The CSTA2 GIPNS 1-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§6.4.5) is coded as:

```
def GIPNS_HPSphere2(p,r):
    # CSTA1 1-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( EV2(p) + Pow(2,-1)*r**2*ei2 )
```

8.1.49 DCSTA GIPNS 2-vector hyperhyperboloid of one sheet

The DCSTA GIPNS 2-vector *hyperhyperboloid of one sheet (hyperpseudosphere)* (§7.3.2) is coded as:

```
def GIPNS_HPSphere(p,r):
    # DCSTA 2-vector hyperpseudosphere (hyperhyperboloid)
    # p is the STA center point
    # r is the radius
    # imaginary radius r=I*|r| makes imaginary hyperpseudosphere
    return ( GIPNS_HPSphere1(p,r)^GIPNS_HPSphere2(p,r) )
```

8.1.50 CSTA1 GIPNS 2-vector plane

The CSTA1 GIPNS 2-vector *plane* (§6.4.10) is coded as:

```
def GIPNS_Plane1(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    # The STA1 plane bivector B is da^db
    p1 = p
    B1 = da^db
    N1 = Pow(sqrt(scalar(B1|(e1*B1.rev()*e1))),-1)*B1
    D1 = ((1*N1*1)|I41.inv())
    return ( D1 - ((p1|D1)^ei1) )
```

8.1.51 CSTA2 GIPNS 2-vector plane

The CSTA2 GIPNS 2-vector *plane* (§6.4.10) is coded as:

```
def GIPNS_Plane2(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    p2 = (p|e1)*e7+(-p|e2)*e8+(-p|e3)*e9+(-p|e4)*e10
    da2 = (da|e1)*e7+(-da|e2)*e8+(-da|e3)*e9+(-da|e4)*e10
    db2 = (db|e1)*e7+(-db|e2)*e8+(-db|e3)*e9+(-db|e4)*e10
    B2 = da2^db2
    N2 = Pow(sqrt(scalar(B2|(e7*B2.rev()*e7))),-1)*B2
    D2 = (N2|I42.inv())
    return ( D2 - ((p2|D2)^ei2) )
```

8.1.52 DCSTA GIPNS 4-vector standard plane

The DCSTA GIPNS 4-vector *standard plane* (§7.3.6) is coded as:

```
def GIPNS_Plane(p,da,db):
    # p is any STA1 point on the plane
    # da is STA1 direction one of plane
    # db is STA1 direction two of plane
    return ( GIPNS_Plane1(p,da,db)^GIPNS_Plane2(p,da,db) )
```

The standard plane can be intersected with all other DCSTA GIPNS surface entities.

8.1.53 CSTA1 GIPNS 3-vector line

The CSTA1 GIPNS 3-vector *line* (§6.4.11) is coded as:

```
def GIPNS_Line1(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    dc = e1*d*e1
    d1 = Pow(sqrt(scalar(d|dc)),-1)*d
    D1 = d1|(-I41)
    return ( D1 + ((p|D1)^ei1) )
```

8.1.54 CSTA2 GIPNS 3-vector line

The CSTA2 GIPNS 3-vector *line* (§6.4.11) is coded as:

```
def GIPNS_Line2(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    p2 = (p|e1)*e7+(-p|e2)*e8+(-p|e3)*e9+(-p|e4)*e10
    dc = e1*d*e1
    d1 = Pow(sqrt(scalar(d|dc)),-1)*d
    d2 = (d1|e1)*e7+(-d1|e2)*e8+(-d1|e3)*e9+(-d1|e4)*e10
    D2 = d2|(-I42)
    return ( D2 + ((p2|D2)^ei2) )
```

8.1.55 DCSTA GIPNS 6-vector standard line

The DCSTA GIPNS 6-vector *standard line* (§7.3.7) is coded as:

```
def GIPNS_Line(p,d):
    # p is any STA1 point on the line
    # d is the STA1 direction of line
    return ( GIPNS_Line1(p,d)^GIPNS_Line2(p,d) )
```

The standard line can be intersected with all other DCSTA GIPNS surface entities.

8.1.56 CSTA1 plane-line intersection

The CSTA1 plane-line intersection (§6.5.5) is coded as:

```
def GIPNS_PlaneLineIntersection1(p,l):
    # Intersect GIPNS_Plane1 p and GIPNS_Line1 l
    plwedge = (p^l)
    if plwedge != 0: return ei1
    plmeet = (((p|I41.inv())^(l|I41.inv()))|I41)
    if plmeet == 0: return l
    return ((e1*plmeet*e1)|p)^l
```

8.1.57 CSTA2 plane-line intersection

The CSTA2 plane-line intersection (§6.5.5) is coded as:

```
def GIPNS_PlaneLineIntersection2(p,l):
    # Intersect GIPNS_Plane2 p and GIPNS_Line2 l
    plwedge = (p^l)
    if plwedge != 0: return ei2
    plmeet = (((p|I42.inv())^(l|I42.inv()))|I42)
    if plmeet == 0: return l
    return ((e7*plmeet*e7)|p)^l
```

8.1.58 CSTA1 GOPNS 2-vector point pair decomposition

The decomposition of a CSTA1 GOPNS 2-vector *point pair* (§6.5.3) is coded as:

```
def GOPNS_PointPairDecomp1(pp,pm):
    # pp is a CSTA1 GOPNS 2-vector point pair
    # pm is -1 or 1 to select a point of the pair
    # returns a CSTA1 null 1-vector point entity
    return ( (pp + pm*sqrt(scalar(pp|pp)))*(-ei1|pp).inv() )
```

8.1.59 CSTA2 GOPNS 2-vector point pair decomposition

The decomposition of a CSTA2 GOPNS 2-vector *point pair* (§6.5.3) is coded as:

```
def GOPNS_PointPairDecomp2(pp,pm):
    # pp is a CSTA2 GOPNS 2-vector point pair
    # pm is -1 or 1 to select a point of the pair
    # returns a CSTA2 null 1-vector point entity
    return ( (pp + pm*sqrt(scalar(pp|pp)))*(-ei2|pp).inv() )
```

8.1.60 CSTA1 GOPNS 2-vector flat point projection

The projection of the point of a CSTA1 GOPNS 2-vector *flat point* (§6.5.5) is coded as:

```
def GOPNS_FlatPointProj1(fp):
    # fp is a CSTA1 GOPNS 2-vector flat point
    # returns the STA1 vector projection of the point
    E = eo1^ei1
    return ( -(fp|eo1)*Pow(scalar(E|fp),-1) - eo1 )
```

8.1.61 CSTA2 GOPNS 2-vector flat point projection

The projection of the point of a CSTA2 GOPNS 2-vector *flat point* (§6.5.5) is coded as:

```
def GOPNS_FlatPointProj2(fp):
    # fp is a CSTA2 GOPNS 2-vector flat point
    # returns the STA2 vector projection of the point
    E = eo2^ei2
    return ( -(fp|eo2)*Pow(scalar(E|fp),-1) - eo2 )
```

8.1.62 SA1, STA1, and CSTA1 2-versor rotor

The CSTA1 2-versor spatial *rotor* (§2.6) is coded as:

```
def Rotor1(axis,angle):
    # Spatial rotor in SA1, STA1, and CSTA1, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees.
    ax1 = Pow(norm(axis),-1)*axis
    ang = pi*Pow(180,-1)*angle
    return (
        cos(ang*Pow(2,-1)) +
        sin(ang*Pow(2,-1))*(ax1|(-I31))
    )
```

8.1.63 SA2, STA2, and CSTA2 2-versor rotor

The CSTA2 2-versor spatial *rotor* (§2.6) is coded as:

```
def Rotor2(axis,angle):
    # Spatial rotor in SA2, STA2, and CSTA2, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees.
    ax1 = Pow(norm(axis),-1)*axis
    ax2 = (-ax1|e2)*e8 + (-ax1|e3)*e9 + (-ax1|e4)*e10
    ang = pi*Pow(180,-1)*angle
    return (
        cos(ang*Pow(2,-1)) +
        sin(ang*Pow(2,-1))*(ax2|(-I32))
    )
```

8.1.64 DCSTA 4-versor rotor

The DCSTA 4-versor spatial *rotor* (§7.7.6) is coded as:

```
def Rotor(axis,angle):
    # Spatial rotor in DCSTA, where
    #  axis is SA1 vector axis of rotation and
    #  angle is scalar angle of rotation in degrees
    return ( Rotor1(axis,angle)^Rotor2(axis,angle) )
```

8.1.65 CSTA1 2-versor line rotor

The CSTA1 2-versor *line rotor* (§6.6.5) for the rotation around a line is coded as:

```
def LRotor1(p,d,a):
    # Rotor around a line l by angle a in degrees
    # line l is given by STA1 point p and direction d
    l = GIPNS_Line1(p,d)
    t = Rational(1,2)*pi*Pow(180,-1)*a
    return ( cos(t) + sin(t)*(-e1|l) )
```

8.1.66 CSTA2 2-versor line rotor

The CSTA2 2-versor *line rotor* (§6.6.5) for the rotation around a line is coded as:

```
def LRotor2(p,d,a):
    l = GIPNS_Line2(p,d)
    t = Rational(1,2)*pi*Pow(180,-1)*a
    return ( cos(t) + sin(t)*(-e7|l) )
```

8.1.67 DCSTA 4-versor line rotor

The DCSTA 4-versor *line rotor* (§7.7.6) for the rotation around a line is coded as:

```
def LRotor(p,d,a):
    return LRotor1(p,d,a)^LRotor2(p,d,a)
```

8.1.68 STA1 and CSTA1 2-versor hyperbolic rotor (boost operator)

The CSTA1 2-versor spacetime *hyperbolic rotor (boost operator)* (§5.2.3) is coded as:

```
def HRotor1(b,d):
    # STA1 and CSTA1 boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    v1 = Pow(sqrt(scalar(-d|d)),-1)*d
    r = atanh(b)
    return ( cosh(Pow(2,-1)*r) + sinh(Pow(2,-1)*r)*(v1^e1) )
```

The `b` is the natural speed $\beta_{\mathbf{v}}$. The `d` is the SA1 spatial velocity direction $\hat{\mathbf{v}}$ that is normalized as `v1`. The spatial velocity of the boost is $\mathbf{v} = \beta_{\mathbf{v}} c\hat{\mathbf{v}} = \|\mathbf{v}\|\hat{\mathbf{v}}$ relative to an observer $\boldsymbol{o}t = ct\mathbf{e}_1$. The `r` is the rapidity $\varphi_{\mathbf{v}} = \operatorname{atanh}(\beta_{\mathbf{v}})$.

8.1.69 STA2 and CSTA2 2-versor hyperbolic rotor (boost operator)

The CSTA2 2-versor spacetime *hyperbolic rotor (boost operator)* (§6.6.8) is coded as:

```
def HRotor2(b,d):
    # STA2 and CSTA2 boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    v1 = Pow(sqrt(scalar(-d|d)),-1)*d
    v2 = (-v1|e2)*e8 + (-v1|e3)*e9 + (-v1|e4)*e10
    r = atanh(b)
    return ( cosh(Pow(2,-1)*r) + sinh(Pow(2,-1)*r)*(v2^e7) )
```

8.1.70 DCSTA 4-versor hyperbolic rotor (boost operator)

The DCSTA 4-versor spacetime *hyperbolic rotor (boost operator)* (§7.7.3) is coded as:

```
def HRotor(b,d):
    # DCSTA boost operator, where
    #  0<=b<=1 is scalar natural speed of boost and
    #  d is SA1 direction vector of boost velocity
    return ( HRotor1(b,d)^HRotor2(b,d) )
```

For an *anisotropic dilation* (§7.7.9) of a *quadric surface* `Q` by factor d in direction `d`, then speed `b` should be set to $\beta_{\mathbf{v}} = \sqrt{1-d^2}$, which may be an imaginary number.

The anisotropic dilation of `Q` by a dilation factor `d` in an SA1 direction `v=vx*e2+vy*e3+vz*e4` is coded as:

```
((
    HRotor(sqrt(1-d**2),v)*
    Q*
    HRotor(sqrt(1-d**2),v).rev()
)|IDS)*IDS.inv()
```

The projection using `IDS` is the space projection (§7.7.2) into $\mathcal{G}_{2,8}$ DCSA, which discards imaginary components that are by-products of the directed scaling operation. A good example to try is `Q=Ellipsoid(px,py,pz,rx,ry,rz)`.

8.1.71 CSTA1 2-versor translator

The CSTA1 2-versor spacetime *translator* (§6.6.4) is coded as:

```
def Translator1(d):
    # CSTA1 spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    d1 = d
    return ( 1 - Pow(2,-1)*(d1^ei1) )
```

8.1.72 CSTA2 2-versor translator

The CSTA2 2-versor spacetime *translator* (§6.6.4) is coded as:

```
def Translator2(d):
    # CSTA2 spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    d2 = (d|e1)*e7 + (-d|e2)*e8 + (-d|e3)*e9 + (-d|e4)*e10
    return ( 1 - Pow(2,-1)*(d2^ei2) )
```

8.1.73 DCSTA 4-versor translator

The DCSTA 4-versor spacetime *translator* (§7.7.7) is coded as:

```
def Translator(d):
    # DCSTA spacetime translator, where
    #  d is an STA1 spacetime displacement vector.
    return ( Translator1(d)^Translator2(d) )
```

8.1.74 CSTA1 2-versor isotropic dilator

The CSTA1 2-versor spacetime *isotropic dilator* (§6.6.6) is coded as:

```
def Dilator1(d):
    # CSTA1 isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Pow(2,-1)*(1+d) + Pow(2,-1)*(1-d)*(ei1^eo1) )
```

8.1.75 CSTA2 2-versor isotropic dilator

The CSTA2 2-versor spacetime *isotropic dilator* (§6.6.6) is coded as:

```
def Dilator2(d):
    # CSTA2 isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Pow(2,-1)*(1+d) + Pow(2,-1)*(1-d)*(ei2^eo2) )
```

8.1.76 DCSTA 4-versor isotropic dilator

The DCSTA 4-versor spacetime *isotropic dilator* (§7.7.8) is coded as:

```
def Dilator(d):
    # DCSTA isotropic dilator, where
    #  d is the scalar dilation factor.
    # Note: dilation factor d=0 is not generally valid.
    return ( Dilator1(d)^Dilator2(d) )
```

The *anisotropic dilator* (§7.7.9) on quadric surface entities is implemented using the *hyperbolic rotor* (§8.1.70).

8.2 Example computations

8.2.1 Reframe to new observer in STA

The observer position is $\boldsymbol{o}t = ct\mathbf{e}_1$, and an observable $\boldsymbol{v}t = (\boldsymbol{o} + \mathbf{v})t$ moves relative to $\boldsymbol{o}$, where

$$\mathbf{v} \;=\; \beta_{\mathbf{v}} c\mathbf{e}_2 = \frac{1}{2} c\mathbf{e}_2. \qquad (8.7)$$

We want to passively transform observable $\boldsymbol{o}$, which is the conventional coordinate time t observer, relative to the rest frame of the observable $\boldsymbol{v}$ with proper time τ. However, we prefer not to passively transform t into τ, and prefer to get a velocity subtraction in the frame of $\boldsymbol{o}$. Solution: use a passive boost operation (§5.2.3) on $\boldsymbol{o}$, followed by a spacetime contraction.

```
o_rel_v = (
    HRotor1( Rational(1,2), e2 ).rev()*
    ( c*t*e1 )*
    HRotor1( Rational(1,2), e2 )
)
normalized = c*t*Pow( scalar(o_rel_v|e1), -1 )*o_rel_v
normalized
```

$$\text{normalized} \;=\; ct\mathbf{e}_1 - \frac{ct}{2}\mathbf{e}_2. \qquad (8.8)$$

Observable $\boldsymbol{o}$ is seen to be moving with velocity $\mathbf{v}=-\frac{1}{2}c\mathbf{e}_2$ relative to observable $\boldsymbol{v}$. The spacetime contraction is a normalization of the conventional observer component $\boldsymbol{o}$ into the normalized spacetime velocity value $c\boldsymbol{\gamma}_0$ or spacetime position value $ct\boldsymbol{\gamma}_0$.

8.2.2 Collinear velocity addition in STA

A particle moves with velocity $\mathbf{u}=\frac{3}{4}c\mathbf{e}_2$ relative to another particle moving with velocity $\mathbf{v}=\frac{1}{2}c\mathbf{e}_2$ relative to an observer $\boldsymbol{o}=c\mathbf{e}_1$. The two velocities $\mathbf{u}$ and $\mathbf{v}$ are collinear, and if we simply add the velocities, we may conclude that $\mathbf{u}$ relative to $\boldsymbol{o}$ is a velocity $\mathbf{v}+\mathbf{u}=\frac{5}{4}c\mathbf{e}_2$. However, this speed is greater than light speed c, which according to the physical theory of special relativity is an impossible speed. Velocities cannot be simply added, and we must use a reframe operation to reframe $\mathbf{u}$ relative to $\boldsymbol{o}$. Relative to $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$, the particle moving with velocity $\mathbf{u}$ is written $\boldsymbol{u}t=\boldsymbol{o}t+\mathbf{u}t=ct\mathbf{e}_1+\mathbf{u}t$, where *this* $\boldsymbol{o}$ *is* $\boldsymbol{v}$ as the observer and *this time* t is its proper time. We want this $\boldsymbol{u}$ reframed relative to observer $\boldsymbol{o}$ of $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$. The solution is to apply to $\boldsymbol{u}$ the operation for the reverse of the reframe relative to $\boldsymbol{v}$ that goes back to relative to its $\boldsymbol{o}$, and this reframe is also seen as *boosting* the particle $\boldsymbol{u}=\boldsymbol{o}+\mathbf{u}$ by the velocity $\mathbf{v}$ relative to $\boldsymbol{o}$.

```
u_rel_o = (
    HRotor1( Rational(1,2), e2 )*
    ( c*t*e1 + Rational(3,4)*c*t*e2 )*
    HRotor1( Rational(1,2), e2 ).rev()
)
normalized = c*t*Pow( scalar(u_rel_o|e1), -1 )*u_rel_o
normalized
```

$$\text{normalized} \;=\; ct\mathbf{e}_1+\frac{10}{11}ct\mathbf{e}_2. \tag{8.9}$$

The result is relativistic velocity addition, where the boost of velocities does not exceed the speed of light c.

8.2.3 Velocity addition in STA

The velocities $\mathbf{u}$ and $\mathbf{v}$ need not be collinear, and the same operation of the previous section (§8.2.2) for collinear velocities can be applied to reframe any velocity $\mathbf{u}$ relative to $\boldsymbol{v}=\boldsymbol{o}+\mathbf{v}$ into $\mathbf{u}$ relative to $\boldsymbol{o}$. The result is the so-called *velocity-addition formula*, which could also be called the *velocity boost formula*,

$$\begin{aligned}
\boldsymbol{u}_{\mathrm{rel}\,\boldsymbol{v}} \to \boldsymbol{u}_{\mathrm{rel}\,\boldsymbol{o}} &= \boldsymbol{o}+\mathbf{u}_{\mathrm{rel}\,\boldsymbol{o}} && (8.10)\\
&= \boldsymbol{o}+\mathbf{u}\oplus\mathbf{v} && (8.11)\\
& c\boldsymbol{\gamma}_0+\frac{\mathbf{u}^{\|\hat{\mathbf{v}}}+\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\,\mathbf{u}^{\perp\hat{\mathbf{v}}}+\mathbf{v}}{1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} && (8.12)
\end{aligned}$$

where $\boldsymbol{o}=c\boldsymbol{\gamma}_0$, and the $\mathcal{G}_{0,3}$ SA metric gives

$$\begin{aligned}
\hat{\mathbf{v}} &= \frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{\mathbf{v}}{\sqrt{-\mathbf{v}^2}} && (8.13)\\
\hat{\mathbf{v}}^2 &= -1 && (8.14)\\
\mathbf{u}\cdot\mathbf{v} &= -\|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta_{\mathbf{uv}}). && (8.15)
\end{aligned}$$

The notation $\mathbf{u}\oplus\mathbf{v}$ can be read "$\mathbf{u}$ boosted by $\mathbf{v}$" since this is the actual operation, but this may be backwards compared to some other literature. In general, $\mathbf{u}\oplus\mathbf{v}\neq\mathbf{v}\oplus\mathbf{u}$. Some other identities are

$$\mathbf{u}=\mathbf{u}^{\|\hat{\mathbf{v}}}+\mathbf{u}^{\perp\hat{\mathbf{v}}} = (\mathbf{u}\cdot\hat{\mathbf{v}}+\mathbf{u}\wedge\hat{\mathbf{v}})\hat{\mathbf{v}}^{-1}=(-\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}+(\hat{\mathbf{v}}\wedge\mathbf{u})\cdot\hat{\mathbf{v}} \tag{8.16}$$

$$\beta_{\mathbf{v}} = \frac{\|\mathbf{v}\|}{c} \tag{8.17}$$

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\beta_{\mathbf{v}}^2}}. \tag{8.18}$$

When the boost velocity approaches light speed $\|\mathbf{v}\|\to c$, we get

$$\mathbf{u}\oplus\mathbf{v} = \frac{\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})\hat{\mathbf{v}}+c\hat{\mathbf{v}}}{1+\frac{\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})}{c}}=\frac{c(\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})+c)\hat{\mathbf{v}}}{c+\|\mathbf{u}\|\cos(\theta_{\mathbf{uv}})}=c\hat{\mathbf{v}}=\mathbf{v}. \tag{8.19}$$

For collinear $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\oplus\mathbf{v}=\alpha\mathbf{v}\oplus\mathbf{v}=\mathbf{v}\oplus\mathbf{u} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{\|\mathbf{u}\|\|\mathbf{v}\|}{c^2}} \tag{8.20}$$

where as the boost velocity approaches light speed $\|\mathbf{v}\|\to c$,

$$\mathbf{u}\oplus\mathbf{v} \to \frac{c(\|\mathbf{u}\|+c)\hat{\mathbf{u}}}{c+\|\mathbf{u}\|}=c\hat{\mathbf{u}}=\mathbf{v}. \tag{8.21}$$

For perpendicular $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u}\oplus\mathbf{v} = \sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}\mathbf{u}+\mathbf{v}=\frac{1}{\gamma_{\mathbf{v}}}\mathbf{u}+\mathbf{v} \tag{8.22}$$

where as the boost velocity approaches light speed $\|\mathbf{v}\|\to c$, $\mathbf{u}/\gamma_{\mathbf{v}}\to 0$ and $\mathbf{u}\oplus\mathbf{v}\to\mathbf{v}$. The velocity-addition formula is derived and discussed more in [9] and (§5.2.3).

8.2.4 Boost of an ellipsoid entity for contraction effect

Any DCSTA GIPNS 2-vector quadric surface entity can be boosted into a velocity in spacetime. Boosting sets the quadric surface into motion at constant velocity and gives the surface a length contraction effect. As an example of the contraction effect, we can boost an ellipsoid to a natural speed $\beta_{\mathbf{v}}=\sqrt{1-d^2}$ for the dilation factor d. A good example is to choose $d=1/2$ to squeeze the ellipsoid into one-half its length in the direction of the velocity.

```
moving_ellipsoid = (
    HRotor( sqrt(1-Rational(1,2)**2), e2 )*
    GIPNS_Ellipsoid(0,0,0,10,10,10)*
    HRotor( sqrt(1-Rational(1,2)**2), e2 ).rev()
)
print( N(V|moving_ellipsoid) )
```

The `moving_ellipsoid` is evaluated at a symbolic point `V`. The full symbolic output can be long, therefore numeric output has been generated using `N()`. The result is printed in plain text using `print()`. Output of this form can be graphed using *Mayavi*. For graphing, it works well to use natural units, where $c=1$, so that the graph can be near the origin. Mayavi seems to work best if graphing can be limited to a small cube around the origin that is about ± 20 units on each axis. If Mayavi is installed and working, a small `mayavi.py` python file can be created to graph this output (copied into `surface`) as:

```
from __future__ import division
from numpy import *
from mayavi import mlab

mlab.figure(bgcolor=(1,1,1))
x, y, z = mgrid[-20:20:100j, -20:20:100j, -20:20:100j]

# axes
cylx = y**2 + z**2 - 1/10
cyly = x**2 + z**2 - 1/10
cylz = y**2 + x**2 - 1/10
mlab.contour3d(x,y,z,cylx,contours=[0],opacity=0.25,color=(1,0,0))
mlab.contour3d(x,y,z,cyly,contours=[0],opacity=0.25,color=(0,1,0))
mlab.contour3d(x,y,z,cylz,contours=[0],opacity=0.25,color=(0,0,1))

# function for rendering a dot somewhere
def dotat(px,py,pz):
    blackdot = (x-px)**2 + (y-py)**2 + (z-pz)**2 - 1/sqrt(5)
    mlab.contour3d(
        x, y, z, blackdot, contours=[0],
        opacity=0.5, color=(0,0,0)
    )
    return

# plot some dots
dotat(5,0,0)
dotat(0,10,0)
dotat(0,0,10)

# Set the light speed (units per second)
#  Use a small unit or else boosted moving surfaces move
#  out of graphing range after only a few time units.
c = 1
# Set the time.
#  Boosted surfaces move natural-speed units per time unit.
#  At t=20, a surface at speed c=1 moves out of graphing range.
t = 0
# The numerical printed output, copied into here:
surface = (
    0.03*c**2*t**2 - 0.0692820323027551*c*t*x +
    0.04*x**2 + 0.01*y**2 + 0.01*z**2 - 0.999999999999999
)
# Mayavi rendering function
mlab.contour3d(
    x, y, z, surface, contours=[0], opacity=0.5,
    color=(0.0, 1.0, 1.0)
)
mlab.show()
```

The `mayavi.py` file is saved and then run from a command line as:

```
ipython mayavi.py
```

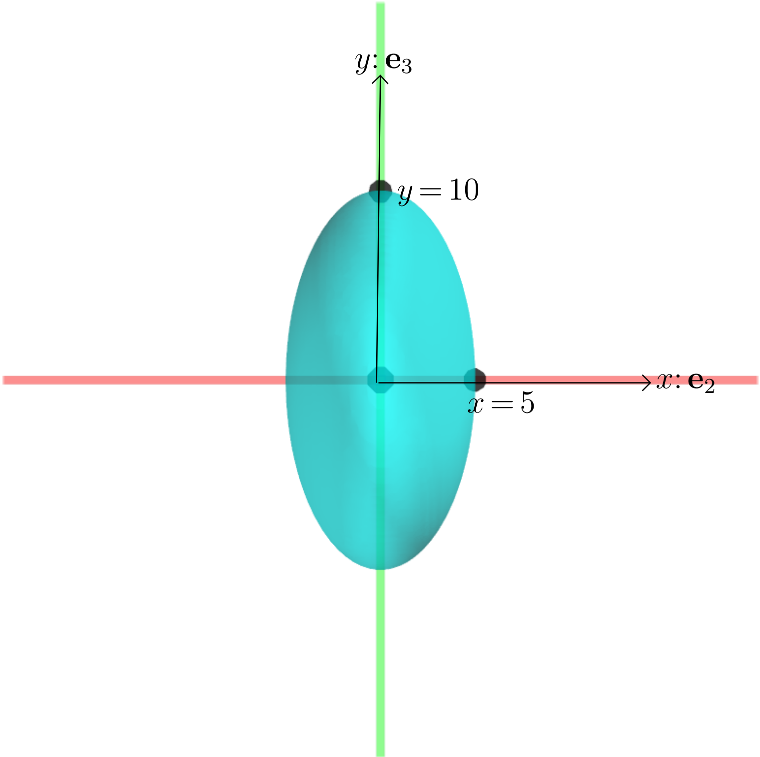

Figure 8.1. Ellipsoid (sphere $r = 10$) boosted to $\beta_{\mathbf{v}} = \sqrt{1 - \left(\frac{1}{2}\right)^2}$ in x-direction

Figure 8.1 shows a boosted ellipsoid at time $t = 0$. The ellipsoid was initially at the origin and spherical with radius $r = r_x = r_y = r_z = 10$. The spherical ellipsoid was boosted into a natural speed $\beta_{\mathbf{v}} = \sqrt{1 - \left(\frac{1}{2}\right)^2}$ for a dilation factor $d = \frac{1}{2}$ in the x-direction $\hat{\mathbf{v}} = \boldsymbol{\gamma}_1 = \mathbf{e}_2$. The boosted sphere is squeezed by the boost into an ellipsoid that is length-contracted to half-size in the x-direction with $r_x = 5$, while the y and z directions hold their sizes with $r_y = r_z = 10$. As the time t is increased, the boosted sphere moves toward the right along the x-axis. For natural units $c = 1$, the boosted sphere moves $\beta_{\mathbf{v}} = \sqrt{3}\,/\,2 \approx 0.866$ units per time unit.

8.2.5 Boost of an ellipsoid entity for dilation

As an example of dilating a quadric surface by dilation factor $d = 2$ using the boost operation, we can take a spherical ellipsoid with radius $r = 5$ centered at $5(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$ and boost it into an imaginary natural speed $\beta_{\mathbf{v}} = \sqrt{1 - d^2}$ in the unit direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)$. Following the boost for dilation, the result is projected onto the spatial subalgebra using the $\mathcal{G}_{2,8}$ anti-DCGA pseudoscalar $\mathbf{I}_{\mathcal{DS}}$. The projection discards imaginary time components and resets the entity at time $t = 0$. The dilation is coded as:

```
dilated_ellipsoid = ((
    HRotor( sqrt(1-2**2), e2+e3+e4 )*
    GIPNS_Ellipsoid(5,5,5,5,5,5)*
    HRotor( sqrt(1-2**2), e2+e3+e4 ).rev()
)|IDS)*IDS.inv()
print( N(V|dilated_ellipsoid) )
```

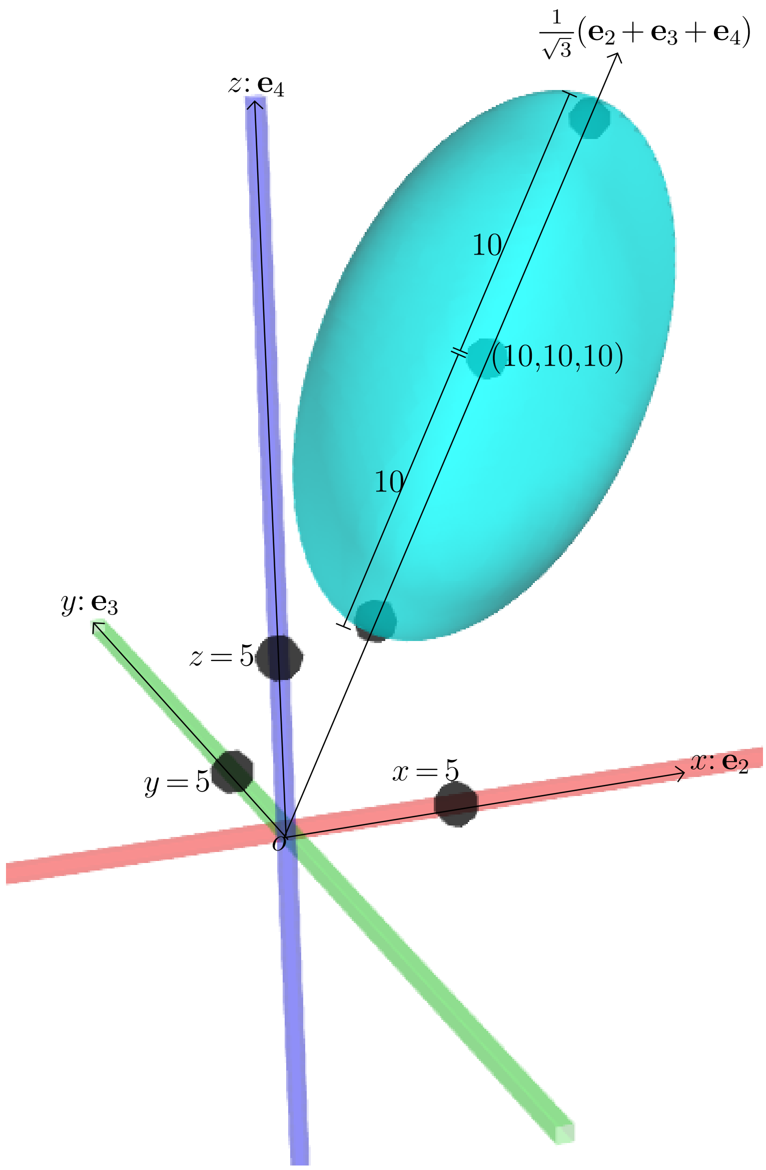

Figure 8.2. Ellipsoid (sphere $r=5$) dilated by factor $d=2$ in direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$

Figure 8.2 shows the graph of the dilated ellipsoid. In the unit direction $\frac{1}{\sqrt{3}}(\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4)$ of dilation by factor $d=2$, the spherical ellipsoid is dilated from a diameter of 10 into a diameter of 20. The spherical diameter remains 10 orthogonal to the direction of dilation. The center point $(5,5,5)$ of the original spherical ellipsoid is dilated into $(10,10,10)$ as the new center point of the dilated ellipsoid.

9 Conclusion

The $\mathcal{G}_{4,8}$ Double Conformal Space-Time Algebra (DCSTA) has been presented in this paper as a straightforward extension of the $\mathcal{G}_{8,2}$ Double Conformal / Darboux Cyclide Geometric Algebra (DCGA) [7][5][6][8]. DCSTA is a large, complicated algebra and this paper may contain some mistakes and has probably overlooked some things that should have been discussed. Nevertheless, this author feels that this paper substantially conveys the basic ideas and concepts of DCSTA. Certainly, much further research can be done into DCSTA and its applications.

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