The Number Pi and the Riemann zeta function

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abstract

In this note we give some formulas related to the number Pi and the Riemann zeta function $\zeta(x)$

ALGUNAS FÓRMULAS QUE INVOLUCRAN EL NÚMERO π Y LA FUNCIÓN ZETA DE RIEMANN $\zeta(n)$

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Resumen
Se muestra una colección de fórmulas que involucran el número $\pi = 3.14159...$, y la función zeta de Riemann $\zeta(n)$.

1. INTRODUCCIÓN

El número π, la función zeta de Riemann $\zeta(s)$, y la función zeta alternada $\zeta^*(s)$, se definen por las series:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad , \quad \Re(s) > 1$$

$$\zeta^*(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} \quad , \quad \Re(s) > 0$$

Las funciones $\zeta(s)$ y $\zeta^*(s)$ se relacionan por la ecuación: $\zeta^*(s) = \left(1-2^{-s}\right)\zeta(s)$, en esta nota se muestra una colección de fórmulas en las que aparece el número π, y la función $\zeta(s)$.

2. FÓRMULAS

2.1.

$$\frac{\pi}{2} - ln(2) = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right)\zeta(n+1)$$

2.2.

$$\frac{\pi\sqrt{3}}{6} + 2\,ln(2) - \frac{3\,ln(3)}{2} = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n\right)\zeta(n+1)$$

2.3.

$$\frac{\pi\sqrt{3}}{2}-\frac{3\ln(3)}{2}=\sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{6}\right)^{n}\right)\zeta(n+1)$$

2.4.

$$\frac{\pi}{2}\left(1-\frac{1}{\sqrt{3}}\right)+\frac{3\ln(3)}{2}-3\ln(2)=\sum_{n=1}^{\infty}\left(\left(\frac{1}{3}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right)\zeta(n+1)$$

2.5.

$$\frac{\pi}{\sqrt{3}}-2\ln(2)=\sum_{n=1}^{\infty}\left(\left(\frac{1}{3}\right)^{n}-\left(\frac{1}{6}\right)^{n}\right)\zeta(n+1)$$

2.6.

$$\frac{\pi}{2}\left(\sqrt{3}-1\right)+\ln(2)-\frac{3\ln(3)}{2}=\sum_{n=1}^{\infty}\left(\left(\frac{1}{4}\right)^{n}-\left(\frac{1}{6}\right)^{n}\right)\zeta(n+1)$$

2.7.

$$\frac{\pi\sqrt{3}}{6}-2\ln(2)+\frac{3\ln(3)}{2}=\sum_{n=1}^{\infty}\left(\left(\frac{2}{3}\right)^{n}-\left(\frac{1}{2}\right)^{n}\right)\zeta(n+1)$$

2.8.

$$\frac{\pi}{\sqrt{3}}=\sum_{n=1}^{\infty}\left(\left(\frac{2}{3}\right)^{n}-\left(\frac{1}{3}\right)^{n}\right)\zeta(n+1)$$

2.9.

$$\frac{\pi}{2}\left(1+\frac{1}{\sqrt{3}}\right)+\frac{3\ln(3)}{2}-3\ln(2)=\sum_{n=1}^{\infty}\left(\left(\frac{2}{3}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right)\zeta(n+1)$$

2.10.

$$\frac{\pi}{2}\left(1-\frac{1}{\sqrt{3}}\right)+3\ln(2)-\frac{3\ln(3)}{2}=\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^{n}-\left(\frac{2}{3}\right)^{n}\right)\zeta(n+1)$$

2.11.

$$\frac{\pi}{2}+\ln(2)=\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^{n}-\left(\frac{1}{2}\right)^{n}\right)\zeta(n+1)$$

2.12.

$$\pi=\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right)\zeta(n+1)$$

2.13.

$$\frac{3\ln(3)}{2}-\frac{\pi\sqrt{3}}{6}=\sum_{n=1}^{\infty}\frac{\zeta(n+1)}{3^n}$$

2.14.

$$3\ln(2)-\frac{\pi}{2}=\sum_{n=1}^{\infty}\frac{\zeta(n+1)}{4^n}$$

2.15.

$$\frac{\pi\sqrt{3}}{6}+\frac{3\ln(2)}{2}=\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\zeta(n+1)$$

2.16.

$$\frac{\pi}{2}+3\ln(2)=\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n\zeta(n+1)$$

2.17.

$$\pi\left(1-\frac{1}{\sqrt{3}}\right)=\sum_{n=1}^{\infty}\left(2\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n-2\left(\frac{1}{4}\right)^n+\left(\frac{1}{6}\right)^n\right)\zeta(n+1)$$

2.18.

$$\pi=\frac{3}{2}\sum_{n=0}^{\infty}\left(\left(\frac{3}{4}\right)^n-\left(\frac{1}{2}\right)^n\right)\zeta(n+2)$$

2.19.

$$\frac{\pi}{4}=1-\frac{\ln(2)}{6}-\sum_{n=1}^{\infty}\frac{\left(3^n-1+(-1)^n\right)\left(1-2^{-n}\right)}{6^{n+1}}\zeta(n+1)$$

2.20.

$$\frac{\pi}{4}=1-\frac{\ln(2)}{6}-\sum_{n=1}^{\infty}\frac{\left(3^{2n-1}-2\right)\left(1-2^{-2n+1}\right)}{6^{2n}}\zeta(2n)$$
$$-\frac{1}{6}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}\left(1-2^{-2n}\right)\zeta(2n+1)$$

2.21.

$$\pi=4-\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{4n-3}\zeta(2n)$$

2.22.

$$\frac{\pi}{\sqrt{3}}=\frac{3}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{2}{3}\right)^n-\left(\frac{1}{3}\right)^n\right)\zeta(n+1)$$

2.23.

$$\frac{\pi\sqrt{3}}{6}+2\ln(2)-\frac{3}{2}\ln(3)=\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{2}{3}\right)^n-\left(\frac{1}{2}\right)^n\right)\zeta(n+1)$$

2.24.

$$\frac{\pi\sqrt{3}}{6}-2\ln(2)+\frac{3}{2}\ln(3)=1+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n\right)\zeta(n+1)$$

2.25.

$$\frac{\pi\left(3+\sqrt{3}\right)}{6}+3\ln(2)-\frac{3}{2}\ln(3)=\frac{5}{2}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{2}{3}\right)^n-\left(\frac{1}{4}\right)^n\right)\zeta(n+1)$$

2.26.

$$\frac{\pi}{\sqrt{3}}+2\ln(2)=3+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{3}\right)^n-\left(\frac{1}{6}\right)^n\right)\zeta(n+1)$$

2.27.

$$\frac{\pi\left(\sqrt{3}-1\right)}{2}-\ln(2)+\frac{3}{2}\ln(3)=2+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{4}\right)^n-\left(\frac{1}{6}\right)^n\right)\zeta(n+1)$$

2.28.

$$\frac{\pi\left(3-\sqrt{3}\right)}{6}-3\ln(2)+\frac{3}{2}\ln(3)=\frac{1}{6}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{3}{4}\right)^n-\left(\frac{2}{3}\right)^n\right)\zeta(n+1)$$

2.29.

$$\frac{\pi}{2}-\ln(2)=\frac{2}{3}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{3}{4}\right)^n-\left(\frac{1}{2}\right)^n\right)\zeta(n+1)$$

2.30.

$$\pi=\frac{8}{3}+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{3}{4}\right)^n-\left(\frac{1}{4}\right)^n\right)\zeta(n+1)$$

2.31.

$$\frac{\pi}{2}+ln(2)=2+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right)\zeta(n+1)$$

2.32.

$$\frac{\pi\sqrt{3}}{2}+\frac{3}{2}ln(3)=4+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{6}\right)^{n}\right)\zeta(n+1)$$

2.33.

$$\frac{\pi\left(3-\sqrt{3}\right)}{6}+3\,ln(2)-\frac{3}{2}ln(3)=1+\sum_{n=1}^{\infty}(-1)^{n-1}\left(\left(\frac{1}{3}\right)^{n}-\left(\frac{1}{4}\right)^{n}\right)\zeta(n+1)$$

2.34.

$$\frac{\pi\sqrt{3}}{6}+\frac{3}{2}ln(3)=3-\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{3}\right)^{n}\zeta(n+1)$$

2.35.

$$\frac{\pi}{2}+3\,ln(2)=4-\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{4}\right)^{n}\zeta(n+1)$$

2.36.

$$3\,ln(2)-\frac{\pi}{2}=\frac{4}{3}-\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{3}{4}\right)^{n}\zeta(n+1)$$

2.37.

$$\frac{3}{2}ln(3)-\frac{\pi\sqrt{3}}{6}=\frac{3}{2}-\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3}\right)^{n}\zeta(n+1)$$

2.38.

$$\pi=\sum_{n=0}^{\infty}\left(3\left(\frac{1}{2}\right)^{n+1}-\left(\frac{1}{2}\right)^{2n+1}\right)\zeta(n+2)$$

2.39.

$$\pi=\frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(-\frac{3}{16}\right)^{k}\binom{n}{k}\zeta(n+k+2)$$

$$\pi=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{4n+1}\sum_{k=0}^{n}(-3)^{k}\,2^{4n-4k}\binom{n}{k}\zeta(n+k+2)$$

3. REFERENCIAS

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