Khmelnik S.I.

The Second Solution of Maxwell's Equations

Annotation

A new solution of Maxwell equations for vacuum is presented. First it must be noted that the proof of the solution's uniqueness is based on the Law of energy conservation which is not observed (for instantaneous values) in the known solution. The presented solution does not violate the Law of energy conservation. Besides, in this solution the electrical and magnetic components of intensity are shifted in phase.

A detailed proof is given for interested readers.

Contents

1. Introduction

Recently criticism of validity of Maxwell equations is heard from all sides. The confidence of critics is created first of all by the violation of the Law of energy conservation. And certainly "*the density of electromagnetic energy flow (the module of Umov-Pointing vector) pulsates harmonically. Doesn't it violate the Law of energy conservation?*" [1]. certainly, it is violated, if the electromagnetic wave satisfies the known solution of Maxwell equations. But there is no other solution: "*The proof of solution's uniqueness in general is as follows. If there are two different solutions, then their difference due to the system's linearity, will also be a solution*, *but for zero charges and currents and for zero initial conditions. Hence, using the expression for electromagnetic field energy we must conclude that the difference between solutions is equal to zero, which means that the solutions are identical. Thus the uniqueness of Maxwell equations solution is proved*" [2]. So, <u>the uniqueness of solution is being proved on the base of using the law which is violated in this solution.</u>

Another result following from the existing solution of Maxwell equations is phase synchronism of electrical and magnetic components of energy in an electromagnetic wave. This is contrary to the idea of constant transformation of electrical and magnetic components of energy in an electromagnetic wave. In [1[, for example, this fact is called "one of the vices of the classical electrodynamics".

Such results following from the known solution of Maxwell equations allow doubting the authenticity of Maxwell equations. However, we must stress that these results follow only from the found solution. But this solution, as has been stated above, can be different.

Further we shall deduct another solution of Maxwell equation, in which the density of electromagnetic energy flow remains constant in time, and electrical and magnetic components of intensities in the electromagnetic wave are shifted in in phase.

2. Solution of Maxwell's Equations

First we shall consider the solution of Maxwell equation for vacuum. These equations in GHC system are as follows [3]:

$$rot(E)+\frac{1}{c}\frac{\partial H}{\partial t}=0,$$

$$rot(H)-\frac{1}{c}\frac{\partial E}{\partial t}=0,$$

$$div(E)=0,$$

$$div(H)=0.$$

In cylindrical coordinates system $r, \ \varphi, \ z$ these equations look as follows:

$$\frac{E_r}{r}+\frac{\partial E_r}{\partial r}+\frac{1}{r}\cdot\frac{\partial E_\varphi}{\partial \varphi}+\frac{\partial E_z}{\partial z}=0, \qquad (1)$$

$$\frac{1}{r}\cdot\frac{\partial E_z}{\partial \varphi}-\frac{\partial E_\varphi}{\partial z}=M_r, \qquad (2)$$

$$\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}=M_\varphi, \qquad (3)$$

$$\frac{E_\varphi}{r}+\frac{\partial E_\varphi}{\partial r}-\frac{1}{r}\cdot\frac{\partial E_r}{\partial \varphi}=M_z, \qquad (4)$$

$$\frac{H_r}{r}+\frac{\partial H_r}{\partial r}+\frac{1}{r}\cdot\frac{\partial H_\varphi}{\partial \varphi}+\frac{\partial H_z}{\partial z}=0, \qquad (5)$$

$$\frac{1}{r}\cdot\frac{\partial H_z}{\partial\varphi}-\frac{\partial H_\varphi}{\partial z}=J_r, \tag{6}$$

$$\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r}=J_\varphi, \tag{7}$$

$$\frac{H_\varphi}{r}+\frac{\partial H_\varphi}{\partial r}-\frac{1}{r}\cdot\frac{\partial H_r}{\partial\varphi}=J_z, \tag{8}$$

$$J=\frac{1}{c}\frac{\partial E}{\partial t}, \tag{9}$$

$$M=-\frac{1}{c}\frac{\partial H}{\partial t}. \tag{10}$$

For the sake of brevity further we shall use the following notations:

$$co=\cos(\alpha\varphi+\chi z+\omega t), \tag{11}$$

$$si=\sin(\alpha\varphi+\chi z+\omega t), \tag{12}$$

where e $\alpha,\ \chi,\ \omega$ – are certain constants. Let us present the unknown functions in the following form:

$$J_r.=j_r(r)co, \tag{13}$$

$$J_\varphi.=j_\varphi(r)si, \tag{14}$$

$$J_z.=j_z(r)si, \tag{15}$$

$$H_r.=h_r(r)co, \tag{16}$$

$$H_\varphi.=h_\varphi(r)si, \tag{17}$$

$$H_z.=h_z(r)si, \tag{18}$$

$$E_r.=e_r(r)si, \tag{19}$$

$$E_\varphi.=e_\varphi(r)co, \tag{20}$$

$$E_z.=e_z(r)co, \tag{21}$$

$$M_r.=m_r(r)co, \tag{21}$$

$$M_\varphi.=m_\varphi(r)si, \tag{22}$$

$$M_z.=m_z(r)si, \tag{23}$$

where $j(r),\ h(r),\ e(r),\ m(r)$ - certain function of the coordinate r.

By direct substitution we can verify that the functions (13-23) transform the equations system (1-10) with three arguments $r,\ \varphi,\ z$ into equations system with one argument r and unknown functions $j(r),\ h(r),\ e(r),\ m(r)$.

In Appendix 1 it is shown that for such a system there **exists** a solution of the following form (in Appendix see (24, 27, 18, 31, 33, 34, 32) respectively):

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\varphi(r)}{r}\alpha = 0, \tag{24}$$

$$\frac{e_\varphi(r)}{r} + e_\varphi'(r) - \frac{e_r(r)}{r}\cdot\alpha = 0, \tag{25}$$

$$\frac{h_r(r)}{r} + h_r'(r) + \frac{h_\varphi(r)}{r}\alpha = 0, \tag{26}$$

$$\frac{h_\varphi(r)}{r} + h_\varphi'(r) + \frac{h_r(r)}{r}\cdot\alpha = 0, \tag{27}$$

$$h_\varphi(r) = e_r(r), \tag{28}$$

$$h_r(r) = -e_\varphi(r), \tag{29}$$

$$\chi = \frac{\omega}{c} \tag{30}$$

Thus we have got a monochromatic solution of the equation system (1-10). A transition to polychromatic solution can be achieved with the aid of Fourier transform.

If it exists in cylindrical coordinate system, then it exists in any other coordinate system. It means that we have got a common solution of Maxwell equations in vacuum.

3. Intensities

The equations system (2.24-2.29) is determined – there are 6 equations for 4 functions $e_r,\ e_\varphi,\ h_r,\ h_\varphi$ and two scalars $\alpha,\ \omega$. Considering this system we can see that it is equivalent to two equations (2.24, 2.25) for the functions $e_r,\ e_\varphi$. The two other functions $h_r,\ h_\varphi$ are determined by (28, 29) and satisfy the equations (26, 27).

The two differential equations (2.24, 2.25) can be solved for the given initial conditions and given α. First we shall consider the equation

$$\frac{ay}{x} + y' = 0, \tag{1}$$

The solutions of this equations is as:

$$y = x^{-a}, \tag{2}$$

$$y = 0. \tag{2a}$$

Equations (2.24, 2.25) can be replaced by equations of the form

$$(e_r + e_\varphi)' + \frac{(e_r + e_\varphi)}{r}(1-\alpha) = 0, \tag{3}$$

$$\left(e_r - e_\varphi\right)' + \frac{\left(e_r - e_\varphi\right)}{r}\left(1+\alpha\right) = 0, \tag{4}$$

In accordance with (1, 2) we find for (3):

$$\left(e_r + e_\varphi\right) = Ar^{\alpha-1}, \tag{5}$$

$$\left(e_r + e_\varphi\right) = 0. \tag{5a}$$

In accordance with (1, 2) we find for (4):

$$\left(e_r - e_\varphi\right) = Ar^{-\alpha-1}. \tag{6}$$

$$\left(e_r - e_\varphi\right) = 0. \tag{6a}$$

where $(A\backslash 2)$ – <u>is the amplitude of intensity</u>. Thus, several solutions acceptable for equations (3, 4). In the future, we consider the solution of (5, 6a). From (5) and (6a) it follows:

$$e_r = e_\varphi = \frac{A}{2} r^{\alpha-1}. \tag{7}$$

From (7) it follows:

$$\left(e_r^2 + e_\varphi^2\right) = A \cdot r^{2(\alpha-1)}. \tag{10}$$

Fig. 1 shows, for example, graphics of functions (7, 10) for $A = -1,\ \alpha = 0.8$.

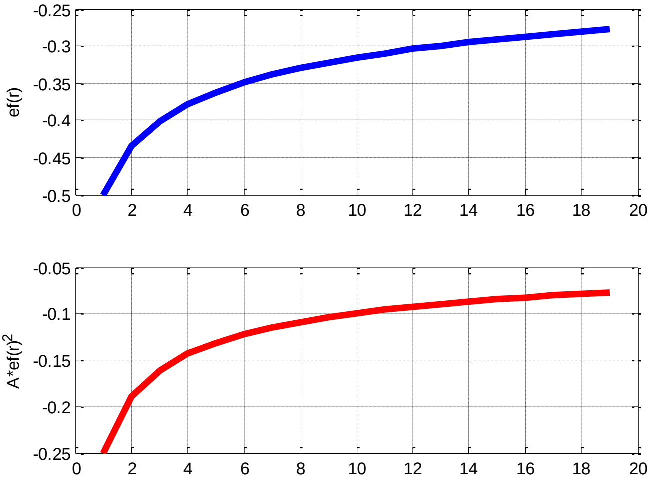


Fig.1. SecondSolMax.m

Fig. 2 shows the vectors of intensities originating from the point $A(r,\varphi)$. Let us remind that $h_\varphi(r) = e_r(r)$ and $h_r(r) = -e_\varphi(r)$ - see (2.28, 2.29). The directions of vectors $e_r(r)$ and $e_\varphi(r)$ are chosen as: $e_r(r) > 0, e_\varphi(r) < 0$. Note that the <u>vectors</u> $E,\ H$ <u>are always</u>

orthogonal. The sum of the modules of these vectors is determined from (2.17, 2.18, 2.20, 2.21, 2.28, 2.29) and is equal to

$$W = E^2 + H^2 = (e_r(r)si)^2 + (e_\varphi(r)si)^2 + (h_r(r)co)^2 + (h_\varphi(r)co)^2$$

or

$$W = (e_r(r))^2 + (e_\varphi(r))^2 \tag{18}$$

- see also (10) and Fig. 1. Thus, the density of electromagnetic wave energy is constant in all points of a circle of this radius.

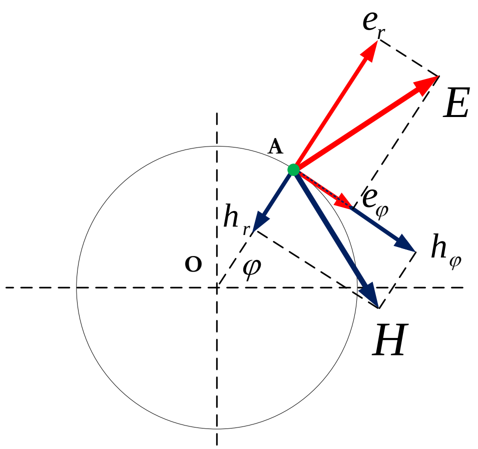


Fig. 2.

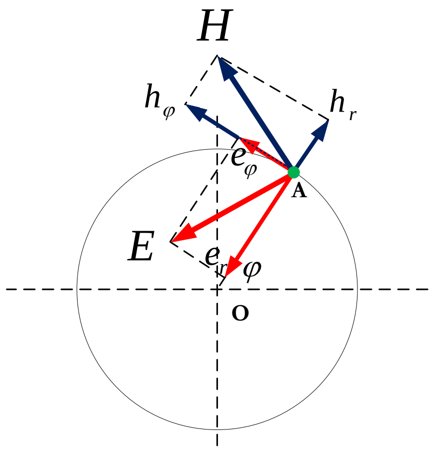


Fig. 3.

The solution exists also for changed signs of the functions (2.11, 2.21). This case is shown on Fig 3. Fig. 2 and Fig. 3 illustrate the fact that there are two possible type of electromagnetic wave circular polarization.

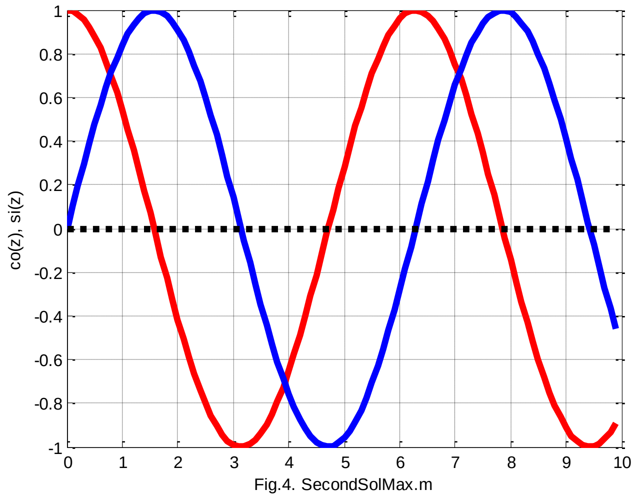


Fig.4. SecondSolMax.m

To demonstrate that the components of the wave (2.13-2.23) are shifted in phase, in Fig. 4 shows the functions

$$co = \cos(\alpha\varphi + \chi z + \omega t), \quad si = \sin(\alpha\varphi + \chi z + \omega t)$$

or equivalent to them at $z = ct$ function

$$co = \cos\left(\alpha\varphi + \frac{2\omega}{c} z\right), \qquad si = \sin\left(\alpha\varphi + \frac{2\omega}{c} z\right).$$

At $\varphi = 0,\ 2\omega/c = 0.1$ these functions take the form $co = \cos(z)$, $si = \sin(z)$ and shown in Fig. 4.

4. Energy Flows

The density of electromagnetic flow is Pointing vector

$$S = \eta E \times H\,, \tag{1}$$

where

$$\eta = c/4\pi. \tag{2}$$

In cylindrical coordinates $r,\ \varphi,\ z$ the density flow of electromagnetic energy has three components $S_r,\ S_\varphi,\ S_z$, directed along вдоль the axis accordingly. They are determined by the formula

$$S = \begin{bmatrix} S_r \\ S_\varphi \\ S_z \end{bmatrix} = \eta(E \times H) = \eta \begin{bmatrix} E_\varphi H_z - E_z H_\varphi \\ E_z H_r - E_r H_z \\ E_r H_\varphi - E_\varphi H_r \end{bmatrix}. \tag{4}$$

From (2.12-2.17, 3.4) follows that the flow passing through a given section of the wave in a given moment, is:

$$\overline{S} = \begin{bmatrix} \overline{S_r} \\ \overline{S_\varphi} \\ \overline{S_z} \end{bmatrix} = \eta \iint\limits_{r,\varphi} \begin{bmatrix} s_r \cdot si^2 \\ s_\varphi \cdot si \cdot co \\ s_z \cdot si \cdot co \end{bmatrix} dr \cdot d\varphi, \tag{5}$$

where

$$\begin{aligned} s_r &= (e_\varphi h_z - e_z h_\varphi) \\ s_\varphi &= (e_z h_r - e_r h_z). \\ s_z &= (e_r h_\varphi - e_\varphi h_r) \end{aligned} \tag{6}$$

In Appendix 1 it is shows that $h_z(r) = 0$, $e_z(r) = 0$. Consequently, $s_r = 0,\ s_\varphi = 0$, i.e. the energy flow extends only along the axis oz and is equal to

$$\overline{S}=\overline{S_z}=\eta\iint_{r,\varphi}[s_z\cdot si\cdot co]dr\cdot d\varphi. \quad (7)$$

We'll find s_z. From (2.28, 2.29), we obtain:

$$e_r h_\varphi = e_r^2, \quad (8)$$

$$e_\varphi h_r = -e_\varphi^2. \quad (9)$$

From (7, 8, 9), we obtain:

$$s_z = \left(e_r^2 + e_\varphi^2\right). \quad (10)$$

In this way,

$$\overline{S}=\eta\iint_{r,\varphi}\left[\left(e_r^2+e_\varphi^2\right)\cdot si\cdot co\right]dr\cdot d\varphi. \quad (11)$$

In Appendix 2 shows that at a constant speed c of propagation of the wave from (11) we obtain

$$\overline{S}=\frac{c}{16\alpha\pi}\left(\cos(4\alpha\pi)-1\right)\int_r\left(\left(e_r^2+e_\varphi^2\right)dr\right). \quad (12)$$

From (10, 3.12), we obtain:

$$\overline{S}=\frac{cA}{16\alpha\pi}\left(1-\cos(4\alpha\pi)\right)\int_r\left(r^{2(\alpha-1)}\right)dr. \quad (12a)$$

Let R be <u>the radius of the circular front of the wave.</u> Then

$$S_{\text{int}}=\int_{r=0}^{R}\left(r^{2(\alpha-1)}\right)dr=\frac{R^{(2\alpha-1)}}{(2\alpha-1)}, \quad (13)$$

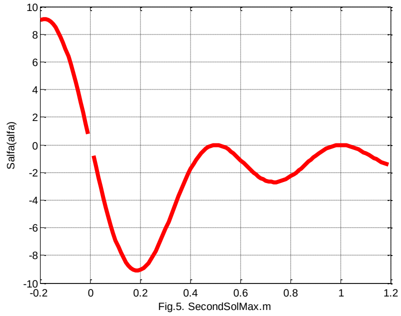
$$S_{alfa}=\frac{1}{\alpha}\left(1-\cos(4\alpha\pi)\right), \quad (14)$$

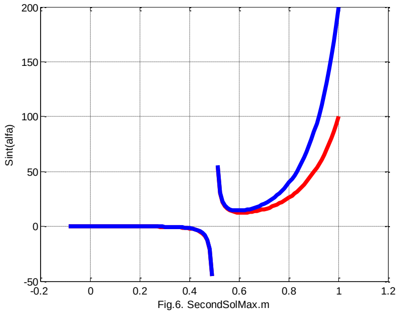
$$\overline{S}=\frac{cA}{16\pi}S_{alfa}S_{\text{int}}. \quad (15)$$

Fig. 5 shows the function $S_{alfa}(\alpha)$ (13) and Fig. 6 shows the function $S_{\text{int}}(\alpha)$. On Fig. 6 the upper and lower curves refer accordingly to $R=200$ and $R=100$. Из формулы (15), рис. 5 и рис 6 видно, что поток энергии является положительным, например, при $A=-1,\ \alpha=0.8$.

Since the energy flow and the energy are related by the expression $S=W\cdot c$, then from (15) we can find the energy of a wavelength unit:

$$\overline{W}=\frac{A}{16\pi}S_{alfa}S_{\text{int}}. \quad (17)$$





In Appendix 2 also shows that the energy flux density on the circle is determined by function of the form

$$\overline{S}_{rz} = \left(e_r^2 + e_\varphi^2\right)\sin\left(2\alpha\varphi + 4\omega z/c\right). \tag{18}$$

From this and from (3.10) we obtain:

$$\overline{S}_{rz} = A \cdot r^{2(\alpha-1)} \sin\left(2\alpha\varphi + 4\omega z/c\right). \tag{19}$$

In Fig. 7 shows these functions, when $A=1,\ \alpha=0.8,\ r=1$, and the second term has two values: 0; 0.5 - see the solid and dashed lines, respectively.

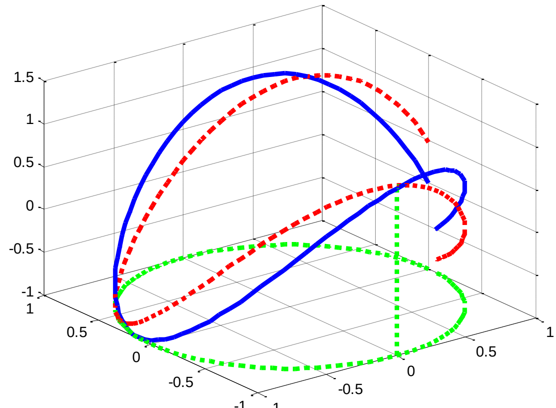


Fig.7. SecondSolMax.m

It follows that

- flux density is unevenly distributed over the flow cross section – there is a picture of the distribution of flow density by the cross section of the wave
- this picture is rotated while moving on the axis OZ;
- the flow of energy (15), passing through the cross-sectional area, not depend on $t,\ \varphi,\ z$; the main thing is that the value does not change with time, and this complies with the Law of energy conservation.

5. Discussion

The resulting solution describes a wave. The main distinctions from the known solution are as follows:

1. Instantaneous (and not average by certain period) energy flow **does not** change with time, which complies with the Law of energy conservation.
2. The energy flow has a positive value
3. The energy flow extends along the wave.
4. Magnetic and electrical intensities on one of the coordinate axes $r,\ \varphi,\ z$ phase-shifted by a quarter of period.
5. The solution for magnetic and electrical intensities is a real value.
6. The solution exists at constant speed of wave propagation.
7. The existence region of the wave **does not** expand, as evidenced by the existence of laser.
8. The vectors of electrical and magnetic intensities are orthogonal.
9. There are two possible types of electromagnetic wave circular polarization.
10. The wave and its energy are determined if the parameters $A,\ \omega,\ R,\ \alpha$ are specified. For given $R,\ \overline{S}$ the parameter α can be found.

Appendix 1

Let us consider the solution of equations (2.1-2.10) in the form of (2.13-2.23). Further the derivatives of r will be designated by strokes. We write the equations (2.1-2.10) in view of (2.11, 2.12) in the form

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\varphi(r)}{r}\alpha - \chi \cdot e_z(r) = 0, \tag{1}$$

$$-\frac{1}{r} \cdot e_z(r)\alpha + e_\varphi(r)\chi = m_r(r), \tag{2}$$

$$e_r(r)\chi - e_z'(r) = m_\varphi(r), \tag{3}$$

$$\frac{e_\varphi(r)}{r} + e_\varphi'(r) - \frac{e_r(r)}{r} \cdot \alpha = m_z(r), \tag{4}$$

$$\frac{h_r(r)}{r} + h_r'(r) - \frac{h_\varphi(r)}{r}\alpha - \chi \cdot h_z(r) = 0, \tag{5}$$

$$-h_r(r)\chi - h_z'(r) = j_\varphi(r), \qquad (7)$$

$$\frac{h_\varphi(r)}{r} + h_\varphi'(r) + \frac{h_r(r)}{r}\cdot\alpha - j_z(r) = 0, \qquad (8)$$

$$j_r = \frac{\omega}{c}e_r, \qquad j_\varphi = -\frac{\omega}{c}e_\varphi, \qquad j_z = -\frac{\omega}{c}e_z, \qquad (9)$$

$$m_r = \frac{\omega}{c}h_r, \qquad m_\varphi = -\frac{\omega}{c}h_\varphi, \qquad m_z = -\frac{\omega}{c}h_z, \qquad (10)$$

We multiply (8) on $(-\chi)$ and take into account (9). Then we get:

$$-\frac{\chi\cdot h_\varphi(r)}{r} - \chi\cdot h_\varphi'(r) - \frac{\chi\cdot h_r(r)}{r}\cdot\alpha + \frac{\chi\omega}{c}\cdot e_z(r) = 0, \qquad (11)$$

or

$$\frac{c\chi}{\omega}\frac{h_\varphi(r)}{r} + \frac{c\chi}{\omega}h_\varphi'(r) + \frac{c\chi}{\omega}\frac{h_r(r)}{r}\cdot\alpha - \chi\cdot e_z(r) = 0, \qquad (12)$$

Comparing (1) and (12), we see that they are the same, if

$$\frac{c\chi}{\omega}h_\varphi(r) = e_r(r), \qquad (13)$$

$$-\frac{c\chi}{\omega}h_r(r) = e_\varphi(r). \qquad (14)$$

It is important to note that this comparison is valid only for. $e_z(r) \neq 0$. Thus, in accordance with (9) and $j_z(r) \neq 0$. In the equations (13, 14) we shall perform substitution according to (9):

$$\chi h_\varphi(r) = j_r(r), \qquad (15)$$

$$-\chi h_r(r) = j_\varphi(r). \qquad (16)$$

Equations (15, 16) coincide with (6, 7) for $h_z(r) = 0$. This implies

<u>Lemma 1.</u> The equation system (1, 5-9) for $e_z(r) \neq 0$ is compatible only if $h_z(r) = 0$.

Let us now consider the case when $e_z(r) = 0$. Then according to (9) , we obtain $j_z(r) = 0$ and the initial system (1, 5-8) will take the form:

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\varphi(r)}{r}\alpha = 0, \qquad (17)$$

$$\frac{h_r(r)}{r} + h_r'(r) + \frac{h_\varphi(r)}{r}\alpha + \chi\cdot h_z(r) = 0, \qquad (18)$$

$$-\frac{1}{r}\cdot h_z(r)\alpha + h_\varphi(r)\chi = j_r(r), \qquad (19)$$

$$-h_r(r)\chi - h_z'(r) = j_\varphi(r), \tag{20}$$

$$\frac{h_\varphi(r)}{r} + h_\varphi'(r) + \frac{h_r(r)}{r}\cdot\alpha = 0, \tag{21}$$

We substitute (9) into (17). Then we get:

$$\frac{j_r(r)}{r} + j_r'(r) - \frac{j_\varphi(r)}{r}\alpha = 0, \tag{22}$$

We substitute (19, 20) into (22). Then we get:

$$-\frac{1}{r^2}\cdot h_z(r)\alpha + \frac{1}{r}\cdot h_\varphi(r)\chi - \frac{1}{r}\cdot h_z'(r)\alpha + h_\varphi'(r)\chi - \left(-h_r(r)\chi - h_z'(r)\right)\frac{\alpha}{r} = 0$$

or

$$-\frac{1}{r^2}\cdot h_z(r)\alpha + \frac{1}{r}\cdot h_\varphi(r)\chi + h_\varphi'(r)\chi + h_r(r)\frac{\chi\alpha}{r} = 0 \tag{23}$$

$$\frac{h_\varphi(r)}{r} + h_\varphi'(r) + \frac{h_r(r)}{r}\cdot\alpha = 0, \tag{21}$$

For the calculation of three intensities we shall get three equations (19, 21, 23). Let us exclude $h_\varphi'(r)$ from (21, 23):

$$-\frac{1}{r^2}\cdot h_z(r)\alpha + \frac{1}{r}\cdot h_\varphi(r)\chi - \left(\frac{1}{r}\cdot h_\varphi(r) + h_r(r)\frac{\alpha}{r}\right)\chi + h_r(r)\frac{\chi\alpha}{r} = 0$$

or $\frac{-1}{r^2}\cdot h_z(r)\alpha = 0$, or $h_z(r) = 0$. Thus, and for $j_z(r) = 0$ the condition $h_z(r) = 0$ must also be complied. Hence there follows

<u>Lemma 2.</u> The equations system (1, 5-9) for $e_z(r) = 0$ is compatible only if $h_z(r) = 0$.

From Lemmas 1 and 2 follows

<u>Lemma 3.</u> The equations system (1, 5-9) is compatible only for $h_z(r) = 0$ and, according to (10), $m_z(r) = 0$. However, there is a case when $e_z(r) \neq 0$ and $j_z(r) \neq 0$.

For $e_z(r) = 0$ and $h_z(r) = 0$ equations (1, 5-9) take the following form - equation (1, 5, 8) are simplified, and the equation (6, 7) are replaced by equations (13, 14):

$$\frac{e_r(r)}{r} + e_r'(r) - \frac{e_\varphi(r)}{r}\alpha = 0, \tag{3.1}$$

$$\frac{h_r(r)}{r} + h_r'(r) + \frac{h_\varphi(r)}{r}\alpha = 0, \tag{3.2}$$

$$\frac{c\chi}{\omega}h_\varphi(r) = e_r(r), \tag{3.3}$$

$$-\frac{c\chi}{\omega}h_r(r)=e_\varphi(r), \tag{3.4}$$

$$\frac{h_\varphi(r)}{r}+h'_\varphi(r)+\frac{h_r(r)}{r}\cdot\alpha=0. \tag{3.5}$$

Similarly we can prove

Lemma 4. The equations system (1-5, 10) is compatible only for $e_z(r)=0$ and, according to (9), $j_z(r)=0$. This is similar to the formulas (24, 28), we obtain the formulas

$$\frac{e_r(r)}{r}+e'_r(r)-\frac{e_\varphi(r)}{r}\alpha=0, \tag{4.1}$$

$$e_\varphi(r)\chi=-\frac{\omega}{c}h_r(r), \tag{4.2}$$

$$e_r(r)\chi=\frac{\omega}{c}h_\varphi(r), \tag{4.3}$$

$$\frac{e_\varphi(r)}{r}+e'_\varphi(r)-\frac{e_r(r)}{r}\cdot\alpha=0, \tag{4.4}$$

$$\frac{h_r(r)}{r}+h'_r(r)+\frac{h_\varphi(r)}{r}\alpha=0. \tag{4.5}$$

From Lemmas 3 and 4 follows

Lemma 5. System (1-10) is compatible only for $h_z(r)=0$, $e_z(r)=0$, $m_z(r)=0$, $j_z(r)=0$.

Consequently, the original system of equations (1-10) takes the form of equations listed in Lemmas 3 and 4. We combine them for the convenience of the reader:

$$\frac{e_r(r)}{r}+e'_r(r)-\frac{e_\varphi(r)}{r}\alpha=0, \tag{24}$$

$$e_\varphi(r)\chi=-\frac{\omega}{c}h_r(r), \tag{25}$$

$$-e_r(r)\chi=-\frac{\omega}{c}h_\varphi(r), \tag{26}$$

$$\frac{e_\varphi(r)}{r}+e'_\varphi(r)+\frac{e_r(r)}{r}\cdot\alpha=0, \tag{27}$$

$$\frac{h_r(r)}{r}+h'_r(r)-\frac{h_\varphi(r)}{r}\alpha=0, \tag{28}$$

$$h_\varphi(r)\chi=\frac{\omega}{c}e_r(r), \tag{29}$$

$$-h_r(r)\chi = \frac{\omega}{c} e_\varphi(r), \tag{30}$$

$$\frac{h_\varphi(r)}{r} + h'_\varphi(r) + \frac{h_r(r)}{r}\cdot\alpha = 0. \tag{31}$$

We multiply (26) on (29). Then we get:

$$-e_r(r)h_\varphi(r)\chi^2 = -\left(\frac{\omega}{c}\right)^2 e_r(r)h_\varphi(r)$$

or

$$\chi = \omega/c. \tag{32}$$

We substitute (32) into (26, 29). Then we get:

$$h_\varphi(r) = e_r(r). \tag{33}$$

Thus, under the condition (32) the equations (26, 29) are equivalent to one equation (33). A similar relationship follows from (25, 30):

$$h_r(r) = -e_\varphi(r), \tag{34}$$

Thus, the system (24-31) is equivalent to the system (24, 27, 28, 31-34).

Appendix 2

In (3.11) it is shown that the energy flow passing through the wave cross-section, is

$$\overline{S} = \eta\iint_{r,\varphi}\left[\left(e_r^2 + e_\varphi^2\right)\cdot si\cdot co\right]dr\cdot d\varphi. \tag{1}$$

Let the speed of wave propagation is constant and equal to c. Then,

$$z = ct. \tag{2}$$

Then from (2, 2.11, 2.12, 2.30), we obtain:

$$co = \cos(\alpha\varphi + \chi z + \omega t) = \cos\left(\alpha\varphi + \left(2\omega/c\right)z\right) \tag{3}$$

and similarly,

$$si = \sin\left(\alpha\varphi + \left(2\omega/c\right)z\right). \tag{4}$$

Due to (3, 4), we can rewrite (1) as:

$$\overline{S} = \frac{1}{2}\eta\iint_{r,\varphi}\left[\left(e_r^2 + e_\varphi^2\right)\sin\left(2\left(\alpha\varphi + \left(2\omega/c\right)z\right)\right)\right]drd\varphi. \tag{5}$$

Thus, the energy flux density on the circle defined by function of the form

$$\overline{S}_{rz} = \left(e_r^2 + e_\varphi^2\right)\sin\left(2\alpha\varphi + 4\omega z/c\right). \tag{5a}$$

When z=0 on the axis oz have:

$$\overline{S} = \frac{1}{2}\eta\iint_{r,\varphi}\left[\left(e_r^2 + e_\varphi^2\right)\sin\left(2\alpha\varphi\right)\right]drd\varphi. \tag{6}$$

Further, from (6) we find:

$$\overline{S} = \frac{\eta}{2}\int_r \left(\left(e_r^2 + e_\varphi^2\right) \left(\int_\varphi \sin(2\alpha\varphi) d\varphi \right) dr \right). \tag{7}$$

We have:

$$\int_\varphi \sin(2\alpha\varphi) d\varphi = \int_0^{2\pi} \sin(2\alpha\varphi) d\varphi = \frac{1}{2\alpha}\left(1 - \cos(4\pi\alpha)\right). \tag{8}$$

From (7, 8) , we obtain:

$$\overline{S} = \frac{\eta}{4\alpha}\left(1 - \cos(4\alpha\pi)\right)\int_r \left(\left(e_r^2 + e_\varphi^2\right) dr\right). \tag{9}$$

Substituting here (3.2), we finally obtain:

$$\overline{S} = \frac{c}{16\alpha\pi}\left(1 - \cos(4\alpha\pi)\right)\int_r \left(\left(e_r^2 + e_\varphi^2\right) dr\right). \tag{10}$$

Obviously, for any choice of the point $z = 0$ on the axis oz last relation is maintained.

References

1. Kann K.B. Electrodynamics. Electromagnetic waves, http://electrodynamics.narod.ru/em-waves.html
2. The bias current and Maxwell's equations, http://www.webpoliteh.ru/subj/dinamo/767-25-tok-smeshheniya-i-sistema-uravnenij-maksvella.html
3. Rozanov N.N. Special sections of mathematical physics. Part 3. The electromagnetic waves in a vacuum. ITMO. St. Petersburg, 2005.