Pi Formulas , Part 3

On Arctangent Relations for Pi

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abstract

In this note we give some formulas related to the constant Pi

π - FÓRMULAS
Sumas Finitas de la Función Arcotangente

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Resumen
Se muestra una colección de sumas finitas de la función arcotangente , para la constante $\pi = 3.1415926....$

1. INTRODUCCIÓN.

En esta nota mostramos una colección de fórmulas generales del tipo Machin para la constante π, y algunos ejemplos particulares.Previamente recordamos algunas fórmulas clásicas de este tipo.

John Machin (1706) :

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

Euler (1750):

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79}$$

Gauss (1810):

$$\frac{\pi}{4} = 12\tan^{-1}\frac{1}{18} + 8\tan^{-1}\frac{1}{57} - 5\tan^{-1}\frac{1}{239}$$

En todos los casos $\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$, $|x| < 1$.

2. COLECCIÓN DE FÓRMULAS GENERALES.

Para $n \in \mathbb{N}$, se tiene:

2.1. $$\frac{\pi}{4} = \tan^{-1}\frac{1}{n+1} + \sum_{k=1}^{n} \tan^{-1}\frac{1}{k^2+k+1}$$

2.2. $$\frac{\pi}{6} = \tan^{-1}\frac{1}{(n+1)\sqrt{3}} + \sum_{K=1}^{n} \tan^{-1}\frac{\sqrt{3}}{3k^2+3k+1}$$

2.3. $$\frac{\pi}{4} = \tan^{-1}\frac{1}{(n+1)^n} + \sum_{K=1}^{n}\tan^{-1}\frac{(k+1)^n - k^n}{k^n(k+1)^n + 1}$$

2.4. $$\frac{\pi}{4} = \tan^{-1}\frac{m}{m+n} + \sum_{k=1}^{n}\tan^{-1}\frac{m}{(m+k)(m+k-1)+m^2} \quad , \quad m>0$$

2.5. $$\frac{\pi}{4} = \frac{1}{n+1}\tan^{-1}\frac{1}{n+1} + \sum_{k=1}^{n}\frac{1}{k(k+1)}\tan^{-1}\frac{1}{k+1} + \sum_{k=1}^{n}\frac{1}{k}\tan^{-1}\frac{1}{k^2+k+1}$$

2.6. $$\frac{\pi}{4} = (n+1)^2\tan^{-1}\frac{1}{n+1} - \sum_{k=1}^{n}(2k+1)\tan^{-1}\frac{1}{k+1} + \sum_{k=1}^{n}k^2\tan^{-1}\frac{1}{k^2+k+1}$$

2.7. $$\frac{\pi}{4} = (n+1)^3\tan^{-1}\frac{1}{n+1} - \sum_{k=1}^{n}\left(3k^2+3k+1\right)\tan^{-1}\frac{1}{k+1} + \sum_{k=1}^{n}k^3\tan^{-1}\frac{1}{k^2+k+1}$$

2.8. $$\frac{\pi}{12} = \tan^{-1}\frac{1}{n+2+\sqrt{3}} + \sum_{k=1}^{n}\tan^{-1}\frac{1}{k^2+3k+6+(2k+3)\sqrt{3}}$$

2.9. $$\frac{\pi}{6} = \tan^{-1}\frac{1}{n+\sqrt{3}} + \sum_{k=1}^{n}\tan^{-1}\frac{1}{k^2-k+4+(2k-1)\sqrt{3}}$$

2.10. $$\frac{\pi}{12} = \tan^{-1}\frac{1}{\left(2+\sqrt{3}\right)^{n+1}} + \sum_{k=1}^{n}\tan^{-1}\frac{\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^k}{\left(2+\sqrt{3}\right)^{2k+1}+1}$$

2.11. $$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{3}\right)^{\frac{n+1}{2}} + \sum_{k=1}^{n}\tan^{-1}\frac{3^{k/2}\left(\sqrt{3}+1\right)}{3^k\sqrt{3}+1}$$

2.12. $$(a+b)\frac{\pi}{4} = (an+b)\tan^{-1}\frac{1}{(n+1)^m} - \sum_{k=1}^{n-1}a\cdot\tan^{-1}\frac{1}{(k+1)^m} +$$

$$+\sum_{k=1}^{n}(ak+b)\tan^{-1}\frac{(k+1)^m-k^m}{k^m(k+1)^m+1} \quad , \quad a>0, b>0, m>0$$

2.13. $$\frac{\pi}{4} = n\tan^{-1}\frac{1}{an+1} - \sum_{k=1}^{n-1}\tan^{-1}\frac{1}{ak+1} + \sum_{k=1}^{n}k\tan^{-1}\frac{a}{a^2k^2+\left(2a-a^2\right)k+2-a}$$

$$0<a<1$$

2.14. $$(a+b)\frac{\pi}{4} = (an+b)\tan^{-1}\frac{1}{n+1} - \sum_{k=1}^{n-1}a\cdot\tan^{-1}\frac{1}{k+1} + \sum_{k=1}^{n}(ak+b)\tan^{-1}\frac{1}{k^2+k+1}$$

$$a>0, b>0$$

2.15. $\frac{\pi}{4}=\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2^{n+1}}+\sum_{k=1}^{n}\tan^{-1}\frac{2^k}{2^{2k+1}+1}$

2.16. $\frac{\pi}{4}=\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{3^{n+1}}+\sum_{k=1}^{n}2\tan^{-1}\frac{2\cdot 3^k}{3^{2k+1}+1}$

2.17. $\frac{\pi}{4}=\tan^{-1}\frac{1}{2^{n+1}}+\tan^{-1}\frac{1}{5^{n+1}}+\tan^{-1}\frac{1}{8^{n+1}}+$

$$+\sum_{k=1}^{n}\left(\tan^{-1}\frac{2^k}{2^{2k+1}+1}+\tan^{-1}\frac{4\cdot 5^k}{5^{2k+1}+1}+\tan^{-1}\frac{7\cdot 8^k}{8^{2k+1}+1}\right)$$

2.18. $\frac{\pi}{4}=4\tan^{-1}\frac{1}{n+5}-\tan^{-1}\frac{1}{n+239}+\sum_{k=1}^{n}4\tan^{-1}\frac{1}{n^2+9n+21}+$

$$-\sum_{k=1}^{n}\tan^{-1}\frac{1}{n^2+477n+56883}$$

2.19. $\frac{\pi}{4}=6\tan^{-1}\frac{1}{n+8}+2\tan^{-1}\frac{1}{n+57}+\tan^{-1}\frac{1}{n+239}+\sum_{k=1}^{n}6\tan^{-1}\frac{1}{n^2+15n+57}+$

$$+\sum_{k=1}^{n}2\tan^{-1}\frac{1}{n^2+113n+3193}+\sum_{k=1}^{n}\tan^{-1}\frac{1}{n^2+477n+56883}$$

3. OTRA CLASE DE FÓRMULAS GENERALES DEL TIPO MACHIN.

3.1. $\frac{\pi}{4}=4\tan^{-1}x-\tan^{-1}y+\tan^{-1}u+4\tan^{-1}\left(\frac{1-5x}{5+x}\right)-\tan^{-1}\left(\frac{1-70y}{70+y}\right)+\tan^{-1}\left(\frac{1-99u}{99+u}\right)$

$$0\le x\le\frac{1}{5},0\le y\le\frac{1}{70},0\le u\le\frac{1}{99}$$

3.2. $\frac{\pi}{4}=5\tan^{-1}x+4\tan^{-1}y+2\tan^{-1}u+5\tan^{-1}\left(\frac{1-7x}{7+x}\right)+4\tan^{-1}\left(\frac{1-53y}{53+y}\right)+$

$$+2\tan^{-1}\left(\frac{1-4443u}{4443+u}\right)$$

$$0\le x\le\frac{1}{7},0\le y\le\frac{1}{53},0\le u\le\frac{1}{4443}$$

3.3. $\frac{\pi}{4}=44\tan^{-1}x+7\tan^{-1}y-12\tan^{-1}u+24\tan^{-1}v+44\tan^{-1}\left(\frac{1-57x}{57+x}\right)+$

$$+7\tan^{-1}\left(\frac{1-239y}{239+y}\right)-12\tan^{-1}\left(\frac{1-682u}{682+u}\right)+24\tan^{-1}\left(\frac{1-12943v}{12943+v}\right)$$

$$0\le x\le\frac{1}{57},0\le y\le\frac{1}{239},0\le u\le\frac{1}{682},0\le v\le\frac{1}{12943}$$

4. EJEMPLOS PARTICULARES.

4.1.En la formula (2.1) ponemos $n=2$:

$$\frac{\pi}{4}=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}$$

4.2.En (2.3) ponemos $n=3$:

$$\frac{\pi}{4}=\tan^{-1}\frac{7}{9}+\tan^{-1}\frac{19}{217}+\tan^{-1}\frac{37}{1729}+\tan^{-1}\frac{1}{64}$$

4.3.En (2.4) ponemos $m=2,n=3$:

$$\frac{\pi}{4}=\tan^{-1}\frac{2}{5}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{12}$$

4.4.En (2.5) ponemos $n=3$:

$$\frac{\pi}{4}=\frac{1}{2}\tan^{-1}\frac{1}{2}+\frac{7}{6}\tan^{-1}\frac{1}{3}+\frac{1}{3}\tan^{-1}\frac{1}{4}+\frac{1}{2}\tan^{-1}\frac{1}{7}+\frac{1}{3}\tan^{-1}\frac{1}{13}$$

4.5.En (2.6) ponemos $n=2$ y $n=5$:

$$\frac{\pi}{4}=-3\tan^{-1}\frac{1}{2}+5\tan^{-1}\frac{1}{3}+4\tan^{-1}\frac{1}{7}$$

$$\frac{\pi}{4}=-3\tan^{-1}\frac{1}{2}-4\tan^{-1}\frac{1}{3}-7\tan^{-1}\frac{1}{4}-9\tan^{-1}\frac{1}{5}+25\tan^{-1}\frac{1}{6}+4\tan^{-1}\frac{1}{7}+$$

$$+9\tan^{-1}\frac{1}{13}+16\tan^{-1}\frac{1}{31}$$

4.6.En (2.12) ponemos $a=b=m=2$ y $n=3$:

$$\pi=4\tan^{-1}\frac{3}{5}-2\tan^{-1}\frac{1}{4}+6\tan^{-1}\frac{5}{37}-2\tan^{-1}\frac{1}{9}+8\tan^{-1}\frac{1}{16}+8\tan^{-1}\frac{7}{145}$$

4.7.En (3.1) ponemos $x=\frac{1}{6},y=\frac{1}{71},u=\frac{1}{100}$:

$$\frac{\pi}{4}=4\tan^{-1}\frac{1}{6}+4\tan^{-1}\frac{1}{31}-\tan^{-1}\frac{1}{71}+\tan^{-1}\frac{1}{100}-\tan^{-1}\frac{1}{4971}+\tan^{-1}\frac{1}{9901}$$

4.8.En (3.2) ponemos $x = \frac{1}{8}, y = \frac{1}{54}, u = \frac{1}{4444}$:

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{8} + 4\tan^{-1}\frac{1}{54} + 5\tan^{-1}\frac{1}{57} + 4\tan^{-1}\frac{1}{2863} + 2\tan^{-1}\frac{1}{4444} + 2\tan^{-1}\frac{1}{19744693}$$

4.9.En (3.3) ponemos $x = \frac{1}{58}, y = \frac{1}{240}, u = \frac{1}{683}, v = \frac{1}{12944}$:

$$\frac{\pi}{4} = 44\tan^{-1}\frac{1}{58} + 44\tan^{-1}\frac{1}{3307} + 7\tan^{-1}\frac{1}{240} + 7\tan^{-1}\frac{1}{57361} - 12\tan^{-1}\frac{1}{683} +$$

$$-12\tan^{-1}\frac{1}{465807} + 24\tan^{-1}\frac{1}{12944} + 24\tan^{-1}\frac{1}{167534193}$$

5. REFERENCIAS.

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