Pi Formulas , Part 1

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abstract

In this note we give some formulas related to the constant Pi

Número Pi , Fórmulas

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10 - 07 - 2007

Resumen – Abstract

Se muestra una fórmula general que involucra la constante Pi : $\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159...$

1. Fórmula

Para $x \in \mathbb{R}; p \in \mathbb{N}; m_1, m_2, ..., m_p \in \mathbb{N}; k_i \in \{0,1,...,2m_i\}, i = 1,..., p$, se tiene

$$\tan^{-1}(x) = \sum_{k_1=0}^{2m_1} ... \sum_{k_p=0}^{2m_p} (-1)^{k_1+...+k_p} t_p(x, m_1, ..., m_p, k_1, ..., k_p)$$

donde

$$t_p(x, m_1, ..., m_p, k_1, ..., k_p) = \int_0^x \frac{y^{2k_1+2(2m_1+1)k_2+2(2m_1+1)(2m_2+1)k_3+...+2(2m_1+1)...(2m_{p-1}+1)k_p}}{1+y^{2(2m_1+1)(2m_2+1)...(2m_p+1)}} dy$$

$$t_p(x, m_1, ..., m_p, k_1, ..., k_p) = \sum_{k_{p+1}=0}^{2m_{p+1}} (-1)^{k_{p+1}} t_{p+1}(x, m_1, ..., m_{p+1}, k_1, ..., k_{p+1})$$

$$t_p(x, m_1, ..., m_p, k_1, ..., k_p) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{bj+a+1}}{bj+a+1} \quad , |x| \leq 1$$

$$a = 2k_1 + 2(2m_1+1)k_2 + 2(2m_1+1)(2m_2+1)k_3 + ... + 2(2m_1+1)...(2m_{p-1}+1)k_p$$

$$b = 2(2m_1+1)(2m_2+1)...(2m_p+1)$$

2. Ejemplos

2.1. $x=1, p=1, m_1 \in \mathbb{N}$

$$\frac{\pi}{4}=\sum_{k_1=0}^{2m_1}\left(-1\right)^{k_1} t_1\left(1,m_1,k_1\right)$$

$$t_1\left(1,m_1,k_1\right)=\int_0^1 \frac{y^{2k_1}}{1+y^{4m_1+2}}\,dy=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j}{\left(4m_1+2\right)j+2k_1+1}$$

caso trivial $m_1=0$

$$\frac{\pi}{4}=t_1\left(1,0,0\right)=\int_0^1 \frac{1}{1+y^2}\,dy=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j}{2j+1}$$

caso $m_1=1$

$$\frac{\pi}{4}=t_1\left(1,1,0\right)-t_1\left(1,1,1\right)+t_1\left(1,1,2\right)$$

$$\frac{\pi}{4}=\int_0^1 \frac{1}{1+y^6}\,dy-\int_0^1 \frac{y^2}{1+y^6}\,dy+\int_0^1 \frac{y^4}{1+y^6}\,dy$$

$$\frac{\pi}{4}=\sum_{j=0}^{\infty}\left(-1\right)^j\left(\frac{1}{6j+1}-\frac{1}{6j+3}+\frac{1}{6j+5}\right)$$

caso $m_1=2$

$$\frac{\pi}{4}=t_1\left(1,2,0\right)-t_1\left(1,2,1\right)+t_1\left(1,2,2\right)-t_1\left(1,2,3\right)+t_1\left(1,2,4\right)$$

$$\frac{\pi}{4}=\int_0^1 \frac{1}{1+y^{10}}\,dy-\int_0^1 \frac{y^2}{1+y^{10}}\,dy+\int_0^1 \frac{y^4}{1+y^{10}}\,dy-\int_0^1 \frac{y^6}{1+y^{10}}\,dy+\int_0^1 \frac{y^8}{1+y^{10}}\,dy$$

$$\frac{\pi}{4}=\sum_{j=0}^{\infty}\left(-1\right)^j\left(\frac{1}{10j+1}-\frac{1}{10j+3}+\frac{1}{10j+5}-\frac{1}{10j+7}+\frac{1}{10j+9}\right)$$

2.2. $x=1, p=2, m_1, m_2 \in \mathbb{N}$

$$\frac{\pi}{4}=\sum_{k_1=0}^{2m_1}\sum_{k_2=0}^{2m_2}\left(-1\right)^{k_1+k_2} t_2\left(1,m_1,m_2,k_1,k_2\right)$$

$$t_2\left(1,m_1,m_2,k_1,k_2\right)=\int_0^1 \frac{y^{2k_1+2\left(2m_1+1\right)k_2}}{1+y^{2\left(2m_1+1\right)\left(2m_2+1\right)}}\,dy$$

$$t_2\left(1,m_1,m_2,k_1,k_2\right)=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j}{2\left(2m_1+1\right)\left(2m_2+1\right)j+2k_1+\left(4m_1+2\right)k_2+1}$$

caso $m_1=m_2=1$

$$\frac{\pi}{4}=\sum_{k_1=0}^{2m_1}\sum_{k_2=0}^{2m_2}\left(-1\right)^{k_1+k_2}t_2\left(1,1,1,k_1,k_2\right)$$

$$\begin{aligned}\frac{\pi}{4}=\;&t_2\left(1,1,1,0,0\right)-t_2\left(1,1,1,0,1\right)+t_2\left(1,1,1,0,2\right)\\&-t_2\left(1,1,1,1,0\right)+t_2\left(1,1,1,1,1\right)-t_2\left(1,1,1,1,2\right)\\&+t_2\left(1,1,1,2,0\right)-t_2\left(1,1,1,2,1\right)+t_2\left(1,1,1,2,2\right)\end{aligned}$$

$$t_2\left(1,1,1,k_1,k_2\right)=\int_0^1\frac{y^{2k_1+6k_2}}{1+y^{18}}\,dy=\sum_{j=0}^{\infty}\frac{\left(-1\right)^j}{18j+2k_1+6k_2+1}\quad,k_1,k_2\in\left\{0,1,2\right\}$$

$$\begin{aligned}\frac{\pi}{4}=\;&\int_0^1\frac{1}{1+y^{18}}\,dy-\int_0^1\frac{y^6}{1+y^{18}}\,dy+\int_0^1\frac{y^{12}}{1+y^{18}}\,dy\\&-\int_0^1\frac{y^2}{1+y^{18}}\,dy+\int_0^1\frac{y^8}{1+y^{18}}\,dy-\int_0^1\frac{y^{14}}{1+y^{18}}\,dy\\&+\int_0^1\frac{y^4}{1+y^{18}}\,dy-\int_0^1\frac{y^{10}}{1+y^{18}}\,dy+\int_0^1\frac{y^{16}}{1+y^{18}}\,dy\end{aligned}$$

$$\begin{aligned}\frac{\pi}{4}=\sum_{j=0}^{\infty}\left(-1\right)^j\Bigg(&\frac{1}{18j+1}-\frac{1}{18j+3}+\frac{1}{18j+5}-\frac{1}{18j+7}+\frac{1}{18j+9}-\frac{1}{18j+11}+\\&+\frac{1}{18j+13}-\frac{1}{18j+15}+\frac{1}{18j+17}\Bigg)\end{aligned}$$

2.3. $0<x<1;\ p\in\mathbb{N};\ m_1,m_2,\ldots,m_p\in\mathbb{N};\ k_i\in\left\{0,1,\ldots,2m_i\right\},\ i=1,\ldots,p$

$$\begin{aligned}\frac{\pi}{4}=\sum_{k_1=0}^{2m_1}\cdots\sum_{k_p=0}^{2m_p}\left(-1\right)^{k_1+\ldots+k_p}\Big(&t_p\left(x,m_1,\ldots,m_p,k_1,\ldots,k_p\right)+\\&+t_p\left(\left(1-x\right)/\left(1+x\right),m_1,\ldots,m_p,k_1,\ldots,k_p\right)\Big)\end{aligned}$$

2.4. $0 < x < 1;\ p, q \in \mathbb{N};$

$$m_i \in \mathbb{N}, k_i \in \left\{0, ..., 2m_i\right\}, i = 1, ..., p; n_j \in \mathbb{N}, s_j \in \left\{0, ..., 2n_j\right\}, j = 1, ..., q$$

$$\frac{\pi}{4} = \sum_{k_1=0}^{2m_1} ... \sum_{k_p=0}^{2m_p} (-1)^{k_1+...+k_p} t_p\left(x, m_1, ..., m_p, k_1, ..., k_p\right) +$$

$$+ \sum_{s_1=0}^{2n_1} ... \sum_{s_q=0}^{2n_q} (-1)^{s_1+...+s_q} t_p\left(\frac{1-x}{1+x}, n_1, ..., n_q, s_1, ..., s_q\right)$$

2.5.

$$\frac{\pi}{4} = \sum_{k_1=0}^{4} (-1)^{k_1} \left(t_1\left(\frac{1}{2}, 2, k_1\right) + t_1\left(\frac{1}{3}, 2, k_1\right)\right)$$

$$\frac{\pi}{4} = \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{2}\right)^{10j+9} \left(\frac{256}{10j+1} - \frac{64}{10j+3} + \frac{16}{10j+5} - \frac{4}{10j+7} + \frac{1}{10j+9}\right) +$$

$$+ \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{3}\right)^{10j+9} \left(\frac{6561}{10j+1} - \frac{729}{10j+3} + \frac{81}{10j+5} - \frac{9}{10j+7} + \frac{1}{10j+9}\right)$$

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