Some Integral Representations for the Constant Pi

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abstract

In this note we show some integrals to the constant Pi

Algunas Representaciones Integrales Para La Constante Pi

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Resumen-Abstract

En este Preprint se muestran algunas fórmulas integrales para la constante Pi:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = 3.14159265358979323846264338\ldots$$

Palabras clave: Número Pi , Integrales , Series

Introducción

■ Master Fórmula

Recordamos una fórmula integral para la constante Pi:

Teorema 1.

$$\pi = 2^n s_n \int_0^\infty \frac{2}{2\,\mathrm{Cosh}[x] + c_n}\,dx\,,\quad n = 1, 2, 3, \ldots \tag{1}$$

donde s_n y c_n se definen como sigue:

$$s_1 = \sqrt{2}\,,\; s_2 = \sqrt{2-\sqrt{2}}\,,\; s_3 = \sqrt{2-\sqrt{2+\sqrt{2}}}\,,\;\; ..\text{etc} \tag{2}$$

En general se tiene:

$$s_n = \sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2\,\mathrm{Sin}\left[\frac{\pi}{2^{n+1}}\right],\; n-\text{radicales}\,,\; n = 1, 2, 3, \ldots \tag{3}$$

$$c_1 = \sqrt{2}\,,\; c_2 = \sqrt{2+\sqrt{2}}\,,\; c_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}\,,\;\; \ldots\text{etc.} \tag{4}$$

En general se tiene:

$$c_n = \sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} = 2\,\mathrm{Cos}\left[\frac{\pi}{2^{n+1}}\right],\; n-\text{radicales}\,,\; n = 1, 2, 3, \ldots \tag{5}$$

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La demostración del Teorema 1, se sigue de la evaluación de la integral:

$$\int_0^\infty \frac{1}{\mathrm{Cosh}[x] + a}\,dx = \frac{2}{\sqrt{1-a^2}}\,\mathrm{ArcTan}\left[\sqrt{\frac{1-a}{1+a}}\right],\; -1 < a < 1 \tag{6}$$

los detalles quedan como ejercicio.

La fórmula (1), tiene varias representaciones equivalentes. A continuación mostramos algunas.

Algunas Representaciones Equivalentes a la fórmula (1)

Teorema 2.

Para $n = 1, 2, 3, \ldots$, se tiene :

$$\pi = 2^n s_n \int_0^\infty \frac{2 e^{-x}}{1 + c_n e^{-x} + e^{-2x}} \, dx \tag{7}$$

$$\pi = 2^n s_n \int_0^1 \frac{2}{1 + c_n x + x^2} \, dx \tag{8}$$

$$\pi = 2^n s_n \int_{-\infty}^\infty \frac{2 e^{x-e^x}}{1 + c_n e^{-e^x} + e^{-2e^x}} \, dx \tag{9}$$

$$\pi = 2^n s_n \int_1^\infty \frac{2}{1 + c_n x + x^2} \, dx \tag{10}$$

$$\pi = 2^n s_n \int_0^{\frac{1}{2+c_n}} 2 \operatorname{ArcCosh}\left[\frac{1}{2x} - \frac{c_n}{2}\right] dx \tag{11}$$

$$\pi = 2^n s_n \int_1^\infty \frac{2}{(2x + c_n)\sqrt{x^2 - 1}} \, dx \tag{12}$$

$$\pi = 2^n s_n \int_{2+c_n}^\infty \frac{2}{x\sqrt{(x - c_n - 2)(x - c_n + 2)}} \, dx \tag{13}$$

$$\pi = 2^n s_n \int_0^{\frac{2}{2+c_n}} \operatorname{ArcCosh}\left[\frac{1}{x} - \frac{c_n}{2}\right] dx \tag{14}$$

$$\pi = 2^n s_n \int_{-1}^1 \frac{4}{5 + c_n + (2 + c_n) x + x^2} \, dx \tag{15}$$

$$\pi = 2^n s_n \left(1 + \int_0^1 \frac{1 - c_n x + x^2}{1 + c_n x + x^2} \, dx\right) \tag{16}$$

$$\pi = 2^n s_n \left(1 - \int_1^\infty \frac{1 + c_n x - x^2}{(1 + c_n x + x^2) x^2} \, dx\right) \tag{17}$$

$$\pi = 2^n s_n \left(1 + \int_0^\infty \frac{1 - c_n e^{-x} - e^{-2x}}{1 + c_n e^{-x} + e^{-2x}} e^{-x} \, dx\right) \tag{18}$$

$$\pi = 2^n s_n \left(1 - c_n + \frac{1}{2} \int_{\frac{2}{2+c_n}}^2 \sqrt{\frac{8}{x} + c_n^2 - 4} \, dx\right) \tag{19}$$

$$\pi = 2^n s_n \int_0^1 \frac{2}{(2 + c_n x)\sqrt{1 - x^2}} \, dx \tag{20}$$

$$\pi = 2^n s_n \int_0^{c_n} \frac{2}{(2 + x)\sqrt{c_n^2 - x^2}} \, dx \tag{21}$$

$$\pi = 2^n s_n \int_{c_n}^{2+c_n} \frac{4}{x^2 + 4 - c_n^2} \, dx \tag{22}$$

$$\pi = 2^n s_n \left(1 + \int_{\frac{c_n}{2}}^{1+\frac{c_n}{2}} \frac{4 + c_n^2 - 4 x^2}{4 - c_n^2 + 4 x^2} \, dx\right) \tag{23}$$

$$\pi = 2^n s_n \int_0^1 \left(\sum_{k=0}^{\infty} \frac{2}{2 \operatorname{Cosh}[x + k] + c_n}\right) dx \tag{24}$$

$$\pi = 2^n s_n \int_0^1 \left(\sum_{k=1}^{\infty} \frac{2}{1 + c_n k + k^2 + (2 k + c_n) x + x^2}\right) dx \tag{25}$$

$$\pi = 2^n s_n \int_{2+c_n}^{\infty} \frac{4}{x^2 + 4 - c_n^2} \, dx \tag{26}$$

$$\pi = 2^n s_n \int_{c_n}^{\infty} \frac{2}{x^2 + 4 - c_n^2} \, dx \tag{27}$$

$$\pi = 2^n s_n \int_1^{\infty} \frac{2 c_n}{4 + c_n^2 (x^2 - 1)} \, dx \tag{28}$$

$$\pi = 2^n s_n \int_0^{\infty} \frac{2 c_n \operatorname{Sinh}[x]}{4 + c_n^2 (\operatorname{Sinh}[x])^2} \, dx \tag{29}$$

$$\pi = 2^n s_n \int_0^{\infty} \frac{2 x}{(4 + x^2) \sqrt{c_n^2 + x^2}} \, dx \tag{30}$$

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Demostración. Fórmula (1) y Algebra de Integrales.

Series Relacionadas

Teorema 3.

Para $n = 1, 2, 3, \ldots,$ se tiene :

$$\pi = 2^n s_n \left\{\frac{4}{3 + c_n} \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{r=0}^{m} \frac{(-1)^m}{m + r + 1} \binom{k}{m} \binom{m}{r} \left(\frac{1 + c_n}{3 + c_n}\right)^{k-m} \left(\frac{2 c_n}{3 + c_n}\right)^m \left(\frac{1}{c_n}\right)^r\right\} \tag{31}$$

$$\pi = 2^n s_n \left\{2 \sum_{k=0}^{\infty} \frac{(-1)^k (1 + c_n)^{-k-1}}{2 k + 1} \operatorname{F}\left(k + 1, 1, 2 k + 2, \frac{c_n}{1 + c_n}\right)\right\} \tag{32}$$

$$\pi = 2^n s_n \left\{2 \sum_{k=0}^{\infty} \frac{(-1)^k (1 + c_n)^{-2 k-1}}{2 k + 1} \operatorname{F}\left(2 k + 1, k + 1, 2 k + 2, \frac{c_n}{1 + c_n}\right)\right\} \tag{33}$$

$$\pi = 2^n s_n \left\{\frac{2 \operatorname{Log}[1 + c_n]}{c_n} + 2 \sum_{k=1}^{\infty} \left(\binom{2 k}{k} \operatorname{Log}[1 + c_n] + \sum_{m=0, m \neq k}^{2 k} \frac{(-1)^{k+m} \left((1 + c_n)^{m-k} - 1\right)}{m - k} \binom{2 k}{m}\right)\right\} \tag{34}$$

$$\pi = 2^n s_n \left\{2 \sum_{k=0}^{\infty} \frac{a_k}{k + 1}\right\} \tag{35}$$

$$\pi = 2^n s_n \left\{2 \sum_{k=0}^{\infty} \frac{a_k y^{k+1}}{k + 1} + \int_y^1 \frac{2}{1 + c_n x + x^2} \, dx\right\}, \quad 0 < y < 1 \tag{36}$$

En las fórmulas (32) y (33) , F(a, b, c, z) es la función hypergeometrica de Gauss , en las fórmulas

(35) y (36) , los números a_k se definen por la recurrencia:

$$a_{k+2} = -c_n\, a_{k+1} - a_k\,,\;\; a_0 = 1,\; a_1 = -c_n,\;\; k = 0,\, 1,\, 2,\, 3,\; \ldots \tag{37}$$

$$\pi = 2^n\, s_n \left\{ \frac{2}{c_n} \sum_{k=0}^{\infty} \frac{(-1)^k}{2\,k+1} \left(\frac{4}{c_n^2} - 1 \right)^k \right\} \tag{38}$$

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Serie de Arcotangentes

Teorema 4.

Para $n = 1,\, 2,\, 3,\; \ldots,$ se tiene :

$$\pi = 2^{n+1} \sum_{k=1}^{\infty} \mathrm{ArcTan}\left[\frac{t_n}{k(k+1) + t_n^2}\right] \tag{39}$$

donde

$$t_n = \frac{s_n}{c_n} = \mathrm{Tan}\left[\frac{\pi}{2^{n+1}}\right] \tag{40}$$

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Otras Fórmulas

$$\pi = \int_0^{\infty} \frac{6}{2\,\mathrm{Cosh}[x] + \sqrt{3}}\, dx \tag{41}$$

$$\pi = \int_0^{\infty} \frac{4\sqrt{2}}{2\,\mathrm{Cosh}[x] + \sqrt{2}}\, dx \tag{42}$$

$$\pi = \int_0^{\infty} \frac{3\sqrt{3}}{2\,\mathrm{Cosh}[x] + 1}\, dx \tag{43}$$

$$\pi = 6 \sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\sqrt{3}\right)^{-k-1}}{2\,k+1}\, \mathrm{F}\left(k+1,\, 1,\, 2\,k+2,\, \frac{\sqrt{3}}{1+\sqrt{3}}\right) \tag{44}$$

$$\pi = 6 \sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\sqrt{3}\right)^{-2\,k-1}}{2\,k+1}\, \mathrm{F}\left(2\,k+1,\, k+1,\, 2\,k+2,\, \frac{\sqrt{3}}{1+\sqrt{3}}\right) \tag{45}$$

$$\pi = 4\sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\sqrt{2}\right)^{-k-1}}{2\,k+1}\, \mathrm{F}\left(k+1,\, 1,\, 2\,k+2,\, \frac{\sqrt{2}}{1+\sqrt{2}}\right) \tag{46}$$

$$\pi = 4\sqrt{2} \sum_{k=0}^{\infty} \frac{(-1)^k \left(1+\sqrt{2}\right)^{-2\,k-1}}{2\,k+1}\, \mathrm{F}\left(2\,k+1,\, k+1,\, 2\,k+2,\, \frac{\sqrt{2}}{1+\sqrt{2}}\right) \tag{47}$$

$$\pi = 3\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^k\, 2^{-k-1}}{2k+1}\,\mathrm{F}\left(k+1,\,1,\,2k+2,\,\frac{1}{2}\right) \tag{48}$$

$$\pi = 3\sqrt{3}\sum_{k=0}^{\infty}\frac{(-1)^k\, 2^{-2k-1}}{2k+1}\,\mathrm{F}\left(2k+1,\,k+1,\,2k+2,\,\frac{1}{2}\right) \tag{49}$$

$$\pi = 3\sqrt{3}\left\{\int_0^y \frac{1}{2\,\mathrm{Cosh}[x]+1}\,dx + \sum_{k=0}^{\infty}\left(\frac{e^{-(3k+1)y}}{3k+1} - \frac{e^{-(3k+2)y}}{3k+2}\right)\right\},\quad y \ge 0 \tag{50}$$

Nota. Las fórmulas se han tomado de la referencia E.

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