A categorical approach for relativity theory

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Abstract

We provide a categorical interpretation for a model unifying the Galilei relativity and the special relativity, which is based on the introduction of two times variables, one associated to the absolute time of the Galilei relativity, and the other to the local time of the special relativity. The relation between these two time variables is the key point for the construction of a natural transformation relating two functors $\overline{G}$ and $\overline{L}$ that translates to the framework of category the role of the Galilei and the Lorentz transformations bringing with them a decomposition of the Lorentz transformation in terms of the Galilei transformation, which in some sense unify both relativities.

1 Introduction

In a previous work [1] we developed a model relating kinematical aspects from the Galilei and the special relativity (SR), namely, given two inertial reference frames S and S' moving with relative velocity $\vec{v}$ we showed how the Galilei transformation of coordinate and velocity given by

$$\vec{x}\,' = \vec{x} - \vec{v}\tau, \quad \tau' = \tau$$
$$\vec{u}\,' = \vec{u} - \vec{v}$$

with $\vec{u} := \frac{d\vec{x}}{d\tau}, \quad \vec{u}\,' := \frac{d\vec{x}\,'}{d\tau}$ induce the corresponding coordinate and velocity transformations of the SR

$$\vec{x}\,' = \vec{x} - (1-\gamma_{\widetilde{v}})\frac{\vec{x}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}} - \gamma_{\widetilde{v}}t\vec{\widetilde{v}}\,, \qquad t' = \gamma_{\widetilde{v}}\Big(t - \frac{\vec{x}\cdot\vec{\widetilde{v}}}{c^2}\Big)$$
$$\vec{\widetilde{u}}\,' = \frac{\vec{\widetilde{u}} - \gamma_{\widetilde{v}}\vec{\widetilde{v}} - (1-\gamma_{\widetilde{v}})\frac{\vec{\widetilde{u}}\cdot\vec{\widetilde{v}}}{\widetilde{v}^2}\vec{\widetilde{v}}}{\gamma_{\widetilde{v}}\big(1 - \frac{\vec{\widetilde{u}}\cdot\vec{\widetilde{v}}}{c^2}\big)}$$

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where $\vec{\tilde{u}} := \frac{d\vec{x}}{dt}$, $\vec{\tilde{u}}\,' := \frac{d\vec{x}\,'}{dt'}$ and $\gamma_{\tilde{v}} = 1/\sqrt{1-\frac{\tilde{v}^2}{c^2}}$. We obtained this result in [1] by reevaluating the role played by the Galilean concept of absolute time, which currently became superseded by the interpretation of time in SR. In fact, in SR there is no concept of absolute time, the latter being understood as any time variable transforming according to $\tau = \tau'$. The closer we can get to such a transformation in SR is when we consider the low speed limit $\tilde{v} \ll c$ of the relative motion between two inertial reference frames, a circunstance where both frames would register the same time for the ocurrence of an event, e.g. $t \simeq t'$, as we can see by neglecting terms of order $\geq \frac{\tilde{v}^2}{c^2}$ in the time transformation law, for example,

$$t' = \gamma_{\tilde{v}}\Big(t - \frac{\vec{x}\cdot\vec{\tilde{v}}}{c^2}\Big) = \Big(1 - \frac{1}{2}\frac{\tilde{v}^2}{c^2} + \dots\Big)\big(t - \frac{1}{c^2}\vec{x}\cdot\vec{\tilde{v}}\big) \simeq t\ .$$

We also obtain low speed limit cases for the other special relativity transformations, for instance

$$\vec{x}\,' = \vec{x} - \vec{\tilde{v}}t, \quad \vec{\tilde{u}}' = \vec{\tilde{u}} - \vec{\tilde{v}}\ ,$$

which are similar to the Galilean laws since in this low velocity limit we identify $\vec{\tilde{u}} = \vec{u}$, $\vec{\tilde{v}} = \vec{v}$, as we have seen in [1].

There is, however, a possible way to give a concrete representation for the absolute time that goes beyond this low speed limit as treated in the SR. This is achieved by making asssumptions borrowing elements from both relativities and consisting essentially on assuming two ways of registering the time, one based on the absolute time τ that obeys the laws of the Galilei relativity, and another based on the local time t that obeys the laws of the SR. As a consequence of these assumptions we have discovered in [1] a class of transformations - the so-called Generalized Lorentz transformation - that includes as a particular case the standard Lorentz transformation of the SR together with other transformations that we denoted by h that work as shifting the main elements of the Galilei relativity, for example, the Galilean coordinate system and the Galilean transformation to the corresponding elements of the Special relativity, for example, the Lorentzian coordinate system and the Lorentz transformation as shown in the diagram below (we review this construction in section 3 and define the concepts of the Galilean and Lorentzian coordinate systems in section 4)

$$\begin{array}{ccc} (\tau,\vec{x}) & \xrightarrow{\text{Galilei}} & (\tau,\vec{x}\,') \\ \big\downarrow h & & \big\downarrow h' \\ (t,\vec{x}) & \xrightarrow{\text{Lorentz}} & (t',\vec{x}\,')\ . \end{array} \tag{1}$$

It is the purpose of our work to investigate if the relations involving the kinematical aspects analyzed in our previous work [1] and represented schematically in diagram (1) reveal some sort of mathematical structure uderlying the structure of the Galilei relativity and the SR. In fact, all our effort it will be show that the vertical maps h, h' shown in diagram (1) define a *natural transformation* between some appropriate functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$, where the categories $\mathcal{S}$ and $\mathcal{C}$ are introduced in order to characterize the distinct but complementary roles played in physics by the concepts of *inertial reference frame* and *coordinate system*. As we will see, these concepts become two stages on a process of modelling the physical world into a categorical way of thinking that elucidates, from a mathematical perspective, a connection between the formalisms of the Galilei relativity and the SR.

Our work is organized as follows. In section 2 we discuss the concepts of event, inertial reference frame, and coordinate system in a sense that will allow us to *categorize* them, i.e. to define the categories of inertial reference frames and of coordinate systems. In section 3 we summarize the formalism we developed in [1] describing axiomatically a model combining aspects of the Galilei and the SR. We discuss the consistency of the axioms and conclude that the time variable as described in SR depends on the state of motion of the observers [1]. In section 4 we introduce two particular perspectives that are useful in the description of physical phenomena: the Galilean and the Lorentzian coordinate systems. In section 5 we define a certain multivalued functor and fix some of its properties. In section 6 we develop a convenient categorical formulation for the elements introduced in sections 3 and 4 in order to reinterpret the model we developed in [1] and to clarify within a categorical perspective the unification of the Galilei relativity and the SR through a sort of multivalued natural transformation uniting the Galilei and the Generalized Lorentz transformation.

In our work the term *relativity* refers indistinctly either to the Galilei relativity or to the special relativity. The constant c always refer to the speed of light in vacuum.

A complete and very readable reference for relativity is given in [3], [4]. A general reference for category is given in [5], [6]. For another treatment involving relativity and category we suggest [7]. As to the notion of multivalued functor, it seems it has not been extensively developed in the literature, therefore we recommend [8] for another reference that treats the same concept but in another context and form.

2 Some physics notions

Relativity is concerned with the description of physical occurrences in space and time and how different observers relate these descriptions among themselves. Then, in order to make our work readable for a large class of readers, we start with a brief exposition

[1]This fact has been also discussed in another context in [2].

of some basic physical concepts such as event, inertial reference frame, and coordinate system.

An *event* is as any occurrence of a physical phenomenon that may be described by the position *where* it takes place and the instant of time *when* it happens. In this definition it is implict that an event is any idealized physical phenomenon that occurs localized in space and time, i.e. it is any occurrence without extension and duration. In this sense a solid body is not an event since it has an extension, while a particle is not an event since it has a duration or, equivalently, it persists in time. However, single processes of creation and annihilation of a particle are both events.

An *inertial reference frame* (for brevity, a reference frame) is any material body very small in size and free of forces that can be used as a reference point relative to which we can determine the position of other material bodies. This reference frame relates to another reference frame in two ways: they are either at rest or in relative motion with constant velocity [2]. We denote reference frames by $S, S', S'', \ldots$.

By *observer* we mean any inteligent person that uses an inertial reference frame (relative to which he is at rest) in order to analyze physical phenomena. In this sense, we will make no distinction between an observer and his reference frame, using interchangeably both terms.

A *coordinate system* (on a reference frame) consists of any set of rules and clocks attached to a reference frame that are used for measuring lengths and intervals of time, together with a prescription that associates these measurements with a 4-upla of numbers (the prescription involves, for instance, adopting rectangular or curvilinear coordinates), which provides a consistent way for registering events. Physically, the establishment of a coordinate system may be thought of as an *idealized* process where we take a reference frame as a material body which has attach a system of three mutually perpendicular axis. We use the rules to form a grid in space, together with a system of clocks rigidly attached to each point of the grid that will register the instant of time of events ocurring at the position where the clock is placed.

By *physical world* we mean a 4-dimensional space whose points represent events. Then the physical world is what an observer describes as the reality surrounding him.

[2]For simplicity, reference frames that are at rest relative to each other will be considered as equivalent in the sense we may take any one of them as representing all.

3 The unified scheme for the Galilei relativity and the special relativity

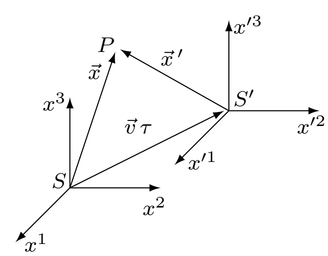
In a previous work [1], we presented a unified scheme for the Galilei and the special relativity. It consisted essentially in describing time through two perspectives, one based in the Galilei relativity, where time has an *absolute* character, and another based on the special relativity, where time has a *local* character. The precise meaning of these terms will be given shortly after examining their transformation properties and how they are derived from assumptions taken from both relativities. Let us denote these two time variables by τ and t, which we call respectively the absolute and the local time.

3.1 The Axioms

We take as axioms the following assumptions:

I. Events are described relative to an inertial reference frame S by means of coordinates $\{\tau, t, \vec{x}\}$ where $\vec{x} = (x^1, x^2, x^3)$ refers to the spatial coordinates marking the *location* where the event occurred, while t, τ refers to the *instant* when the event occurred.

II. *The Galilei relativity law*
Relative to two inertial reference frames S, S' moving with uniform velocity $\vec{v}$ as shown in the figure



any event P is described by coordinates sets $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ where

$$\vec{x}\,' = \vec{x} - \vec{v}\tau \tag{2}$$

$$\tau' = \tau \; . \tag{3}$$

III. *The invariance of the quadratic form* $Q(ct, x) := c^2t^2 - \vec{x}^2$
Given an event P described by $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$ relative to two inertial reference frames we have

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}\,'^{\,2} \; . \tag{4}$$

IV. The relation between the local times t and t' is of the type

$$t' = at + b\,\vec{v} \cdot \vec{x} \tag{5}$$

with a and b two arbitrary real parameters [3].

3.2 The coordinate transformation

Working with axioms I to IV, we showed in [1] that by fixing $b = \frac{\sqrt{a^2-1}}{vc}$ we obtain the following transformation between $\{\tau, t, \vec{x}\}$, $\{\tau', t', \vec{x}\,'\}$

$$(t, \vec{x}) \overset{L(a(v),\vec{v})}{\longrightarrow} (t', \vec{x}\,') : \begin{cases} \vec{x}\,' = \vec{x} - \frac{1}{v^2}(1 - a(v))\vec{x} \cdot \vec{v}\,\vec{v} - \frac{c}{v}\sqrt{a(v)^2 - 1}\; t\vec{v} \\ t' = a(v)t - \frac{1}{cv}\sqrt{a(v)^2 - 1}\;\vec{x} \cdot \vec{v} \end{cases} \tag{6}$$

where $a(v)$ is an arbitrary function of v restricted only by the condition that $|a(v)| > 1$. We call this transformation the *Generalized Lorentz Transformation* (GLT) and it depends on two parameters: $a(v)$ and the speed v. For any event $(t, \vec{x})$ we also obtained that the absolute time associated to the occurrence of the event is given by the expression

$$\tau = (1 - a(v))\frac{\vec{x} \cdot \vec{v}}{v^2} + \sqrt{a(v)^2 - 1}\,\frac{c}{v}\,t = (1 - a(v'))\frac{\vec{x}\,' \cdot \vec{v}\,'}{v'^2} + \sqrt{a(v')^2 - 1}\,\frac{c}{v'}\,t' = \tau' \tag{7}$$

with $\vec{v}\,' = -\vec{v}$ (note that $a(v') = a(v)$). Equation (7) becomes an operational definition for the absolute time τ.

The GLT satisfies the following property

3.2.1 Theorem: Given a GLT $L(a(v), \vec{v})$ depending on an arbitrary function $a(v)$, with $|a(v)| > 1$, there is defined a choice for velocity, $\vec{\tilde{v}}$, such that $L(a(\tilde{v}), \vec{\tilde{v}})$ becomes the ordinary Lorentz transformation.
Proof: Let us take

$$\vec{\tilde{v}} := c\frac{\sqrt{a(v)^2 - 1}}{a(v)}\frac{\vec{v}}{v}\;. \tag{8}$$

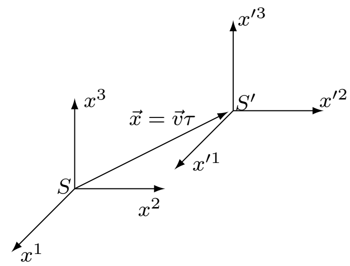
Then, we obtain $a(\tilde{v}) = \frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}} \equiv \gamma_{\tilde{v}}$ and $\frac{\vec{v}}{v} = \frac{\vec{\tilde{v}}}{\tilde{v}}$. A straightforward calculation then shows that equation (6) becomes

$$\begin{cases} \vec{x}\,' := \vec{x} - (1 - \gamma_{\tilde{v}})\frac{\vec{x}\cdot\vec{\tilde{v}}}{\tilde{v}^2}\vec{\tilde{v}} - \gamma_{\tilde{v}} t\vec{\tilde{v}} \\ t' := \gamma_{\tilde{v}}\big(t - \frac{\vec{x}\cdot\vec{\tilde{v}}}{c^2}\big)\;, \end{cases} \tag{9}$$

which defines the ordinary Lorentz transformation, henceforth denoted by $\mathcal{L}(\vec{\tilde{v}})$. ■

3.2.2 Remark: There is an important physical distinction to be made between $\vec{v}$ and $\vec{\tilde{v}}$ that is not apparent in the discussion of the previous result. In fact, let us consider in axiom II a sucession of events represented by the movement of the origin of frame S' as seen by frame S as shown in the figure below

[3] The parameters a and b may depend on the velocity.



Here, we have $\vec{x}\,' = 0$ and from (2) we get $\vec{x} = \vec{v}\tau$, which gives $\vec{v} := \frac{d\vec{x}}{d\tau}$. Then, identifying $\tilde{\vec{v}} := \frac{d\vec{x}}{dt}$ and using (7) we obtain the relation between $\vec{v}$ and $\tilde{\vec{v}}$ as given in (8). Now, since the parameter a in the GLT is an arbitrary function of v the explicit form between $\tilde{\vec{v}}$ and $\vec{v}$ will be determined only when we fix a particular form for $a(v)$. However, despite the form $a(v)$ may have, when $a(v)$ is expressed in terms of $\tilde{\vec{v}}$ it always produce $a(\widetilde{v}) = 1/\sqrt{1 - \frac{\widetilde{v}^2}{c^2}}$.

3.3 The velocity transformation

Theorem **3.2.1** has a counterpart in the case of velocity. Let us consider $\vec{u} = \frac{d\vec{x}}{d\tau}$ and $\tilde{\vec{u}} = \frac{d\vec{x}}{dt}$. Then,

$$\tilde{\vec{u}} = \frac{d\vec{x}}{dt} = \frac{d\vec{x}}{d\tau}\frac{d\tau}{dt} \Rightarrow \tilde{\vec{u}} = \Big[(1-a)\frac{\tilde{\vec{u}}\cdot\vec{v}}{v^2} + \sqrt{a^2-1}\frac{c}{v}\Big]\vec{u}$$

or equivalently

$$\tilde{\vec{u}} = \frac{\sqrt{a^2-1}}{\Big[1-(1-a)\frac{\vec{u}\cdot\vec{v}}{v^2}\Big]}\frac{c}{v}\vec{u}\ . \tag{10}$$

Considering in a similar way $\vec{u}\,' = \frac{d\vec{x}\,'}{d\tau}$ and $\tilde{\vec{u}}\,' = \frac{d\vec{x}\,'}{dt}$ we obtain

$$\tilde{\vec{u}}\,' = \frac{\sqrt{a^2-1}}{\Big[1+(1-a)\frac{\vec{u}\,'\cdot\vec{v}}{v^2}\Big]}\frac{c}{v}\vec{u}\,'\ . \tag{11}$$

Now, taking the derivative relative to the absolute time in axiom II we obtain the velocity law of the Galilei relativity, $\vec{u}\,' = \vec{u} - \vec{v}$, and using (10, 11) in this expression we obtain the *Generalized Lorentz Transformation for Velocity* (GLTV), which we denote by $L^*(a(v), \vec{v})$:

$$\tilde{\vec{u}} \overset{L^*(a(v),\vec{v})}{\longrightarrow} \tilde{\vec{u}}\,' = \frac{\tilde{\vec{u}} - \sqrt{a^2-1}\frac{c}{v}\vec{v} - (1-a)\frac{\tilde{\vec{u}}\cdot\vec{v}}{v^2}\vec{v}}{a - \sqrt{a^2-1}\frac{1}{cv}\tilde{\vec{u}}\cdot\vec{v}}\ . \tag{12}$$

We also have a result similar to theorem **3.2.1**,

3.3.1 Theorem: Given a GLTV $L^*(a(v), \vec{v})$ depending on an arbitrary function $a(v)$ with $|a(v)| > 1$, there is defined a velocity $\tilde{\vec{v}}$ such that $L^*(a(\widetilde{v}), \tilde{\vec{v}})$ is the ordinary Lorentz transformation for velocity.

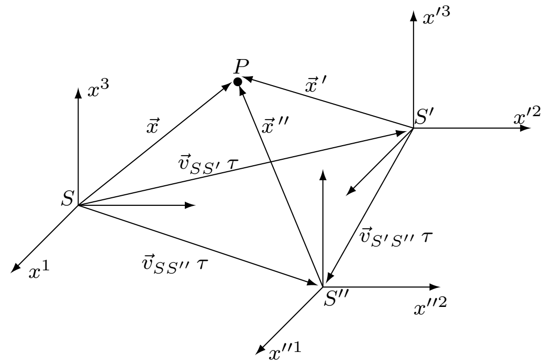
Proof: We have here the same velocity $\tilde{\vec{v}}$ as given in (8). Then, in this case we also get $a(v) = \gamma_{\widetilde{v}}$ and $\frac{\tilde{\vec{v}}}{\widetilde{v}} = \frac{\vec{v}}{v}$, and by a direct calculation we obtain from (12) that

$$\tilde{\vec{u}} \overset{\mathcal{L}^*(\tilde{\vec{v}})}{\longrightarrow} \tilde{\vec{u}}\,' = \frac{\tilde{\vec{u}} - \gamma_{\widetilde{v}}\tilde{\vec{v}} - (1-\gamma_{\widetilde{v}})\frac{\tilde{\vec{u}}\cdot\tilde{\vec{v}}}{\widetilde{v}^2}\tilde{\vec{v}}}{\gamma_{\widetilde{v}}\big(1 - \frac{\tilde{\vec{u}}\cdot\tilde{\vec{v}}}{c^2}\big)} \tag{13}$$

that corresponds to the law of transformation for velocities found in SR. ■

3.4 A consistency requirement for axioms II and IV leading to the dependence of the local time with the state of motion

Now, we wish to analyze the following situation. Given an event P, let it be recorded as $(\tau, \vec{x})$ by S, as $(\tau', \vec{x}\,')$ by S', and as $(\tau'', \vec{x}\,'')$ by S'', where S, S', S'' are three inertial observers moving according to the picture shown below



The main assumption about time made in the Galilei relativity is expressed by equation (3) and it gives $\tau = \tau' = \tau''$, therefore, from the movement of the frames represented above we can write

$$\vec{x}\,' = \vec{x} - \vec{v}_{SS'}\tau \tag{14}$$

$$\vec{x}\,'' = \vec{x}\,' - \vec{v}_{S'S''}\tau \tag{15}$$

$$\vec{x}\,'' = \vec{x} - \vec{v}_{SS''}\tau \ . \tag{16}$$

If the data $\vec{x}, \vec{x}\,', \vec{x}\,''$ appearing in these equations represent values measured independently by each observer, we expect these equations to be compatible, therefore, replacing $\vec{x}\,'$ given in (14) into (15) we obtain $\vec{x}\,'' = \vec{x} - (\vec{v}_{SS'} + \vec{v}_{S'S''})\tau$, which comparing with (16) gives

$$\vec{v}_{SS''} = \vec{v}_{SS'} + \vec{v}_{S'S''} \tag{17}$$

that is the common expression for the addition of velocities in the Galilei relativity.

Now, let us consider again the same situation as before, this time with the observers S, S', S'' recording the event P in terms of the following data $(t, \vec{x}), (t', \vec{x}\,'), (t'', \vec{x}\,'')$. From (2, 5) we have the following relations

$$\vec{x}\,' = \vec{x} - \vec{v}_{SS'}\tau \ , \qquad t' = a(v_{SS'})t + b\,\vec{v}_{SS'} \cdot \vec{x} \tag{18}$$

$$\vec{x}\,'' = \vec{x}\,' - \vec{v}_{S'S''}\tau \ , \qquad t'' = a(v_{S'S''})t' + b\vec{v}_{S'S''} \cdot \vec{x}\,' \tag{19}$$

$$\vec{x}\,'' = \vec{x} - \vec{v}_{SS''}\tau \ , \qquad t'' = a(v_{SS''})t + b\vec{v}_{SS''} \cdot x \tag{20}$$

Replacing $t', \vec{x}\,'$ given in (18) into (19), and using (17), we obtain the following expression

$$\vec{x}\,'' = \vec{x} - \vec{v}_{SS''}\tau \tag{21}$$

$$t'' = a(v_{S'S''})a(v_{SS'})t + b\big(a(v_{S'S''})\vec{v}_{SS'} + \vec{v}_{S'S''}\big) \cdot \vec{x} - b\,\vec{v}_{S'S''} \cdot \vec{v}_{SS'}\tau \ . \tag{22}$$

Comparison of equations (22) and (20) shows there is no choice for a and b that makes the two expressions for t'' to agree. This suggests the local time is defined only with respect to a pair of frames, or as investigated by Horwitz, Arshansky and Elitzur, we can also say that *"in relativity then, the time at which an event occurs depends on the state of motion of the frame (and the clocks attached to it)"*([2], pag. 1163). Then, with this view, the local times t'' and t appearing in equation (20) would be better written as $t'' \equiv t''_{SS''}$ and $t \equiv t_{SS''}$. The same applies to the local time appearing in the other equations. With this prescription the previous inconsistency disappears because $t' \equiv t'_{SS'}$ occuring in equation (18) and $t' \equiv t'_{S'S''}$ occurring in equation (19) are not the same, then, we cannot perform the previous calculation of replacing the t' given in (18) for the t' given in (19). Here, in order to ensure that the local time is always set in a way that depends on the state of motion of one observer relative to another, it is convenient to view the coordinate system $(t, \vec{x})$ in two complementary forms writing

$$(t, \vec{x}) \to (t, \vec{x}, \{\vec{\beta}\}) \ \text{ or } \ (t, \vec{x}) \to (t, \vec{x}, \vec{v}) \tag{23}$$

where in the notation $(t, \vec{x}, \{\vec{\beta}\})$, the reference made to $\{\vec{\beta}\}$ indicates all possible inertial observers S' moving with velocity $\vec{\beta}$ relative to S, while in the notation $(t, \vec{x}, \vec{v})$ we indicate that there is already specified one observer S' moving with velocity $\vec{v} = \vec{v}_{SS'}$ relative to S. In this latter case, if S' uses t' as his local time, and S uses t as his local time, then we have t and t' related by (6). In section 6.3 we will provide a concrete realization for $(t, \vec{x}, \{\vec{\beta}\})$ that will arise as the codomain of a certain multivalued functor.

When dealing with the notation $(t, \vec{x}, \vec{v})$ we rewrite the GLT given in (6) as

$$(t, \vec{x}, \vec{v}) \overset{L(a(v),\vec{v})}{\longrightarrow} (t', \vec{x}\,', \vec{v}\,') : \begin{cases} \vec{x}\,' = \vec{x} - \frac{1}{v^2}(1 - a(v))\vec{x} \cdot \vec{v}\,\vec{v} - \frac{c}{v}\sqrt{a(v)^2 - 1}\, t\vec{v} \\ t' = a(v)t - \frac{1}{cv}\sqrt{a(v)^2 - 1}\,\vec{x} \cdot \vec{v} \end{cases} \tag{24}$$

and we extend the GLT to $(t, \vec{x}, \{\vec{\beta}\})$ in terms of the previous one given above (24)

$$(t, \vec{x}, \{\vec{\beta}\}) \overset{L(a(v),\vec{v})}{\longrightarrow} (t', \vec{x}\,', \{\vec{\beta}\,'\}) := (t, \vec{x}, \vec{v}) \overset{L(a(v),\vec{v})}{\longrightarrow} (t', \vec{x}\,', \vec{v}\,'); \ \ \vec{v}\,' = -\vec{v} \tag{25}$$

where it is implicit that we have assumed the transformation $L(a(v), \vec{v})$ acts on $(t, \vec{x}, \{\vec{\beta}\})$ selecting $\vec{\beta} = \vec{v} = -\vec{\beta}'$. Since every coordinate system is subordinated to a reference frame, the variables t and t' appearing in equation (25) corresponds to a pair of frames having $\vec{v}$ as their relative velocity.

4 The Galilean and Lorentzian systems

In section 2 we introduced a coordinate system on a reference frame by means of a set of rules and clocks allowing us to register the ocurrence of events. Since we assumed two different ways for registering the time associated to the occurence of an event, using either the local time or the absolute time, we may expect to have two distinguished coordinate systems associated to the *same* reference frame, each one providing a convenient description for the physical world, which is proper to the view of the special relativity or to the Galilei relativity. In order to distinguish clearly the nature of these two reference systems we proceed to refine our notation as follows. Let S be an inertial reference frame.

A *Galilean coordinate system* on S is a coordinate system where the points of the physical world are described by

$$(x_G^0, x_G^i) := (\tau, \vec{x}) \ . \tag{26}$$

Given another reference frame S' endowed with a coordinate system $(x_G'^0, x_G'^i) := (\tau', \vec{x}\,')$ and moving with velocity $\vec{v}$ relative to S, the relation between the coordinate systems on S and S' is given by the Galilei transformation as referred to in axiom II

$$(x_G^0, x_G^i) \xrightarrow{G(\vec{v})} (x_G'^0, x_G'^i) : \begin{cases} \vec{x}\,'_G = \vec{x}_G - \vec{v}x_G^0 \\ x_G'^0 = x_G^0 \ . \end{cases} \tag{27}$$

A *Lorentzian coordinate system* on S is a coordinate system where the points of the physical world are described by

$$(x_L^0, x_L^i, \vec{v}) := (t, \vec{x}, \vec{v}) \tag{28}$$

where we have borrowed the same view given on (23) by assuming the time t depends on the relative state of motion between two observers. Given another frame $S\,'$ endowed with a coordinate system

$$(x_L'^0, x_L'^i, \vec{v}\,') := (t', \vec{x}\,', \vec{v}\,') \tag{29}$$

with $\vec{v}\,' = -\vec{v}$ (that means, S' is moving with velocity $\vec{v}$ relative to S) the relation between the coordinate systems on S and S' is given by a expression similar to the GLT of equation (24), which in our notation is written as

$$(x_L^0, x_L^i, \vec{v}) \xrightarrow{L(a(v),\vec{v})} (x_L'^0, x_L'^i, \vec{v}\,') : \begin{cases} x_L'^0 = a(v)x_L^0 - \frac{1}{vc}\sqrt{a(v)^2-1}\,\vec{x}_L\cdot\vec{v} \\ \vec{x}\,'_L = \vec{x}_L - \frac{1}{v^2}(1-a(v))\,\vec{x}_L\cdot\vec{v}\,\vec{v} - \frac{c}{v}\sqrt{a(v)^2-1}\,x_L^0\,\vec{v} \ . \end{cases} \tag{30}$$

Now, we notice that equation (7) expresses a relation between the absolute and the local times registered by an observer that uses a Galilean and a Lorentzian system of coordinates on the same frame S. We also have $\vec{x}_G = \vec{x}_L = \vec{x}$, therefore we define

$$(x^0_G, x^i_G) \xrightarrow{h_{\vec{v}}} (x^0_L, x^i_L, \vec{v}) : \begin{cases} x^0_L = \frac{v}{c\sqrt{a(v)^2-1}}\left(x^0_G - \frac{1}{v^2}(1-a(v))\vec{x}_G\cdot\vec{v}\right) \\ \vec{x}_L = \vec{x}_G \end{cases} \tag{31}$$

which represents the relation between the Galilean and the Lorentzian coordinate systems set on the same frame S.

Given another frame S' endowed respectively with Galilean and Lorentzian coordinates $(x'^0_G, x'^i_G) = (\tau', \vec{x}\,')$ and $(x'^0_L, x'^i_L, \vec{v}\,') = (t', \vec{x}\,', \vec{v}\,')$ there is a similar transformation

$$(x'^0_G, x'^i_G) \xrightarrow{h_{\vec{v}'}} (x'^0_L, x'^i_L, \vec{v}\,') : \begin{cases} x'^0_L = \frac{v'}{c\sqrt{a(v')^2-1}}\left(x'^0_G - \frac{1}{v'^2}(1-a(v'))\vec{x}\,'_G\cdot\vec{v}\,'\right) \\ \vec{x}\,'_L = \vec{x}\,'_G\ . \end{cases} \tag{32}$$

with $\vec{v}\,' = -\vec{v}$.

Now we observe that the maps $h_{\vec{v}}$ and $h_{\vec{v}'}$ together with $L(a(v), \vec{v})$ and $G(\vec{v})$ make the following diagram commutative

$$\begin{array}{ccc} (x^0_G, \vec{x}_G) & \xrightarrow{G(\vec{v})} & (x'^0_G, \vec{x}\,'_G) \\ \downarrow h_{\vec{v}} & & \downarrow h_{\vec{v}\,'} \\ (x^0_L, \vec{x}_L, \vec{v}) & \xrightarrow{L(a(v),\vec{v})} & (x'^0_L, \vec{x}\,'_L, \vec{v}\,') \end{array} \tag{33}$$

i.e. they satisfy

$$h_{\vec{v}'} \circ G(\vec{v}) = L(a(v), \vec{v}) \circ h_{\vec{v}}\ . \tag{34}$$

In fact, it is straightforward to check that

$$h_{\vec{v}'} \circ G(\vec{v})(x^0_G, \vec{x}_G) = (x'^0_L, \vec{x}\,'_L, \vec{v}\,') = L(a(v), \vec{v}) \circ h_{\vec{v}}(x^0_G, \vec{x}_G)$$

with

$$\begin{aligned} x'^0_L &= \frac{a(v_{SS'})}{\sqrt{a^2(v_{SS'})-1}}\frac{v_{SS'}}{c}\, x^0_G + \frac{(1-a(v_{SS'}))}{\sqrt{a^2(v_{SS'})-1}}\frac{1}{cv_{SS'}}\, \vec{x}_G\cdot\vec{v}_{SS'} \\ \vec{x}'_L &= \vec{x}_G - \vec{v}_{SS'}x^0_G\ . \end{aligned}$$

Finally, considering $h^{-1}_{\vec{v}}$ we obtain

$$L(a(v), \vec{v}) = h_{\vec{v}'} \circ G(\vec{v}) \circ h^{-1}_{\vec{v}} \tag{35}$$

and using the redefinition of $\vec{v}$ in terms of $\vec{\tilde{v}}$ given in (8) we obtain that

$$\mathcal{L}(\vec{\tilde{v}}) = h_{\vec{\tilde{v}}'} \circ G(\vec{\tilde{v}}) \circ h^{-1}_{\vec{\tilde{v}}}\ , \tag{36}$$

which shows how to generate the Lorentz transformation from the Galilei transformation. We must notice that $G(\vec{\tilde{v}}) := G(\vec{v}(\vec{\tilde{v}}))$ where $v(\tilde{v})$ is obtained from $a(v) = \frac{1}{\sqrt{1-\frac{\tilde{v}^2}{c^2}}}$.

Later, in section §6.4 we will show how $h_{\vec{v}}$ arises associated to a natural transformation between two functors, $\overline{G}$ and $\overline{L}$ that categorizes the Galilei and the Lorentz transformation.

5 Some category notions

We now introduce some of the concepts we will need in order to provide a categorical interpretation for the GLT and the Galilei transformation. Given a category $\mathcal{U}$ we denote by $\mathrm{Obj}_{\mathcal{U}}$ the class of objects of $\mathcal{U}$ and by $\mathrm{Morf}_{\mathcal{U}}$ we denote the class of morphisms of $\mathcal{U}$.

Let $\mathcal{S}$ and $\mathcal{C}$ be small categories. Given $S \in \mathrm{Obj}_{\mathcal{S}}$ we introduce the set

$$\mathrm{Morf}_{\mathcal{S}}(S, \cdot) := \{S^* \in \mathrm{Obj}_{\mathcal{S}} : \mathrm{Morf}_{\mathcal{S}}(S, S^*) \neq \emptyset\} \tag{37}$$

and we consider multivalued maps defined between the objects of $\mathcal{S}$ and $\mathcal{C}$ having the general form

$$\begin{aligned} F : \mathrm{Obj}_{\mathcal{S}} &\to \mathrm{Obj}_{\mathcal{C}} \\ S &\to F(S) \subset \mathrm{Obj}_{\mathcal{C}} \end{aligned} \tag{38}$$

with $F(S) \simeq \mathrm{Morf}_{\mathcal{S}}(S, \cdot)$ (isomorphism). Then, we assume there is defined a bijection associating to every $S^* \in \mathrm{Morf}_{\mathcal{S}}(S, \cdot)$ an element of $F(S)$ that we denote by C_{SS^*}.

Two elements $C, C^* \in \mathrm{Obj}_{\mathcal{C}}$ are said to be *F-equivalent*, denoted by $C \simeq C^*$, if there is $S \in \mathrm{Obj}_{\mathcal{S}}$ with $C, C^* \in F(S)$ or, in a similar way, if there are $S, S', S'' \in \mathrm{Obj}_C$ such that we identify $C \equiv C_{SS'}$ and $C^* \equiv C_{SS''}$. In this way, all elements of $F(S)$ are equivalent to each other.

We assume that for any pair of equivalent objects C, C^* there exists particular morphisms called *transition maps*, $k_{CC^*} : C \to C^*$ and $k_{C^*C} : C^* \to C$, such that $k_{CC^*}\, k_{C^*C} = I_C$ and $k_{C^*C}\, k_{CC^*} = I_{C^*}$, with I_C and I_{C^*} identities.

With this notion of equivalent objects and transition maps we can give the following prescription to compose morphisms whose domain and codomain though not equal are equivalent objects: whenever we have morphisms of the type $\chi : C \to C^*$, $\psi : C^{**} \to C^{***}$ with $C^* \simeq C^{**}$ we define the composition $\psi \circ \chi$ as

$$\psi \circ \chi := \psi\, k_{C^*C^{**}}\, \chi : C \to C^{***} \tag{39}$$

with the case $C^* = C^{**}$ corresponding to the usual composition of morphisms established in the standard definition of category.

Two morphisms $\chi : C \to C^*$ and $\psi : C^{**} \to C^{***}$ are said to be *F-equivalent*, denoted by $\chi \simeq \psi$, if $C \simeq C^{**}$, $C^* \simeq C^{***}$ and if under this equivalence the diagram below is commutative

$$\begin{array}{ccc} C & \xrightarrow{\ \chi\ } & C^* \\ {\scriptstyle k_{CC^{**}}}\downarrow & & \uparrow{\scriptstyle k_{C^{***}C^*}} \\ C^{**} & \xrightarrow{\ \psi\ } & C^{***}\,. \end{array} \tag{40}$$

We introduce another multivalued map, which by an abuse of notation we denote by the same letter F used in (38)

$$\begin{aligned} F : \mathrm{Morf}_{\mathcal{S}}(S, S') &\longrightarrow \mathrm{Morf}_{\mathcal{C}}(F(S), F(S')) \\ S \stackrel{f}{\to} S' &\longrightarrow F(S) \stackrel{F(f)}{\to} F(S') \end{aligned} \tag{41}$$

with $F(f)$ having for elements all maps that are equivalent to a certain morphism $C_S \stackrel{f^*}{\to} C_{S'}$ with $C_S \in F(S)$ and $C_{S'} \in F(S')$, i.e.

$$F(f) = \{k_{C_{S'}C'}\ f^*\ k_{CC_S} : C \in F(S), C' \in F(S')\}\ . \tag{42}$$

Since the elements of $F(f)$ are equivalent morphisms we can take any of them as representing the class. Then, we can compose $F(S') \stackrel{F(g)}{\to} F(S'')$ with $F(S) \stackrel{F(f)}{\to} F(S')$ by taking the composition of representative morphisms in $F(g)$ and $F(f)$, for instance, if $C_S \stackrel{f^*}{\to} C_{S'} \in F(S)$ and $B_{S'} \stackrel{g^*}{\to} B_{S''} \in F(g)$ then we can compose an arbitrary element $F(f) \ni k_{C_{S'}C'} f^* k_{CC_S} : C \to C'$ with another arbitrary element $F(g) \ni k_{B_{S''}B''} g^* k_{B'B_{S'}} : B' \to B''$ by means of the transition function $k_{C'B'}$

$$k_{B_{S''}B''} g^* k_{B'B_{S'}} k_{C'B'} k_{C_{S'}C'} f^* k_{CC_S} : C \to B''\ . \tag{43}$$

Now, we define a *covariant multivalued functor* between the categories $\mathcal{S}$ and $\mathcal{C}$ as a pair of maps like the ones described in (38, 41) with the property that for any two morphisms in $\mathcal{S}$ having the form $S \stackrel{f}{\to} S'$, $S' \stackrel{g}{\to} S''$ we have satisfied

$$F\big((S' \stackrel{g}{\to} S'') \circ (S \stackrel{f}{\to} S')\big) = F(S' \stackrel{g}{\to} S'') \circ F(S \stackrel{f}{\to} S')\ . \tag{44}$$

Given $K : \mathcal{S} \to \mathcal{C}$ a functor, and $F : \mathcal{S} \to \mathcal{C}$ a multivalued functor we define a natural transformation $\alpha : K \to F$ as a multivalued map
$\alpha : \mathrm{Obj}_{\mathcal{S}} \to \mathrm{Morf}_{\mathcal{C}}$
$\qquad S \to \alpha_S \in \mathrm{Morf}_{\mathcal{C}}(K(S), F(S))$
such that for any morphism $S \stackrel{f}{\to} S'$ the diagram below is commutative

$$\begin{array}{ccc} K(S) & \xrightarrow{K(f)} & K(S') \\ \downarrow{\scriptstyle \alpha_S} & & \downarrow{\scriptstyle \alpha_{S'}} \\ F(S) & \xrightarrow{F(f)} & F(S') \end{array}$$

i.e.

$$F(f)\alpha_S = \alpha_{S'} K(f)\ .$$

6 Relativity under a categorical perspective

Now, we will interpret the physical elements we have previously introduced using the language of categories. We do this in four stages.

6.1 The category of inertial reference frames

Here we establish the categorical equivalent of the physical notion of inertial reference frame.

6.6.1 Def.: The category of *inertial reference frames* is a small category $\mathcal{S} := (\mathrm{Obj}\,_{\mathcal{S}}, \mathrm{Morf}\,_{\mathcal{S}}, \circ)$ where

$\lrcorner$ $\mathrm{Obj}\,_{\mathcal{S}}$ is the set of *objects* of $\mathcal{S}$ and comprises all inertial reference frames $S, S', S'' \ldots$

$\lrcorner$ $\mathrm{Morf}\,_{\mathcal{S}}$ is the set of *morphisms* of $\mathcal{S}$ and it is written as $\mathrm{Morf}\,_{\mathcal{S}} = \cup_{S,S' \in \mathrm{Obj}\,_{\mathcal{S}}} \mathrm{Morf}\,_{\mathcal{S}}(S, S')$, with $\mathrm{Morf}\,_{\mathcal{S}}(S, S')$ being a set with only one element that we denote by $S \xrightarrow{\vec{v}_{SS'}} S'$ and we identify as the velocity of the frame S' relative to the frame S. For ease of notation we sometimes write the morphism $S \xrightarrow{\vec{v}_{SS'}} S'$ as $\vec{v}_{SS'}$.

$\lrcorner$ Composition $\circ$ is an operation, $\circ$: $\mathrm{Morf}\,_{\mathcal{S}}(S', S'') \times \mathrm{Morf}\,_{\mathcal{S}}(S, S') \rightarrow \mathrm{Morf}\,_{\mathcal{S}}(S, S'')$, defined by

$$\left(S' \xrightarrow{\vec{v}_{S'S''}} S''\right) \circ \left(S \xrightarrow{\vec{v}_{SS'}} S'\right) = \left(S \xrightarrow{\vec{v}_{SS''}} S''\right) \tag{45}$$

where we identify $\vec{v}_{SS''}$ in terms of the Galilei relativity law of velocities we have obtained in (17): $\vec{v}_{SS''} = \vec{v}_{SS'} + \vec{v}_{S'S''}$.

$\lrcorner$ To the object S we associate

$$S \xrightarrow{\vec{v}_{SS}} S \tag{46}$$

with $\vec{v}_{SS} = 0$.

We note that the composition is associative and the morphism $S \xrightarrow{\vec{v}_{SS}} S$ given in (46) behaves as the identity for the composition $\circ$, which shows that $\mathcal{C}$ is a category.

6.2 The category of coordinate systems

Now we wish to endow the inertial reference frames introduced in §6.1 with a quantitative way to describe the physical world allowing us to register events representing the occurrence of physical phenomena.

6.2.1 Def.: The category of *coordinate systems* is a small category $\mathcal{C} := (\mathrm{Obj}\,_{\mathcal{C}}, \mathrm{Morf}\,_{\mathcal{C}}, \circ)$ where

$\lrcorner$ $\mathrm{Obj}\,_{\mathcal{C}}$ is the set of all coordinate systems of the type $(x^0_G, \vec{x}_G)$, $(x'^0_G, \vec{x}\,'_G) \ldots$, or $(x^0_L, \vec{x}_L, \vec{v})$, $(x'^0_L, \vec{x}\,'_L, \vec{v}\,'), \ldots$, defined in (26, 28).

$\lrcorner$ $\mathrm{Morf}\,_{\mathcal{C}}$ is the set of morphisms of $\mathcal{C}$ and it comprises any coordinate transformations among inertial frames, in particular, the maps $G(\vec{v}), L(a(v), \vec{v})$ and $h_{\vec{v}}$ given in (27, 30, 31).

⌟ The composition $f \circ g$ of morphisms $f, g \in \mathrm{Morf}_{\mathcal{C}}$ is possible whenever we have domain $f =$ codomain g, for instance,

$$\big[(x'^0_G, \vec{x}\,'_G) \stackrel{G(\vec{v}\,')}{\longrightarrow} (x''^0_G, \vec{x}\,''_G)\big] \circ \big[(x^0_G, \vec{x}_G) \stackrel{G(\vec{v})}{\longrightarrow} (x'^0_G, \vec{x}\,'_G)\big] = (x^0_G, \vec{x}_G) \stackrel{G(\vec{v}\,'')}{\longrightarrow} (x''^0_G, \vec{x}\,''_G) \tag{47}$$

with $\vec{v}\,'' = \vec{v} + \vec{v}\,'$, and

$$\big[(x'^0_L, \vec{x}\,'_L, \vec{v}\,') \stackrel{L(a(v'),\vec{v}\,'=-\vec{v})}{\longrightarrow} (x^0_L, \vec{x}_L, \vec{v})\big] \circ \big[(x^0_L, \vec{x}_L, \vec{v}) \stackrel{L(a(v),\vec{v}=-\vec{v}\,')}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \vec{v}\,')\big] = Id_{(x^0_L, \vec{x}_L, \vec{v})} \ ,$$

and so on. Here, the composition is also associative and $\mathcal{C}$ is a category.

6.3 Introducing the Galilei and the Lorentz functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$

The category $\mathcal{C}$ introduced in §6.2 is too large for our purposes, therefore, in this third stage we define two functors that will distinguish within the category $\mathcal{C}$ two classes of coordinate systems and coordinate transformations that form the core of relativity: the Galilei transformation and the Lorentz transformation.

6.3.1 Def.: The Galilei functor
The *Galilei functor* $\overline{G} : \mathcal{S} \to \mathcal{C}$ is defined as follows.
⌟ $\mathrm{Obj}_{\mathcal{S}} \stackrel{\overline{G}}{\to} \mathrm{Obj}_{\mathcal{C}}$

$$S \stackrel{\overline{G}}{\to} \overline{G}(S) := (x^0_G, \vec{x}_G)$$

⌟ $\mathrm{Morf}_{\mathcal{S}}(S, S') \stackrel{\overline{G}}{\longrightarrow} \mathrm{Morf}_{\mathcal{C}}(\overline{G}(S), \overline{G}(S'))$

$$S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S' \stackrel{\overline{G}}{\to} \overline{G}(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S') := (x^0_G, \vec{x}_G) \stackrel{G(\vec{v}_{SS'})}{\longrightarrow} (x'^0_G, \vec{x}\,'_G)$$

with $G(\vec{v}_{SS'})$ given by (27).

We notice that $\overline{G}$ is a covariant functor from

$$\begin{aligned} \overline{G}\Big(\big(S' \stackrel{\vec{v}_{S'S''}}{\longrightarrow} S''\big) \circ \big(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S'\big)\Big) &\stackrel{(45)}{=} \overline{G}(S \stackrel{\vec{v}_{SS''}}{\longrightarrow} S'') \\ &= (x^0_G, \vec{x}_G) \stackrel{G(\vec{v}_{SS''})}{\longrightarrow} (x''^0_G, \vec{x}\,''_G) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \overline{G}(S' \stackrel{\vec{v}_{S'S''}}{\longrightarrow} S'') \circ \overline{G}(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S') &= \big[\overline{G}(S') \stackrel{\overline{G}(\vec{v}_{S'S''})}{\longrightarrow} \overline{G}(S'')\big] \circ \big[\overline{G}(S) \stackrel{\overline{G}(v_{SS'})}{\longrightarrow} \overline{G}(S')\big] \\ &= \big[(x'^0_G, \vec{x}\,'_G) \stackrel{G(\vec{v}_{S'S''})}{\longrightarrow} (x''^0_G, \vec{x}\,''_G)\big] \circ \big[(x^0_G, \vec{x}_G) \stackrel{G(\vec{v}_{SS'})}{\longrightarrow} (x'^0_G, \vec{x}\,'_G)\big] \\ &\stackrel{(47)}{=} (x^0_G, \vec{x}_G) \stackrel{G(\vec{v}_{SS''})}{\longrightarrow} (x''^0_G, \vec{x}\,''_G) \end{aligned} \tag{49}$$

as the equality of the right hand side of (48) and (49) gives

$$\overline{G}\Big(\big(S' \stackrel{\vec{v}_{S'S''}}{\longrightarrow} S''\big) \circ \big(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S'\big)\Big) = \overline{G}(S' \stackrel{\vec{v}_{S'S''}}{\longrightarrow} S'') \circ \overline{G}(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S') \ .$$

Physically, the role of the Galilei functor $\overline{G}$ is to discriminate in $\mathcal{C}$ a class of coordinate systems where the events are registered as points $(\tau, \vec{x}) \in \mathrm{Obj}_{\mathcal{C}}$, with τ being the absolute time, together with a class of morphisms that constitute the Galilei transformations $(\tau, \vec{x}) \stackrel{\overline{G}(\vec{v}_{SS'})}{\longrightarrow} (\tau', \vec{x}\,')$.

6.3.2. Def.: The Lorentz functor

The *Lorentz functor* $\overline{L} : \mathcal{S} \to \mathcal{C}$ is a multivalued functor defined in the sense of the definition established in (38, 41) as follows.

$\lrcorner$ $\overline{L}$ acts on objects as

$$\begin{aligned} &\mathrm{Obj}_{\mathcal{S}} \stackrel{\overline{L}}{\to} \mathrm{Obj}_{\mathcal{C}} \\ &\quad S \stackrel{\overline{L}}{\to} \overline{L}(S) \subset \mathrm{Obj}_{\mathcal{C}} \\ &\qquad \overline{L}(S) \equiv (x^0_L, \vec{x}_L, \{\vec{\beta}\}) := \{(t, \vec{x}, \vec{\beta}) : \vec{\beta} \in \mathrm{Morf}_{\mathcal{S}}(S, \cdot)\} \end{aligned} \tag{50}$$

$\lrcorner$ $\overline{L}$ acts on morphisms as

$\mathrm{Morf}_{\mathcal{S}}(S, S') \stackrel{\overline{L}}{\to} \mathrm{Morf}_{\mathcal{C}}(\overline{L}(S), \overline{L}(S'))$

$S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S' \stackrel{\overline{L}}{\to} \overline{L}(S \stackrel{\vec{v}_{SS'}}{\longrightarrow} S') \equiv (x^0_L, \vec{x}_L, \{\vec{\beta}\}) \stackrel{\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \{\vec{\beta}\,'\})$

where $\overline{L}(\vec{v}_{SS'})$ is the same map defined in (25), e.g.

$$(x^0_L, \vec{x}_L, \{\vec{\beta}\}) \stackrel{\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \{\vec{\beta}\,'\}) := (x^0_L, \vec{x}_L, \vec{v}_{SS'}) \stackrel{L(a(v_{SS'}), \vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S}) \,. \tag{51}$$

Recall from section 5 that two objects $(x^0_L, \vec{x}_L, \vec{v})$, $(x'^0_L, \vec{x}\,'_L, \vec{v}\,') \in \mathrm{Obj}_{\mathcal{C}}$ are $\overline{L}$-*equivalent* if there is a common frame S such that $(x^0_L, \vec{x}_L, \vec{v}), (x'^0_L, \vec{x}\,'_L, \vec{v}\,') \in \overline{L}(S)$. Here, from the structure of $\overline{L}(S)$ given in (50) we must have $x'^0_L = x^0_L$, $\vec{x}\,'_L = \vec{x}_L$ and two other frames S_1, S_2 such that $\vec{v} = \vec{v}_{SS_1}$, $\vec{v}\,' = \vec{v}_{SS_2}$. Identifying $C \equiv (x^0_L, \vec{x}_L, \vec{v})$ and $C^* \equiv (x^0_L, \vec{x}_L, \vec{v}\,')$ and considering the map given in (31) we define the transition map k_{CC^*} giving the equivalence of objects as

$$k_{CC^*} \equiv h_{\vec{v}_{SS_2}} h^{-1}_{\vec{v}_{SS_1}} \tag{52}$$

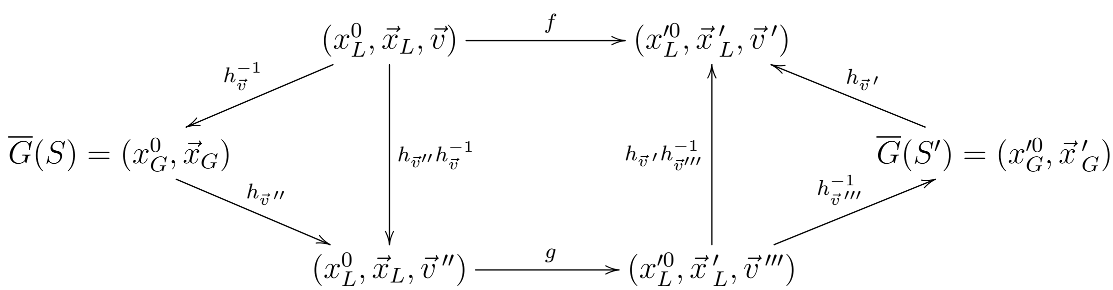
as it is indicated in the diagram below

$$\begin{array}{ccc} & & (x^0_L, \vec{x}_L, \vec{v} = \vec{v}_{SS_1}) \\ & \swarrow^{h^{-1}_{\vec{v}_{SS_1}}} & \downarrow{\scriptstyle h_{\vec{v}_{SS_2}} h^{-1}_{\vec{v}_{SS_1}}} \\ \overline{G}(S) = (x^0_G, \vec{x}_G) & & \\ & \searrow_{h_{\vec{v}_{SS_2}}} & \\ & & (x^0_L, \vec{x}_L, \vec{v}\,' = \vec{v}_{SS_2}) \,. \end{array} \tag{53}$$

According to the definition of equivalent morphisms given in (40), we say that two morphisms f and g

$$(x^0_L, \vec{x}_L, \vec{v}) \xrightarrow{f} (x'^0_L, \vec{x}\,'_L, \vec{v}\,') \in \mathrm{Morf}_{\mathcal{C}}\big((x^0_L, \vec{x}_L, \vec{v}), (x'^0_L, \vec{x}\,'_L, \vec{v}\,')\big)$$
$$(x^0_L, \vec{x}_L, \vec{v}\,'') \xrightarrow{g} (x'^0_L, \vec{x}\,'_L, \vec{v}\,''') \in \mathrm{Morf}_{\mathcal{C}}\big((x^0_L, \vec{x}_L, \vec{v}\,''), (x'^0_L, \vec{x}\,'_L, \vec{v}\,''')\big)$$

are $\overline{L}$-*equivalent* if there are frames S, S' such that the square diagram below is commutative



(54)

i.e.

$$f = h_{\vec{v}\,'} h^{-1}_{\vec{v}\,'''}\, g\, h_{\vec{v}\,''} h^{-1}_{\vec{v}} \ .$$

Now, let us analyze how to compose morphisms under $\overline{L}$. Let us consider the maps

$$(x^0_L, \vec{x}_L, \{\vec{\beta}\}) \overset{\overline{L}(\vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \{\vec{\beta}\,'\}) \equiv (x^0_L, \vec{x}_L, \vec{v}_{SS'}) \overset{L(a(v_{SS'}), \vec{v}_{SS'})}{\longrightarrow} (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$$
$$(x'^0_L, \vec{x}\,'_L, \{\vec{\beta}\,'\}) \overset{\overline{L}(\vec{v}_{S'S''})}{\longrightarrow} (x''^0_L, \vec{x}\,''_L, \{\vec{\beta}\,''\}) \equiv (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S''}) \overset{L(a(v_{S'S''}), \vec{v}_{S'S''})}{\longrightarrow} (x''^0_L, \vec{x}\,''_L, \vec{v}_{S''S'}) \ .$$

Here, the codomain of $L(a(v_{SS'}), \vec{v}_{SS'})$ is $(x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$ and the domain of $L(a(v_{S'S''}), \vec{v}_{S'S''})$ is $(x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S''})$. As we have seen from (39), if the velocities are different we cannot compose the maps $L(a(v_{SS'}), \vec{v}_{SS'})$, $L(a(v_{S'S''}), \vec{v}_{S'S''})$ directly, since $x'^0_L(\vec{v}_{S'S})$ and $x'^0_L(\vec{v}_{S'S''})$ are not the same. Then, we must use apropriate transition functions in order to compose them, in this case, if we identify $C^* = (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S})$ and $C^{**} = (x'^0_L, \vec{x}\,'_L, \vec{v}_{S'S''})$ we write from (52) that $k_{C^*C^{**}} = h_{\vec{v}_{S'S''}} h^{-1}_{\vec{v}_{S'S}}$ and following (39) the composition $\overline{L}(\vec{v}_{S'S''}) \circ \overline{L}(\vec{v}_{SS'})$ is defined by

$$\overline{L}(\vec{v}_{S'S''}) \circ \overline{L}(\vec{v}_{SS'}) := L(a(v_{S'S''}), \vec{v}_{S'S''}) h_{\vec{v}_{S'S''}} h^{-1}_{\vec{v}_{S'S}} L(a(v_{SS'}), \vec{v}_{SS'}) \ . \qquad (55)$$

Explicitly we have

$$\begin{aligned}
x''^0_L &= \frac{\sqrt{a^2(v_{SS'})-1}}{\sqrt{a^2(v_{S'S''})-1}}\left\{a(v_{S'S''})\frac{v_{S'S''}}{v_{SS'}} - \big(1-a(v_{S'S''})\big)\frac{\vec{v}_{SS'}\cdot\vec{v}_{S'S''}}{v_{SS'}v_{S'S''}}\right\}x^0_L + \\
&+\frac{\big(1-a(v_{SS'})\big)}{\sqrt{a^2(v_{S'S''})-1}}\left\{a(v_{S'S''})\frac{v_{S'S''}}{cv^2_{SS'}} - \big(1-a(v_{S'S''})\big)\frac{\vec{v}_{SS'}\cdot\vec{v}_{S'S''}}{cv_{S'S''}v^2_{SS'}}\right\}\vec{x}_L\cdot\vec{v}_{SS'} + \\
&+\frac{\big(1-a(v_{S'S''})\big)}{\sqrt{a^2(v_{S'S''})-1}}\,\frac{1}{cv_{S'S''}}\,\vec{x}_L\cdot\vec{v}_{S'S''} \qquad (56)\\
\vec{x}\,''_L &= \vec{x}_L - \sqrt{a^2(v_{SS'})-1}\,\frac{c}{v_{SS'}}\,x^0_L(\vec{v}_{SS'}+\vec{v}_{S'S''}) + \\
&-\big(1-a(v_{SS'})\big)\frac{1}{v^2_{SS'}}\vec{x}_L\cdot\vec{v}_{SS'}(\vec{v}_{SS'}+\vec{v}_{S'S''})\ .
\end{aligned}$$

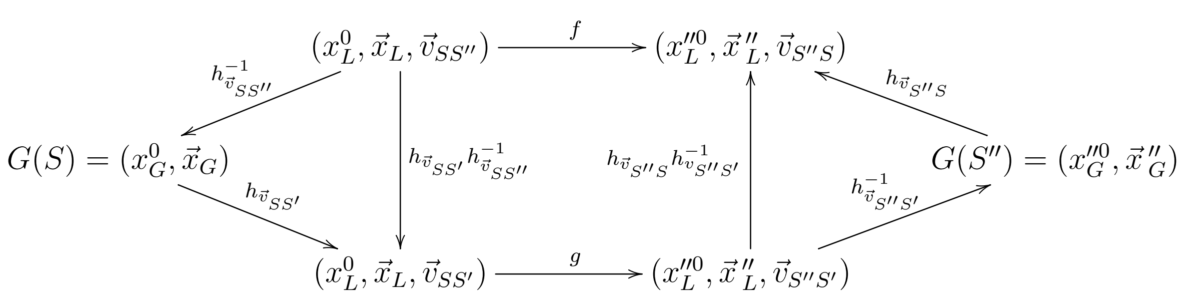
Having defined the above prescription on how to compose maps, let us verify if the covariant transformation law given in (44) is true. From (45) we have that

$$\begin{aligned}
\overline{L}\Big(\big(S'\xrightarrow{\vec{v}_{S'S''}}S''\big)\circ\big(S\xrightarrow{\vec{v}_{SS'}}S'\big)\Big) &= \overline{L}(S\xrightarrow{\vec{v}_{SS''}}S'') \\
&= (x^0_L,\vec{x}_L,\{\vec{\beta}\})\xrightarrow{\overline{L}(\vec{v}_{SS''})}(x''^0_L,\vec{x}\,''_L,\{\vec{\beta}\,''\}) \\
&= (x^0_L,\vec{x}_L,\vec{v}_{SS''})\xrightarrow{L(a(v_{SS''}),\vec{v}_{SS''})}(x''^0_L,\vec{x}\,''_L,\vec{v}_{S''S})\ . \qquad (57)
\end{aligned}$$

We have also seen from (55) that

$$\begin{aligned}
&\overline{L}\big(S'\xrightarrow{\vec{v}_{S'S''}}S''\big)\circ\overline{L}\big(S\xrightarrow{\vec{v}_{SS'}}S'\big) = \\
&= \big((x'^0_L,\vec{x}\,'_L,\{\vec{\beta}\,'\})\xrightarrow{\overline{L}(\vec{v}_{S'S''})}(x''^0_L,\vec{x}\,''_L,\{\vec{\beta}\,''\})\big)\circ\big((x^0_L,\vec{x}_L,\{\vec{\beta}\})\xrightarrow{\overline{L}(\vec{v}_{SS'})}(x'^0_L,\vec{x}\,'_L,\{\vec{\beta}\,'\})\big) \\
&= (x^0_L,\vec{x}_L,\vec{v}_{SS'})\xrightarrow{L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'})}(x''^0_L,\vec{x}\,''_L,\vec{v}_{S''S'})\ . \qquad (58)
\end{aligned}$$

The morphisms given in (57), (58) cannot be directly compared since the domain/codomain of one differs from the domain/codomain of the other. However, we have a similar diagram as the one shown in (54)



with $f = L(a(v_{SS''}),\vec{v}_{SS''})$ and $g = L(a(v_{S'S''}),\vec{v}_{S'S''})h_{\vec{v}_{S'S''}}h^{-1}_{\vec{v}_{S'S}}L(a(v_{SS'}),\vec{v}_{SS'})$, which indicates that the maps given on the rhs of equations (57) and (58) are equivalent, then we have

$$\overline{L}\Big(\big(S'\xrightarrow{\vec{v}_{S'S''}}S''\big)\circ\big(S\xrightarrow{\vec{v}_{SS'}}S'\big)\Big) = \overline{L}\big(S'\xrightarrow{\vec{v}_{S'S''}}S''\big)\circ\overline{L}\big(S\xrightarrow{\vec{v}_{SS'}}S'\big)$$

and we conclude that $\overline{L}$ is a covariant multivalued functor in the sense we have defined in (44).

6.4 Unifying the Galilei and the Lorentz transformation through a natural transformation

In this fourth stage we give a precise meaning for the maps $h_{\vec{v}}$ introduced in (31). Given the functors $\overline{G}, \overline{L} : \mathcal{S} \to \mathcal{C}$ we define a map

$$\begin{aligned} h : \mathrm{Obj}_{\mathcal{S}} &\to \mathrm{Morf}_{\mathcal{C}} \\ S &\to h_S \in \mathrm{Morf}_{\mathcal{C}}(\overline{G}(S), \overline{L}(S)) \end{aligned}$$

with

$$(x^0_G, \vec{x}_G) \stackrel{h_S}{\to} (x^0_L, \vec{x}_L, \{\vec{\beta}\}) := \left\{(x^0_G, \vec{x}_G) \stackrel{h_{\vec{v}_{SS'}}}{\longrightarrow} (x^0_L, \vec{x}_L, \vec{v}_{SS'}) : S' \in \mathrm{Morf}_{\mathcal{S}}(S, \cdot)\right\} \quad (59)$$

or, in a compact form, we simply write $h_S = \{h_{\vec{v}_{SS'}} : S' \in \mathrm{Morf}_{\mathcal{S}}(S, \cdot)\}$. Since $h_S \in \mathrm{Morf}_{\mathcal{C}}\big(\overline{G}(S), \overline{L}(S)\big)$ is class of maps we prescribe that $\overline{L}(\vec{v}) \circ h_S = \overline{L}(\vec{v}) \circ h_{\vec{v}}$, and $h_{S'} \circ \overline{G}(\vec{v}) := h_{\vec{v}\,'} \circ \overline{G}(\vec{v})$ with $\vec{v}\,' = -\vec{v}$. Having set these prescriptions it is straightforward to check that the diagram below is commutative

$$\begin{array}{ccc} \overline{G}(S) = (x^0_G, \vec{x}_G) & \xrightarrow{\overline{G}(\vec{v}_{SS'})} & \overline{G}(S') = (x'^0_G, \vec{x}\,'_G) \\ \Big\downarrow{\scriptstyle h_S} & & \Big\downarrow{\scriptstyle h_{S'}} \\ \overline{L}(S) = (x^0_L, \vec{x}_L, \{\vec{\beta}\}) & \xrightarrow{\overline{L}(\vec{v}_{SS'})} & \overline{L}(S') = (x'^0_L, \vec{x}\,'_L, \{\vec{\beta}'\}) \end{array} \quad (60)$$

In fact, with the previous prescription on composing h_S and $h_{S'}$ we obtain that

$$\begin{aligned} h_{S'} \circ \overline{G}(\vec{v}_{SS'}) &= h_{\vec{v}_{S'S}} \circ G(\vec{v}_{SS'}) \\ \overline{L}(\vec{v}_{SS'}) \circ h_S &= L(a(v_{SS'}), \vec{v}_{SS'}) \circ h_{\vec{v}_{SS'}} \end{aligned}$$

and since $\vec{v}_{S'S} = -\vec{v}_{SS'}$ we obtain from (34) that

$$h_{S'} \circ \overline{G}(\vec{v}_{SS'}) = \overline{L}(\vec{v}_{SS'}) \circ h_S$$

which proves the commutativity of the diagram (60) and then that $h_S : \overline{G} \to \overline{L}$ is a natural transformation.

7 Conclusion

In our work we provided an interpretation for the relation (36)

$$\mathcal{L}(\vec{\tilde{v}}) = h_{-\vec{\tilde{v}}} \circ G(\vec{\tilde{v}}) \circ h_{\vec{\tilde{v}}}^{-1}$$

that shows how the Lorentz transformation decomposes in terms of the Galilei transformation through maps $h_{\vec{v}}$ given in (31). This decomposition presents a new aspect between the Galilei and the Lorentz transformation that, as far as we know, we have not seen in the literature. The map $h_{\vec{v}}$ is built essentialy in terms of a relation involving the absolute and the local time (7), but it goes far beyond this as $h_{\vec{v}}$ also relates two coordinate systems for the *same* frame S, and this was the key element for us to interpret $h : \overline{G} \to \overline{L}$ as a natural transformation of the type $h : \mathrm{Obj}_{\mathcal{S}} \to \mathrm{Morf}_{\mathcal{C}}$.

Now, this categorical framework for unifying the Galilei relativity and the special relativity offers a new perspective for investigation. In fact, the transition from the special relativity to the general relativity is performed as a shift from the infinitesimal line element $ds^2 \equiv \eta_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - d\vec{x}^2$ to the more general one $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Then, introducing this line element as a new data in our physical world it suggest us to search for what new structure could be endowed to the Galilei relativity in order to obtain a model for a general Galilei relativity as indicated in the diagram below

$$\begin{array}{ccc} \text{Galilei relativity} & \xrightarrow{\quad ? \quad} & \text{“general Galilei relativity”} \\ \big\downarrow & & \big\downarrow {\scriptstyle ?} \\ \text{special relativity}, ds_L^2 = \eta_{\mu\nu} dx_L^\mu dx_L^\nu & \longrightarrow & \text{general relativity}, ds^2 = g_{\mu\nu} dx^\mu dx^\nu \end{array}$$

Here, if we think on the level of ds^2 we would write for the Galilei relativity $ds_G^2 = h_{mn} dx_G^m dx_G^n$ with

$$h_{mn} = -g_{mn} + \frac{g_{0m} g_{0n}}{g_{00}} .$$

Here, our main idea is to think on what could be a suitable ds^2 for a “general Galilei relativity” model, built on the basis of making the unknow arrows in the above diagram satisfy the same construction as seen in the diagram (1). Perhaps, the right approach may be using instead of ds^2 the invariant structure given by the relation bewteen the absolute and the local time (7)

$$\tau = (1 - a(v)) \frac{\vec{x} \cdot \vec{v}}{v^2} + \sqrt{a(v)^2 - 1}\, \frac{c}{v}\, t = (1 - a(v')) \frac{\vec{x}\,' \cdot \vec{v}\,'}{v'^2} + \sqrt{a(v')^2 - 1}\, \frac{c}{v'}\, t' = \tau'$$

that in the general Galilei relativity would lead us to search for a more general relation $\tau = \tau(t, \vec{x})$. In any situation, it seems this categorical way of thinking provide us with new insigths towards the generalizaton of some current models.

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