The Number Pi and the Lerch Transcendent function

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abstract

In this note we give some formulas related to the number Pi and the Lerch transcendent function $\Phi(z,s,u)$

EL NÚMERO π Y LA FUNCIÓN $\Phi(z,s,u)$

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Resumen
Se muestra una colección de fórmulas que involucran la constante $\pi = 3.14159...$, y la función $\Phi(z,s,u)$.

1. INTRODUCCIÓN.

La función $\Phi(z,s,u)$ se define por:

$$\Phi(z,s,u) = \sum_{n=0}^{\infty} \frac{z^n}{(n+u)^s}$$

$$u > 0, |z| < 1, s \in \mathbb{C} \, ; \quad u > 0, |z| = 1, \Re(s) > 1$$

Una representación Integral para la función Φ es:

$$\Phi(z,s,u) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-(u-1)t}}{e^t - z} t^{s-1} dt$$

$$u > 0, \Re(s) > 0, |z| < 1 ; u > 0, \Re(s) > 1, |z| = 1$$

Casos particulares de la función $\Phi(z,s,u)$ son:

La función zeta de Riemann $\zeta(s) = \Phi(1,s,1)$

La función zeta de Hurwitz $\zeta(s,u) = \Phi(1,s,u)$

La función Polilogaritmo $Li_n(z) = z\,\Phi(z,n,1)$

En esta nota se muestra una colección de fórmulas que involucran la función $\Phi(z,s,u)$, y el número $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

2. FÓRMULAS.

2.1.

$$\pi+\sqrt{3}\,ln\left(\frac{\left(x+1\right)^{2}}{x^{2}-x+1}\right)+6\,tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)=2\sqrt{3}\Phi\left(-x^{3},1,\frac{1}{3}\right)$$

$$-1<x\leq 1$$

2.2. Para $m\in\mathbb{N}_0=\{0,1,2,3,...\}$, se tiene:

$$\pi=3+\sum_{n=0}^{m}\left(\frac{1}{2}\right)^{2n+1}\Phi\left(-1,2n+3,\frac{3}{2}\right)+\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(n+1\right)\left(n+2\right)\left(2n+3\right)^{2m+3}}$$

2.3. Para $m\in\mathbb{N}_0=\{0,1,2,3...\}$, y $0<a<1/4$, se tiene:

$$\pi^{2}=2\sum_{n=0}^{m}\left(-a\right)^{n}\Phi\left(1,n+1,\frac{1}{2}-a\right)+$$
$$+2^{m+3}\left(-a\right)^{m+1}\sum_{n=0}^{\infty}\frac{1}{\left(2n+1\right)\left(2n+1-2a\right)^{m+1}}$$

2.4. Para $m\in\mathbb{N}_0=\{0,1,2,3...\}$, se tiene:

$$\pi=8\sum_{n=0}^{m}\frac{1}{\left(4n+1\right)\left(4n+3\right)}+\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2^{n+1}}\Phi\left(1,n+2,\frac{4m+5}{4}\right)$$

2.5. Para $1\leq x\leq 5/4$, se tiene:

$$\pi=\sum_{n=0}^{\infty}\left[\left(x-\frac{1}{4}\right)^{n+1}-\left(x-\frac{3}{4}\right)^{n+1}\right]\Phi\left(1,n+2,x\right)$$

2.6. Para $0<x\leq 5/4$, se tiene:

$$\pi=\frac{8}{3}+\sum_{n=0}^{\infty}\left[\left(x-\frac{1}{4}\right)^{n+1}-\left(x-\frac{3}{4}\right)^{n+1}\right]\left(\Phi\left(1,n+2,x\right)-x^{-n-2}\right)$$

2.7.

$$\pi=\frac{8}{3}+\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{2^{n+1}}\Phi\left(1,n+2,\frac{5}{4}\right)$$

$$\pi = \sum_{n=0}^{\infty} \frac{1}{2^{4n+1}} \Phi\left(1,4n+2,\frac{1}{2}\right)$$

2.8. Para $0 < x < 1/4$, se tiene:

$$\pi = \sqrt{3} \sum_{n=0}^{\infty} (-x)^n \Phi\left(-\frac{1}{3},n+1,\frac{1}{2}-x\right)$$

$$\pi = 2 \sum_{n=0}^{\infty} (-x)^n \Phi\left(-1,n+1,\frac{1}{2}-x\right)$$

2.9. Para $m \in \mathbb{N} = \{1,2,3,...\}$,se tiene:

$$\pi^2 = \frac{6}{m^2} \sum_{n=1}^{m} \Phi\left(1,2,\frac{n}{m}\right)$$

2.10.

$$\pi = \sqrt{3}\, \Phi\left(-\frac{1}{3},1,\frac{1}{2}\right)$$

$$\pi = 2\Phi\left(-1,1,\frac{1}{2}\right)$$

$$\pi = 3 + \frac{1}{2}\Phi\left(-1,1,\frac{5}{6}\right) - \frac{1}{2}\Phi\left(-1,1,\frac{7}{6}\right)$$

$$\pi = \frac{1}{2}\Phi\left(-\frac{1}{4},1,\frac{1}{4}\right) + \frac{1}{2}\Phi\left(-\frac{1}{4},1,\frac{1}{2}\right) + \frac{1}{4}\Phi\left(-\frac{1}{4},1,\frac{3}{4}\right)$$

$$\pi = \frac{1}{2}\Phi\left(\frac{1}{16},1,\frac{1}{8}\right) - \frac{1}{4}\Phi\left(\frac{1}{16},1,\frac{1}{2}\right) - \frac{1}{8}\Phi\left(\frac{1}{16},1,\frac{5}{8}\right) - \frac{1}{8}\Phi\left(\frac{1}{16},1,\frac{3}{4}\right)$$

$$\pi\sqrt{3} = \Phi\left(-\frac{1}{27},1,\frac{1}{6}\right) - \frac{1}{3}\Phi\left(-\frac{1}{27},1,\frac{1}{2}\right) + \frac{1}{9}\Phi\left(-\frac{1}{27},1,\frac{5}{6}\right)$$

$$\frac{2\sqrt{3}}{3}\pi = \Phi\left(\frac{1}{9},1,\frac{1}{4}\right) - \frac{1}{3}\Phi\left(\frac{1}{9},1,\frac{3}{4}\right)$$

$$\frac{4\pi}{\sqrt{3}} = \Phi\left(\frac{1}{81},1,\frac{1}{8}\right) - \frac{1}{3}\Phi\left(\frac{1}{81},1,\frac{3}{8}\right) + \frac{1}{9}\Phi\left(\frac{1}{81},1,\frac{5}{8}\right) - \frac{1}{27}\Phi\left(\frac{1}{81},1,\frac{7}{8}\right)$$

$$\pi\sqrt{2}=\frac{2}{3}\Phi\left(-\frac{1}{8},1,\frac{1}{6}\right)+\frac{1}{6}\Phi\left(-\frac{1}{8},1,\frac{1}{2}\right)+\frac{1}{6}\Phi\left(-\frac{1}{8},1,\frac{5}{6}\right)$$

2.11.

$$\frac{\pi^2}{6}-\left(\ln 2\right)^2=\Phi\left(\frac{1}{2},2,1\right)$$

$$=\frac{1}{4}\Phi\left(\frac{1}{4},2,\frac{1}{2}\right)+\frac{1}{8}\Phi\left(\frac{1}{4},2,1\right)$$

$$=\frac{1}{9}\Phi\left(\frac{1}{8},2,\frac{1}{3}\right)+\frac{1}{18}\Phi\left(\frac{1}{8},2,\frac{2}{3}\right)+\frac{1}{36}\Phi\left(\frac{1}{8},2,1\right)$$

2.12.

$$\pi^2=4+\Phi\left(1,2,\frac{1}{2}\right)+\Phi\left(1,2,\frac{3}{2}\right)$$

$$\pi^2=\frac{1}{2}\Phi\left(\frac{1}{64},2,\frac{1}{6}\right)-\frac{3}{4}\Phi\left(\frac{1}{64},2,\frac{1}{3}\right)-\frac{1}{4}\Phi\left(\frac{1}{64},2,\frac{1}{2}\right)+$$
$$-\frac{3}{16}\Phi\left(\frac{1}{64},2,\frac{2}{3}\right)+\frac{1}{32}\Phi\left(\frac{1}{64},2,\frac{5}{6}\right)$$

2.13.

$$\pi=\frac{1}{4}\Phi\left(\frac{1}{16},1,\frac{1}{8}\right)+\frac{1}{4}\Phi\left(\frac{1}{16},1,\frac{1}{4}\right)+\frac{1}{8}\Phi\left(\frac{1}{16},1,\frac{3}{8}\right)+$$
$$-\frac{1}{16}\Phi\left(\frac{1}{16},1,\frac{5}{8}\right)-\frac{1}{16}\Phi\left(\frac{1}{16},1,\frac{3}{4}\right)-\frac{1}{32}\Phi\left(\frac{1}{16},1,\frac{7}{8}\right)$$

$$\pi=-\frac{1}{8}\Phi\left(-\frac{1}{1024},1,\frac{1}{4}\right)-\frac{1}{256}\Phi\left(-\frac{1}{1024},1,\frac{3}{4}\right)+\frac{2}{5}\Phi\left(-\frac{1}{1024},1,\frac{1}{10}\right)+$$
$$-\frac{1}{10}\Phi\left(-\frac{1}{1024},1,\frac{3}{10}\right)-\frac{1}{160}\Phi\left(-\frac{1}{1024},1,\frac{1}{2}\right)+$$
$$-\frac{1}{160}\Phi\left(-\frac{1}{1024},1,\frac{7}{10}\right)+\frac{1}{640}\Phi\left(-\frac{1}{1024},1,\frac{9}{10}\right)$$

2.14. Para $m\in\mathbb{N}_0=\{0,1,2,3,...\}$, se tiene:

$$\frac{\pi}{\sqrt{3}}=\frac{1}{m+1}\sum_{n=0}^{m}\left(-\frac{1}{3}\right)^n\Phi\left(\left(-\frac{1}{3}\right)^{m+1},1,\frac{2n+1}{2m+2}\right)$$

$$\pi = \frac{6\left(2-\sqrt{3}\right)}{m+1}\sum_{n=0}^{m}\left(-1\right)^{n}\left(2-\sqrt{3}\right)^{2n}\Phi\left(\left(-1\right)^{m+1}\left(2-\sqrt{3}\right)^{2m+2},1,\frac{2n+1}{2m+2}\right)$$

2.15.

$$\pi = \frac{3\sqrt{3}}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n}\sum_{k=0}^{n}\frac{\left(-1\right)^{k}}{2k+1}\binom{n}{k}$$

$$\pi = \frac{4}{5}\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\left(\frac{2}{4k+1}+\frac{1}{2k+1}+\frac{1}{4k+3}\right)$$

$$\pi = \frac{9}{4\sqrt{3}}\sum_{n=0}^{\infty}\left(-\frac{1}{8}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\left(\frac{3}{4k+1}-\frac{1}{4k+3}\right)$$

$$\pi = \frac{9}{14\sqrt{3}}\sum_{n=0}^{\infty}\left(\frac{1}{28}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\left(\frac{9}{6k+1}-\frac{1}{2k+1}+\frac{1}{6k+5}\right)$$

$$\pi = \frac{8}{15}\sum_{n=0}^{\infty}\left(-\frac{1}{15}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\left(\frac{8}{8k+1}-\frac{1}{2k+1}-\frac{2}{8k+5}-\frac{1}{4k+3}\right)$$

2.16. Para $m \in \mathbb{N}_0 = \{0,1,2,3,..\}$, se tiene:

$$\pi = \frac{2\sqrt{3}\cdot 3^{m+1}}{3^{m+1}+\left(-1\right)^{m}}\sum_{n=0}^{\infty}\left(\frac{\left(-1\right)^{m}}{3^{m+1}+\left(-1\right)^{m}}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\sum_{s=0}^{m}\frac{\left(-1\right)^{s}}{3^{s}\left(\left(2m+2\right)k+2s+1\right)}$$

$$\pi = A\left(m\right)\sum_{n=0}^{\infty}\left(\frac{\left(-1\right)^{m}}{\left(2+\sqrt{3}\right)^{2m+2}+\left(-1\right)^{m}}\right)^{n}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}\sum_{s=0}^{m}\frac{\left(-1\right)^{s}\left(2-\sqrt{3}\right)^{2s}}{\left(2m+2\right)k+2s+1}$$

$$A\left(m\right) = \frac{12\left(2+\sqrt{3}\right)^{2m+1}}{\left(2+\sqrt{3}\right)^{2m+2}+\left(-1\right)^{m}}$$

3. REFERENCIAS.

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